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# The domination number of Cartesian product of two directed paths

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## Abstract

Let  $\gamma(P_m \square P_n)$  be the domination number of the Cartesian product of directed paths  $P_m$  and  $P_n$  for  $m, n \geq 2$ . In [13] Liu and al. determined the value of  $\gamma(P_m \square P_n)$  for arbitrary  $n$  and  $m \leq 6$ . In this work we give the exact value of  $\gamma(P_m \square P_n)$  for any  $m, n$  and exhibit minimum dominating sets.

**AMS Classification[2010]:**05C69,05C38.

**Keywords:** Directed graph, digraph, Cartesian product, Domination number, Paths.

## 1 Introduction and definitions

Let  $G = (V, E)$  be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex  $u$  *dominates* a vertex  $v$  if  $u = v$  or  $uv \in E$ . A set  $S \subset V$  is a *dominating set* of  $G$  if any vertex of  $G$  is dominated by at least a vertex of  $S$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$  is the minimum cardinality of a dominating set. The set  $V$  is a dominating set thus  $\gamma(G)$  is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

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For two digraphs,  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , the *Cartesian product*  $G_1 \square G_2$  is the digraph with vertex set  $V_1 \times V_2$  and  $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$  if and only if  $x_1 y_1 \in E_1$  and  $x_2 = y_2$  or  $x_2 y_2 \in E_2$  and  $x_1 = y_1$ . Note that  $G \square H$  is isomorphic to  $H \square G$ .

The domination number of Cartesian product of two directed cycles have been recently investigated ([11], [12], [14], [15]). Even more recently, Liu and al.([13]) began the study of the domination number of the Cartesian product of two directed paths  $P_m$  and  $P_n$ . They proved the following result

**Theorem 1** *Let  $n \geq 2$ . Then*

- $\gamma(P_2 \square P_n) = n$
- $\gamma(P_3 \square P_n) = n + \lceil \frac{n}{4} \rceil$
- $\gamma(P_4 \square P_n) = n + \lceil \frac{2n}{3} \rceil$
- $\gamma(P_5 \square P_n) = 2n + 1$
- $\gamma(P_6 \square P_n) = 2n + \lceil \frac{n+2}{3} \rceil$ .

In this paper we are able to give a complete solution of the problem. In Theorem 2 we determine the value of  $\gamma(P_m \square P_n)$  for any  $m, n \geq 2$ . When  $m$  grows, the cases approach appearing in the proof of Theorem 1 seems to be more and more complicated. We proceed by a different and elementary method, but will assume that Theorem 1 is already obtained (at least for  $m \leq 5$  and arbitrary  $n$ ). In the next section we describe three dominating sets of  $P_m \square P_n$  corresponding to the different values of  $m$  modulo 3. In the last section we prove that these dominating sets are minimum and deduce our main result:

**Theorem 2** *Let  $n \geq 2$ . Then*

- $\gamma(P_{3k} \square P_n) = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$  for  $k \geq 2$  and  $n \neq 3$
- $\gamma(P_{3k+1} \square P_n) = k(n+1) + \lceil \frac{2n-3}{3} \rceil$  for  $k \geq 1$  and  $n \neq 3$
- $\gamma(P_{3k+2} \square P_n) = k(n+1) + n$  for  $k \geq 0$  and  $n \neq 3$
- $\gamma(P_3 \square P_n) = \gamma(P_n \square P_3) = n + \lceil \frac{n}{4} \rceil$ .

We will follow the notations used by Liu and al. and refer to their paper for a more complete description of the motivations. Let us recall some of these notations.

We denote the vertices of a directed path  $P_n$  by the integers  $\{0, 1, \dots, n-1\}$ . For any  $i$  in  $\{0, 1, \dots, n-1\}$ ,  $P_m^i$  is the subgraph of  $P_m \square P_n$  induced by the vertices  $\{(k, i) / k \in \{0, 1, \dots, m-1\}\}$ . Note that  $P_m^i$  is isomorphic to  $P_m$ . Notice also that  $P_m \square P_n$  is isomorphic to  $P_n \square P_m$  thus  $\gamma(P_m \square P_n) = \gamma(P_n \square P_m)$ . A vertex  $(a, b) \in P_m^b$  can be dominated by  $(a, b)$ ,  $(a-1, b) \in P_m^b$  (if  $a \geq 1$ ),  $(a, b-1) \in P_m^{b-1}$  (if  $b \geq 1$ ).

