Theoretical Study of an Abstract Bubble Vibration Model

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Abstract. We present the theoretical study of a hyperbolic-elliptic system of equations called *Abstract Bubble Vibration* (Abv) model. This simplified system is derived under non-physical assumptions from a model describing a diphasic low Mach number flow. It is thus aimed at providing mathematical properties of the coupling between the hyperbolic transport equation and the elliptic Poisson equation.

We prove an existence and uniqueness result including the approximation of the time of existence for any smooth initial condition. In particular, we obtain a global-in-time existence result for small initial data. We then pay attention to properties of solutions (depending of their smoothness) such as maximum properties or evenness. In particular, an explicit formula of the mean value of solutions is given.

1. Introduction

Over the past two centuries, several systems of equations have been proposed to model motion of fluids. The most general formulation is the compressible Navier-Stokes system that consists of conservation laws for variables such as density, momentum and energy. This system may be expressed differently including equations on the temperature or pressure variables. Then, the equations may be simplified through physical considerations. For instance, in this particular study, we are interested in the modelling of a diphasic flow – which can be assimilated as a nonmiscible 2-fluid flow – where the Mach number relative to each fluid is very small. In other words, the ratio of the fluid velocity to the sound speed is supposed to be negligible, which enables us to make a formal asymptotic expansion with respect to this small parameter. We shall also underline that our model does not allow for phase change and surface tension even if we consider these phenomena as perspectives for further studies. The system we obtain is called *Diphasic Low Mach Number* (Dlmn) [3]. A numerical study is carried out in [4]. We also underline that similar systems appear in other physical frameworks, like in the Kull-Anisimov instability model [8].

Finally, after non-physical simplifications, we obtain a 2-equation system decoupled from temperature and pressure laws [5]. This system – called *Abstract Bubble Vibration* (Abv) model – consists of a Poisson equation for the velocity field assumed to be potential and a transport equation for the mass fraction of gas, together with initial and boundary conditions. Mathematically speaking, this model has an interesting structure since it couples elliptic and hyperbolic effects in a nonlinear way. Despite a lack of physical meaning and although thermodynamic variables are not involved anymore, we get relevant numerical results by means of

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an efficient numerical algorithm. Nevertheless, one of the most difficult issues raised by fluid mechanics is the numerical handling of interfaces. That is why an accurate resolution requires an adaptive mesh refinement technique to avoid any diffusion of the interface. Concerning these numerical aspects, refer to [12].

The investigations of simpler models provide reliable theoretical and numerical results. They also turn out to be useful for more general studies carried out on the full set of equations since they allow a better understanding of the overall process of the motion of bubbles.

In this paper, we prove different properties of the Abv model. In Section 2, we describe the derivation of this model from the compressible Navier-Stokes set of equations while in Section 3, we get interested in theoretical results including existence and uniqueness issues and properties that solutions must satisfy. At last, we conclude with a lemma that provides an explicit expression for the mean value of solutions that can be interpreted as the volume of a bubble in the case of nonsmooth solutions (more precisely indicator functions of subdomains).

2. Derivation of the model

As bubbles may appear in an operating reactor, we deal with a compressible diphasic flow in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. While many formulations are based on a set of equations for each phase, our model consists of a single system in which variables are global and not specific to one phase or the other.

The compressible Navier-Stokes equations for a viscous compressible diphasic flow under gravity read in conservative variables:

$$\begin{cases} \partial_t(\rho Y_1) + \nabla \cdot (\rho Y_1 \boldsymbol{u}) = 0, & \text{(1a)} \\ \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, & \text{(1b)} \\ \partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P + \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g}, & \text{(1c)} \\ \partial_t(\rho E) + \nabla \cdot (\rho E \boldsymbol{u}) = -\nabla \cdot (P\boldsymbol{u}) + \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\boldsymbol{\sigma} \boldsymbol{u}) + \rho \boldsymbol{g} \cdot \boldsymbol{u}. & \text{(1d)} \end{cases}$$

Here and in the sequel, $\boldsymbol{u}$ denotes the velocity field, ρ the density, $\boldsymbol{g}$ the gravity field, $E = \varepsilon + |\boldsymbol{u}|^2/2$ the total energy, ε the internal energy, κ the thermal conductivity, T the temperature and P the pressure. As shown in the course of the derivation [3], we can formally approximate P by a function depending only on time when the Mach number is close to 0. We note $\boldsymbol{\sigma}$ the linearized Cauchy stress tensor that reads under the linear elasticity assumption as:

$$\boldsymbol{\sigma} = \mu \left(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u} \right) + \lambda (\nabla \cdot \boldsymbol{u}) \mathcal{I}_d.$$

λ and μ are the Lamé coefficients (see [10] for example). Finally, Y_1 is the mass fraction of gas. Hence, Y_1 can be assimilated to the indicator function of the domain $\Omega_1(t)$ occupied by gas. Then, $\Omega_2(t) = \Omega \backslash \Omega_1(t)$ is the liquid domain and $\Sigma(t) = \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t)$ is the location of the interface between liquid and gas. It corresponds to the discontinuity of the function Y_1 because the two phases are non-miscible. The system is closed as soon as the physical coefficients ρ, κ, λ and μ are known (through equations of state and constitutive laws).

After a singular perturbation analysis with respect to the Mach number $\mathcal{M}_* = \mathcal{U}_*/\sqrt{P_*/\rho_*} \ll 1$ applied on the non-conservative form of Syst. (1), the latter boils down to:

$$\begin{cases} \partial_t Y_1 + \boldsymbol{u} \cdot \nabla Y_1 = 0, & \text{(2a)} \\ \nabla \cdot \boldsymbol{u} = \mathcal{G}, & \text{(2b)} \\ \rho\big[\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big] = -\nabla\pi + \nabla \cdot \big[\mu(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u})\big] + \rho\boldsymbol{g}, & \text{(2c)} \\ \rho c_p(\partial_t T + \boldsymbol{u} \cdot \nabla T) = \alpha T P'(t) + \nabla \cdot (\kappa \nabla T), & \text{(2d)} \\ P'(t) = \mathcal{H}(t). & \text{(2e)} \end{cases}$$

For more details about the derivation of this system – called *Diphasic Low Mach Number* (Dlmn) system – please refer to [3, 4]. The methods used to derive Dlmn as well as its theoretical study are based on works from Majda and Embid [7, 9]. α and c_p are thermodynamic variables, π is a dynamic pressure while $\mathcal{G}$ and $\mathcal{H}$ are nonlinear functional of Y_1, T and P.

