Stability analysis of periodic orbits in the framework of Galerkin approximations

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Abstract

In dynamical systems, when periodic orbits are derived using approximation methods and more specifically Galerkin weighted residuals, their stability analysis usually requires a subsequent and sometimes uncorrelated treatment. The trouble is that this additional analysis may involve another level of approximation which, if not consistent with that used in the solution procedure, may lead to incorrect results. This paper suggests a general framework for stability analysis of periodic solutions derived using Galerkin approximation methods that systematically recasts the problem of finding periodic solutions into a fixed point problem in the spirit of the averaging method. It follows that approximations are consistent between the solution derivation and the stability analysis and that virtually any kind of projection basis could be used in a similar fashion. In this initial version of the paper, the approach is illustrated on a Duffing oscillator with strongly nonlinear dynamics.

1 Introduction

The methodology for stability analysis of periodic solutions in dynamical systems usually depends on the way the solutions are actually calculated. For instance, in perturbation methods (e.g. multiple scales, averaging), an approximate dynamical system is derived from the original one, which is then used for both solution derivation and stability analysis. This is interesting because it ensures that the level of approximation used in the solution procedure and in the stability analysis are consistent. However, in the framework of Galerkin methods (such as harmonic balance method), the picture is not that clear and there appears to be several different approaches to analyze the stability of solutions which do not always ensure such a consistency. Most techniques introduce a perturbation defined in the time domain and derive a linearized differential system with periodic coefficients to be treated using the Floquet theory. Then, one can distinguish two main classes of methods depending on the way the monodromy matrix is handled. The first is straightforward and evaluates the monodromy matrix using some numerical integration technique, the eigenvalues of the latter being used to determine the stability of the solution [see Cardona et al., 1998, for instance]. The second uses the so-called Hill method to *approximate* this perturbation using a Fourier series and then reformulate the matrix into an infinite determinant of Hill type.

The effects of approximations in the solution method on the stability analysis appears to have been first raised by Hamdan and Burton [1993] who showed that the direct Floquet approach may lead to incorrect results regarding the stability of periodic solutions if the latter were obtained in an approximate manner (typically harmonic balance solutions). The point is that examining the *exact* dynamics in the neighborhood of an *approximate* solution may be a questionable approach.

The authors hence advocate for the use of an approximate method for stability analysis in such cases. Later in the same vein Hassan [1996] suggested that the use of a truncated Fourier expansion of the perturbation in the stability analysis would fulfil this requirement and that choosing consistent levels of approximation (i.e. in the case of harmonic balance method, the same number of harmonics) between the solution procedure and the stability analysis would lead to correct results as confirmed elsewhere [Szemplinska-Stupnicka, 1988, Al-Qaisia and Hamdan, 2001, von Groll and Ewins, 2001]. Despite these conclusions, in many cases, if the approximation is good enough (i.e. if a sufficient number of harmonics is retained in the case of harmonic balance method), the direct Floquet approach can lead to correct results as demonstrates its (still) important popularity [see e.g. Bauchau and Nikishkov, 2001; Grolet and Thouverez, 2010, in an harmonic balance framework or Bauchau and Hong [1988]; Demailly et al., 2004, in a time finite-element approach]. Moreover, the treatment of approximation methods which do not involve Fourier series is not straightforward in the Hill method.

Here, we follow the aforementioned conclusions and suggest an alternative formulation to derive the stability of periodic orbits in the framework of Galerkin approximation methods that ensures a consistent level of approximation between the solution and its stability.

- The main idea (Section 2) is to systematically recast the problem of finding a periodic orbit into a fixed point problem and to analyze the stability of the former using methods typical of the latter, hence eliminating the need for theoretical tools specific to each kind of generalized orbits.

  We would like to highlight the generality of the proposed approach with respect to two aspects mainly. First, it does not assume a particular form for the basis functions in the Galerkin procedure; for instance, global (e.g. Fourier series) or local (e.g. finite-elements) functions could be used in a similar fashion. Second, althought not covered in the present paper, the approach could intuitively be used for other kinds of non-trivial orbits, such as quasi-periodic solutions.

- While starting from a generic first order dynamical system, we also address the case of second order systems by introducing an ad hoc compatibility condition between state variables (Section 2.2) which leads to similar linearized systems whatever the order of the problem.

- The method is finally illustrated (Section 3) on a Duffing system, featuring a strongly dynamics, which highlights the validity of the approach.

This paper is the result of a preliminary investigation, which ought to be extended in a near future.

2 Stability analysis of approximate periodic orbits

This section gathers the theoretical derivations of the proposed approach for stability analysis of periodic orbits in Galerkin approximations. The presentation is general in that no particular Galerkin method is assumed. In particular the basis functions are not specified explicitly. The general case of first order dynamical system is first considered, followed by another derivation for second order systems as those are often encountered in practical applications.