## 2 Three Dominating sets

We will first study  $P_{3k} \square P_n$  for  $k \geq 1$  and  $n \geq 2$ . Consider the following sets of vertices of  $P_{3k}$ .

- 66 •  $X = \{0, 1, 3, 4, \dots, 3k - 3, 3k - 2\} = \{3i/i \in \{0, 1, \dots, k - 1\}\} \cup \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- 68 •  $Y = \{2, 5, 8, \dots, 3k - 1\} = \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$
- $I = \{0, 3, 6, \dots, 3k - 3\} = \{3i/i \in \{0, 1, \dots, k - 1\}\}$
- 70 •  $J = \{1, 4, 7, \dots, 3k - 2\} = \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$ .

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Let  $D_n$  (see Figure 1) be the set of vertices of  $P_{3k} \square P_n$  consisting of the vertices

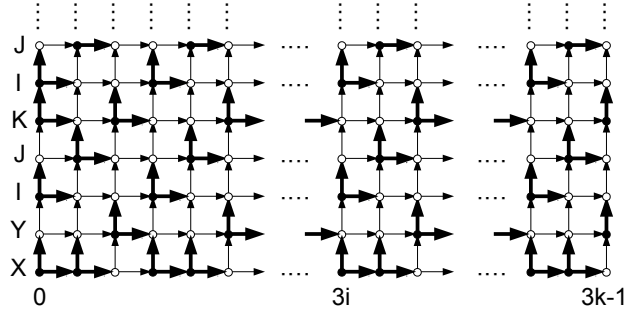


Figure 1: The dominating set  $D_n$

- 74 •  $(a, 0)$  for  $a \in X$
- $(a, 1)$  for  $a \in Y$
- 76 •  $(a, b)$  for  $b \equiv 2 \pmod{3}$  ( $2 \leq b < n$ ) and  $a \in I$
- $(a, b)$  for  $b \equiv 0 \pmod{3}$  ( $3 \leq b < n$ ) and  $a \in J$
- 78 •  $(a, b)$  for  $b \equiv 1 \pmod{3}$  ( $4 \leq b < n$ ) and  $a \in K$ .

80 **Lemma 3** For any  $k \geq 1$ ,  $n \geq 2$  the set  $D_n$  is a dominating set of  $P_{3k} \square P_n$  and  $|D_n| = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$ .

**Proof :** It is immediate to verify that

- 82 • All vertices of  $P_{3k}$  are dominated by the vertices of  $X$
- The vertices of  $P_{3k}$  not dominated by some of  $Y$  are  $\{0, 1, 4, \dots, 3k - 2\} \subset X$
- 84 • The vertices of  $P_{3k}$  not dominated by some of  $I$  are  $\{2, 5, \dots, 3k - 1\} = Y \subset K$
- 86 • The vertices of  $P_{3k}$  not dominated by some of  $J$  are  $\{0, 3, 6, \dots, 3k - 3\} \subset I$

- The vertices of  $P_{3k}$  not dominated by some of  $K$  are  $\{4, 7, \dots, 3k - 2\} \subset J$ .

88 Therefore any vertex of some  $P_{3k}^i$  is dominated by a vertex in  $P_{3k}^i \cap D_n$  or in  
90  $P_{3k}^{i-1} \cap D_n$  (if  $i \geq 1$ ). Furthermore  $|X| = 2k$ ,  $|Y| = |I| = |J| = k$ , and  $|K| = k + 1$   
thus  $|D_n| = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$ .  $\square$

92 Let us study now  $P_{3k+1} \square P_n$  for  $k \geq 1$  and  $n \geq 2$ . Consider the following sets of  
vertices of  $P_{3k+1}$ .

- 94 •  $X = \{0, 2, 4, 5, 7, 8, \dots, 3k - 2, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\} \cup$   
 $\{3i + 1/i \in \{1, \dots, k - 1\}\}$
- 96 •  $I = \{0, 3, 6, \dots, 3k\} = \{3i/i \in \{0, 1, \dots, k\}\}$
- $J = \{1, 4, 7, \dots, 3k - 2\} = \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- 98 •  $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$ .

