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FABER POLYNOMIALS AND SPECTRUM LOCALISATION

OSCAR DEVYS

ABSTRACT. Let K be a compact connected subset of the complex plane, of non-void interior, and whose complement in the extended complex plane is connected. Denote by F_n the n -th Faber polynomial associated with K . The aim of this note is to find suitable Banach spaces of complex sequences, \mathcal{R} , such that statements of the following type hold true: if T is a bounded linear operator acting on the Banach space X such that $(\langle F_n(T)x, x^* \rangle)_{n \geq 1} \in \mathcal{R}$ for each pair $(x, x^*) \in X \times X^*$, then the spectrum of T is included in the interior of K . Generalisations of some results due to W. Mlak, N. Nikolski and J. van Neerven are thus obtained and several applications are given. An interesting feature of these generalisations is the influence of the geometry of K and the regularity of its boundary.

1. INTRODUCTION

Faber polynomials are now classical objects of study in complex analysis, function theory and approximation theory. We use in this paper Faber polynomials as a basic tool to study the following “spectrum localisation problem” in operator theory: identify conditions under which the spectrum of a Banach space bounded linear operator is included in a given domain of the complex plane. Other instances of the use of Faber polynomials for various problems in operator theory are [1, 2, 3, 15].

Computing or estimating the spectrum of a large matrix, or, more generally, of a bounded linear operator acting on a complex Banach space is an important problem in spectral theory, with possible applications in numerical analysis (see for instance [10]). The classical spectral radius formula ([4])

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n},$$

going back to Arne Beurling and Israel Gelfand, allows to find or estimate the radius $r(T)$ of the smallest closed disc in the plane centered in the origin and containing the spectrum $\sigma(T)$ of the given operator T . The condition $r(T) < 1$, corresponding to the exponential stability of T , is an important one for applications. In order to better localise the spectrum and to find inclusions of the form $\sigma(T) \subset K$ or $\sigma(T) \subset \text{int}(K)$ for some simply connected compact K in the complex plane, the right tool to consider is the behaviour of $F_n(T)$, where F_n is the Faber polynomial of degree n associated with K . We are interested to generalise in this vein (replacing the unit disc by more general domains) several existing results in the literature [9, 17, 11, 12, 16]. An interesting feature of our

study is the influence of the geometry of the domain and the regularity of its boundary on the results obtained about the localisation of the spectrum.

The present paper is organized as follows. Section 2 is dedicated to the definition and basic properties of Faber polynomials. We present the influence of the regularity of the boundary of the domain on the behaviour of these polynomials and present known applications in operator theory. In Sections 3 and 4 we present two approaches leading to different criteria for the spectrum to be included in an open domain. Some generalisations of results of Mlak ([9]), Weiss ([17]), van Neerven ([16]) and Nikolski ([11, 12]) are given here. We give in Section 5 some explicit estimates based upon the preceding criteria.

2. PRELIMINARIES ON FABER POLYNOMIALS

2.1. Definition and examples. We present here some classical results about Faber polynomials which will be used later on. We follow the setting of [14]. The reader is referred to [14] or [7] for more details about Faber polynomials and related topics.

Let K be a compact connected subset of the complex plane \mathbb{C} , different from a singleton, and whose complement is connected. From the Riemann mapping theorem, we know that there exists a unique conformal bijective map $\psi : \mathbb{D}^c \rightarrow K^c$ such that

$$\psi(\infty) = \infty \quad \text{and} \quad \psi'(\infty) > 0.$$

The map ψ has a Laurent expansion for $|w| > 1$ of the form

$$\psi(w) = \beta w + \beta_0 + \beta_1 w^{-1} + \dots + \beta_k w^{-k} + \dots$$

where $\beta > 0$ is the transfinite diameter, or (logarithmic) capacity, of K .

Let ϕ be the inverse function of ψ . The map $\phi : K^c \rightarrow \mathbb{D}^c$ has a Laurent expansion in a neighborhood of infinity of the form

$$\phi(z) = \frac{1}{\beta} z + b_0 + b_1 z^{-1} + \dots + b_k z^{-k} + \dots$$

For $n \in \mathbb{N}$, the polynomial part of the Laurent expansion of $\phi(z)^n$ is called the *Faber polynomial of order n* and is denoted by F_n . The Faber polynomial of order n has degree n and principal coefficient $1/\beta^n$. We consider the function ω_n defined by the following equation

$$\phi(z)^n = F_n(z) + \omega_n(z), \quad (z \in K^c).$$

Then $z \mapsto \omega_n(z)$ is an analytic bounded function on K^c which tends to 0 at infinity.

