



HAL
open science

The domination number of Cartesian product of two directed paths

Michel Mollard

► **To cite this version:**

Michel Mollard. The domination number of Cartesian product of two directed paths. *Journal of Combinatorial Optimization*, 2014, 27 (1), pp.144-151. hal-00655403

HAL Id: hal-00655403

<https://hal.science/hal-00655403>

Submitted on 29 Dec 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The domination number of Cartesian product of two directed paths

Michel Mollard*

Institut Fourier
100, rue des Maths
38402 St martin d'hères Cedex FRANCE
`michel.mollard@ujf-grenoble.fr`

Abstract

Let $\gamma(P_m \square P_n)$ be the domination number of the Cartesian product of directed paths P_m and P_n for $m, n \geq 2$. In [13] Liu and al. determined the value of $\gamma(P_m \square P_n)$ for arbitrary n and $m \leq 6$. In this work we give the exact value of $\gamma(P_m \square P_n)$ for any m, n and exhibit minimum dominating sets.

AMS Classification[2010]:05C69,05C38.

Keywords: Directed graph, digraph, Cartesian product, Domination number, Paths.

1 Introduction and definitions

Let $G = (V, E)$ be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex u *dominates* a vertex v if $u = v$ or $uv \in E$. A set $S \subset V$ is a *dominating set* of G if any vertex of G is dominated by at least a vertex of S . The *domination number* of G , denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. The set V is a dominating set thus $\gamma(G)$ is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

*CNRS Université Joseph Fourier

For two digraphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian product* $G_1 \square G_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$ if and only if $x_1 y_1 \in E_1$ and $x_2 = y_2$ or $x_2 y_2 \in E_2$ and $x_1 = y_1$. Note that $G \square H$ is isomorphic to $H \square G$.

The domination number of Cartesian product of two directed cycles have been recently investigated ([11], [12], [14], [15]). Even more recently, Liu and al.([13]) began the study of the domination number of the Cartesian product of two directed paths P_m and P_n . They proved the following result

Theorem 1 *Let $n \geq 2$. Then*

- $\gamma(P_2 \square P_n) = n$
- $\gamma(P_3 \square P_n) = n + \lceil \frac{n}{4} \rceil$
- $\gamma(P_4 \square P_n) = n + \lceil \frac{2n}{3} \rceil$
- $\gamma(P_5 \square P_n) = 2n + 1$
- $\gamma(P_6 \square P_n) = 2n + \lceil \frac{n+2}{3} \rceil$.

In this paper we are able to give a complete solution of the problem. In Theorem 2 we determine the value of $\gamma(P_m \square P_n)$ for any $m, n \geq 2$. When m grows, the cases approach appearing in the proof of Theorem 1 seems to be more and more complicated. We proceed by a different and elementary method, but will assume that Theorem 1 is already obtained (at least for $m \leq 5$ and arbitrary n). In the next section we describe three dominating sets of $P_m \square P_n$ corresponding to the different values of m modulo 3. In the last section we prove that these dominating sets are minimum and deduce our main result:

Theorem 2 *Let $n \geq 2$. Then*

- $\gamma(P_{3k} \square P_n) = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$ for $k \geq 2$ and $n \neq 3$
- $\gamma(P_{3k+1} \square P_n) = k(n+1) + \lceil \frac{2n-3}{3} \rceil$ for $k \geq 1$ and $n \neq 3$
- $\gamma(P_{3k+2} \square P_n) = k(n+1) + n$ for $k \geq 0$ and $n \neq 3$
- $\gamma(P_3 \square P_n) = \gamma(P_n \square P_3) = n + \lceil \frac{n}{4} \rceil$.

We will follow the notations used by Liu and al. and refer to their paper for a more complete description of the motivations. Let us recall some of these notations.