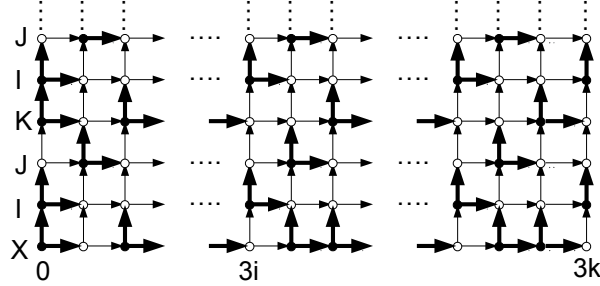


Figure 2: The dominating set  $E_n$

100 Let  $E_n$  (see Figure 2) be the set of vertices of  $P_{3k+1} \square P_n$  consisting of the vertices

- $(a, 0)$  for  $a \in X$
- 102 •  $(a, b)$  for  $b \equiv 1 \pmod{3}$  ( $1 \leq b < n$ ) and  $a \in I$
- $(a, b)$  for  $b \equiv 2 \pmod{3}$  ( $2 \leq b < n$ ) and  $a \in J$
- 104 •  $(a, b)$  for  $b \equiv 0 \pmod{3}$  ( $3 \leq b < n$ ) and  $a \in K$ .

106 **Lemma 4** For any  $k \geq 1$ ,  $n \geq 2$  the set  $E_n$  is a dominating set of  $P_{3k+1} \square P_n$  and  
 $|E_n| = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$ .

**Proof :** It is immediate to verify that

- 108 • All vertices of  $P_{3k+1}$  are dominated by the vertices of  $X$

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- The vertices of  $P_{3k+1}$  not dominated by some of  $I$  are  $\{2, 5, \dots, 3k - 1\} \subset K \subset X$

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- The vertices of  $P_{3k+1}$  not dominated by some of  $J$  are  $\{0, 3, 6, \dots, 3k\} = I$
- The vertices of  $P_{3k+1}$  not dominated by some of  $K$  are  $\{4, 7, \dots, 3k - 2\} \subset J$ .

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Therefore any vertex of some  $P_{3k+1}^i$  is dominated by a vertex in  $P_{3k+1}^i \cap E_n$  or in  $PP_{3k+1}^{i-1} \cap E_n$  (if  $i \geq 1$ ). Furthermore  $|X| = 2k$ ,  $|I| = |K| = k + 1$ , and  $|J| = k$  thus  $|E_n| = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$ .  $\square$

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The last case will be  $P_{3k+2} \square P_n$  for  $k \geq 0$  and  $n \geq 2$ . Consider the following sets of vertices of  $P_{3k+2}$ .

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- $X = \{0, 1, 3, 4, \dots, 3k, 3k + 1\} = \{3i/i \in \{0, 1, \dots, k\}\} \cup \{3i + 1/i \in \{0, 1, \dots, k\}\}$
- $Y = \{2, 5, 8, \dots, 3k - 1\} = \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$
- $I = \{0, 3, 6, \dots, 3k\} = \{3i/i \in \{0, 1, \dots, k\}\}$
- $J = \{1, 4, 7, \dots, 3k + 1\} = \{3i + 1/i \in \{0, 1, \dots, k\}\}$
- $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$ .

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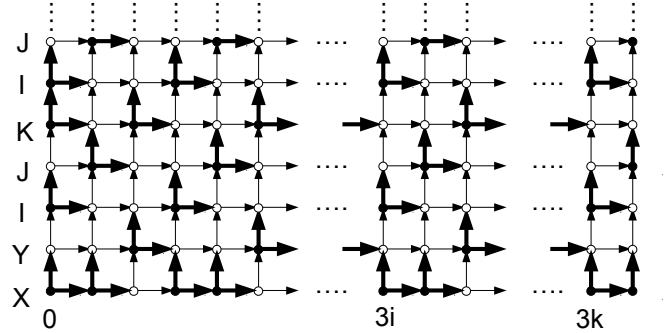


Figure 3: The dominating set  $F_n$

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Let  $F_n$  (see Figure 3) be the set of vertices of  $P_{3k+2} \square P_n$  consisting of the vertices

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- $(a, 0)$  for  $a \in X$
- $(a, 1)$  for  $a \in Y$
- $(a, b)$  for  $b \equiv 2 \pmod{3}$  ( $2 \leq b < n$ ) and  $a \in I$
- $(a, b)$  for  $b \equiv 0 \pmod{3}$  ( $3 \leq b < n$ ) and  $a \in J$
- $(a, b)$  for  $b \equiv 1 \pmod{3}$  ( $4 \leq b < n$ ) and  $a \in K$ .

130

**Lemma 5** For any  $k \geq 0$ ,  $n \geq 2$ , the set  $F_n$  is a dominating set of  $P_{3k+2} \square P_n$  and  $|F_n| = k(n+1) + n$ .

