Existence of global Solutions to the 1D Abstract Bubble Vibration Model

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GLOBAL SOLUTIONS
TO THE 1D ABSTRACT BUBBLE VIBRATION MODEL

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Abstract

The *Abstract Bubble Vibration* model (Abv) is a set of two PDEs consisting of a transport equation and a Poisson equation. It has been derived in order to provide a better understanding of hyperbolic-elliptic couplings which are involved in a more general low Mach number system modelling bubbles.

While a local existence theorem has already been proven in any dimension for the Abv model, we get interested in this paper in the one-dimensional case, where we prove the existence of global solutions no matter how smooth the data. We also provide explicit expressions of the solutions thanks to the method of characteristics that we apply to the transport equation despite the coupling with the Poisson equation. We then compute these solutions by means of a numerical scheme based on a discrete version of the method of characteristics.

Keywords Explicit global smooth and irregular solutions, method of characteristics, numerical simulations, ABV

AMS Subject Classification 35A02, 35B65, 35A01, 35L03.

1 Introduction

Aiming at modelling bubbles in low Mach number flows, we study a set of partial differential equations obtained from an asymptotic expansion with respect to the Mach number applied to the diphasic compressible Navier-Stokes equations. This simpler system is called the *Diphasic Low Mach Number* system (Dlmn – [3]). To analyze the mathematical structure of the Dlmn system, we made further simplifications based on a potential velocity hypothesis and a linearization of the right-hand-side of one equation. The resulting system is named the *Abstract Bubble Vibration* model (ABV) and couples the hyperbolic linear transport equation to an elliptic Poisson equation. It was first introduced in [4] in the framework of the modelling of bubbles. Similar systems can be found in the chemotaxis field of bio-mathematics [17] or in astrophysics with the Smoluchovski-Poisson equation [2].

Theoretical studies have been carried out on the ABV model in [5] where a local existence theorem is proven *via* a Picard iterates method. A lower bound for the time of existence has been given in [15], as well as properties that solutions must satisfy depending on the regularity of initial data. Indeed, the smoothness of the initial mass fraction influences the methods we may use in theoretical investigations. The main assumption in our local existence result is that the initial datum belongs to a Sobolev space with index large enough. Nevertheless, since bubbles are modelled by indicator functions, this first approach does not enable to conclude about the existence of weak (bounded) solutions.

In this paper, we address the one-dimensional case where it is possible to derive a solution to the ABV model for a bubble-kind initial condition. A similar solution also exists for radial cases in 2D and 3D. In the smooth case, Th. 3.1 [15] ensures that there exists a unique solution on a certain time interval but it does not provide any expression. Here, we construct a solution for any bounded initial datum *via* the method of characteristics which turns out to be global in time but semi-explicit. Then, we work out the link between smooth and nonsmooth solutions. The last part relates to 1D numerical simulations of the two kinds of solutions. We use a different numerical scheme for each one taking into account potential sharp transitions. Note that in [14], we have proposed an AMR-type algorithm to handle discontinuities in 2D and 3D.

2 A simplified model in one space dimension

The Dlmn system [3] is a 5-equation model on a smooth bounded open domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. This system includes a reformulation of the mass conservation law, which is rewritten as $\nabla \cdot \boldsymbol{u} = \mathcal{G}$, where $\boldsymbol{u}$ is the velocity field and $\mathcal{G}$ a thermodynamic term. When $\boldsymbol{u}$ is a gradient field ($\boldsymbol{u} = \nabla\phi$), the latter relation reads $\Delta\phi = \mathcal{G}$. The right hand side couples the Poisson equation to other equations involving temperature and pressure. Overlooking thermodynamic effects lead us to replace $\mathcal{G}$ by a new right hand side depending only on the mass fraction of the vapor phase and satisfying a null mean-value condition over the domain Ω. The remaining equations form the Abv model, which reads:[1]

$$\begin{cases} \partial_t Y(t,\boldsymbol{x}) + \nabla\phi(t,\boldsymbol{x}) \cdot \nabla Y(t,\boldsymbol{x}) = 0, \\ Y(t=0,\boldsymbol{x}) = Y^0(\boldsymbol{x}), \\ \Delta\phi(t,\boldsymbol{x}) = \psi(t) \left[Y(t,\boldsymbol{x}) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega Y(t,\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \right], \\ \nabla\phi \cdot \boldsymbol{n}_{|\partial\Omega} = 0. \end{cases} \tag{1}$$

The unknowns are the mass fraction of the vapor phase Y and the potential ϕ, while functions $Y^0 = Y^0(\boldsymbol{x})$ and $\psi = \psi(t)$ are given data. The "pulse" ψ is assumed to be continuous on $[0, +\infty)$. Here, $\boldsymbol{n}$ is the normal outward unit vector to the boundary $\partial\Omega$.

When the dimension is $d = 1$, Syst. (1) is simpler due to the fact that every function is a gradient field. Hence, we consider the following formulation over the domain $\Omega = (-L, L)$, $L > 0$:

$$\begin{cases} \partial_t Y(t,x) + u(t,x)\partial_x Y(t,x) = 0, & \text{(2a)} \\ Y(0,x) = Y^0(x), & \text{(2b)} \\ u(t,x) = \psi(t) \left[\displaystyle\int_{-L}^{x} Y(t,y)\, \mathrm{d}y - \dfrac{x+L}{2L} \displaystyle\int_{-L}^{L} Y(t,y)\, \mathrm{d}y \right]. & \text{(2c)} \end{cases}$$

[1] In the framework of bounded solutions, we rewrite the first equation in (1) under the conservative form $\partial_t Y + \nabla \cdot (Y\nabla\phi) = \psi(t)Y(Y-\mu)$ to allow a more natural definition of weak solutions, μ being the mean value of Y given by (3a).

This system may be rewritten as a unique nonlinear integro-differential equation on the variable Y or as the unique second-order PDE:

$$\partial_{tx}^2 R + \psi(t) R \partial_{xx}^2 R = -\mu'(t),$$

with μ given (3a) and R being the antiderivative of $Y - \mu$ vanishing at $x = -L$. Nevertheless, we shall use the latter formulation (2) in the sequel.

Applying results from [15] in the particular case $d = 1$, we know that:

① (Th. 3.1 and 3.2, [15]) If $Y^0 \in H^2(-L, L)$, there exists a unique solution to Syst. (2) in the space

$$\mathscr{C}^0\left([0, \mathscr{T}_*], L^2(-L, L)\right) \cap L^\infty\big([0, \mathscr{T}_*], H^2(-L, L)\big),$$

where $\mathscr{T}_* > 0$ is prescribed by:

$$\left\| Y^0 - \frac{1}{2L}\int_{-L}^{L} Y^0(y)\,\mathrm{d}y \right\|_2 \int_0^{\mathscr{T}_*} |\psi(\tau)|\,\mathrm{d}\tau \le \frac{1}{e \cdot C_*(L)}.$$

Here, $C_*(L) > 0$ is a universal constant.[2]

② (Lemmas 3.4 and 3.7, [15]) When the initial datum Y^0 is not regular enough to apply Th. 3.1, the existence of a solution is not ensured but there exists at most one solution $Y \in L^\infty\big([0, \mathscr{T}], W^{1,\infty}(-L, L)\big)$ for some $\mathscr{T} > 0$ and this solution is an even function (almost everywhere) if Y^0 is even (a.e.).

③ (Prop. 3.9, [15]) If there exists a (weak) solution $Y \in L^\infty\big([0, \mathscr{T}] \times [-L, L]\big)$ for a certain $\mathscr{T} > 0$ with Y^0 taking values in $[0, 1]$, then the mean value $\mu(t)$ of Y over $[-L, L]$ is explicitly known:

$$\mu(t) := \frac{1}{2L}\int_{-L}^{L} Y(t, y)\,\mathrm{d}y = \frac{\displaystyle\int_{-L}^{L} Y^0(x)\exp\big[\Psi(t)Y^0(x)\big]\,\mathrm{d}x}{\displaystyle\int_{-L}^{L} \exp\big[\Psi(t)Y^0(x)\big]\,\mathrm{d}x}, \tag{3a}$$

[2]According to the notations in [15], $C_*(L) = C_{abv}(s = 2, d = 1, \Omega = (-L, L))$.

with Ψ being the antiderivative of ψ vanishing at $t = 0$, and:

$$\exp\left[\int_0^t \psi(\tau)\mu(\tau)\,\mathrm{d}\tau\right] = \frac{1}{2L}\int_{-L}^{L} \exp\big[\Psi(t)Y^0(x)\big]\,\mathrm{d}x. \tag{3b}$$

Moreover, there is a weak maximum principle (Lemma 3.10, [15]): If $Y^0 : [-L, L] \to [0, 1]$, then $Y(t, \cdot) : [-L, L] \to [0, 1]$ for almost all $t \in [0, \mathscr{T}]$.