2.1 First order dynamical systems

Consider a first order dynamical system governed by the following differential equation:

$$\dot{x} = f(x, \mu) \tag{1}$$

in which μ is a parameter and f is a sufficiently smooth mapping. We are interested in the stability of periodic orbits of system (1), which are defined by the following condition:

$$\exists\, T > 0, \quad x(t+T) = x(t) \quad \forall t \in \mathbb{R} \tag{2}$$

where $T = 2\pi/\omega$ is the fundamental period of the motion. The determination of such solutions is done in the framework of a Galerkin weighted residuals procedure which consists of the following steps.

First, approximate periodic orbits are thought in the following form:

$$x_h(\tau) = \sum_{i=1}^{N} \varphi_i(\tau) x_i, \quad \tau = \omega t \tag{3}$$

where $\{\varphi_i\}_{1 \leq i \leq N}$ are admissible basis functions with respect to boundary condition (2).

Then, this approximated solution is introduced in Eq. (1) to derive a residue:

$$r_h(t) = f(x_h, \mu) - \sum_{i=1}^{N} \omega {\varphi_i}' x_i \tag{4}$$

which, once projected on the basis functions as per the Galerkin weighted residuals approach, must be zero, leading to a set of algebraic equations:

$$\langle f(x_h, \mu), \varphi_j \rangle - \sum_{i=1}^{N} \omega \langle {\varphi_i}', \varphi_j \rangle x_i = 0 \quad \forall \varphi_j \quad j = 1, \ldots, N \tag{5}$$

where $\langle f, g \rangle = \int_0^{2\pi} fg \mathrm{d}\tau$ is a scalar product. System (5) may be rewritten in the following vector form:

$$F(\mathbf{X}, \mu) - \omega \mathcal{D}_h^{(1)} \mathbf{X} = 0 \tag{6}$$

in which we have introduce a differentiation matrix:

$$\mathcal{D}_h^{(1)} = (\langle {\varphi_i}', \varphi_j \rangle)_{i=1,\ldots,N, j=1,\ldots,N} \tag{7}$$

and the projected nonlinear term:

$$F(\mathbf{X}, \mu) = \left(\langle f(\sum_{i=1}^{N} \varphi_i x_i, \mu), \varphi_j \rangle \right)_{j=1,\ldots,N} \tag{8}$$

System (6) may finally be solved numerically.

Now focusing on the dynamics in the neighborhood of the approximate solution, let us introduce a perturbation of the latter so that the state variable x is thought in the following form:

$$x_h(\tau, \eta) = \sum_{i=1}^{N} \varphi_i(\tau) x_i(\eta) \tag{9}$$

which features two *independent* time scales, namely:

- $\tau = \omega t$, the time scale of the periodic motion, as introduced earlier and,
- $\eta = \epsilon t$, the time scale of perturbations.

Following a similar derivation as above, one defines a residue:

$$r_h(\tau, \eta) = f(x_h, \mu) - \sum_{i=1}^{N} (\omega \varphi_i{}' x_i + \epsilon \varphi_i x_i{}') \tag{10}$$

and performs a Galerkin weighted residuals projection, still on $\{\varphi_i\}$, leading to:

$$\sum_{i=1}^{N} \epsilon \langle \varphi_i, \varphi_j \rangle x_i{}' = \langle f(x_h, \mu), \varphi_j \rangle - \sum_{i=1}^{N} \omega \langle \varphi_i{}', \varphi_j \rangle x_i \quad \forall \varphi_j \quad j = 1, \dots, N \tag{11}$$

Eq. (11) defines a new dynamical system, in which state variables are the degrees-of-freedom of the Galerkin projection (9). As above, it can be rewritten as:

$$\epsilon \mathcal{D}_h^{(0)} \mathbf{X}' = F(\mathbf{X}, \mu) - \omega \mathcal{D}_h^{(1)} \mathbf{X} \tag{12}$$

in which another differentiation matrix appears:

$$\mathcal{D}_h^{(0)} = (\langle \varphi_i, \varphi_j \rangle)_{i=1,\dots,N, j=1,\dots,N} \tag{13}$$

Clearly, fixed points of Eq. (12) are solutions of Eq. (6), namely approximated periodic orbits of the original dynamical system governed by Eq. (1). One would then naturally infer that the stability of these orbits may be derived by studying the underlying linearized system derived from (12). This means that the stability of *approximate* periodic orbits obtained in the framework of a Galerkin projection can be studied in a similar fashion to that of equilibria, namely those of Eq. (12). Note that the same reasoning is probably applicable for other kind of non-trivial orbits (e.g. quasi-periodic orbits), provided that one is able to think of an appropriate projection basis.