We denote the vertices of a directed path P_n by the integers $\{0, 1, \dots, n-1\}$. For any i in $\{0, 1, \dots, n-1\}$, P_m^i is the subgraph of $P_m \square P_n$ induced by the vertices $\{(k, i) / k \in \{0, 1, \dots, m-1\}\}$. Note that P_m^i is isomorphic to P_m . Notice also that $P_m \square P_n$ is isomorphic to $P_n \square P_m$ thus $\gamma(P_m \square P_n) = \gamma(P_n \square P_m)$. A vertex $(a, b) \in P_m^b$ can be dominated by (a, b) , $(a-1, b) \in P_m^b$ (if $a \geq 1$), $(a, b-1) \in P_m^{b-1}$ (if $b \geq 1$).

2 Three Dominating sets

We will first study $P_{3k} \square P_n$ for $k \geq 1$ and $n \geq 2$. Consider the following sets of vertices of P_{3k} .

- 66 • $X = \{0, 1, 3, 4, \dots, 3k - 3, 3k - 2\} = \{3i/i \in \{0, 1, \dots, k - 1\}\} \cup \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- 68 • $Y = \{2, 5, 8, \dots, 3k - 1\} = \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$
- $I = \{0, 3, 6, \dots, 3k - 3\} = \{3i/i \in \{0, 1, \dots, k - 1\}\}$
- 70 • $J = \{1, 4, 7, \dots, 3k - 2\} = \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$.

72 .

Let D_n (see Figure 1) be the set of vertices of $P_{3k} \square P_n$ consisting of the vertices

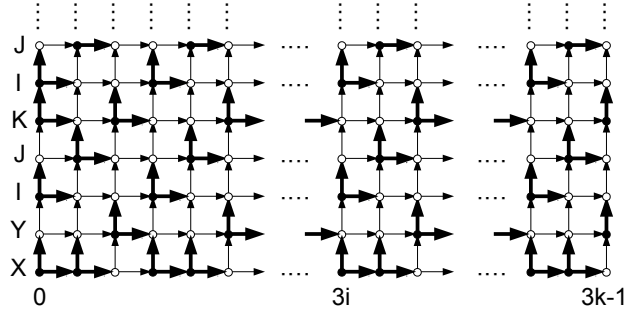


Figure 1: The dominating set D_n

- 74 • $(a, 0)$ for $a \in X$
- $(a, 1)$ for $a \in Y$
- 76 • (a, b) for $b \equiv 2 \pmod{3}$ ($2 \leq b < n$) and $a \in I$
- (a, b) for $b \equiv 0 \pmod{3}$ ($3 \leq b < n$) and $a \in J$
- 78 • (a, b) for $b \equiv 1 \pmod{3}$ ($4 \leq b < n$) and $a \in K$.

80 **Lemma 3** For any $k \geq 1$, $n \geq 2$ the set D_n is a dominating set of $P_{3k} \square P_n$ and $|D_n| = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$.

Proof : It is immediate to verify that

- 82 • All vertices of P_{3k} are dominated by the vertices of X
- The vertices of P_{3k} not dominated by some of Y are $\{0, 1, 4, \dots, 3k - 2\} \subset X$
- 84 • The vertices of P_{3k} not dominated by some of I are $\{2, 5, \dots, 3k - 1\} = Y \subset K$
- 86 • The vertices of P_{3k} not dominated by some of J are $\{0, 3, 6, \dots, 3k - 3\} \subset I$

- The vertices of P_{3k} not dominated by some of K are $\{4, 7, \dots, 3k - 2\} \subset J$.

88 Therefore any vertex of some P_{3k}^i is dominated by a vertex in $P_{3k}^i \cap D_n$ or in
90 $P_{3k}^{i-1} \cap D_n$ (if $i \geq 1$). Furthermore $|X| = 2k$, $|Y| = |I| = |J| = k$, and $|K| = k + 1$
thus $|D_n| = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$. \square

92 Let us study now $P_{3k+1} \square P_n$ for $k \geq 1$ and $n \geq 2$. Consider the following sets of
vertices of P_{3k+1} .