Within the next paragraphs, we shall derive an implicit global solution for $Y^0 \in L^\infty(-L, L)$. This solution can be explicited in the particular case $Y^0 = \mathbf{1}_{[\alpha_0,\beta_0]}$ using two different methods. In the smooth case, this result shows that the time $\mathscr{T}_*$ introduced previously is not optimal in 1D. We have instead $\mathscr{T} = +\infty$.

3 Specifying solutions

We tackle the issue of expressing solutions whether their existence be guaranteed by the local existence theorem or not. In the first subsection, we proved a global existence theorem for bounded initial data with values in $[0, 1]$. In the second subsection, we pay attention to an initial datum representing the initial mass fraction of the vapor phase in a diphasic flow, *i.e.* a function with values in $\{0, 1\}$. Although the previous result applies to this irregular case, we present another proof consisting in direct calculations.

3.1 Bounded initial data

We address in this paragraph the problem of bounded initial data. Note that functions in $H^2(-L, L)$ are regular enough so that we can apply the local existence theorem, which ensures that there exists a unique solution on a certain time interval. Nevertheless, this result does not provide any expression for the solution. We determine here a way to express the solution depending on the domain Ω, the pulse ψ and the initial datum Y^0. It is based on the *method of characteristics* (see for instance [10, 13]) in the smooth case. The derived function turns out to be still a solution for $Y^0 \in L^\infty(-L, L)$.

Proposition 1 *Let Y^0 be a function in $L^\infty(-L,L)$ with $Y^0(x) \in [0,1]$ for almost all $x \in [-L,L]$. Then, there exists a global solution to the* Abv *model (2), given by:*

$$\mathcal{Y}(t,x) = Y^0\big(\Theta_t^{-1}(x)\big),$$

where Θ_t^{-1} is the x-inverse of the function:

$$\Theta(t,x_0) = 2L \frac{\displaystyle\int_{-L}^{x_0} e^{\Psi(t)Y^0(y)}\,\mathrm{d}y}{\displaystyle\int_{-L}^{L} e^{\Psi(t)Y^0(y)}\,\mathrm{d}y} - L. \tag{4}$$

We shall make a comment about uniqueness. As $\Theta_t^{-1} \in \mathscr{C}^0\big([0,\mathscr{T}]\times[-L,L]\big)$, the corresponding solution $\mathcal{Y}$ belongs to the space $L^\infty\big([0,\mathscr{T}]\times(-L,L)\big)$. But Lemma 3.4 [15] about uniqueness does not apply to $\mathcal{Y}$ because of the smoothness requirement in $L^\infty\big([0,\mathscr{T}], W^{1,\infty}(-L,L)\big)$. Nevertheless, as soon as $Y^0 \in W^{1,\infty}(-L,L)$, we can state that the solution induced by **Prop. 1** is unique.

Proof. We first characterize the function Θ as defined in (4). Since $Y^0 \in L^\infty(-L,L)$ and $\Psi \in \mathscr{C}^1(0,+\infty)$, then $\Theta \in \mathscr{C}^1\big([0,\mathscr{T}], W^{1,\infty}(-L,L)\big)$ for all $\mathscr{T} > 0$. In particular, for $t \in [0,\mathscr{T}]$, $\Theta(t,\cdot)$ is continuous and monotone-increasing from $[-L,L]$ into itself. Hence, $\Theta(t,\cdot)$ is a homeomorphism on $[-L,L]$ with inverse Θ_t^{-1}. Moreover, the estimate:

$$\big|\Theta(t,x_1) - \Theta(t,x_2)\big| \le |x_1 - x_2| \cdot \exp\big(2 \sup_{[0,\mathscr{T}]} |\Psi| \cdot \|Y^0\|_\infty\big)$$

shows that $\Theta(t,\cdot)$ is Lipschitz-continuous. Then, we have the following Lipschitz change of variable result [9]:

$$\forall\ f \in L^1(-L,L),\ \int_{-L}^{L} f(x)\,\mathrm{d}x = \int_{-L}^{L} f \circ \Theta(t,x_0)\,|\partial_{x_0}\Theta(t,x_0)|\,\mathrm{d}x_0. \tag{5}$$

Given this remark, we now define *weak solutions* of Syst. (2). For $\mathscr{T} > 0$, let $\Omega_{\mathscr{T}}$ be the domain $[0, \mathscr{T}) \times [-L, L]$. Let Y be a function in $L^\infty(\Omega_{\mathscr{T}})$. Note that u from relation (2c) is continuous w.r.t (t, x). We say that Y is a weak solution of (2) if, for all $\varphi \in \mathscr{C}_0^\infty(\Omega_{\mathscr{T}})$:

$$\iint_{\Omega_{\mathscr{T}}} Y\left[\partial_t\varphi + u\partial_x\varphi + \psi(t)\varphi\big(Y - \mu(t)\big)\right](t,x)\,\mathrm{d}t\mathrm{d}x + \int_{-L}^{L} Y^0(x)\varphi(0,x)\,\mathrm{d}x = 0. \tag{6}$$

We shall prove that $\mathcal{Y} : (t,x) \mapsto Y^0\big(\Theta_t^{-1}(x)\big)$ satisfies Eq. (6). We note $\mathcal{U}$ the associated velocity to $\mathcal{Y}$ via (2c). As we have:

$$\partial_{x_0}\Theta(t,x_0) = \frac{2L}{\displaystyle\int_{-L}^{L} e^{\Psi(t)Y^0(y)}\mathrm{d}y} e^{\Psi(t)Y^0(x_0)},$$

we deduce that the characteristic equation $\partial_t\Theta = \mathcal{U}(t,\Theta)$ is satisfied despite a loss of regularity:[3]

$$\begin{aligned}
\partial_t\Theta(t,x_0) &= 2L\psi(t)\frac{\displaystyle\int_{-L}^{x_0} Y^0(y)e^{\Psi(t)Y^0(y)}\mathrm{d}y}{\displaystyle\int_{-L}^{L} e^{\Psi(t)Y^0(y)}\mathrm{d}y} \\
&\qquad - 2L\psi(t)\frac{\displaystyle\int_{-L}^{x_0} e^{\Psi(t)Y^0(y)}\mathrm{d}y}{\displaystyle\int_{-L}^{L} e^{\Psi(t)Y^0(y)}\mathrm{d}y}\,\frac{\displaystyle\int_{-L}^{L} Y^0(y)e^{\Psi(t)Y^0(y)}\mathrm{d}y}{\displaystyle\int_{-L}^{L} e^{\Psi(t)Y^0(y)}\mathrm{d}y}, \\
&= \psi(t)\left[\int_{-L}^{x_0} Y^0(y)\partial_{x_0}\Theta(t,y)\,\mathrm{d}y - \big(\Theta(t,x_0)+L\big)\mu(t)\right], \\
&= \psi(t)\left[\int_{-L}^{\Theta(t,x_0)} Y^0\big(\Theta_t^{-1}(x)\big)\,\mathrm{d}x - \big(\Theta(t,x_0)+L\big)\mu(t)\right], \\
&= \mathcal{U}\big(t,\Theta(t,x_0)\big).
\end{aligned} \tag{7}$$

[3] $\mathcal{U}$ is not smooth enough to apply the *Cauchy-Lipschitz* theorem to the ODE $\Theta' = \mathcal{U}(t,\Theta)$. Here, $\mathcal{U} \in L^\infty\big([0,\mathscr{T}], W^{1,\infty}(-L,L)\big)$.