**Proof :** It is immediate to verify that

- All vertices of  $P_{3k+2}$  are dominated by the vertices of  $X$
- The vertices of  $P_{3k+2}$  not dominated by some of  $Y$  are  $\{0, 1, 4, \dots, 3k+1\} \subset X$
- The vertices of  $P_{3k+2}$  not dominated by some of  $I$  are  $\{2, 5, \dots, 3k-1\} = Y \subset K$
- The vertices of  $P_{3k+2}$  not dominated by some of  $J$  are  $\{0, 3, \dots, 3k\} = I$
- The vertices of  $P_{3k+2}$  not dominated by some of  $K$  are  $\{4, 7, \dots, 3k+1\} \subset J$ .

Therefore any vertex of some  $P_{3k+2}^i$  is dominated by a vertex in  $P_{3k+2}^i \cap F_n$  or in  $P_{3k+2}^{i-1} \cap F_n$  (if  $i \geq 1$ ). Furthermore  $|X| = 2k+2$ ,  $|Y| = k$  and  $|I| = |J| = |K| = k+1$ , thus  $|F_n| = k(n+1) + n$ .  $\square$

### 3 Optimality of the three sets

The structure of  $P_m \square P_n$  implies the following strong property.

**Proposition 6** Let  $S$  be a dominating set of  $P_m \square P_n$ . For any  $n' \leq n$  consider

$$S_{n'} = \bigcup_{i=0, \dots, n'-1} P_m^i \cap S.$$

Then  $S_{n'}$  is a dominating set of  $P_m \square P_{n'}$ .

Notice that the three sets  $D_n$ ,  $E_n$ ,  $F_n$  satisfy, for example,  $(D_n)_{n'} = D_{n'}$  therefore we can use the same notation without ambiguity.

If  $S$  is a dominating set of  $P_m \square P_n$ , for any  $i$  in  $\{0, 1, \dots, n-1\}$  let  $s_i = |P_m^i \cap S|$ . We have thus  $|S| = \sum_{i=0}^{n-1} s_i$ .

**Proposition 7** Let  $S$  be a dominating set of  $P_m \square P_n$ . Let  $i \in \{1, 2, \dots, n-1\}$  then  $s_{i-1} + 2s_i \geq m$ .

**Proof :** Any vertex of  $P_m^i$  must be dominated by some vertex of  $P_m^i \cap S$  or of  $P_m^{i-1} \cap S$ . A vertex in  $P_m^i \cap S$  dominates at most two vertices of  $P_m^i$  and a vertex in  $P_m^{i-1} \cap S$  dominates a unique vertex of  $P_m^i$ .  $\square$

**Lemma 8** Let  $k \geq 0$  and  $n \geq 2$ ,  $n \neq 3$ , then  $\gamma(P_{3k+2} \square P_n) = k(n+1) + n$ .

**Proof :** The case  $n = 2$  is immediate by Theorem 1.

Let  $S$  be a dominating set of  $P_{3k+2} \square P_n$  with  $n \geq 4$ .

By Proposition 7,  $s_i \leq k$  implies  $s_{i-1} + s_i \geq m - s_i \geq 2k+2$ . Therefore for any  $i \in \{2, \dots, n-1\}$  we get  $s_i \geq k+1$  or  $s_{i-1} + s_i \geq 2(k+1)$ .

Apply the following algorithm:

```

164  $I := \emptyset; J := \emptyset; i := n - 1;$ 
while  $i \geq 5$  do
  if  $s_i \geq k + 1$  then
166  $I := I \cup \{i\}; i := i - 1$ 
  else
168  $J := J \cup \{i, i - 1\}; i := i - 2$ 
  end if
170 end while

```

If  $n = 4$  or  $n = 5$  the algorithm only sets  $I$  and  $J$  to  $\emptyset$ . In the general case, the algorithm stop when  $i = 3$  or  $i = 4$  and we get two disjoint sets  $I, J$  with  $\{0, 1, \dots, n - 1\} = \{0, 1, 2, 3\} \cup I \cup J$  or  $\{0, 1, \dots, n - 1\} = \{0, 1, 2, 3, 4\} \cup I \cup J$ . Furthermore  $\sum_{i \in I} s_i \geq |I|(k + 1)$  and  $\sum_{i \in J} s_i \geq |J|(k + 1)$ . We have thus one of the two inequalities