The elliptic equation (2b) is a reformulation of the mass conservation law to highlight the compressibility of the system despite the low Mach number. Furthermore, it leads to Eq. (2e) by ensuring the compatibility with the boundary condition $\boldsymbol{u}_{|\partial\Omega} = 0$ [2]. Syst. (2) is thus closed.

Since all the coefficients appearing in Syst. (2) depend implicitly on Y_1, T and P through the equations of state, the Dlmn system is highly nonlinear. To tackle this issue, we derive, as a preliminary, a simplified model based on the potential assumption that consists in stating that $\boldsymbol{u}$ is a gradient field. Let ϕ be the potential (known up to a constant), *i.e.* $\boldsymbol{u} = \nabla\phi$. Eq. (2c) is overlooked and ϕ is determined by means of a Poisson equation which is coupled to the mass fraction, temperature and pressure equations due to the dependance of $\mathcal{G}$ w.r.t. (Y_1, T, P). To decouple the velocity equation from temperature and pressure evolution laws, we replace $\mathcal{G}$ by a nonphysical term depending only on Y_1 (linearly). The simplified model – called *Abstract Bubble Vibration* (Abv) model – reads [5]:

$$\begin{cases} \partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0, & \text{(3a)} \\ Y_1(0, \boldsymbol{x}) = Y^0(\boldsymbol{x}), & \text{(3b)} \\ \Delta\phi = \psi(t) \left[Y_1(t, \boldsymbol{x}) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega Y_1(t, \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}' \right], & \text{(3c)} \\ \nabla\phi \cdot \boldsymbol{n}_{|\partial\Omega} = 0. & \text{(3d)} \end{cases}$$

Y^0 and ψ are given functions of $\boldsymbol{x}$ and t respectively, with ψ continuous on $[0, +\infty)$. We additionally assume that Ω **is smooth enough** to allow the existence of the normal unit vector $\boldsymbol{n}_{|\partial\Omega}$ and to provide elliptic regularity results for the Poisson equation. Note that the global system (3) is still non-linear.

In spite of the loss of physical meaning due to the lack of thermodynamic effects, Syst. (3) is interesting for its structure: It consists of a transport equation for Y_1 (hyperbolic) and a Poisson equation for ϕ (elliptic). The same nonlinear hyperbolic-elliptic coupling appears in Syst. (2) when $\boldsymbol{u}$ is written as $\boldsymbol{w} + \nabla\phi$ (Hodge decomposition), which legitimates theoretical and numerical studies of the Abv model.

3. Theoretical results

We present in this section results in a smooth case that is interesting from a mathematical point of view as well as properties that less regular solutions must satisfy. The last paragraph deals with estimates for the volume of bubbles.

3.1. **Preliminary.** Before any investigation, we make two remarks about the potential ϕ. On the one hand, it is obvious that the potential cannot be unique, except up to a constant. We chose the following gauge for ϕ:

$$\int_\Omega \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0. \tag{4}$$

On the other hand, as Eq. (3c) is stationary, the potential necessarily satisfies the initial condition:

$$\begin{cases} \Delta\phi^0 = \psi(0)\left[Y^0(\boldsymbol{x}) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Y^0(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\right], \\ \nabla\phi^0\cdot\boldsymbol{n}_{|\partial\Omega} = 0. \end{cases} \tag{5}$$

Eqs. (4-5) will be implicitly included in Syst. (3) in the sequel even if they are not referred to. Any initial condition satisfying (5) is called well-prepared.

We introduce the functional space related to this problem (and more specifically to advection problems) defined for $\mathscr{T} > 0$ by:

$$\mathcal{W}_{s,\mathscr{T}}(\Omega) = \mathscr{C}^0\left([0,\mathscr{T}], L^2(\Omega)\right) \cap L^\infty\big([0,\mathscr{T}], \mathcal{H}^s(\Omega)\big).$$

The set $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ is a Banach space when equipped with the norm:

$$\|f\|_{s,\mathscr{T}} = \sup_{t\in[0,\mathscr{T}]} \|f(t,\cdot)\|_s.$$

The injection from $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ to $\mathscr{C}^0\big([0,\mathscr{T}], \mathcal{H}^{s'}(\Omega)\big)$ is continuous for any $s' < s$. So does the injection from $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ to $\mathscr{C}^0\big([0,\mathscr{T}]\times\overline{\Omega}\big)$ when $s > d/2$ (see **Lemma 10** and [5]).

3.2. **Short time existence theorem. Th. 1** below was first published in [5]. Nevertheless, we present here a proof that enables to specify an approximation of the time interval (**Th. 2**) and that leads to a global existence in time result for a certain class of initial data (**Cor. 1**). Let s_0 be the integer $s_0 = \lfloor d/2 \rfloor + 1$.

Theorem 1. *Assume $Y^0 \in \mathcal{H}^s(\Omega)$ with s an integer such that $s \geq s_0 + 1$ and $\psi \in \mathscr{C}^0(0,+\infty)$. Then there exists $\mathscr{T}_0 > 0$ depending on ψ and $\|Y^0\|_s$ such that System (3) has a unique classical solution $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ for $\mathscr{T}$ at least greater than $\mathscr{T}_0$.*

The time of existence $\mathscr{T}_0$ is not said to be optimal: It is prescribed by the way we prove **Th. 1**, namely the combination of a boundedness property in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and a strong convergence in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$. In the course of the proof, we derive the constraint (8) for $\mathscr{T}_0$ that we improve to obtain the following lower bound:

Theorem 2. *Under the same assumptions as in Th. 1, we have $Y_1 \in \mathcal{W}_{s,\mathscr{T}_0}(\Omega)$ for any $\mathscr{T}_0 > 0$ such that:*

$$\int_0^{\mathscr{T}_0} |\psi(\tau)|\,\mathrm{d}\tau \leq \frac{1}{e\cdot C_{abv}(s,d,\Omega)}\left\|Y^0 - \mu(Y^0)\right\|_s^{-1}, \tag{6}$$

where $\mu(Y^0) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Y^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and C_{abv} is a universal constant.

We note that the left hand side in (6) is monotone-increasing w.r.t. $\mathscr{T}_0$. Thus, the greater $\left\|Y^0 - \mu(Y^0)\right\|_s$, the lower $\mathscr{T}_0$.

Furthermore, if $Y^0 \equiv 1 \in \mathcal{H}^\infty(\Omega)$ – which corresponds to a bubble occupying the whole domain – (resp. $Y^0 \equiv 0$), the unique solution is trivially given by $Y_1 \equiv 1$ (resp. $Y_1 \equiv 0$) without restriction on the time of existence. Likewise, for $\psi \equiv 0$, $Y_1 \equiv Y^0$ is a global solution. In those three cases, (6) is optimal.