Let $\mathbf{X}(\eta) = \mathbf{X}_0 + \mathbf{Y}(\eta)$ be a small perturbation of the equilibrium $\mathbf{X}_0$ of Eq. (12). Introducing this perturbation into Eq. (12) and linearizing yield:

$$\epsilon \mathcal{D}_h^{(0)} \mathbf{Y}' = \left(\nabla_{\mathbf{X}} F(\mathbf{X}_0, \mu) - \omega \mathcal{D}_h^{(1)} \right) \mathbf{Y} \tag{14}$$

The solution of Eq. (14) is determined by the eigenvalues and eigenvectors of the underlying generalized eigenvalue problem. These eigenvalues are actually defined as the *characteristic exponents.* If all eigenvalues have strictly negative real parts, the corresponding periodic orbit is stable, if at least one eigenvalue has a positive real part, it is unstable and if there is a pair of purely imaginary eigenvalues, the equilibrium gives birth to a *limit cycle*, which means that the periodic orbit degenerates into a *quasi-periodic* orbit.

2.2 Second order dynamical system

In many applications, dynamical systems are governed by a second-order differential equation. For instance, the canonical motion equation in mechanics is $M(q)\ddot{q} = f(t, \dot{q}, q)$ in which $M(q)$ is the mass matrix at q of the considered system and f here accounts for both internal and external forces. Applying the previous derivation straightforward to such a system is obviously doable but this would result in a second order dynamical system similar to (12) which might pose some difficulties when it comes to examining the results of the linearized system. We thus propose an alternative derivation of higher order dynamical systems, illustrated on the case of a second-order one, that leads to a similar formulation to that of first order systems.

Consider then a second order dynamical system governed by the following differential equation:

$$\ddot{x} = g(\dot{x}, x, \mu) \tag{15}$$

Periodic solutions are defined by the following conditions:

$$\exists\, T > 0, \quad x(t+T) = x(t) \quad \text{and} \quad \dot{x}(t+T) = \dot{x}(t) \quad \forall t \in \mathbb{R} \tag{16}$$

As above, perturbations of approximate periodic solutions are thought in the form of Eq. (9). Then, instead of deriving the first and second derivatives of x from Eq. (16), a *compatibility condition*, defining the approximation of first derivative of x, is introduced as:

$$\dot{x}_h(\tau, \eta) = \sum_{i=1}^{N} \omega {\varphi_i}'(\tau) x_i(\eta) \tag{17}$$

In turns, the second derivative is:

$$\ddot{x}_h(\tau, \eta) = \sum_{i=1}^{N} \omega^2 {\varphi_i}''(\tau) x_i(\eta) + \omega\epsilon {\varphi_i}'(\tau) {x_i}'(\eta) \tag{18}$$

Introducing the latter in Eq. (15) and averaging over the time scale of periodic motion yield the following dynamical system:

$$\epsilon\omega \mathcal{D}_h^{(1)} \mathbf{X}' = G(\mathbf{X}, \omega, \mu) - \omega^2 \mathcal{D}_h^{(2)} \mathbf{X} \tag{19}$$

in which another differentiation matrix appears:

$$\mathcal{D}_h^{(2)} = (\langle {\varphi_i}'', \varphi_j \rangle)_{i=1,\ldots,N, j=1,\ldots,N} \tag{20}$$

The stability of periodic orbits of the second order dynamical system is obtained in a similar manner as for the first order one, by linearizing Eq. (19):

$$\epsilon\omega \mathcal{D}_h^{(1)} \mathbf{Y}' = \left(\nabla_{\mathbf{X}} G(\mathbf{X}, \omega, \mu) - \omega^2 \mathcal{D}_h^{(2)} \right) \mathbf{Y} \tag{21}$$

and solving the underlying eigenvalue problem.

Remark 2.1 *The interest of the above derivation is the introduction the compatibility condition* (17) *which yields a first order dynamical system despite the original system is of order two. This is particularly interesting since the stability analysis simply involves a standard eigenvalue problem instead of a second order one.*

Remark 2.2 *Higher order differentiation matrices ($\mathcal{D}^{(n)}$, $n > 0$) can be defined in different manners. So far, only the classical (straightforward) form has been presented. Alternative forms can be derived using integration by parts which, in particular, make it possible to lower the smoothness conditions on basis functions φ_i. E.g. considering the second order differentiation matrix, an alternative form is:*

$$\mathcal{D}_h^{(2)} = -\left(\langle {\varphi_i}', {\varphi_j}' \rangle\right)_{i=1,\ldots,N,j=1,\ldots,N} \tag{22}$$

accounting for the fact that basis functions φ_i are admissible ($\varphi_i(2\pi) = \varphi_i(0)$).