We have used Eqs.(3a) for μ and (5) for the change of variable. We introduce the notation:

$$\mathcal{A}(\mathcal{Y}) := \iint_{\Omega_{\mathscr{T}}} \mathcal{Y}\big[\partial_t \varphi + \mathcal{U}\partial_x \varphi\big](t,x)\,\mathrm{d}t\mathrm{d}x.$$

By virtue of the change of variable result (5), we have:

$$\mathcal{A}(\mathcal{Y}) = \iint_{\Omega_{\mathscr{T}}} Y^0(x_0)\big[\partial_t \varphi + \mathcal{U}\partial_x \varphi\big]\big(t,\Theta(t,x_0)\big)\partial_{x_0}\Theta(t,x_0)\,\mathrm{d}t\mathrm{d}x_0.$$

By remarking that $(t,x_0) \mapsto \varphi\big(t,\Theta(t,x_0)\big)$ belongs to $\mathscr{C}^1\big([0,\mathscr{T}],\mathscr{C}^0(-L,L)\big)$, then the equality $\frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi\big(t,\Theta(t,x_0)\big)\big] = \big[\partial_t\varphi + \partial_t\Theta(t,x_0)\partial_x\varphi\big]\big(t,\Theta(t,x_0)\big)$ holds in a strong sense. Taking into account (7) leads to:

$$\mathcal{A}(\mathcal{Y}) = \int_{-L}^{L} Y^0(x_0)\int_0^{\mathscr{T}} \frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi\big(t,\Theta(t,x_0)\big)\big]\partial_{x_0}\Theta(t,x_0)\,\mathrm{d}t\mathrm{d}x_0.$$

As $(t,x_0) \mapsto \partial_{x_0}\Theta(t,x_0)$ lies in $\mathscr{C}^1\big([0,\mathscr{T}],L^\infty(-L,L)\big)$, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\big[\varphi\big(t,\Theta(t,x_0)\big)\big]\partial_{x_0}\Theta(t,x_0)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big[\varphi\big(t,\Theta(t,x_0)\big)\partial_{x_0}\Theta(t,x_0)\big] - \varphi\big(t,\Theta(t,x_0)\big)\partial_t\partial_{x_0}\Theta(t,x_0).\end{aligned}$$

Hence:

$$\begin{aligned}\mathcal{A}(\mathcal{Y}) = \int_{-L}^{L} Y^0(x_0)\Big[&\overbrace{\varphi\big(\mathscr{T},\Theta(\mathscr{T},x_0)\big)}^{=0}\partial_{x_0}\Theta(\mathscr{T},x_0) - \varphi(0,x_0)\Big]\,\mathrm{d}x_0\\ &- \int_{-L}^{L} Y^0(x_0)\int_0^{\mathscr{T}} \varphi\big(t,\Theta(t,x_0)\big)\partial_t\partial_{x_0}\Theta(t,x_0)\,\mathrm{d}t\mathrm{d}x_0.\end{aligned}$$

Differentiating Θ with respect to x_0 then to t enables to show the identity[4] $\partial_t \partial_{x_0} \Theta = \partial_x \mathcal{U}(t,\Theta) \cdot \partial_{x_0}\Theta$. We thus obtain:

$$\mathcal{A}(\mathcal{Y}) = -\int_{-L}^{L} Y^0(x_0)\varphi(0,x_0)\,\mathrm{d}x_0 \\ - \int_{-L}^{L}\int_0^{\mathscr{T}} Y^0(x_0)(\varphi\partial_x\mathcal{U})\big(t,\Theta(t,x_0)\big)\partial_{x_0}\Theta(t,x_0)\,\mathrm{d}t\mathrm{d}x_0.$$

Applying (5) to the second term, we have the expected result, namely:

$$\mathcal{A}(\mathcal{Y}) = -\int_{-L}^{L} Y^0(x_0)\varphi(0,x_0)\,\mathrm{d}x_0 \\ - \iint_{\Omega_{\mathscr{T}}} \psi(t)\mathcal{Y}(t,x)\varphi(t,x)\big(\mathcal{Y}(t,x)-\mu(t)\big)\,\mathrm{d}t\mathrm{d}x,$$

and (6) holds.

This completes the proof: $\mathcal{Y}$ is a weak solution to the ABV model on $[0,\mathscr{T}]$ for all $\mathscr{T} > 0$. ∎

A first consequence of **Prop. 1** is that if $\Psi(t_0) = 0$ with $t_0 > 0$, then $\Theta(t_0,x_0) = x_0$. **Thus, if Ψ vanishes, we recover the initial configuration of the bubble**. Even if the expression of the global solution is not explicit (because inverting Θ is not straightforward), it has enabled to prove the global existence. In addition, we easily prove that if Y^0 is even almost everywhere, $\Theta(t,\cdot)$ is odd and the resulting solution $\mathcal{Y}$ is even a.e., as stated in Lemma 3.7, [15].

We shall now explain the derivation of the expression of Θ. Considering $Y^0 \in H^2(-L,L)$, let Y be the solution given by Th. 3.1 in [15] and u be the associated velocity. We note $\mathscr{T} \le +\infty$ its time of existence. We know that Y satisfies the maximum principle and that the mean value of Y over $[-L,L]$ is given by (3a).

[4] For Y^0 smooth enough to apply the Schwarz lemma which would ensure that $\partial^2_{t,x_0}\Theta = \partial^2_{x_0,t}\Theta$, we could recover the jacobian equation by differentiating (7) w.r.t. x_0.

In the framework of the method of characteristics, we introduce the ODE:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = u\big(t, \mathcal{X}(t)\big), \\ \\ \mathcal{X}(0) = x_0 \in [-L, L]. \end{cases} \tag{8}$$

By means of Sobolev embeddings, we show that $Y \in \mathscr{C}^1\big([0,\mathscr{T}] \times [-L, L]\big)$, from which we deduce that $u \in \mathscr{C}^1\big([0,\mathscr{T}], \mathscr{C}^2(-L, L)\big)$. By virtue of the *Cauchy-Lipschitz* theorem, ODE (8) has a unique solution depending continuously on x_0. We shall note $\mathcal{X}(t; x_0)$ its solution. Its time of existence is equal to $\mathscr{T}$ since the boundary condition $\nabla\phi \cdot \boldsymbol{n} = 0$ ensures that the trajectories stay in $[-L, L]$. $\mathcal{X}(t; x_0)$ is the position at time t of a particle which was in x_0 at time 0. Then, the method is based on the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[Y\big(t, \mathcal{X}(t; x_0)\big)\right] = 0,$$

which implies:

$$Y\big(t, \mathcal{X}(t; x_0)\big) = Y(0, x_0) = Y^0(x_0). \tag{9}$$

Furthermore, as the velocity field is smooth enough, the jacobian of $\mathcal{X}$ satisfies the following ODE [13]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_{x_0}\mathcal{X}(t; x_0) = \partial_x u\big(t, \mathcal{X}(t; x_0)\big) \cdot \partial_{x_0}\mathcal{X}(t; x_0).$$

Due to the fact that $\partial_{x_0}\mathcal{X}(0; x_0) = 1$, we solve the ODE as:

$$\begin{aligned} \partial_{x_0}\mathcal{X}(t; x_0) &= \exp\left[\int_0^t \partial_x u\big(\tau, \mathcal{X}(\tau; x_0)\big)\,\mathrm{d}\tau\right], \\ &= \exp\left[\int_0^t \psi(\tau)\left[Y\big(\tau, \mathcal{X}(\tau; x_0)\big) - \mu(\tau)\right]\,\mathrm{d}\tau\right], \\ &= \exp\left[\Psi(t) Y^0(x_0)\right] \exp\left[-\int_0^t \psi(\tau)\mu(\tau)\,\mathrm{d}\tau\right], \end{aligned} \tag{10}$$

according to (9). Given (3b), we know that the exponential term is independant from Y and exists for all $t \in [0, \mathscr{T}]$. (10) thus demonstrates that $\partial_{x_0}\mathcal{X}(t; x_0) > 0$ for all $t \in [0, \mathscr{T}]$. Moreover, we notice that $\mathcal{X}(t; -L) = -L$ and $\mathcal{X}(t; L) = L$. Thus, $\mathcal{X}(t, \cdot)$ is a bijective mapping from $[-L, L]$ into itself for all $t \in [0, \mathscr{T}]$. Let $\mathcal{X}_t^{-1}$ its x-inverse. Then, for all $(t, x) \in [0, \mathscr{T}] \times [-L, L]$, $Y(t, x) = Y^0\left(\mathcal{X}_t^{-1}(x)\right)$.