For any $R > 1$, let Γ_R be the analytic Jordan curve $\{\psi(w) : |w| = R\}$. Denote by G_R its interior. If z is in G_R then we have

$$F_n(z) = \frac{1}{2i\pi} \int_{\Gamma_R} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta = \frac{1}{2i\pi} \int_{|w|=R} \frac{w^n \psi'(w)}{\psi(w) - z} dw.$$

One can deduce from this relation that the Faber polynomials satisfy the following asymptotic relations

$$\lim_{n \rightarrow \infty} |F_n(z)|^{1/n} = |\phi(z)| \text{ for all } z \notin K \text{ and } \lim_{n \rightarrow \infty} |F_n(z)|^{1/n} \leq 1 \text{ for all } z \in K.$$

It also implies that for any fixed z in G_R , $(F_n(z))_n$ is the sequence of the Laurent coefficients of the map

$$w \mapsto \frac{\psi'(w)}{\psi(w) - z}, \quad |w| > R$$

in the neighborhood of the point $w = \infty$. Therefore the generating function of the Faber polynomials is given by

$$(2.1) \quad \frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad (|w| > R, z \in G_R).$$

We can deduce from this relation that Faber polynomials satisfy the following recurrence formula. Setting $\beta_{-1} := \beta$, we have

$$(2.2) \quad -k\beta_k = \sum_{s=0}^{k+1} \beta_{k-s} F_s(z) - zF_k(z), \quad (k \geq 0).$$

The Faber polynomial of order n of the disk centered at z_0 and of radius R are given by the formula $F_n(z) = ((z - z_0)/R)^n$. The series of Faber polynomials of an analytic function in the neighborhood of the compact K is a generalisation of the Taylor expansion of an analytic function in an open disc. More precisely, we have the following theorem (cf [14, Theorem 3.2.2]).

Theorem 2.1 ([14]). *An analytic function f in a neighborhood of the compact K can be uniquely expanded in a series of Faber polynomials with uniform convergence on V a neighborhood of K . That is to say, there exist complex numbers $a_n = a_n(f)$ such that for every z in V we have*

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z).$$

Examples. (1) If $K = [-1; 1]$, the map ψ is given by

$$\psi(w) = \frac{1}{2}(w + 1/w).$$

From (2.2), it follows that the sequence of Faber polynomials satisfies the 3-term recurrence formula

$$2zF_{n+1}(z) = F_{n+2}(z) + F_n(z), \quad (n \geq 1),$$

with $F_0(z) = 1$, $F_1(z) = 2z$ and $F_2(z) = 2(2z^2 - 1)$. We can deduce that for any $n \geq 1$, we have $F_n = 2C_n$, where C_n is the Chebychev polynomial of order n ,

$$C_n(t) = \cos(n \arccos(t)), \quad (t \in [-1; 1]).$$

(2) Let $m \in \mathbb{N}$ with $m \geq 2$. We consider the m -cusped hypocycloid included in $\overline{\mathbb{D}}$, denoted by H_m and delimited by the parametric curve of equation

$$z = \frac{m-1}{m} \exp(i\theta) + \frac{1}{m} \exp(-i(m-1)\theta), \quad 0 \leq \theta < 2\pi.$$

The exterior conformal map associated with H_m is

$$\psi(w) = \frac{m-1}{m}w + \frac{1}{m}w^{-m+1}.$$

The sequence of Faber polynomials is given by the recurrence formula

$$F_{n+m}(z) = \frac{m}{m-1}zF_{n+m-1}(z) - \frac{1}{m-1}F_n(z), \quad (n \geq 1),$$

and

$$F_k(z) = \left(\frac{m}{m-1}z\right)^k, \quad k = 1, \dots, m-1 \quad \text{and} \quad F_m = \left(\frac{m}{m-1}z\right)^m - \frac{m}{m-1}.$$

We refer to [8] and [6] for more information about the Faber polynomials of m -cusped hypocycloids.