- 94 • $X = \{0, 2, 4, 5, 7, 8, \dots, 3k - 2, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\} \cup$
 $\{3i + 1/i \in \{1, \dots, k - 1\}\}$
- 96 • $I = \{0, 3, 6, \dots, 3k\} = \{3i/i \in \{0, 1, \dots, k\}\}$
- $J = \{1, 4, 7, \dots, 3k - 2\} = \{3i + 1/i \in \{0, 1, \dots, k - 1\}\}$
- 98 • $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$.

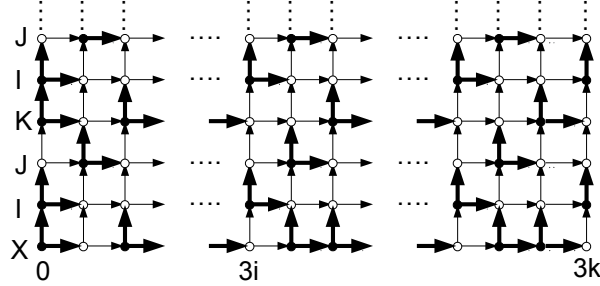


Figure 2: The dominating set E_n

100 Let E_n (see Figure 2) be the set of vertices of $P_{3k+1} \square P_n$ consisting of the vertices

- $(a, 0)$ for $a \in X$
- 102 • (a, b) for $b \equiv 1 \pmod{3}$ ($1 \leq b < n$) and $a \in I$
- (a, b) for $b \equiv 2 \pmod{3}$ ($2 \leq b < n$) and $a \in J$
- 104 • (a, b) for $b \equiv 0 \pmod{3}$ ($3 \leq b < n$) and $a \in K$.

106 **Lemma 4** For any $k \geq 1$, $n \geq 2$ the set E_n is a dominating set of $P_{3k+1} \square P_n$ and
 $|E_n| = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$.

Proof : It is immediate to verify that

- 108 • All vertices of P_{3k+1} are dominated by the vertices of X

110 • The vertices of P_{3k+1} not dominated by some of I are $\{2, 5, \dots, 3k - 1\} \subset K$
 $\subset X$

• The vertices of P_{3k+1} not dominated by some of J are $\{0, 3, 6, \dots, 3k\} = I$

112 • The vertices of P_{3k+1} not dominated by some of K are $\{4, 7, \dots, 3k - 2\} \subset J$.

114 Therefore any vertex of some P_{3k+1}^i is dominated by a vertex in $P_{3k+1}^i \cap E_n$ or
in $PP_{3k+1}^{i-1} \cap E_n$ (if $i \geq 1$). Furthermore $|X| = 2k$, $|I| = |K| = k + 1$, and $|J| = k$
thus $|E_n| = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$. \square

116

118 The last case will be $P_{3k+2} \square P_n$ for $k \geq 0$ and $n \geq 2$. Consider the following sets
of vertices of P_{3k+2} .

• $X = \{0, 1, 3, 4, \dots, 3k, 3k + 1\} = \{3i/i \in \{0, 1, \dots, k\}\} \cup \{3i + 1/i \in \{0, 1, \dots, k\}\}$

120 • $Y = \{2, 5, 8, \dots, 3k - 1\} = \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$

• $I = \{0, 3, 6, \dots, 3k\} = \{3i/i \in \{0, 1, \dots, k\}\}$

122 • $J = \{1, 4, 7, \dots, 3k + 1\} = \{3i + 1/i \in \{0, 1, \dots, k\}\}$

• $K = \{0, 2, 5, 8, \dots, 3k - 1\} = \{0\} \cup \{3i + 2/i \in \{0, 1, \dots, k - 1\}\}$.