We also infer that given $\mathscr{T} > 0$ and $\psi \in \mathscr{C}^0(0, \mathscr{T})$, there exists a local solution $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ for every Y^0 s.t. $\left\|Y^0 - \mu(Y^0)\right\|_s \le \left[e \cdot C_{abv} \cdot \|\psi\|_{L^1(0,\mathscr{T})}\right]^{-1}$. Consequently, if ψ also belongs to $L^1(0, +\infty)$, we have a global-in-time existence result:

Corollary 1. *Let ψ be a function in $\mathscr{C}^0(0, +\infty) \cap L^1(0, +\infty)$. Then there exists a unique solution Y_1 global in time for any $Y^0 \in \mathcal{H}^{s_0+1}(\Omega)$ provided:*

$$\left\|Y^0 - \mu(Y^0)\right\|_{s_0+1} \le \frac{1}{e \cdot C_{abv}(s, d, \Omega) \cdot \|\psi\|_{L^1}}.$$

Proof of Theorem 1. For the proof of uniqueness, see **Lemma 1** below. For the existence part, we consider the Picard iterates for Syst. (3). More precisely, we introduce the sequences $(Y^{(k)})$ and $(\phi^{(k)})$ defined by induction as follows:

① $Y^{(k=0)} = Y^0$.

② Given $Y^{(k)}$, we compute $\phi^{(k)}$ as the solution of:

$$\begin{cases} \Delta\phi^{(k)}(t, \boldsymbol{x}) = \psi(t) \left(Y^{(k)}(t, \boldsymbol{x}) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega Y^{(k)}(t, \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}' \right), \\ \nabla\phi^{(k)} \cdot \boldsymbol{n}_{|\partial\Omega} = 0. \end{cases} \tag{7a}$$

③ Then, $Y^{(k+1)}$ satisfies:

$$\begin{cases} \partial_t Y^{(k+1)} + \nabla\phi^{(k)} \cdot \nabla Y^{(k+1)} = 0, \\ Y^{(k+1)}(0, \cdot) = Y^0. \end{cases} \tag{7b}$$

We shall show that the sequence $(Y^{(k)})$ is bounded in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and converges strongly in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$. Applying **Lemma 8** to Eq. (7b) and **Lemma 9** to Eq. (7a), we get:[1]

$$\begin{aligned} \|Y^{(k+1)}\|_{s,\mathscr{T}} &\le \|Y^0\|_s \exp\left[C_{adv}(s) \int_0^{\mathscr{T}} \|\mathrm{Hess}(\phi^{(k)})\|_{s-1}(t) \, \mathrm{d}t \right] \\ &\le \|Y^0\|_s \exp\left[C_{adv}(s) \cdot C_{ell}(s-1) \cdot \|Y^{(k)}\|_{s-1,\mathscr{T}} \int_0^{\mathscr{T}} |\psi(t)| \, \mathrm{d}t \right] \\ &\le \|Y^0\|_s \exp\left[C_{abv}(s) \cdot \|Y^{(k)}\|_{s-1,\mathscr{T}} \cdot \overline{\Psi}(\mathscr{T}) \right], \end{aligned}$$

where $C_{abv}(s) = C_{adv}(s) \cdot C_{ell}(s-1)$ and $\overline{\Psi}$ is s.t. $\overline{\Psi}' = |\psi|$ and $\overline{\Psi}(0) = 0$. We introduce the sequence (u_k) defined by $u_0 = C_{abv}(s) \cdot \left\|Y^0\right\|_s \cdot \overline{\Psi}(\mathscr{T})$ and $u_{k+1} = u_0 \exp u_k$. Thus, we have $C_{abv}(s) \cdot \|Y^{(k)}\|_{s,\mathscr{T}} \cdot \overline{\Psi}(\mathscr{T}) \le u_k$.

[1]We emphasize dependancies on s for the constants appearing in the proof and we omit other dependancies as specified in the annex.

It is easy to prove that (u_k) converges iff $u_0 \leq e^{-1}$. Then, the limit is the lowest solution[2] x_0 of the equation $x\exp(-x) = u_0$ and we have $u_k \leq u_{k+1} \leq x_0$. Hence, under the assumption:

$$e \cdot C_{abv}(s) \cdot \|Y^0\|_s \cdot \overline{\Psi}(\mathscr{T}) \leq 1, \tag{8}$$

the sequence $(Y^{(k)})$ is uniformly bounded in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ with upper bound $e^{x_0}\|Y^0\|_s$. In particular, this result implies that the sequence $\big(\|Y^{(k)}(t,\cdot)\|_0\big)$ is equicontinuous and uniformly bounded in $\mathscr{C}^0\big([0,\mathscr{T}]\big)$. The *Arzelà-Ascoli* theorem yields the existence of a subsequence $(Y^{(k')})$ that converges strongly in $\mathscr{C}^0\big([0,\mathscr{T}], L^2(\Omega)\big)$. Likewise, the boundedness property in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ also provides the weak-$\star$ convergence of a subsequence $(Y^{(k'')})$ of $(Y^{(k')})$ in $L^\infty\big([0,\mathscr{T}], \mathcal{H}^s(\Omega)\big)$. We still note $(Y^{(k)})$ the weak-$\star$ convergent subsequence in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and $\tilde{Y} \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ its limit.

We shall prove now that the sequence $(Y^{(k)})$ converges strongly in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ thanks to a contraction inequality. Indeed, we deduce from Eq. (7b):

$$\begin{cases} \partial_t(Y^{(k+1)} - Y^{(k)}) + \nabla\phi^{(k)} \cdot \nabla(Y^{(k+1)} - Y^{(k)}) = -(\nabla\phi^{(k)} - \nabla\phi^{(k-1)}) \cdot \nabla Y^{(k)}, \\ (Y^{(k+1)} - Y^{(k)})(0,\cdot) = 0. \end{cases}$$