3 Application

This section concerns an application of the previously description methodology for stability analysis of periodic orbits to a Duffing oscillator, described by the following equation:

$$\ddot{x} + \mu\dot{x} + \omega_0^2 x + \Gamma x^3 = F\cos\omega t \tag{23}$$

The parameters are chosen so that the dynamics is *strongly* non-linear: $\mu = 0.05$, $\omega_0 = 1$, $\Gamma = 1$ and $F = 1$. For the record, the same system was studied recently in [Lazarus and Thomas, 2010].

Here, a Fourier-Galerkin approximation is used, which means that a truncated Fourier series is used in the expansion (3). In this framework, if N harmonics are kept in the Fourier series, the differentiation matrices are:

$$\mathcal{D}_h^{(0)} = \mathcal{I}, \quad \mathcal{D}_h^{(1)} = \text{diag}\left([0, 1, \ldots, N]\right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D}_h^{(2)} = -\text{diag}\left([0, 1, \ldots, N]\right)^2 \otimes \mathcal{I}_2 \tag{24}$$

(in which $\otimes$ is the Kronecker product). The nonlinear restoring force approximation, given by Eq. (8), is obtained using a discrete Fourier transform.

The dynamic response of the oscillator is computed within the frequency range of $\omega = [0, 5]$ rad/s and a (basic) pseudo-arclength continuation algorithm is used (no bifurcation detection is performed, only stability analysis). Nine harmonics are kept in the approximation and the discrete Fourier transform uses 500 points.

Fig. 1 depicts the resonance response of this system with unstable regions marked with dots while Fig. 2 shows the characteristic exponents arising from the eigenvalue problem underlying the linearized system (21). A general view and a zoom on a super-harmonic resonance are shown.

Regions of stable and unstable periodic response appear to be correctly predicted. In particular, the saddle node bifurcations for the primary and super-harmonic resonances correspond to turning points of the resonance response. Overall, results compare well with those of the literature [e.g. Lazarus and Thomas, 2010].

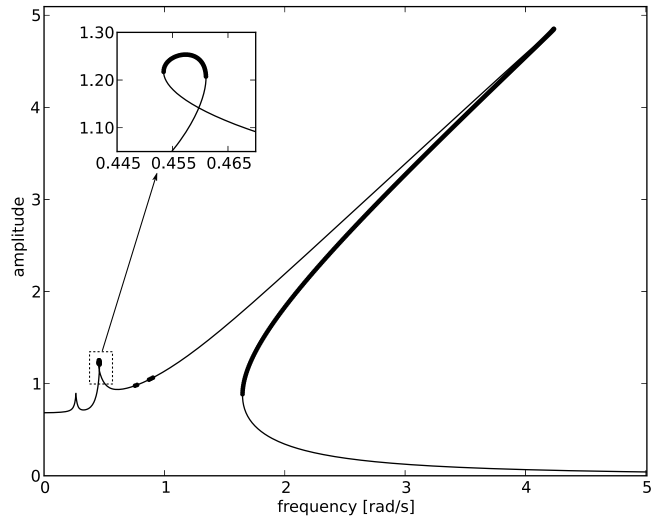

Figure 1: Frequency response of the Duffing oscillator; dots indicate unstable response.

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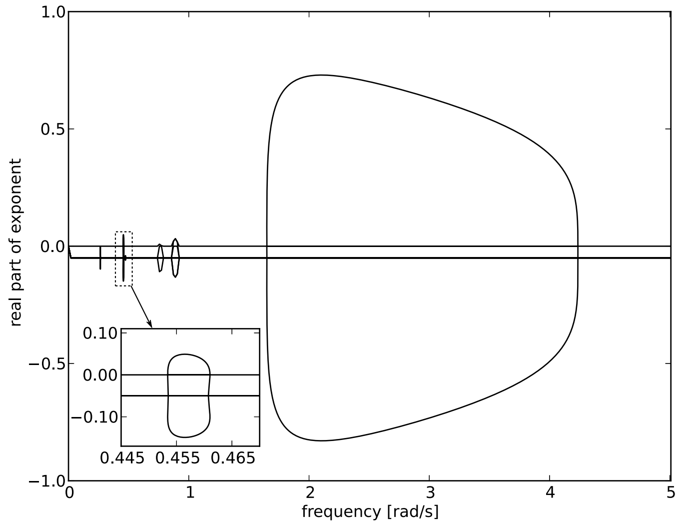

Figure 2: Characteristic exponents of the Duffing oscillator.

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