The advantage of dimension 1 is that we can derive an explicit expression for $\mathcal{X}$, which will enable to prove that the solution $Y(t, x) = Y^0\left(\mathcal{X}_t^{-1}(x)\right)$ is global. Integrating Eq. (10) w.r.t. x_0, we obtain thanks to (3b):

$$\mathcal{X}(t; x_0) = 2L \frac{\displaystyle\int_{-L}^{x_0} e^{\Psi(t) Y^0(y)}\, \mathrm{d}y}{\displaystyle\int_{-L}^{L} e^{\Psi(t) Y^0(y)}\, \mathrm{d}y} + \xi(t).$$

$\mathcal{X}(t; -L) = -L$ leads to $\xi(t) \equiv -L$. We have thus found out the solution to ODE (8). We conclude by remarking that this expression holds for all $t \geq 0$ since Ψ is continuous on $[0, +\infty)$ and Y^0 is bounded on $[-L, L]$.

As we showed before, this expression holds for less smooth Y^0: We described the way to derive the expression given in **Prop. 1** where we proved that this expression is still valid for $Y^0 \in L^\infty(-L, L)$.

3.2 Irregular initial conditions

We come back in this part to the modelling of bubbles which are assimilated as indicator functions of subdomains of $(-L, L)$. Although this case is included in the statement in **Prop. 1** (since characteristic functions are in L^∞ with values in $[0, 1]$), we present another proof of the derivation of the solution based on the *Rankine-Hugoniot* relation.

Proposition 2 *Let $\alpha_0 < \beta_0$ be two real numbers in $[-L, L]$. One* ***global-in-time*** *solution to the* Abv *model (2) with $Y^0(x) = \mathbf{1}_{[\alpha_0,\beta_0]}(x)$ is given by $\underline{Y}(t,x) = \mathbf{1}_{[\alpha(t),\beta(t)]}(x)$, with:*

$$\alpha(t) = \frac{\frac{\beta_0+\alpha_0}{2} - \frac{\beta_0-\alpha_0}{2}e^{\Psi(t)}}{\frac{\beta_0-\alpha_0}{2L}e^{\Psi(t)} + 1 - \frac{\beta_0-\alpha_0}{2L}} \quad \textit{and} \quad \beta(t) = \frac{\frac{\beta_0+\alpha_0}{2} + \frac{\beta_0-\alpha_0}{2}e^{\Psi(t)}}{\frac{\beta_0-\alpha_0}{2L}e^{\Psi(t)} + 1 - \frac{\beta_0-\alpha_0}{2L}}.$$

Proof. Assume there exists a solution $Y(t,x) = \mathbf{1}_{[\alpha(t),\beta(t)]}(x)$ with α and β to be determined such that $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$. On the one hand, Eq. (3a), which corresponds in this irregular case to Lemma 1.1 in [5], provides the relation:

$$\mathcal{V}(t) := \frac{\beta(t) - \alpha(t)}{2L} = \frac{e^{\Psi(t)}(\beta_0 - \alpha_0)}{e^{\Psi(t)}(\beta_0 - \alpha_0) + 2L - (\beta_0 - \alpha_0)}. \tag{11}$$

On the other hand, the *Rankine-Hugoniot* jump condition across the discontinuity at $x = \alpha(t)$ gives:

$$[\![u(t,\cdot)Y(t,\cdot)]\!]_{x=\alpha(t)} = \alpha'(t)[\![Y(t,\cdot)]\!]_{x=\alpha(t)}.$$

As Y is bounded, Eq. (2c) shows that u is continuous in space and the jump condition becomes:

$$u\big(t, \alpha(t)\big) = \alpha'(t). \tag{12}$$

Allowing for the fact that $Y(t,x) = \mathbf{1}_{[\alpha(t),\beta(t)]}(x)$, Eq. (2c) reduces to:

$$\forall\, x \in [-L, \alpha(t)],\ u(t,x) = -\psi(t)\mathcal{V}(t)(x + L).$$

Hence, Eq. (12) reads:

$$\alpha'(t) = -\psi(t)\mathcal{V}(t)\big(\alpha(t) + L\big).$$

Taking Eq. (11) into account, the ODE can be easily integrated to obtain:

$$\alpha(t) = -L + \frac{2L(\alpha_0 + L)}{e^{\Psi(t)}(\beta_0 - \alpha_0) + 2L - (\beta_0 - \alpha_0)} = \frac{\frac{\beta_0+\alpha_0}{2} - \frac{\beta_0-\alpha_0}{2}e^{\Psi(t)}}{\mathcal{V}(0)e^{\Psi(t)} + 1 - \mathcal{V}(0)}. \tag{13a}$$

We then deduce from Eqs. (11) and (13a) that:

$$\beta(t) = \frac{\frac{\beta_0+\alpha_0}{2} + \frac{\beta_0-\alpha_0}{2} e^{\Psi(t)}}{\mathcal{V}(0)e^{\Psi(t)} + 1 - \mathcal{V}(0)}. \tag{13b}$$

We remark that α and β are defined (and differentiable) for all $\mathscr{T} > 0$ insofar as Ψ is continuous on $[0, +\infty)$. That is why Y seems to be a global solution to Syst. (2).

Conversely, given α and β from Eqs. (13a-13b), we show that $\underline{Y}(t,x) = \mathbf{1}_{[\alpha(t),\beta(t)]}(x)$ and $\underline{u}$ deduced from Eq. (2c) are solutions to Syst. (2). For all $\mathscr{T} > 0$ and $\chi \in \mathscr{C}_0^\infty\big([0,\mathscr{T}) \times [-L,L]\big)$, we have:[5]

$$\begin{aligned}
&\iint_{[0,\mathscr{T}]\times[-L,L]} \underline{Y}(t,x)\big(\partial_t\chi(t,x) + \partial_x(\underline{u}\chi)(t,x)\big)\,\mathrm{d}x\mathrm{d}t + \int_{-L}^{L} Y^0(x)\chi(0,x)\,\mathrm{d}x \\
&= \int_{\alpha(\mathscr{T})}^{\beta(\mathscr{T})} \chi(\mathscr{T},x)\,\mathrm{d}x - \int_{\alpha_0}^{\beta_0} \chi(0,x)\,\mathrm{d}x + \int_{-L}^{L} Y^0(x)\chi(0,x)\,\mathrm{d}x \\
&+ \int_0^{\mathscr{T}} \big[\chi\big(t,\beta(t)\big)\left\{\underline{u}\big(t,\beta(t)\big) - \beta'(t)\right\} - \chi\big(t,\alpha(t)\big)\left\{\underline{u}\big(t,\alpha(t)\big) - \alpha'(t)\right\}\big]\,\mathrm{d}t \\
&= 0,
\end{aligned}$$

after having checked that $\underline{u}\big(t,\beta(t)\big) = \beta'(t)$ and $\underline{u}\big(t,\alpha(t)\big) = \alpha'(t)$. Thus, $\underline{Y}$ satisfies Eq. (2a) in the sense of distributions, which means that **we have obtained a global solution to the** Abv **model in 1D** for a nonsmooth initial datum. ■

We shall make a few comments about the solution based on Eqs. (13a-13b). First, we do not prove that this solution is unique.[6] Nevertheless, the derivation of $\underline{Y}$ goes to show that it is the only solution of the form $\mathbf{1}_{\tilde{\Omega}(t)}$ with $\tilde{\Omega}(t) \subset [-L,L]$ simply connected.