124

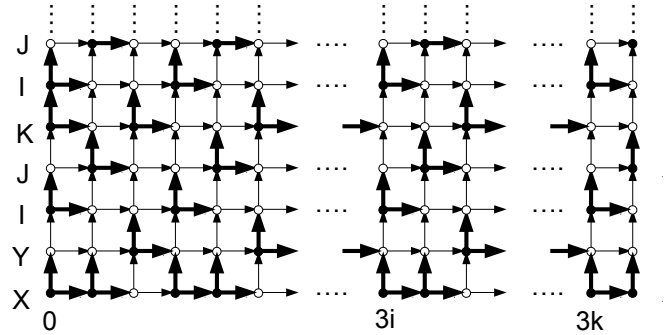


Figure 3: The dominating set F_n

Let F_n (see Figure 3) be the set of vertices of $P_{3k+2} \square P_n$ consisting of the vertices

126 • $(a, 0)$ for $a \in X$

• $(a, 1)$ for $a \in Y$

128 • (a, b) for $b \equiv 2 \pmod{3}$ ($2 \leq b < n$) and $a \in I$

• (a, b) for $b \equiv 0 \pmod{3}$ ($3 \leq b < n$) and $a \in J$

130 • (a, b) for $b \equiv 1 \pmod{3}$ ($4 \leq b < n$) and $a \in K$.

Lemma 5 For any $k \geq 0$, $n \geq 2$, the set F_n is a dominating set of $P_{3k+2} \square P_n$ and $|F_n| = k(n+1) + n$.

Proof : It is immediate to verify that

- All vertices of P_{3k+2} are dominated by the vertices of X
- The vertices of P_{3k+2} not dominated by some of Y are $\{0, 1, 4, \dots, 3k+1\} \subset X$
- The vertices of P_{3k+2} not dominated by some of I are $\{2, 5, \dots, 3k-1\} = Y \subset K$
- The vertices of P_{3k+2} not dominated by some of J are $\{0, 3, \dots, 3k\} = I$
- The vertices of P_{3k+2} not dominated by some of K are $\{4, 7, \dots, 3k+1\} \subset J$.

Therefore any vertex of some P_{3k+2}^i is dominated by a vertex in $P_{3k+2}^i \cap F_n$ or in $P_{3k+2}^{i-1} \cap F_n$ (if $i \geq 1$). Furthermore $|X| = 2k+2$, $|Y| = k$ and $|I| = |J| = |K| = k+1$, thus $|F_n| = k(n+1) + n$. \square

3 Optimality of the three sets

The structure of $P_m \square P_n$ implies the following strong property.

Proposition 6 Let S be a dominating set of $P_m \square P_n$. For any $n' \leq n$ consider

$$S_{n'} = \bigcup_{i=0, \dots, n'-1} P_m^i \cap S.$$

Then $S_{n'}$ is a dominating set of $P_m \square P_{n'}$.

Notice that the three sets D_n , E_n , F_n satisfy, for example, $(D_n)_{n'} = D_{n'}$ therefore we can use the same notation without ambiguity.

If S is a dominating set of $P_m \square P_n$, for any i in $\{0, 1, \dots, n-1\}$ let $s_i = |P_m^i \cap S|$. We have thus $|S| = \sum_{i=0}^{n-1} s_i$.

Proposition 7 Let S be a dominating set of $P_m \square P_n$. Let $i \in \{1, 2, \dots, n-1\}$ then $s_{i-1} + 2s_i \geq m$.

Proof : Any vertex of P_m^i must be dominated by some vertex of $P_m^i \cap S$ or of $P_m^{i-1} \cap S$. A vertex in $P_m^i \cap S$ dominates at most two vertices of P_m^i and a vertex in $P_m^{i-1} \cap S$ dominates a unique vertex of P_m^i . \square

Lemma 8 Let $k \geq 0$ and $n \geq 2$, $n \neq 3$, then $\gamma(P_{3k+2} \square P_n) = k(n+1) + n$.

Proof : The case $n = 2$ is immediate by Theorem 1.

Let S be a dominating set of $P_{3k+2} \square P_n$ with $n \geq 4$.

By Proposition 7, $s_i \leq k$ implies $s_{i-1} + s_i \geq m - s_i \geq 2k+2$. Therefore for any $i \in \{2, \dots, n-1\}$ we get $s_i \geq k+1$ or $s_{i-1} + s_i \geq 2(k+1)$.