The energy estimate given by **Lemma 8** reads:

$$\begin{aligned} e^{-\chi_0^{(k)}(t)} & \left\| \left(Y^{(k+1)} - Y^{(k)}\right)(t,\cdot) \right\|_0 \\ & \leq \int_0^t e^{-\chi_0^{(k)}(\tau)} \left\| \left(\nabla\phi^{(k)} - \nabla\phi^{(k-1)}\right) \cdot \nabla Y^{(k)}(\tau,\cdot) \right\|_0 \mathrm{d}\tau, \\ & \leq C_M(0,s-1,d) \cdot \|Y^{(k)}\|_{s,\mathscr{T}} \int_0^t e^{-\chi_0^{(k)}(\tau)} \left\| \left(\nabla\phi^{(k)} - \nabla\phi^{(k-1)}\right)(\tau,\cdot) \right\|_0 \mathrm{d}\tau, \\ & \leq \underbrace{C_M \cdot e^{x_0}\|Y^0\|_s \cdot C_{PW} \sup_{t\in[0,\mathscr{T}]} |\psi(t)|}_{C_{abv,2}} \int_0^t e^{-\chi_0^{(k)}(\tau)} \left\| \left(Y^{(k)} - Y^{(k-1)}\right)(\tau,\cdot) \right\|_0 \mathrm{d}\tau, \end{aligned}$$

using **Lemma 9** and the Moser inequality (**Lemma 10**). Here, the exponent is given by $\chi_0^{(k)}(t) = \dfrac{1}{2}\displaystyle\int_0^t \|\Delta\phi^{(k)}(\tau,\cdot)\|_\infty \,\mathrm{d}\tau$. Using the boundedness property and the Sobolev embedding inequality (see **Lemma 10**), we have:

$$\chi_0^{(k)}(t) \leq \int_0^t |\psi(\tau)| \cdot \|Y^{(k)}(\tau,\cdot)\|_\infty \,\mathrm{d}\tau \leq \chi(t),$$

with $\chi(t) = e^{x_0}\|Y^0\|_s \cdot C_{sob}(s) \cdot \overline{\Psi}(t)$. We can thus replace $\chi_0^{(k)}$ by χ in the energy estimate:[3]

$$e^{-\chi(t)} \left\| \left(Y^{(k+1)} - Y^{(k)}\right)(t,\cdot) \right\|_0 \leq C_{abv,2} \int_0^t e^{-\chi(\tau)} \left\| \left(Y^{(k)} - Y^{(k-1)}\right)(\tau,\cdot) \right\|_0 \mathrm{d}\tau.$$

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[2] x_0 satisfies $x_0 \in [0,1]$ and $x_0 = 1$ iff (8) is an equality.

[3] See Lemma 8: the exponent may be replaced by any upper bound (in the differential form).

Iterating the process, we obtain:

$$\begin{aligned}
&e^{-\chi(t)}\left\|\left(Y^{(k+1)}-Y^{(k)}\right)(t,\cdot)\right\|_0\\
&\quad\leq C^k_{abv,2}\int_0^t e^{-\chi(\tau)}\left\|\left(Y^{(1)}-Y^{(0)}\right)(\tau,\cdot)\right\|_0\frac{(t-\tau)^{k-1}}{(k-1)!}\,\mathrm{d}\tau\\
&\quad\leq \frac{C^k_{abv,2}\,t^k}{k!}\|Y^{(1)}-Y^{(0)}\|_{0,\mathscr{T}}.
\end{aligned}$$

Thus:

$$\|Y^{(k+1)}-Y^{(k)}\|_{0,\mathscr{T}}\leq\frac{(C_{abv,2}\,\mathscr{T})^k}{k!}e^{\chi(\mathscr{T})}\|Y^{(1)}-Y^{(0)}\|_{0,\mathscr{T}}.$$

The series $\sum_k\|Y^{(k+1)}-Y^{(k)}\|_{0,\mathscr{T}}$ is convergent, which shows that the sequence $Y^{(k)}$ converges in the complete space $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ to $Y\in\mathcal{W}_{0,\mathscr{T}}(\Omega)$. By uniqueness of the limit, the weak-$\star$ limit $\tilde{Y}$ is necessarily equal to Y. Therefore, $Y\in\mathcal{W}_{s,\mathscr{T}}(\Omega)$ even if there is no proof that $Y^{(k)}$ tends to Y in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ (strongly). However, we can show by means of an interpolation inequality [11] that the convergence is strong in $\mathcal{W}_{s',\mathscr{T}}(\Omega)$ for any $s'<s$.

Likewise, we prove that $\nabla\phi^{(k)}$ converges in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ to $\Phi\in\mathcal{W}_{s+1,\mathscr{T}}(\Omega)$. Indeed, applying **Lemma 9**, we have:

$$\begin{aligned}
\|\nabla\phi^{(k)}\|_{s+1,\mathscr{T}}&\leq C_{ell}(s)\sup_{t\in[0,\mathscr{T}]}|\psi(t)|\cdot\|Y^{(k)}\|_{s,\mathscr{T}},\\
\left\|\nabla\left(\phi^{(k)}-\phi^{(k-1)}\right)\right\|_{0,\mathscr{T}}&\leq C_{PW}\sup_{t\in[0,\mathscr{T}]}|\psi(t)|\cdot\left\|\left(Y^{(k)}-Y^{(k-1)}\right)\right\|_{0,\mathscr{T}}.
\end{aligned}$$

The previous results for $Y^{(k)}$ provides the convergence for $\nabla\phi^{(k)}$. Moreover, there exists $\phi\in\mathcal{W}_{s+2,\mathscr{T}}(\Omega)$ such that $\Phi=\nabla\phi$ because the gradient field space is closed.

It remains to prove that Y and $\nabla\phi$ are solutions to Syst. (3). To do so, we rewrite (7b-7a) as:

$$\begin{aligned}
Y^{(k+1)}&=Y^0-\int_0^t\nabla\phi^{(k)}\cdot\nabla Y^{(k+1)}\,\mathrm{d}\tau,\\
\forall\,\varphi\in\mathcal{H}^1(\Omega),\ \int_\Omega\nabla\varphi\cdot\nabla\phi^{(k)}\,\mathrm{d}\boldsymbol{x}&=-\psi(t)\int_\Omega\varphi\left(Y^{(k)}-\frac{1}{|\Omega|}\int_\Omega Y^{(k)}\,\mathrm{d}\boldsymbol{x}'\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

As each function involved in the latter relations belongs to $\mathscr{C}^0\big([0,\mathscr{T}]\times\overline{\Omega}\big)$ due to the embedding $\mathcal{W}_{s,\mathscr{T}}(\Omega)\subset\mathscr{C}^0\big([0,\mathscr{T}],\mathcal{H}^{s'}(\Omega)\big)$ for $s'<s$ (see Lemma A.1, [5]), we apply the dominated convergence theorem to obtain the integral form of (3). In particular, we have:

$$Y_1(t,\boldsymbol{x})=Y^0(\boldsymbol{x})-\int_0^t\nabla\phi\cdot\nabla Y_1(\tau,\boldsymbol{x})\,\mathrm{d}\tau.$$