(3) We consider now the ellipse with the foci at the points 1 and -1 and semi-axes

$$a = \frac{1}{2}(R + 1/R) \quad \text{and} \quad b = \frac{1}{2}(R - 1/R),$$

where $R > 1$. The equation of this ellipse can be written in the form

$$z = \frac{1}{2} \left(Re^{i\theta} + \frac{1}{Re^{i\theta}} \right), \quad 0 \leq \theta < 2\pi.$$

Hence, the exterior conformal map associated with this ellipse is given by

$$\psi(w) = \frac{1}{2} \left(Rw + \frac{1}{Rw} \right).$$

The corresponding Faber polynomials are

$$F_n(z) = \frac{2}{R^n} C_n(z), \quad (n \geq 1).$$

2.2. Consequences of the regularity of the boundary Γ . If Γ , the boundary of K , is a Jordan curve, its regularity has influence on the behaviour of the map ψ and on the asymptotic behaviour of the sequence of Faber polynomials (F_n) on Γ .

A function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a *modulus of continuity* if it satisfies:

- (1) ρ is increasing;
- (2) $\lim_{t \rightarrow 0} \rho(t) = 0$;
- (3) ρ is subadditive, i.e.,

$$\rho(t+s) \leq \rho(t) + \rho(s).$$

A modulus of continuity is said to satisfy *the Alper condition* if

$$\int_0^\varepsilon \frac{\rho(x)}{x} \log\left(\frac{1}{x}\right) dx < \infty,$$

for some ε .

A Jordan curve Γ is said to be of class C^ρ if it has a parametrization $\tau : [0; 1) \rightarrow \Gamma$ that is differentiable, with $\tau'(x) \neq 0$ of all $x \in [0, 1)$ and τ' satisfying a continuity condition

$$|\tau'(x_1) - \tau'(x_2)| \leq \rho(|x_1 - x_2|), \quad x_1, x_2 \in [0; 1).$$

A Jordan arc is of class C^ρ if it is a subarc of a Jordan curve of class C^ρ .

The following theorem is a simplified version of [13, Theorem 1.1].

Theorem 2.2 ([13]). *Let Γ be a rectifiable Jordan curve. For $z_0 \in \Gamma$, assume that Γ has an exterior angle of opening $\alpha\pi$ at z_0 , with $0 < \alpha \leq 2$, formed by C^ρ arcs, where ρ is a modulus of continuity satisfying the Alper condition. Then,*

$$F_n(z_0) = \alpha\phi(z_0)^n + o(1), \quad \text{as } n \rightarrow \infty.$$

Remark. The case $\alpha = 1$ corresponds to a point where the boundary form an arc of regularity C^ρ , the case $\alpha = 0$ to an inside pointing cusp and the case $\alpha = 2$ to an outside pointing cusp.

It is often sufficient to consider $\rho(x) = x^\beta$ for some $\beta > 0$. The class C^ρ is then the Hölder class $C^{1,\beta}$.

The following definition will be used in the next sections as a basic geometric condition about the regularity of the boundary.

Definition. A subset T of the plan \mathbb{C} will be called a C^ρ curve with angles if it is a rectifiable Jordan curve and for any $\zeta \in T$, T either form an arc of regularity C^ρ at ζ , or has an exterior angle of opening $\alpha\pi$ at ζ , $0 < \alpha \leq 2$, formed by C^ρ arcs, where ρ is a modulus of continuity satisfying the Alper condition.

We want to stress here that C^ρ curves with angles do not have inside pointing cusps.

We also record for further use the following result about analytic Jordan curves.

Theorem 2.3 ([14]). *Suppose that Γ is an analytic Jordan curve. Then ψ has an analytic univalent extension in $\overline{r\mathbb{D}^c}$ for some $r \in (0; 1)$.*

2.3. Faber polynomials and operators on a Banach space.

Notation. For X a complex Banach space, we denote by $B(X)$ the Banach algebra of the bounded linear operators from X into itself and by $I = I_X$ the identity on X . The spectrum of $T \in B(X)$ will be denoted by

$$\sigma(T) := \{ \lambda \in \mathbb{C}; (T - \lambda I) \text{ is not invertible} \}.$$

Lemma 2.4. *Let X be a Banach space. Let $T \in B(X)$ be such that $\sigma(T) \subset K$. Then, for $\lambda \in \mathbb{C}$, $|\lambda| > 1$, we have*

$$(\psi(\lambda)I - T)^{-1} = \psi'(\lambda)^{-1} \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(T).$$

Proof. Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Let $R = |\lambda|$. Define

$$f(z) = \frac{\psi'(\lambda)}{\psi(\lambda) - z}, \quad z \in G_R.$$

It follows from (2.1) that for every $z \in G_R$ we have

$$f(z) = \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(z).$$

As f is an analytic function on G_R , we have from Theorem 2.1 the uniform convergence of the series in any compact subset of G_R .