Apply the following algorithm:

```

164  $I := \emptyset; J := \emptyset; i := n - 1;$ 
while  $i \geq 5$  do
  if  $s_i \geq k + 1$  then
166    $I := I \cup \{i\}; i := i - 1$ 
  else
168    $J := J \cup \{i, i - 1\}; i := i - 2$ 
  end if
170 end while

```

If $n = 4$ or $n = 5$ the algorithm only sets I and J to \emptyset . In the general case, the algorithm stop when $i = 3$ or $i = 4$ and we get two disjoint sets I, J with $\{0, 1, \dots, n - 1\} = \{0, 1, 2, 3\} \cup I \cup J$ or $\{0, 1, \dots, n - 1\} = \{0, 1, 2, 3, 4\} \cup I \cup J$. Furthermore $\sum_{i \in I} s_i \geq |I|(k + 1)$ and $\sum_{i \in J} s_i \geq |J|(k + 1)$. We have thus one of the two inequalities

$$|S| - (s_0 + s_1 + s_2 + s_3) \geq (n - 4)(k + 1)$$

or

$$|S| - (s_0 + s_1 + s_2 + s_3 + s_4) \geq (n - 5)(k + 1).$$

In the first case by Proposition 6 and Theorem 1 we get $s_0 + s_1 + s_2 + s_3 \geq \gamma(P_{3k+2} \square P_4) = \gamma(P_4 \square P_{3k+2}) = 3k + 2 + \lceil \frac{6k+4}{3} \rceil = 5k + 4$. Thus $|S| \geq (n + 1)k + n$. In the second case we get $s_0 + s_1 + s_2 + s_3 + s_4 \geq \gamma(P_5 \square P_{3k+2}) = 6k + 5$. Thus again $|S| \geq (n + 1)k + n$.

Therefore for any $n \geq 4$ we have $\gamma(P_{3k+2} \square P_n) \geq k(n + 1) + n$ and the equality occurs by Lemma 5. \square

Notice that, by Theorem 1, $\gamma(P_{3k+2} \square P_3) = 3k + 2 + \lceil \frac{3k+2}{4} \rceil \neq 4k + 3$ for $k \geq 1$.

Lemma 9 *Let $k \geq 1$ and $n \geq 2, n \neq 3$, then $\gamma(P_{3k+1} \square P_n) = k(n + 1) + \lceil \frac{2n-3}{3} \rceil$.*

Proof: Consider some fixed $k \geq 1$. Notice first that by Theorem 1, $\gamma(P_{3k+1} \square P_2) = 3k + 1$, $\gamma(P_{3k+1} \square P_4) = 5k + 2$ and $\gamma(P_{3k+1} \square P_5) = 6k + 3$ thus the result is true for $n \leq 5$.

We knows, by Lemma 4, that for any $n \geq 2$ the set E_n is a dominating set of $P_{3k+1} \square P_n$ and $|E_n| = (n + 1)k + \lceil \frac{2n-3}{3} \rceil$.

We will prove now that E_n is a minimum dominating set .

If this is not true consider n minimum, $n \geq 2$, such that there exists a dominating set S of $P_{3k+1} \square P_n$ with $|S| < |E_n|$. We knows that $n \geq 6$.

For $n' \leq n$ let $S_{n'} = \cup_{i=0, \dots, n'-1} P_{3k+1}^i \cap S$ and $s_{n'} = |P_{3k+1}^{n'} \cap S|$.

188

Case 1 $n = 3p, p \geq 2$.

Notice first that $|E_n| - |E_{n-1}| = k$ and $|E_n| - |E_{n-2}| = 2k + 1$. We have also by hypothesis $|S| \leq |E_n| - 1$. By minimality of n , E_{n-1} is minimum thus $|S_{n-1}| \geq |E_{n-1}|$. Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k - 1$. On the other hand, by Proposition 7, $s_{n-2} + 2s_{n-1} \geq 3k + 1$ thus $s_{n-2} + s_{n-1} \geq (3k + 1) - (k - 1) = 2k + 2$. This implies $|S_{n-2}| \leq |S_n| - 2k - 2 \leq |E_n| - 2k - 3 < |E_n| - 2k - 1 = |E_{n-2}|$, thus E_{n-2} is not minimum in contradiction with n minimum.