The previous embedding results show that ∇Y_1 and $\nabla\phi$ are continuous. This fact implies that $Y_1\in\mathscr{C}^1\big([0,\mathscr{T}]\times\overline{\Omega}\big)$ and we recover the differential form of Eq. (3a). Similarly, $\nabla\phi\in\mathcal{W}_{s+1,\mathscr{T}}(\Omega)\subset\mathscr{C}^0\big([0,\mathscr{T}],\mathscr{C}^1(\overline{\Omega})\big)$, which means that the weak formulation above is equivalent to Eq. (3c) in the strong sense. ■

Proof of Theorem 2. Let $Y^0\in\mathcal{H}^s$ with $s>d/2+1$ and $\psi\in\mathscr{C}^0(0,+\infty)$. Then there exists $Y_1\in\mathcal{W}_{s,\mathscr{T}}(\Omega)$ with $\mathscr{T}$ prescribed by (8) (as great as possible, maybe

$\mathscr{T} = +\infty$). Let c_0 be a constant such that $\|Y^0 - c_0\|_s = \min\limits_{c\in\mathbb{R}} \|Y^0 - c\|_s$, *i.e.*:

$$c_0 = \frac{1}{|\Omega|}\int_\Omega Y^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Hence, we have $\|Y^0 - c_0\|_s \leq \|Y^0\|_s$. We consider Syst. (3) with initial condition $Z(0,\cdot) = Y^0 - c_0 \in \mathcal{H}^s$ to which we apply **Th. 1**. There exists a unique solution $Z \in \mathcal{W}_{s,\mathscr{T}'}(\Omega)$ for $\mathscr{T}'$ satisfying:

$$\int_0^{\mathscr{T}'} |\psi(t)|\,\mathrm{d}t \leq \frac{1}{e\cdot C_{abv}\cdot \|Y^0 - c_0\|_s}.$$

Hence, we can choose $\mathscr{T}' \geq \mathscr{T}$ with a strict inequality iff $c_0 \neq 0$ and $\mathscr{T} < +\infty$. Thus, $Y_1 = Z + c_0$ is a solution to Syst. (3) on $[0,\mathscr{T}']$. ■

3.3. **Other properties.** In this paragraph, we give some lemmas about solutions under weaker assumptions than in **Th. 1**. Set $\mathcal{Z}_\mathscr{T}(\Omega) = L^\infty\big([0,\mathscr{T}], W^{1,\infty}(\Omega)\big)$ for some $\mathscr{T} > 0$. Note that $\mathcal{W}_{s,\mathscr{T}}(\Omega) \subset \mathcal{Z}_\mathscr{T}(\Omega)$, which means that the lemmas below can be applied to the classical solution induced by **Th. 1**. We do not state any existence result in $\mathcal{Z}_\mathscr{T}(\Omega)$ but any solution must satisfy the following properties.

First, we shall wonder whether $\mathcal{Z}_\mathscr{T}(\Omega)$ is a suitable functional space for solutions to Syst. (3). Let $Y_1 \in \mathcal{Z}_\mathscr{T}(\Omega)$ be a solution of:

$$\begin{cases} Y_1 = Y^0 - \displaystyle\int_0^t \nabla\phi\cdot\nabla Y_1\,\mathrm{d}\tau, \\ \Delta\phi = \psi(t)\big(Y_1 - \mu(Y_1)\big). \end{cases} \tag{9}$$

As $Y_1 \in L^\infty\big([0,\mathscr{T}]\times\Omega\big)$, elliptic regularity results guarantee that the solution $\nabla\phi$ of the Poisson equation in (9) belongs to $L^\infty\big([0,\mathscr{T}],\mathscr{C}^0(\Omega)\big)$. Knowing that $\nabla Y_1 \in L^\infty\big([0,\mathscr{T}]\times\Omega\big)$, the term $\nabla\phi\cdot\nabla Y_1$ is in $L^\infty\big([0,\mathscr{T}],L^2(\Omega)\big) \subset L^1\big([0,\mathscr{T}],L^2(\Omega)\big)$. Thus, the integral in (9) is continuous w.r.t. t and differentiable for almost all t (see § II.4.1, [1]) and Y_1 satisfies (3a) in $L^2(\Omega)$ and thus almost everywhere in Ω, which legitimates the following calculus.

Lemma 1. *There exists at most one solution in the space $\mathcal{Z}_\mathscr{T}(\Omega)$.*

Proof. Let (Y_1,ϕ_1) and (Y_2,ϕ_2) be two solutions. Combining the two equations with the notation $\delta* = *_1 - *_2$, we have: $\partial_t\delta Y + \nabla\phi_1\cdot\nabla\delta Y = -\nabla Y_2\cdot\nabla\delta\phi$.

Multiplying by δY and integrating by parts, we get by virtue of the Cauchy-Schwarz inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta Y\|_0 \leq \frac{1}{2}\|\Delta\phi_1\|_\infty\|\delta Y\|_0 + \|\nabla Y_2\|_\infty\|\nabla\delta\phi\|_0.$$

We apply **Lemma 9** to the last term and the Grönwall's inequality to obtain $\|\delta Y\|_0 = 0$ due to the fact that $\|\delta Y(0,\cdot)\|_0 = 0$. ■

Lemma 2. *Assume Y_1 is a solution in the space $\mathcal{Z}_\mathscr{T}(\Omega)$. Then, Y_1 keeps the same upper and lower bounds as Y^0 almost everywhere.*

Proof. We first prove that if $Y^0 \geq 0$, then $Y_1 \geq 0$. Multiplying Eq. (3a) by $Y_1^- = \min(Y_1, 0) \in \mathcal{Z}_\mathscr{T}(\Omega)$ and integrating by parts, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y_1^-\|_0^2 = \frac{1}{2}\int_\Omega \big(Y_1^-\big)^2(t,\boldsymbol{x})\Delta\phi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

As $\Delta\phi \in \mathcal{Z}_{\mathscr{T}}(\Omega) \subset L^\infty\big([0,\mathscr{T}]\times\Omega\big)$, the Grönwall's inequality yields $\|Y_1^-\|_0 = 0$ allowing for the fact that $\|Y_1^-(0,\cdot)\|_0 = 0$. Thus $Y_1 \geq 0$ a.e.