In addition, $\underline{Y} \in \mathscr{C}^0\big([0,\mathscr{T}], L^\infty(-L,L)\big)$ and satisfies properties which were only proven in the space $L^\infty\big([0,\mathscr{T}], W^{1,\infty}(-L,L)\big)$ in [15]: Indeed, $\underline{Y}$ takes values in $[0,1] = [\min Y^0, \max Y^0]$;

[5] $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\alpha(t)}^{\beta(t)} \chi(t,x)\,\mathrm{d}x\right) = \beta'(t)\chi\big(t,\beta(t)\big) - \alpha'(t)\chi\big(t,\alpha(t)\big) + \int_{\alpha(t)}^{\beta(t)} \partial_t\chi(t,x)\,\mathrm{d}x$ since α and β are differentiable.

[6] Uniqueness has only been proven in $L^\infty\big([0,\mathscr{T}], W^{1,\infty}(-L,L)\big)$. See Lemma 3.4, [15].

Likewise, if Y^0 is even, which can be expressed equivalently as $\alpha_0 = -\beta_0$, we have $\alpha(t) = -\beta(t)$ and $\underline{Y}$ is even too.

In addition, we check that if $\alpha_0 > -L$ (resp. $\beta_0 < L$), then $\alpha(t) > -L$ (resp. $\beta(t) < L$) for all $t \geq 0$. This fact implies that this bubble cannot reach the boundary in finite time.

Lastly, Eqs. (13a-13b) highlight the influence of the pulse ψ: If ψ is positive, then α decreases and β increases, which means that the bubble expands. Similarly, it contracts if ψ is negative. If $\psi(t)$ tends to infinity when $t \to +\infty$, the bubble tends to occupy the whole domain since $\mathcal{V}(t) \to 1$. Moreover, if ψ is periodic with a null mean value over its period, then the bubble is periodic.

We could have applied **Prop. 1** directly to $Y^0 = \mathbf{1}_{[\alpha_0,\beta_0]}$, where Θ reads:

$$\Theta(t,x_0) = -L + \frac{2L}{2L + (\beta_0 - \alpha_0)(e^{\Psi(t)} - 1)} \begin{cases} x_0 + L & x_0 \in [-L, \alpha_0], \\ L + \alpha_0 + (x_0 - \alpha_0)e^{\Psi(t)} & x_0 \in (\alpha_0, \beta_0), \\ L + x_0 + (\beta_0 - \alpha_0)(e^{\Psi(t)} - 1) & x_0 \in [\beta_0, L]. \end{cases}$$

Resolving Eq. $\Theta(t,x_0) = x$ enables to determine Θ_t^{-1}, which leads to the stated conlusion: $Y^0\left(\Theta_t^{-1}(x)\right) = \underline{Y}(t,x)$.

We can also study radial cases in 2D and 3D circular geometries: If Ω is an open ball with radius $R > 0$ in dimension d and $Y^0(\boldsymbol{x}) = \mathbf{1}_{\{\|\boldsymbol{x}\| \leq \rho_0\}}(\boldsymbol{x})$ with $0 < \rho_0 < R$, there exists a solution $Y(t,\boldsymbol{x}) = \mathbf{1}_{\{\|\boldsymbol{x}\| \leq \rho(t)\}}(\boldsymbol{x})$, with (see A):

$$\rho(t) = \frac{\rho_0}{\sqrt[d]{R^d - \rho_0^d}} \frac{R \exp\left[\dfrac{1}{d}\Psi(t)\right]}{\sqrt[d]{1 + \dfrac{\rho_0^d}{R^d - \rho_0^d} \exp \Psi(t)}}. \tag{14}$$

The previous 1D result is the particular case $d = 1$ in this formula.

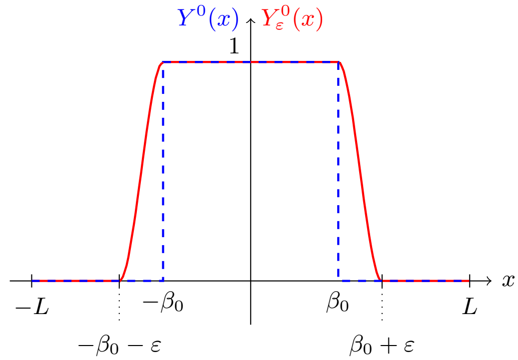


Figure 1: Graphs of functions Y^0 and Y^0_ε

3.3 Convergence

Considering results from the two propositions, we may wonder whether smooth and nonsmooth solutions match up with each other and in what sense. We recall that it is assumed in **Prop. 1** that Y^0 belongs to $L^\infty(-L,L)$ but uniqueness only holds in $W^{1,\infty}$. For the sake of simplicity and without loss of generality, we consider in the sequel even initial data (which implies even solutions).

Let Y^0 be the function $Y^0 = \mathbf{1}_{[-\beta_0,\beta_0]}$ with $\beta_0 \in (0,L)$. We introduce $\varepsilon \in (0, L-\beta_0)$ and Y^0_ε a regularization of Y^0 as shown on Fig. 1. For $x \in [\beta_0, \beta_0+\varepsilon]$, $Y^0_\varepsilon(x)$ is given by:

$$Y^0_\varepsilon(x) = \Lambda\left(2\frac{\beta_0 - x}{\varepsilon} + 1\right),$$

where Λ is a continuous bijective function from $[-1,1]$ to $[0,1]$, e.g.:

$$\begin{aligned}
\Lambda_1(z) &= \frac{z+1}{2}, && \implies Y^0_{\varepsilon,1} \in W^{1,\infty}(-L,L),\\
\Lambda_3(z) &= \frac{-1}{4}\left(z^3 - 3z - 2\right), && \implies Y^0_{\varepsilon,3} \in H^2(-L,L),\\
\Lambda_\infty(z) &= \frac{\displaystyle\int_{-1}^{z} \lambda(\hat{z})\,\mathrm{d}\hat{z}}{\displaystyle\int_{-1}^{1} \lambda(\hat{z})\,\mathrm{d}\hat{z}},\ \lambda(z) = e^{\frac{-z^2}{1-z^2}} && \implies Y^0_{\varepsilon,\infty} \in H^\infty(-L,L).
\end{aligned}$$

We aim at comparing solutions coming from initial data Y^0 and Y_ε^0 in the limit as $\varepsilon \to 0$. In the irregular case, one solution is given by $\underline{Y}(t,x) = \mathbf{1}_{[-\beta(t),\beta(t)]}(x)$ (*cf.* **Prop. 2**) with:

$$\beta(t) = \frac{L\beta_0 \exp \Psi(t)}{L - \beta_0 + \beta_0 \exp \Psi(t)},$$

while in the smooth case (at least $W^{1,\infty}$), there exists a unique solution Y_ε given by **Prop. 1**.

We first remark that **we have** $Y_{\varepsilon=0} = \underline{Y}$ **when** $Y^0_{\varepsilon=0} = Y^0$ (*cf.* **Prop. 2**).

Then, we may wonder whether Y_ε, $\varepsilon > 0$, converges to $\underline{Y}$ in the limit as $\varepsilon \to 0$. We first describe the profile of Y_ε. We know that $\Theta(t,\cdot)$ is bijective:

- from $[-L, -\beta_0 - \varepsilon]$ into $[-L, \Theta(t, -\beta_0 - \varepsilon)]$,
- from $[-\beta_0, \beta_0]$ into $[\Theta(t,-\beta_0), \Theta(t,\beta_0)]$
- and from $[\beta_0 + \varepsilon, L]$ into $[\Theta(t, \beta_0 + \varepsilon), L]$.

As Y_ε is constant along the characteristics and Y_ε is even, then $Y_\varepsilon(t,x) = 0$ for $x \in [-L, -\delta_\varepsilon(t)] \cup [\delta_\varepsilon(t), L]$ and $Y_\varepsilon(t,x) = 1$ for $x \in [-\beta_\varepsilon(t), \beta_\varepsilon(t)]$, where:

$$\delta_\varepsilon(t) = \Theta(t, \beta_0 + \varepsilon) \quad \text{ and } \quad \beta_\varepsilon(t) = \Theta(t, \beta_0).$$

Moreover, $Y_\varepsilon(t,x) \in [0,1]$ for $x \in [-\delta_\varepsilon(t), -\beta_\varepsilon(t)] \cup [\beta_\varepsilon(t), \delta_\varepsilon(t)]$ according to the maximum principle (Lemma 3.5, [15]). Hence, Y_ε has a similar profile to Y_ε^0.