194

196 **Case 2** $n = 3p + 1, p \geq 2$.

198 In this case we have $|E_n| - |E_{n-1}| = k + 1$ and $|E_n| - |E_{n-2}| = 2k + 1$. We have
 200 also by hypothesis $|S| \leq |E_n| - 1$. By minimality of n , E_{n-1} is minimum thus
 202 $|S_{n-1}| \geq |E_{n-1}|$. Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k$. On the
 other hand, by Proposition 7, $s_{n-2} + 2s_{n-1} \geq 3k + 1$ thus $s_{n-2} + s_{n-1} \geq 2k + 1$.
 This implies $|S_{n-2}| \leq |S_n| - 2k - 1 < |E_n| - 2k - 1 = |E_{n-2}|$, thus E_{n-2} is not
 minimum in contradiction with n minimum.

204 **Case 3** $n = 3p + 2, p \geq 2$.

206 In this case, $|E_n| - |E_{n-2}| = 2k + 2$ and we cannot proceed like case 1 and case 2.
 Hopefully, by Lemma 8, $\gamma(P_{3k+1} \square P_{3p+2}) = \gamma(P_{3p+2} \square P_{3k+1}) = p(3k+2) + 3k + 1 =$
 $k(3p+3) + 2p + 1$. Therefore, since $n + 1 = 3p + 3$ and $\lceil \frac{2n-3}{3} \rceil = 2p + 1$, E_n is
 208 minimum. \square

Lemma 10 *Let $k \geq 2$ and $n \geq 2, n \neq 3$ then $\gamma(P_{3k} \square P_n) = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$.*

210 **Proof :**

Case 1 $n = 3p + 1, p \geq 1$.

212 By Lemma 9, $\gamma(P_{3k} \square P_{3p+1}) = \gamma(P_{3p+1} \square P_{3k}) = p(3k+1) + 2k - 1 = k(3p+2) + p - 1$.
 We obtain the conclusion since $3p + 2 = n + 1$ and $\lfloor \frac{n-2}{3} \rfloor = p - 1$.

214 **Case 2** $n = 3p + 2, p \geq 0$.

216 By Lemma 8, $\gamma(P_{3k} \square P_{3p+2}) = \gamma(P_{3p+2} \square P_{3k}) = p(3k+1) + 3k = k(3p+3) + p$. We
 obtain again the conclusion since $3p + 3 = n + 1$ and $\lfloor \frac{n-2}{3} \rfloor = p$.

218 **Case 3** $n = 3p, p \geq 2$.

220 We know, by Lemma 3, that the set D_n is a dominating set of $P_{3k} \square P_n$ and $|D_n| =$
 $k(n+1) + \lfloor \frac{n-2}{3} \rfloor$.

222 If D_n is not a minimum dominating set let S be a dominating set with $|S| < |D_n|$.
 For $n' \leq n$ let $S_{n'} = \cup_{i=0, \dots, n'-1} P_{3k}^i \cap S$ and $s_{n'} = |P_{3k}^{n'} \cap S|$.

224 Because $n = 3p$ and $p \geq 2$ we get $|D_n| - |D_{n-1}| = k$ and $|D_n| - |D_{n-2}| = 2k + 1$. We
 have also by hypothesis $|S| \leq |D_n| - 1$. Notice that, by Lemma 8, $\gamma(P_{3k} \square P_{n-1}) =$
 $\gamma(P_{3p-1} \square P_{3k}) = (p-1)(3k+1) + 3k = kn + \lfloor \frac{n-3}{3} \rfloor = |D_{n-1}|$ thus D_{n-1} is minimum
 and $|S_{n-1}| \geq |D_{n-1}|$.