If $Y^0 \leq 1$, we apply the previous result to the variable $Z = 1 - Y_1$ which is a solution to (3) with initial condition $Z(0,\cdot) = 1 - Y^0 \geq 0$. Hence $Z \geq 0$ and $Y_1 \leq 1$. The general case $Y^0 \in [a,b]$ can be inferred from the positivity of variables $Y_1 - a$ and $b - Y_1$, which are solutions to Syst. (3) with suitable initial data. ■

Lemma 3. *The system is time-reversible in $\mathcal{Z}_{\mathscr{T}}(\Omega)$.*

Proof. Let Y_1 be a solution to Syst. (3) in the class $\mathcal{Z}_{\mathscr{T}}(\Omega)$ for a certain $\mathscr{T} > 0$. The question adressed in the lemma is to determine whether starting from $Y_1(\mathscr{T},\cdot)$, one recovers the initial condition Y^0 by "inverting" the time scale by means of the transformation $t \mapsto \mathscr{T} - t$. With $\hat{\psi}(t) = -\psi(\mathscr{T} - t)$, we check out that $\big(\hat{Y}_1, \hat{\phi}\big) = \big(Y_1(\mathscr{T} - t, \boldsymbol{x}), -\phi(\mathscr{T} - t, \boldsymbol{x})\big)$ is a solution to the system on $[0,\mathscr{T}]$. By the uniqueness lemma 1, $\big(\hat{Y}_1, \hat{\phi}\big)$ is the unique solution and $\hat{Y}_1(\mathscr{T},\cdot) = Y_1(0,\cdot) = Y^0$. ■

The last lemma deals with the symmetric case where Ω satisfies:

- $\boldsymbol{x} \in \Omega \Longrightarrow (-\boldsymbol{x}) \in \Omega$;
- $\forall\, \boldsymbol{x} \in \partial\Omega$, $\boldsymbol{n}(-\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x})$.

Lemma 4. *If Ω is symmetric and Y^0 is even, then any solution in the space $\mathcal{Z}_{\mathscr{T}}(\Omega)$ is also even.*

Proof. Denoting $\tilde{Y}_1(t,\boldsymbol{x}) = Y_1(t,-\boldsymbol{x})$ and $\tilde{\phi}(t,\boldsymbol{x}) = \phi(t,-\boldsymbol{x})$, we remark that:

$$\int_\Omega \tilde{Y}_1(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega Y_1(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

which shows that $(\tilde{Y}_1, \tilde{\phi})$ is a solution to Syst. (3) with the same initial datum $Y^0(-\boldsymbol{x}) = Y^0(\boldsymbol{x})$ and the same boundary condition. The uniqueness lemma 1 provides $Y_1(t,\boldsymbol{x}) = \tilde{Y}_1(t,\boldsymbol{x}) = Y_1(t,-\boldsymbol{x})$. The velocity field is odd. ■

3.4. **Volume.** We consider in this paragraph a more general case, namely $Y_1 \in L^\infty\big([0,\mathscr{T}]\times\Omega\big)$ and Y^0 bounded in $[0,1]$: this case corresponds to the modelling of bubbles in which Y_1 is the mass fraction of gas. For miscible fluids, Y_1 takes values between 0 and 1 while in the present study (without phase change), Y_1 is equal to 0 or 1. In the latter case, the mean value of Y_1 is equal to the volume of the bubble. We present in this section a general result about mean values (**Prop. 1**) and its application to a more physical case (**Lemma 7**).

Let $\mu_n(t)$ be the mean value of $Y_1^n(t,\cdot)$ over Ω. In the class $\mathcal{Z}_{\mathscr{T}}(\Omega)$, when Y^0 takes values in $[0,1]$, so does Y_1 according to **Lemma 2**. The sequence $\big(\mu_n(t)\big)_n$ is bounded (in $[0,1]$) and monotone-decreasing (pointwise). Thus, $\mu_n(t)$ converges to $\mu_\infty(t) := |\Omega_1(t)|/|\Omega|$ where $\Omega_1(t) = \{\boldsymbol{x} \in \Omega \,:\, Y_1(t,\boldsymbol{x}) = 1\}$ since $(Y_1)_{|\Omega_1(t)} = (Y_1^n)_{|\Omega_1(t)} = 1$. Nonetheless, these considerations do not enable to conclude about the convergence of μ_n in the weaker case $L^\infty\big([0,\mathscr{T}]\times\Omega\big)$. This is achieved thanks to **Prop. 1**, which provides an explicit expression for μ_n and a new proof for a maximum principle restricted to $[0,1]$ (**Lemma 6**).

We first establish an ODE to which μ_n is a solution.

Lemma 5. *The sequence $(\mu_n)_n$ satisfies the following ODE:*

$$\mu_n'(t) = \psi(t)\big(\mu_{n+1}(t) - \mu_1(t)\mu_n(t)\big). \tag{10}$$

Proof. If Y_1 is a weak solution of Eq. (3a), then Y_1^n satisfies $\partial_t Y_1^n + \nabla\phi\cdot\nabla Y_1^n = 0$ according to the *renormalisation* principle [13]. That means:

$$\forall\, \xi \in \mathscr{C}_0^\infty\left([0,\mathscr{T}]\times\overline{\Omega}\right),\ \int_0^{\mathscr{T}}\int_\Omega Y_1^n\big(\partial_t\xi + \nabla\cdot(\xi\nabla\phi)\big)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0.$$

Taking $\xi(t,\boldsymbol{x}) = \zeta(t)\xi_p(\boldsymbol{x})$ with $\zeta \in \mathscr{C}_0^\infty\big(0,\mathscr{T}\big)$ and $\xi_p \in \mathscr{C}_0^\infty\left(\overline{\Omega}\right)$ converging pointwise to $\mathbf{1}_\Omega$, the last equality can be rewritten as:

$$\int_0^{\mathscr{T}}\zeta'\int_\Omega Y_1^n\xi_p\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^{\mathscr{T}}\zeta\int_\Omega Y_1^n\nabla\cdot(\xi_p\nabla\phi)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0.$$

In the limit as $p\to+\infty$ through the dominated convergence theorem, the equation reduces to:

$$\forall\,\zeta\in\mathscr{C}_0^\infty\big(0,\mathscr{T}\big),$$
$$\int_0^{\mathscr{T}}\zeta'(t)\mu_n(t)\,\mathrm{d}t + \int_0^{\mathscr{T}}\frac{\zeta(t)\psi(t)}{|\Omega|}\int_\Omega Y_1^n(t,\boldsymbol{x})\big(Y_1(t,\boldsymbol{x}) - \mu_1(t)\big)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0,$$

which is ODE (10) in the sense of distributions. Since ψ is continuous and μ_n bounded for all n, the right hand side in (10) is bounded. We deduce that μ_n is continuous, which provides the continuity of the right hand side. ODE (10) thus holds in a classical sense. ■