$$|S| - (s_0 + s_1 + s_2 + s_3) \geq (n - 4)(k + 1)$$

or

$$|S| - (s_0 + s_1 + s_2 + s_3 + s_4) \geq (n - 5)(k + 1).$$

In the first case by Proposition 6 and Theorem 1 we get  $s_0 + s_1 + s_2 + s_3 \geq \gamma(P_{3k+2} \square P_4) = \gamma(P_4 \square P_{3k+2}) = 3k + 2 + \lceil \frac{6k+4}{3} \rceil = 5k + 4$ . Thus  $|S| \geq (n + 1)k + n$ . In the second case we get  $s_0 + s_1 + s_2 + s_3 + s_4 \geq \gamma(P_5 \square P_{3k+2}) = 6k + 5$ . Thus again  $|S| \geq (n + 1)k + n$ .

Therefore for any  $n \geq 4$  we have  $\gamma(P_{3k+2} \square P_n) \geq k(n + 1) + n$  and the equality occurs by Lemma 5.  $\square$

Notice that, by Theorem 1,  $\gamma(P_{3k+2} \square P_3) = 3k + 2 + \lceil \frac{3k+2}{4} \rceil \neq 4k + 3$  for  $k \geq 1$ .

**Lemma 9** *Let  $k \geq 1$  and  $n \geq 2, n \neq 3$ , then  $\gamma(P_{3k+1} \square P_n) = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$ .*

**Proof:** Consider some fixed  $k \geq 1$ . Notice first that by Theorem 1,  $\gamma(P_{3k+1} \square P_2) = 3k + 1$ ,  $\gamma(P_{3k+1} \square P_4) = 5k + 2$  and  $\gamma(P_{3k+1} \square P_5) = 6k + 3$  thus the result is true for  $n \leq 5$ .

We knows, by Lemma 4, that for any  $n \geq 2$  the set  $E_n$  is a dominating set of  $P_{3k+1} \square P_n$  and  $|E_n| = (n + 1)k + \lceil \frac{2n-3}{3} \rceil$ .

We will prove now that  $E_n$  is a minimum dominating set .

If this is not true consider  $n$  minimum,  $n \geq 2$ , such that there exists a dominating set  $S$  of  $P_{3k+1} \square P_n$  with  $|S| < |E_n|$ . We knows that  $n \geq 6$ .

For  $n' \leq n$  let  $S_{n'} = \cup_{i=0, \dots, n'-1} P_{3k+1}^i \cap S$  and  $s_{n'} = |P_{3k+1}^{n'} \cap S|$ .

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**Case 1**  $n = 3p, p \geq 2$ .

Notice first that  $|E_n| - |E_{n-1}| = k$  and  $|E_n| - |E_{n-2}| = 2k + 1$ . We have also by hypothesis  $|S| \leq |E_n| - 1$ . By minimality of  $n$ ,  $E_{n-1}$  is minimum thus  $|S_{n-1}| \geq |E_{n-1}|$ . Therefore  $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k - 1$ . On the other hand, by Proposition 7,  $s_{n-2} + 2s_{n-1} \geq 3k + 1$  thus  $s_{n-2} + s_{n-1} \geq (3k + 1) - (k - 1) = 2k + 2$ . This implies  $|S_{n-2}| \leq |S_n| - 2k - 2 \leq |E_n| - 2k - 3 < |E_n| - 2k - 1 = |E_{n-2}|$ , thus  $E_{n-2}$  is not minimum in contradiction with  $n$  minimum.

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196 **Case 2**  $n = 3p + 1, p \geq 2$ .

198 In this case we have  $|E_n| - |E_{n-1}| = k + 1$  and  $|E_n| - |E_{n-2}| = 2k + 1$ . We have  
 200 also by hypothesis  $|S| \leq |E_n| - 1$ . By minimality of  $n$ ,  $E_{n-1}$  is minimum thus  
 202  $|S_{n-1}| \geq |E_{n-1}|$ . Therefore  $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k$ . On the  
 other hand, by Proposition 7,  $s_{n-2} + 2s_{n-1} \geq 3k + 1$  thus  $s_{n-2} + s_{n-1} \geq 2k + 1$ .  
 This implies  $|S_{n-2}| \leq |S_n| - 2k - 1 < |E_n| - 2k - 1 = |E_{n-2}|$ , thus  $E_{n-2}$  is not  
 minimum in contradiction with  $n$  minimum.

204 **Case 3**  $n = 3p + 2, p \geq 2$ .