Noting $\tilde{\Lambda}(t) = \dfrac{1}{2}\displaystyle\int_{-1}^{1} \exp\left[\Psi(t)\Lambda(x)\right] \mathrm{d}x$, we deduce from (4):

$$\delta_\varepsilon(t) = \frac{L\left(\beta_0 e^{\Psi(t)} + \varepsilon\tilde{\Lambda}(t)\right)}{L - \beta_0 - \varepsilon + \beta_0 e^{\Psi(t)} + \varepsilon\tilde{\Lambda}(t)}, \tag{15a}$$

$$\beta_\varepsilon(t) = \frac{L\beta_0 e^{\Psi(t)}}{L - \beta_0 - \varepsilon + \beta_0 e^{\Psi(t)} + \varepsilon\tilde{\Lambda}(t)}. \tag{15b}$$

These expressions show that δ_ε and β_ε have a limit when $\varepsilon \to 0$ and this limit is β, which proves that Y_ε converges pointwise to $\underline{Y}$.

4 Numerical comparisons

This section relates to numerical simulations of the ABV model with initial data $Y^0 = \mathbf{1}_{[-\beta_0,\beta_0]}$, $\beta_0 \in [0, L]$, and Y_ε^0 for Λ_1 and Λ_3. The former case is treated with a nondiffusive scheme while in the latter case, we discretize the method of characteristics to construct a scheme able to simulate solutions from **Prop. 1**. We use in both cases a uniform space grid $x_j = -L + (j-1)\Delta x$, $j \in \{1, \ldots, N+1\}$ and $\Delta x = 2L/N$ with $L = 1$. We note t^n the time sampling (with time size $\Delta t^n = t^{n+1} - t^n$) and Y_i^n the approximation of the solution at time t^n and position x_i. The pulse is here $\psi(t) = \cos(10\,t)$ in order to emphasize periodic motions and stability phenomena.

4.1 Simulations of indicator functions

A major issue in numerically solving a transport equation is to preserve discontinuities. In 2D simulations, an AMR-type algorithm has been developped in [14]. In this paper, we use a uniform mesh and we choose the antidiffusive scheme proposed in [6] by Després and Lagoutière for the transport equation. This scheme mixes advantages from the upwind (stability) and downwind (nondiffusivity) schemes. It has been proven that the numerical diffusion is controlled uniformly in time. The velocity field is computed by means of the first order rectangular integration formula. The flux are computed in each cell using the local velocity at the previous time step.

The stability is ensured by means of a CFL-kind condition. The time size Δt^n is chosen such that $\Delta t^n = \min(\Delta t_{max}, \lambda \cdot \Delta x / \max_i |u_i^n|)$ with $\lambda = 9 \cdot 10^{-1}$ and $\Delta t_{max} = 5 \cdot 10^{-2}$. This parameter is necessary in case of vanishing velocities and is tuned so as to be lower than the period of ψ (here $2\pi/10$).

We represent on Fig. 3 the exact solution from **Prop. 2** and numerical solutions from the antidiffusive and upwind schemes. We see that the antidiffusive scheme provides more reliable results than the upwind scheme for $Y^0 = \mathbf{1}_{[-0.25,0.25]}$. The numerical diffusion process is restricted to 2 cells (over 300 cells in this case) after 100 iterations.

4.2 Discrete method of characteristics for smooth solutions

The antidiffusive scheme turns out to be less accurate in the smooth case: It is coherent with the application for which it has been designed, namely the nondiffusivity of discontinuities. Here, Y^0 is a smoothened profile as described in § 3.3, Fig. 1. We thus have to construct a suitable scheme for smooth solutions based on the method of characteristics that has provided the expression of the solution $\mathcal{Y}$ (see **Prop. 1**). We outline in the sequel the derivation of the scheme MOC2. For further details, please refer to [16].

Compared to § 3.1 where the origin of time is 0, we introduce the characteristic flow $\mathcal{X}(t; s, x)$ originated from time s as the solution to:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = \mathcal{U}\big(t, \mathcal{X}(t)\big), \\ \mathcal{X}(s) = x, \end{cases} \tag{16}$$

for $s \geq 0$ and $x \in [-L, L]$. By virtue of the uniqueness result in the *Cauchy-Lipschitz* theorem, we have the identity:

$$\mathcal{X}\big(t; s, \mathcal{X}(s; t, x)\big) = x. \tag{17}$$

In particular, we deduce that the solution $\mathcal{Y}$ satisfies the equalities:

$$\mathcal{Y}(t, x) = Y^0\big(\mathcal{X}(0; t, x)\big) = \mathcal{Y}\big(s, \mathcal{X}(s; t, x)\big).$$

Given the discrete solution at time t^n, we approximate the solution at time t^{n+1} as:

$$Y_i^{n+1} \approx \mathcal{Y}\big(t^n, \mathcal{X}(t^n; t^{n+1}, x_i)\big).$$

There are two major issues in numerical methods of characteristics. The first one is about the computation of $\xi_i^n := \mathcal{X}(t^n; t^{n+1}, x_i)$ which is the foot of the characteristic curve going through x_i at time t^{n+1} (called the upstream point of x_i) – see Fig. 2(a). The second one concerns the calculation of $\mathcal{Y}(t^n, \xi_i^n)$ while we only know values of the numerical solution at nodes (x_j) and ξ_i^n is generally not a mesh node.

In most schemes, the first step is achieved thanks to a linearization of the form $\hat{\xi}_{i,1}^n = x_i - u^* \Delta t$ for some value u^*. In [11], u^* results from an interpolation between values of the velocity (assumed to be a datum) at times t^n and t^{n+1}, which are not available in our case. Note that in [11], the second step is treated by means of a polynomial (Holly-Preissmann scheme) of the third order, whose coefficients are calculated using the values Y_j^n and Y_{j+1}^n of the solution as well as its derivatives when the characteristic curve goes through the cell $[x_j, x_{j+1}]$. In [7], the authors first rewrite the transport equation with a directional derivative. Then, they use an Euler scheme to obtain a semi-discrete (in time) equation that they solved with a finite-difference or a finite-element method for the discretization in space (MMOC for *Modified Method of Characteristics*). But this procedure fails to preserve the mass conservation. That is why a new method has been proposed in [8], where the velocity field used to compute the first order upstream point is chosen so as to conserve mass. However, this method is derived for advection-diffusion equations.

As we consider in this paper a simple advection equation, we propose another way to simulate the solution: We first improve the accuracy with a second-order calculation for the upstream point. Concerning the second step, we use a second-order polynomial involving three values depending on the local properties of the solution (monotonicity, convexity). As the velocity field is prescribed by Eq. (2c), the numerical scheme described in [16] reads:

$$\hat{\xi}_{i,2}^n = x_i - \frac{\Delta t^n}{2}(3u_i^n - u_i^{n-1}) + \frac{(\Delta t^n)^2}{2}\psi(t^n)u_i^n(Y_i^n - \mu^n). \tag{18}$$

The mean value $\mu^n \approx \mu(t^n)$ is computed by means of the second-order trapezoidal formula. Then, we find out the interval $[x_j, x_{j+1})$ from which the characteristic curve issues at time t^n. Let θ_{ij}^n be the position of $\hat{\xi}_{i,2}^n$ in this interval, *i.e.* $\theta_{ij}^n = (x_{j+1} - \hat{\xi}_{i,2}^n)/\Delta x$. Hence $\theta_{ij}^n \in (0, 1]$ and $\theta_{ij}^n = 1$ for $\hat{\xi}_{i,2}^n = x_j$. A natural choice to compute $Y_i^{n+1} \approx \mathcal{Y}(t^n, \xi_i^n)$ is to use a linear interpolation of Y_j^n and Y_{j+1}^n. But accuracy rapidly deteriorates, which accounts for our interest for second-order interpolation schemes. Depending on the values to take into account, we could use:

$$① \ Y_i^{n+1} = -\frac{\theta_{ij}^n(1-\theta_{ij}^n)}{2}Y_{j-1}^n + \theta_{ij}^n(2-\theta_{ij}^n)Y_j^n + (1-\theta_{ij}^n)\left(1-\frac{\theta_{ij}^n}{2}\right)Y_{j+1}^n \ ;$$

$$= \frac{(\theta_{ij}^n)^2}{2}\left(Y_{j-1}^n - 2Y_j^n + Y_{j+1}^n\right) - \frac{\theta_{ij}^n}{2}\left(Y_{j-1}^n - 4Y_j^n + 3Y_{j+1}^n\right) + Y_{j+1}^n \ ;$$

$$② \ Y_i^{n+1} = \frac{\theta_{ij}^n(1+\theta_{ij}^n)}{2}Y_j^n + \left(1-(\theta_{ij}^n)^2\right)Y_{j+1}^n - \frac{\theta_{ij}^n(1-\theta_{ij}^n)}{2}Y_{j+2}^n \ ;$$

$$= \frac{(\theta_{ij}^n)^2}{2}\left(Y_{j+2}^n - 2Y_{j+1}^n + Y_j^n\right) - \frac{\theta_{ij}^n}{2}\left(Y_{j+2}^n - Y_j^n\right) + Y_{j+1}^n \ ;$$