The main consequence is that we can derive an explicit expression for μ_n in terms of ψ and Y^0:

Proposition 1. *Let Ψ be s.t. $\Psi' = \psi$ and $\Psi(0) = 0$. Then:*

$$\mu_n(t) = \frac{\displaystyle\int_\Omega [Y^0(\boldsymbol{x})]^n\exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}{\displaystyle\int_\Omega\exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}. \tag{11}$$

Proof. Since $\mu_n' + \psi\mu_1\mu_n$ can be expressed as:

$$\left[\mu_n(t)\exp\int_0^t\mu_1(\tau)\psi(\tau)\,\mathrm{d}\tau\right]'\exp\left(-\int_0^t\mu_1(\tau)\psi(\tau)\,\mathrm{d}\tau\right),$$

ODE (10) can be rewritten under the integral form:

$$M_N(t) = \mu_N(0) + \int_0^t\psi(\tau)M_{N+1}(\tau)\,\mathrm{d}\tau, \tag{12}$$

with $M_N(t) = \mu_N(t)\exp\displaystyle\int_0^t\psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau$. By induction, we show that:

$$M_1(t) = \sum_{k=1}^N\mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!} + \int_0^t\psi(\tau)M_{N+1}(\tau)\frac{[\Psi(t)-\Psi(\tau)]^N}{N!}\,\mathrm{d}\tau.$$

Since Ψ is continuous and the sequence $\mu_k(0)$ is uniformly bounded ($Y^0 \in [0,1]$), the series $\sum_k \mu_{k+1}(0)\Psi^k(t)/k!$ is normally convergent on every compact set. Furthermore, the last term reads:

$$\frac{1}{|\Omega|}\int_0^t \int_\Omega \frac{\big[Y_1(\tau,\boldsymbol{x})\big(\Psi(t)-\Psi(\tau)\big)\big]^N}{N!} Y_1(\tau,\boldsymbol{x})\psi(\tau)e^{\int_0^\tau \psi(\sigma)\mu_1(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\tau.$$

In the limit as $N \to +\infty$, the integral tends to 0 by virtue of the dominated convergence theorem. Thus:

$$M_1(t) = \mu_1(t)\exp\left(\int_0^t \psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau\right) = \sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!}.$$

Multiplying by ψ and integrating, we obtain:

$$\exp\left(\int_0^t \psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau\right) = 1 + \sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^k}{k!}.$$

The combination of the last two equalities leads to:

$$\mu_1(t) = \frac{\displaystyle\sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!}}{\displaystyle 1+\sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^k}{k!}} = \frac{\displaystyle\int_\Omega Y^0(\boldsymbol{x})\exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}{\displaystyle\int_\Omega \exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}.$$

The last equality is obtained by inverting integral and sum symbols, as the series $\sum_k [Y^0(\boldsymbol{x})\Psi(t)]^k/k!$ converges normally. Then, we show (11) by induction: For $n=2$, we differentiate the expression of M_1 as well as Eq. (12), and so on. ■

This result holds for any solution to Syst. (3) given ψ and Y^0 at least bounded. Moreover, it enables to extend **Lemma 2** (when $Y^0 \in [0,1]$) to the bounded case:

Lemma 6. *Let Y_1 be a solution to Syst.* (3) *belonging to $L^\infty\big([0,\mathscr{T}]\times\Omega\big)$ for a certain $\mathscr{T}>0$. If $Y^0 \in [0,1]$, then Y_1 also takes values in $[0,1]$ (almost everywhere).*

Proof. First note that Eq. (11) shows that $\mu_n(t)$ converges for all t since $Y^0 \in [0,1]$. Considering the definition of μ_n, that is:

$$\mu_n(t) = \frac{1}{|\Omega|}\int_\Omega Y_1^n(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

we shall prove that Y_1 cannot take values outside $[0,1]$. Indeed, assume there exists $\omega(t)\subset\Omega$ s.t. $|\omega(t)|\neq 0$ and $Y_1(t,\boldsymbol{x})>1$ for almost all $x\in\omega(t)$. Writing μ_{2n} as:

$$\mu_{2n}(t) = \underbrace{\frac{1}{|\Omega|}\int_{\Omega\setminus\omega(t)} Y_1^{2n}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{\geq 0} + \underbrace{\frac{1}{|\Omega|}\int_{\omega(t)} Y_1^{2n}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{\xrightarrow[n\to+\infty]{}+\infty},$$

we show that μ_n cannot converge, which is contradictory to what we stated above. Thus, $Y_1 \leq 1$ a.e. Likewise, if $Y_1 < 0$ on a positive measure set, we consider the solution $Z = 1-Y_1$ associated to the initial condition $Z^0 = 1-Y^0$. Necessarily, $Z\leq 1$ as shown previously and $Y_1 \geq 0$ a.e. ■

Although **Prop. 1** and **Lemma 6** have been proven for Y^0 taking values in $[0,1]$, they still hold for general bounded Y^0 in $[a,b]$ by considering $Z = \dfrac{Y-a}{b-a}$.

When Y_1 is the mass fraction of gas as described in § 2, **Prop. 1** has the following simpler formulation:

Lemma 7. (Lemma 1.1, [5]) *Assume there exists a solution of the type* $Y_1(t,\boldsymbol{x}) = \mathbf{1}_{\Omega_1(t)}(\boldsymbol{x})$ *where* $\Omega_1(t) \subset \Omega$, $t \in [0,\mathscr{T}]$ *for a certain* $\mathscr{T} > 0$. *Let* V *be the "volume of the bubble",* i.e. $V(t) = |\Omega_1(t)|$. *Then* V *is explicitly known as:*

$$V(t) = \frac{1}{\left(\dfrac{1}{V(0)} - \dfrac{1}{|\Omega|}\right)\exp\left[-\Psi(t)\right] + \dfrac{1}{|\Omega|}}. \tag{13}$$

Proof. This lemma was first proven in [5]. Here, the proof is based on **Prop. 1**. In this irregular case, $\mu_N(0) = \mu_1(0) = V(0)/|\Omega|$ for all N and (11) leads to (13). ■

Remark 1. *Eq.* (13) *turns out to be interesting from a numerical point of view. As it is an exact formula for the volume, we can compute this volume so as to compare it to the numerical approximation. Thus we can check out the accuracy of the numerical scheme involved in the simulations* [12].