206 In this case,  $|E_n| - |E_{n-2}| = 2k + 2$  and we cannot proceed like case 1 and case 2.  
 Hopefully, by Lemma 8,  $\gamma(P_{3k+1} \square P_{3p+2}) = \gamma(P_{3p+2} \square P_{3k+1}) = p(3k+2) + 3k + 1 =$   
 $k(3p+3) + 2p + 1$ . Therefore, since  $n + 1 = 3p + 3$  and  $\lceil \frac{2n-3}{3} \rceil = 2p + 1$ ,  $E_n$  is  
 208 minimum.  $\square$

**Lemma 10** *Let  $k \geq 2$  and  $n \geq 2, n \neq 3$  then  $\gamma(P_{3k} \square P_n) = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$ .*

210 **Proof :**

**Case 1**  $n = 3p + 1, p \geq 1$ .

212 By Lemma 9,  $\gamma(P_{3k} \square P_{3p+1}) = \gamma(P_{3p+1} \square P_{3k}) = p(3k+1) + 2k - 1 = k(3p+2) + p - 1$ .  
 We obtain the conclusion since  $3p + 2 = n + 1$  and  $\lfloor \frac{n-2}{3} \rfloor = p - 1$ .

214 **Case 2**  $n = 3p + 2, p \geq 0$ .

216 By Lemma 8,  $\gamma(P_{3k} \square P_{3p+2}) = \gamma(P_{3p+2} \square P_{3k}) = p(3k+1) + 3k = k(3p+3) + p$ . We  
 obtain again the conclusion since  $3p + 3 = n + 1$  and  $\lfloor \frac{n-2}{3} \rfloor = p$ .

218 **Case 3**  $n = 3p, p \geq 2$ .

220 We know, by Lemma 3, that the set  $D_n$  is a dominating set of  $P_{3k} \square P_n$  and  $|D_n| =$   
 $k(n+1) + \lfloor \frac{n-2}{3} \rfloor$ .

222 If  $D_n$  is not a minimum dominating set let  $S$  be a dominating set with  $|S| < |D_n|$ .  
 For  $n' \leq n$  let  $S_{n'} = \cup_{i=0, \dots, n'-1} P_{3k}^i \cap S$  and  $s_{n'} = |P_{3k}^{n'} \cap S|$ .

224 Because  $n = 3p$  and  $p \geq 2$  we get  $|D_n| - |D_{n-1}| = k$  and  $|D_n| - |D_{n-2}| = 2k + 1$ . We  
 have also by hypothesis  $|S| \leq |D_n| - 1$ . Notice that, by Lemma 8,  $\gamma(P_{3k} \square P_{n-1}) =$   
 $\gamma(P_{3p-1} \square P_{3k}) = (p-1)(3k+1) + 3k = kn + \lfloor \frac{n-3}{3} \rfloor = |D_{n-1}|$  thus  $D_{n-1}$  is minimum  
 and  $|S_{n-1}| \geq |D_{n-1}|$ .

226 Therefore  $s_{n-1} = |S| - |S_{n-1}| \leq |D_n| - 1 - |D_{n-1}| = k - 1$ . By Proposi-  
 228 tion 7,  $s_{n-2} + 2s_{n-1} \geq 3k$  thus  $s_{n-2} + s_{n-1} \geq 2k + 1$ . This implies  $|S_{n-2}| \leq$   
 $|S| - 2k - 1 < |D_n| - 2k - 1 = |D_{n-2}|$ . On the other hand, by Lemma 9,  
 $\gamma(P_{3k} \square P_{3p-2}) = \gamma(P_{3p-2} \square P_{3k}) = (p-1)(3k+1) + 2k - 1 = k(n-1) + \lfloor \frac{n-4}{3} \rfloor = |D_{n-2}|$   
 230 thus  $D_{n-2}$  is minimum, a contradiction.  $\square$

232 Notice that, by Theorem 1,  $\gamma(P_{3k} \square P_3) = 3k + \lceil \frac{3k}{4} \rceil \neq 4k$  for  $k \geq 3$ .

## 234 4 Conclusions

Putting together Lemma 8, Lemma 9, Lemma 10 and the case  $m = 3$  or  $n = 3$ ,  
 236 we obtain  $\gamma(P_m \square P_n)$  for any  $m, n$  (Theorem 2).

As a conclusion, notice that the minimum dominating sets we build for  $P_5 \square P_n$  and

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$P_6 \square P_n$  are different than those proposed by Liu and al.([13]). An open problem would be to characterize all minimum dominating sets of  $P_m \square P_n$ .

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