226 Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |D_n| - 1 - |D_{n-1}| = k - 1$. By Proposi-
 228 tion 7, $s_{n-2} + 2s_{n-1} \geq 3k$ thus $s_{n-2} + s_{n-1} \geq 2k + 1$. This implies $|S_{n-2}| \leq$
 $|S| - 2k - 1 < |D_n| - 2k - 1 = |D_{n-2}|$. On the other hand, by Lemma 9,
 $\gamma(P_{3k} \square P_{3p-2}) = \gamma(P_{3p-2} \square P_{3k}) = (p-1)(3k+1) + 2k - 1 = k(n-1) + \lfloor \frac{n-4}{3} \rfloor = |D_{n-2}|$
 230 thus D_{n-2} is minimum, a contradiction. \square

232 Notice that, by Theorem 1, $\gamma(P_{3k} \square P_3) = 3k + \lceil \frac{3k}{4} \rceil \neq 4k$ for $k \geq 3$.

234 4 Conclusions

236 Putting together Lemma 8, Lemma 9, Lemma 10 and the case $m = 3$ or $n = 3$,
 we obtain $\gamma(P_m \square P_n)$ for any m, n (Theorem 2).

As a conclusion, notice that the minimum dominating sets we build for $P_5 \square P_n$ and

238
240
242
244
246
248
250
252
254
256
258
260
262
264
266
268
270
272

$P_6 \square P_n$ are different than those proposed by Liu and al.([13]). An open problem would be to characterize all minimum dominating sets of $P_m \square P_n$.

References

- [1] T.Y. Chang, W.E. Clark: “The Domination numbers of the $5 \times n$ and $6 \times n$ grid graphs”, *J.Graph Theory*, **17** (1993) 81-107.
- [2] M. El-Zahar, C.M. Pareek: “Domination number of products of graphs”, *Ars Combin.*, **31** (1991) 223-227.
- [3] M. El-Zahar, S. Khamis, Kh. Nazzal: “On the Domination number of the Cartesian product of the cycle of length n and any graph”, *Discrete App. Math.*, **155** (2007) 515-522.
- [4] R.J. Faudree, R.H. Schelp: “The Domination number for the product of graphs”, *Congr. Numer.*, **79** (1990) 29-33.
- [5] S. Gravier, M. Mollard: “On Domination numbers of Cartesian product of paths”, *Discrete App. Math.*, **80** (1997) 247-250.
- [6] B. Hartnell, D. Rall: “On dominating the Cartesian product of a graph and K_2 ”, *Discuss. Math. Graph Theory*, **24(3)** (2004) 389-402.
- [7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater: *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.
- [8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater eds.: *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.
- [9] M.S. Jacobson, L.F. Kinch: “On the Domination number of products of graphs I”, *Ars Combin.*, **18** (1983) 33-44.
- [10] S. Klavžar, N. Seifter: “Dominating Cartesian products of cycles”, *Discrete App. Math.*, **59** (1995) 129-136.
- [11] J. Liu, X.D. Zhang, X. Chen, J.Meng: “The Domination number of Cartesian products of directed cycles”, *Inf. Process. Lett.*, **110(5)** (2010) 171-173.
- [12] J. Liu, X.D. Zhang, X. Chen, J.Meng: “On Domination number of Cartesian product of directed cycles”, *Inf. Process. Lett.*, **111(1)** (2010) 36-39.
- [13] J. Liu, X.D. Zhang, X. Chen, J.Meng: “On Domination number of Cartesian product of directed paths”, *J. Comb. Optim.*, **22(4)** (2011) 651-662.
- [14] R.S. Shaheen: “Domination number of toroidal grid digraphs”, *Utilitas Mathematica* **78**(2009) 175-184.
- [15] M.Mollard: “On Domination of Cartesian product of directed cycles”, submitted (2011). Manuscript available on line: <http://hal.archives-ouvertes.fr/hal-00576481/fr/> .