Formulae (11) and (13) are global in time, which tends to show that there is no blow-up in finite time, even if it is still an open problem. Moreover, they show the influence of ψ: If ψ is positive, the bubble grows and conversely. Likewise, if ψ is periodic and has a zero mean value over the period, the volume is periodic too.

Another remark is the dependance w.r.t. $|\Omega|$: the same bubble inside two domains of different sizes will evolve differently. It is the influence of the Poisson equation and more particularly the boundary condition. We recall that the Abv model is derived from a low Mach number system (Dlmn, [3, 4]). In that case, the acoustic waves have an infinite speed of propagation which gives an elliptic character to the Dlmn system.

Finally, Eq. (13) shows that the bubble cannot reach the boundary in finite time.

4. Conclusion

The mathematical coupling in the Abv model between a transport equation and a Poisson equation turned out to be an interesting problem balancing hyperbolic properties and elliptic effects. Despite an underlying physical context, we got interested in various cases according to the regularity of the initial datum. In the smooth case, we proved both short time existence and uniqueness of the solution of the Abv-model. In less regular situations, we established properties of possible solutions. However, the study of periodicity is still in progress: If ψ is periodic and has a zero mean value over the period, we proved that the mean value of solutions is periodic but the periodicity of the solution itself remains unsolved.

Among open problems, it remains to prove existence of solutions in weaker functional spaces satisfying physical constraints. There exists an explicit solution in 1D for a bubble-kind initial datum, which tends to show that there exist solutions in the general bounded case even if we did not prove either existence or uniqueness yet in higher dimensions. A possibility may be to extend to the space of functions with bounded variations. The question of the time interval is still open. We derived in this article an approximation of the time of existence which may be improved. The fact remains that this study forms a relevant starting point for the analysis of the Dlmn system [3, 4].

A. Annex

We recall in this part some functional results about hyperbolic and elliptic regularity as well as classical inequalities.

Lemma 8 (See Lemma 2.4, [6] and Lemma 3.1, [5])**.** *Assume that* $Y^0 \in \mathcal{H}^s(\Omega)$, $\boldsymbol{u} \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ *such that* $\boldsymbol{u} \cdot \boldsymbol{n}_{|\partial\Omega} = 0$ *and* $f \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ *with* $\mathscr{T} > 0$ *and* s *an integer s.t.* $s \geq s_0 + 1$*. Then, the transport equation:*

$$\begin{cases} \partial_t Y + \boldsymbol{u} \cdot \nabla Y = f, \\ Y(0, \boldsymbol{x}) = Y^0(\boldsymbol{x}), \end{cases}$$

has a unique classical solution $Y \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ *satisfying the energy estimates:*

$$\begin{aligned} \|Y(t,\cdot)\|_0 &\leq e^{\chi_0(t)} \left(\left\|Y^0\right\|_0 + \int_0^t e^{-\chi_0(\tau)} \|f(\tau,\cdot)\|_0 \,\mathrm{d}\tau \right), \\ \|Y(t,\cdot)\|_s &\leq e^{\chi_s(t)} \left(\left\|Y^0\right\|_s + \int_0^t e^{-\chi_s(\tau)} \|f(\tau,\cdot)\|_s \,\mathrm{d}\tau \right), \end{aligned}$$

for all $t \in [0, \mathscr{T}]$ *and any functions* χ_0 *and* χ_s *such that:*

$$\chi_0'(t) \geq \frac{1}{2} \|\nabla \cdot \boldsymbol{u}(t,\cdot)\|_\infty \qquad \text{and} \qquad \chi_s'(t) \geq C_{adv}(s,d,\Omega) \|\nabla \boldsymbol{u}(t,\cdot)\|_{s-1}.$$

Lemma 9 (See Lemma 3.2, [5] and Th. III.5.3, [1])**.** *Suppose* $\psi \in \mathscr{C}^0(0,+\infty)$ *and* $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ *for* $\mathscr{T} > 0$ *and* $s \in \mathbb{N}$*. There exists a unique solution to the system:*

$$\begin{cases} \Delta\phi(t,\boldsymbol{x}) = \psi(t) \left(Y_1(t,\boldsymbol{x}) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega Y_1(t,\boldsymbol{x}') \,\mathrm{d}\boldsymbol{x}' \right), \\ \nabla\phi \cdot \boldsymbol{n}_{|\partial\Omega} = 0, \qquad \displaystyle\int_\Omega \phi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0. \end{cases}$$

This solution satisfies $\nabla\phi \in \mathcal{W}_{s+1,\mathscr{T}}(\Omega)$ *and:*[4]

$$\begin{aligned} \|\nabla\phi(t,\cdot)\|_0 &\leq C_{PW}(d,\Omega) \cdot |\psi(t)| \cdot \|Y_1(t,\cdot)\|_0, \\ \|\nabla\phi(t,\cdot)\|_{s+1} &\leq C_{ell}(s,d,\Omega) \cdot |\psi(t)| \cdot \|Y_1(t,\cdot)\|_s. \end{aligned}$$

Lemma 10. *We recall that* $s_0 = \lfloor d/2 \rfloor + 1$*.*

(1) *Let* s_1 *and* s_2 *be two integers satisfying* $s_1 + s_2 \geq s_0$*. Assume* $f \in \mathcal{H}^{s_1}$ *and* $g \in \mathcal{H}^{s_2}$*. Then* $fg \in \mathcal{H}^{s_3}$ *with* $s_3 = \min(s_1, s_2, s_1 + s_2 - s_0)$*. Moreover, there exists* $C_M = C_M(s_1, s_2, d)$ *s.t. for all* f *and* g *as above:*

$$\|fg\|_{s_3} \leq C_M \|f\|_{s_1} \|g\|_{s_2}.$$

(2) *(Sobolev embeddings)* s_0 *is the lowest integer* s *such that* $\mathcal{H}^s(\Omega) \subset L^\infty(\Omega)$*:* $\forall\ s \geq s_0,\ \exists\ C_{sob}(s,d,\Omega) > 0,\ \forall\ f \in \mathcal{H}^s(\Omega),\ \|f\|_\infty \leq C_{sob}\|f\|_s$*. Likewise, we have* $\mathcal{H}^s(\Omega) \subset \mathscr{C}^m(\overline{\Omega})$ *as soon as* $s > m + d/2$*.*

[4] C_{PW} and C_{ell} denote the constants involved in the *Poincaré-Wirtinger* inequality and in elliptic regularity results.

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