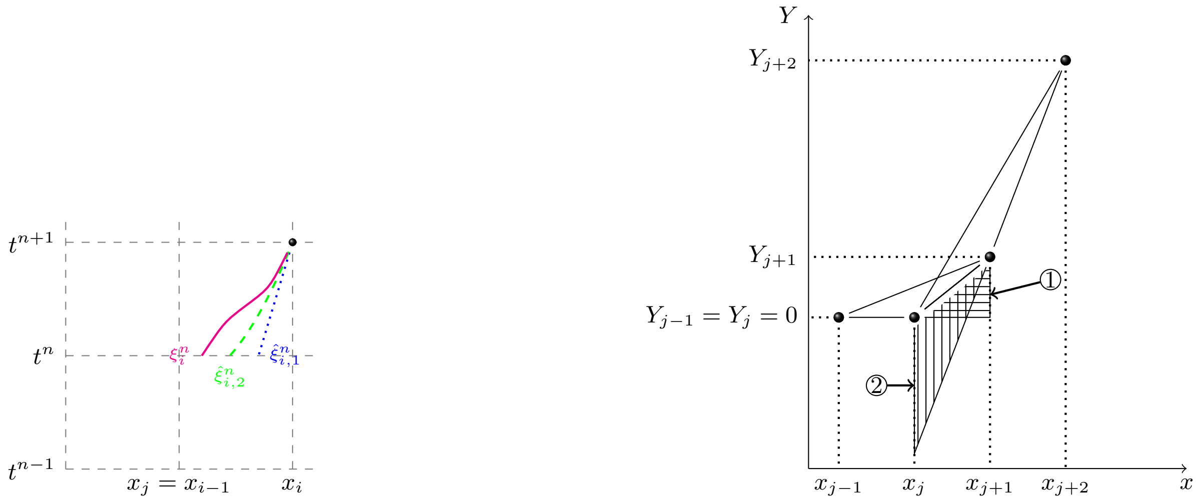
These formula result from a Lagrange interpolation of $Y(\hat{\xi}_i^n)$ involving the nodes x_{j-1}, x_j and x_{j+1} (resp. x_j, x_{j+1} and x_{j+2}). These two possibilities thus differ from the stencil and the sign of the weights: In both cases, Y_j^n and Y_{j+1}^n have positive coefficients unlike the third value (Y_{j-1}^n or Y_{j+2}^n) which has a negative weight. This proves that each scheme (of 2nd-order) is not monotonicity-preserving (as stated by Godunov [12, Th. 16.1]). We see on Fig. 2(b) that the linear combination in the second scheme may provide a negative value. Other configurations may imply negative values with the first scheme.

The idea of our algorithm is to combine the two schemes in order to ensure the maximum principle. We name "MOC2" the coupling between (18) and Sch. ①-②.

Two major properties of this strategy are the **explicitness** of the scheme and the fact that Δx and Δt can be chosen small enough independantly from each other: **In other words, there is no constraint deduced from stability considerations**.

To estimate the accuracy of our scheme, we get interested in initial datum $Y^0_{\varepsilon=0.2}$ for $\Lambda = \Lambda_1$. The unique solution is given by $\mathcal{Y}$ in **Prop. 1**. In this particular case, we can invert Θ:

$$\Theta_t^{-1}(x) = \begin{cases} \Upsilon_\varepsilon(t)e^{-\Psi(t)}\dfrac{x}{L}, & x \in [0, \beta_\varepsilon(t)], \\ \beta_0 + \varepsilon - \dfrac{\varepsilon}{\Psi(t)}\ln\left[1 + \dfrac{\Psi(t)}{\varepsilon}\left[\Upsilon_\varepsilon(t)\left(1-\dfrac{x}{L}\right) - L + \beta_0 + \varepsilon\right]\right], & \\ \qquad\qquad\qquad\qquad\qquad x \in [\beta_\varepsilon(t), \delta_\varepsilon(t)], & \\ L - \Upsilon_\varepsilon(t)\left[1 - \dfrac{x}{L}\right], & x \in [\delta_\varepsilon(t), L], \end{cases}$$



(a) 1^{st} order and 2^{nd} order upstream points

(b) Configuration of nodes and influence of interpolation schemes

Figure 2: Method of characteristics: Computation of the upstream point ξ_i^n and interpolation at (t^n, ξ_i^n)

with:

$$\Upsilon_\varepsilon(t) = L - \beta_0 - \varepsilon + \beta_0 e^{\Psi(t)} + \varepsilon \frac{e^{\Psi(t)} - 1}{\Psi(t)},$$

and δ_ε, β_ε given by (15a-15b). We recall that $\Theta_t^{-1}(-x) = \Theta_t^{-1}(x)$. On Fig. 4, we compare the exact solution to the numerical simulation with "MOC2". Here, there are 300 nodes, 1200 iterations and Δt is tuned as $1 \cdot 10^{-2}$. Pointwise, there is a slight diffusion located at the angles which can be explained by the smoothness requirements of the scheme. Indeed, the Taylor expansion resulting in Sch. ①-② requires a $\mathscr{C}^2$-smoothness of the function, which is not the case with a Λ_1-regularization (Λ_1 is the only polynomial such that Θ is analytically invertible). However, numerical results are still better than those obtained with the classical upwind scheme (see Fig. 4(b)). As for the mean value error, we use the exact expression (3a) for $\mu(t^n)$ to compare to the second-order approximation $(\Delta x/2) \sum (Y_i + Y_{i+1})$ of the integral of the numerical solution. The error is of order $(\Delta t)^2$.

For $\Lambda = \Lambda_3$, the solution is smooth enough in regard to the derivation of the scheme but Θ cannot be inverted explicitly. Nevertheless, we explicitly know the top β_ε and the bottom δ_ε of the transition zone between 1 and 0 (*cf.* § 3.3). We thus estimate on Fig. 5 the expression $\mathcal{E}_3 = |Y_j^{n=100} - 1| + |Y_k^{n=100}|$ where $\beta_\varepsilon(t^n = 1) \in [x_j, x_{j+1}]$ and $\delta_\varepsilon(t^n = 1) \in [x_k, x_{k+1}]$. It emphasizes the global order 2 in space of the method.

5 Conclusion

We have studied the particular case of dimension 1 for the Abstract Bubble Vibration model, which is an initial-boundary value problem for a coupled system of two PDEs. Based on properties previoulsy proven, we have found out global-in-time solutions for smooth and nonsmooth initial data. This tends to show that the approximation of the time of existence obtained in [15] is not optimal in 1D.

The derivation of the solution is based on the method of characteristics, which also enables to construct a numerical scheme to simulate smooth solutions. This scheme has been proven to be explicit and accurate in the special case of a single "smoothened" bubble. On the one hand, it remains to prove theoretically the convergence of the scheme. On the other hand, the scheme has to be extended to 2D and 3D cases with cartesian meshes: The calculation of the upstream point will be similar but we shall have to pay attention to the interpolating step. Finally, to simulate radial cases for which we have an explicit solution, cartesian meshes may not be convenient, which would require further investigations in the derivation of the scheme on other kinds of meshes.

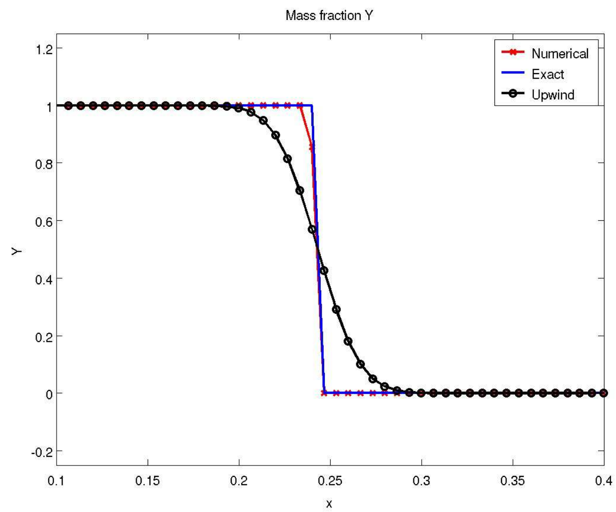


Figure 3: Upwind and Després-Lagoutière schemes for indicator functions (100 iterations)

A Bubbles in radial geometries

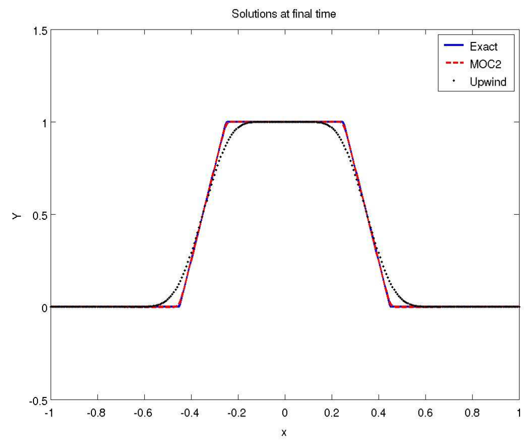
Assume $\Omega = \mathcal{D}_o(0,R) \subset \mathbb{R}^2$ with radius $R > 0$. The *Abstract Bubble Vibration* model reads in polar coordinates:

$$\begin{cases} \partial_t Y + \partial_r Y \cdot \partial_r \phi + \frac{1}{r^2} \partial_\theta Y \cdot \partial_\theta \phi = 0\,, \\ Y(t=0,\cdot,\cdot) = Y^0, \\ \dfrac{1}{r}\partial_r \left(r \partial_r \phi \right) + \dfrac{1}{r^2} \partial^2_{\theta^2} \phi = \psi(t) \left(Y - \dfrac{1}{\pi R^2} \displaystyle\int_0^{2\pi} \int_0^R Y(t, r', \theta')\, r' \mathrm{d}r' \mathrm{d}\theta' \right), \\ \partial_r \phi(t, r=R, \theta) = 0\,. \end{cases}$$

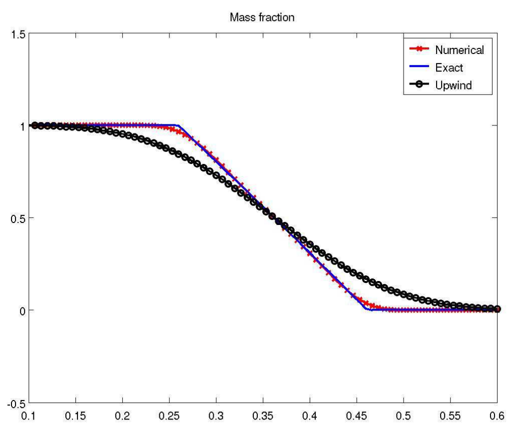
In addition, we suppose that $Y^0(t,r,\theta) = \mathbf{1}_{\{r \le \rho_0\}}(r)$, $0 < \rho_0 < R$. We are then looking for radial weak solutions, rewriting the transport equation in the following conservative form:

$$\partial_t Y + \frac{1}{r}\partial_r (r Y \partial_r \phi) = \psi Y \left(Y - \frac{2}{R^2} \int_0^R Y(t, r') r' \, \mathrm{d}r' \right),$$

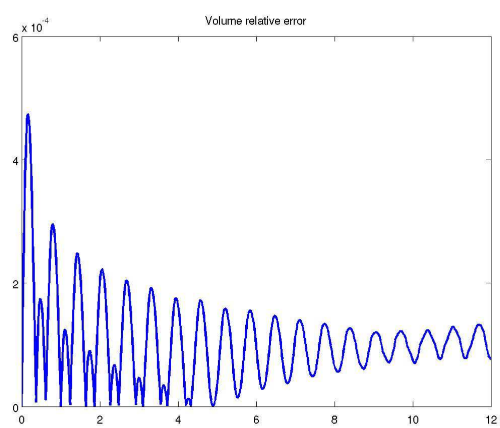
so as to define weak solutions.



(a) Solution at time $t = 12$



(b) Solution at time $t = 12$ (close to the regularization)



(c) Mean value error

Figure 4: Numerical solutions with upwind and "MOC2" schemes for the smooth Λ_1-case

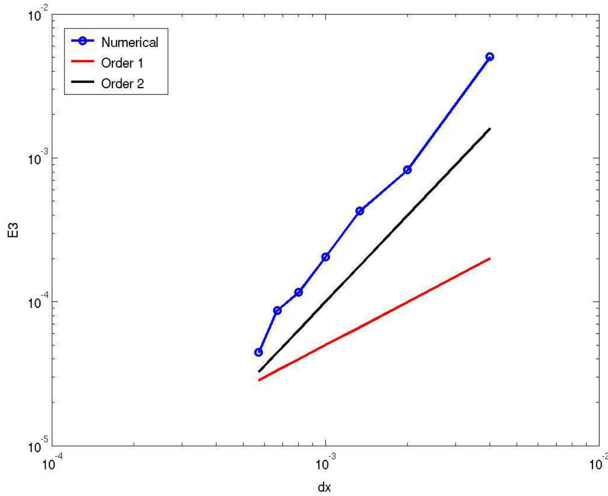


Figure 5: Order of convergence for $\mathcal{E}_3$ with the "MOC2" schemes in the smooth Λ_3-case

The weak formulation reads:

$$\begin{aligned}
&\forall\, \varphi \in \mathscr{C}_0^\infty\big([0,\mathscr{T})\times(0,R]\big),\\
&\qquad \int_0^{\mathscr{T}}\int_0^R Y\left[\partial_t\varphi + \partial_r\phi\cdot\partial_r\varphi + \psi(t)\varphi\left(Y - \frac{2}{R^2}\int_0^R Y(t,r')r'\,\mathrm{d}r'\right)\right]\, r\mathrm{d}r\mathrm{d}\theta\\
&\qquad\qquad + \int_0^{\rho_0}\varphi(0,r)\, r\mathrm{d}r = 0.
\end{aligned}$$

If there exists a weak radial solution $Y(t,r,\theta) = \mathbf{1}_{\{r\le\rho(t)\}}(r)$, then the previous expression reduces in the particular case $\varphi(t,r) = \varphi(t) \in \mathscr{C}_0^\infty\big([0,\mathscr{T})\big)$ to:

$$\int_0^{\mathscr{T}} \frac{\rho^2(t)}{2}\left[\varphi'(t) + \varphi(t)\psi(t)\left(1 - \frac{\rho^2(t)}{R^2}\right)\right]\,\mathrm{d}t + \varphi(0)\frac{\rho_0^2}{2} = 0.$$

This implies that $\zeta(t) = \rho^2(t)/R^2 \in L^\infty(0,\mathscr{T})$ satisfies in the sense of distributions the following ODE:

$$\zeta' = \psi\zeta(1-\zeta).$$

As the right hand side is bounded, then ζ is continuous and almost evereywhere differentiable, which means that the ODE holds in a strong sense. Moreover, as $\zeta(0) \in (0,1)$, then $\zeta(t) \in (0,1)$ almost everywhere. We deduce that:

$$\zeta(t) = \frac{\zeta(0)\exp\Psi(t)}{1-\zeta(0)+\zeta(0)\exp\Psi(t)},$$

which leads to (14) for $d=2$ coming back to variable ρ. The case $d=3$ is treated analogously.

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