Using $\sigma(T) \subset K$, the functional calculus gives

$$(\psi(\lambda)I - T)^{-1} = \psi'(\lambda)^{-1} \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(T).$$

□

In their article [1], A. Atzmon, A. Eremenko and M. Sodin have proved that for any compact set $K \subset \mathbb{C}$ with connected complement, there exists a sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ and a positive number r such that if T is a bounded operator on X , its spectrum is included in K if and only if

$$\limsup_{n \rightarrow \infty} \|P_n(T)\|^{1/n} \leq r.$$

If K is connected, this can be expressed in terms of Faber polynomials. For the convenience of the reader we give a simple proof.

Theorem 2.5 ([1]). *Let X be a Banach space and $T \in B(X)$. Then $\sigma(T) \subset K$ if and only if*

$$\limsup_{n \rightarrow \infty} \|F_n(T)\|^{1/n} \leq 1.$$

Proof. We have that

$$\lim_{|z| \rightarrow \infty} |\phi(z)| = +\infty,$$

so there exists a real $R > 0$ such that

$$\sigma(T) \subset \overline{G_R}.$$

Then ϕ/R is the unique conformal representation of $\overline{G_R}^c$ onto $\overline{\mathbb{D}}^c$ such that $(\phi/R)(\infty) = \infty$ and $(\phi/R)'(\infty) > 0$.

We deduce that the sequence of Faber polynomials associated with the compact $\overline{G_R}$ is $(F_n/R^n)_{n \geq 0}$. Lemma 2.4 implies that

$$R\psi'(Rz) (\psi(Rz)I - T)^{-1} = \sum_{n=0}^{\infty} \frac{F_n(T)}{R^n z^{n+1}}.$$

Therefore, for every $|z| > R$,

$$\psi'(z) (\psi(z)I - T)^{-1} = \sum_{n=0}^{\infty} \frac{F_n(T)}{z^{n+1}}.$$

If $\limsup_{n \rightarrow \infty} \|F_n(T)\|^{1/n} \leq 1$, we get that the function g defined by

$$g(z) = \sum_{n=0}^{\infty} \frac{F_n(T)}{z^{n+1}}$$

is analytic on $\overline{\mathbb{D}}^c$. The maps

$$z \mapsto (\psi(z)I - T) \frac{g(z)}{\psi'(z)}$$

and

$$z \mapsto \frac{g(z)}{\psi'(z)} (\psi(z)I - T)$$

are analytic on $\overline{\mathbb{D}}^c$ and equal to I on $\{z; |z| > R\}$. Thus they are constant and equal to I on $\overline{\mathbb{D}}^c$.

Therefore, for every z with $|z| > 1$ we have

$$(\psi(z)I - T) \frac{g(z)}{\psi'(z)} = \frac{g(z)}{\psi'(z)} (\psi(z)I - T) = I.$$

Thus $\psi(z) \notin \sigma(T)$ if $|z| > 1$ and $\sigma(T) \subset K$.

If $\sigma(T) \subset K$, then the Laurent expansion of the function g holomorphic on $\overline{\mathbb{D}}^c$,

$$g(z) = \sum_{n=0}^{\infty} \frac{F_n(T)}{z^{n+1}}$$

gives $\limsup_{n \rightarrow \infty} \|F_n(T)\|^{1/n} \leq 1$.

□

3. CONDITION ON THE WEAK FABER ORBIT OF AN OPERATOR

Suppose that the spectrum $\sigma(T)$ of an operator T is included in the closure of a given domain G . The aim of this section is to propose some conditions, in the spirit of the Tauberian conditions of [11, 12], implying that $\sigma(T)$ is included in G . Such results have been proved in the case when G is the open unit disc by Mlak [9], Weiss [17], van Neerven [16] and Nikolski [11, 12].

In this section we consider K , a compact and simply connected subset of \mathbb{C} with non empty interior G . Let (F_n) be the sequence of Faber polynomials associated with K .

Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space such that

- (1) For any $k \in \mathbb{N}$, the map $(a_n) \mapsto a_k$ is bounded.
- (2) The set of finitely supported sequences, denoted by $\mathbb{C}^{(\mathbb{N})}$, is a dense subset of \mathcal{R} .
- (3) $\limsup_{n \rightarrow \infty} \|e_n\|_{\mathcal{R}}^{1/n} \leq 1$, where (e_n) denotes the canonical algebraic basis of $\mathbb{C}^{(\mathbb{N})}$.

As $\mathbb{C}^{(\mathbb{N})}$ is dense in \mathcal{R} , each element φ in \mathcal{R}^* is characterized by the sequence $(\varphi_n)_{n \geq 0}$, where

$$\varphi_n = \langle e_n; \varphi \rangle_{\mathcal{R}}.$$

This allows us to identify \mathcal{R}^* with a subset of $\mathbb{C}^{\mathbb{N}}$. Furthermore, for any finitely supported sequence (a_n) , we get from (1) that there exists an element $\varphi \in \mathcal{R}^*$ such that $\hat{\varphi}(n) = a_n$ for any $n \in \mathbb{N}$. We shall consider $\mathbb{C}^{(\mathbb{N})}$ included in \mathcal{R}^* via this identification.

Proposition 3.1. • *The condition (3) is equivalent to the following one:*

- (a) $(a_n) \in \mathcal{R}$ as soon as $\limsup_{n \rightarrow \infty} |a_n|^{1/n} < 1$.
- *if $\varphi \in \mathcal{R}^*$ and $f \in \mathbb{C}^{(\mathbb{N})}$, the duality mapping is given by*

$$\langle f; \varphi \rangle_{\mathcal{R}} = \sum_n f_n \varphi_n.$$

If $f \in \mathcal{R}$ and $\varphi \in \mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^$, this equality still holds.*

Proof. Let $\varphi \in \mathcal{R}^*$ be finitely supported and let n_0 be such that $\langle e_n; \varphi \rangle = 0$ whenever $n \geq n_0$. From (2), we know that there exists a sequence (f_k) of elements in $\mathbb{C}^{(\mathbb{N})}$ such that

$$f_k \xrightarrow[k \rightarrow \infty]{} f \quad \text{in } \mathcal{R}.$$

Hence,

$$\langle f; \varphi \rangle_{\mathcal{R}} = \lim_{k \rightarrow \infty} \langle f_k; \varphi \rangle_{\mathcal{R}}.$$

We also have $\langle f_k; \varphi \rangle_{\mathcal{R}} = \sum_{n=0}^{n_0} f_k(n) \varphi_n$. Therefore, using the hypothesis (1), we get that $f_k(n) \xrightarrow[k \rightarrow \infty]{} f(n)$, for every n .

We obtain

$$\langle f; \varphi \rangle_{\mathcal{R}} = \lim_{k \rightarrow \infty} \langle f_k; \varphi \rangle_{\mathcal{R}} = \sum_n f(n) \varphi_n.$$

□

The crucial axiom needed for our Tauberian conditions is the following one.

- (4) For any $\zeta \in \partial K$, the map V_{ζ} , defined in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$ by

$$V_{\zeta} : \varphi \mapsto \sum_{n \geq 0} \varphi_n F_n(\zeta)$$

is not bounded.

The following definition extends the corresponding unit-disc analogue from [12].

Definition. Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space and $T \in B(X)$. The operator T is said to be of *weak type* \mathcal{R} if for every pair $(x; y) \in X \times X^*$ we have

$$(\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \in \mathcal{R}.$$

Theorem 3.2. Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space satisfying conditions (1) to (4) for K . Let $T \in B(X)$ be a weak type \mathcal{R} operator with $\sigma(T) \subset K$. Then, in fact,

$$\sigma(T) \subset G.$$

Lemma 3.3. Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space satisfying the condition (1). Suppose that $T \in B(X)$ is a weak type \mathcal{R} operator. Then

$$\sup \left\{ \left\| (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \right\|_{\mathcal{R}} ; \|y\| \leq 1, \|x\| \leq 1 \right\} < \infty.$$

Proof. Let us prove that for any $x \in X$, the map

$$\begin{aligned} f_x : X^* &\rightarrow \mathcal{R} \\ y &\mapsto (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \end{aligned}$$

is bounded.

Indeed, f_x is well defined because T is a weak type \mathcal{R} operator. Let $(y_n)_{n \in \mathbb{N}} \subset X^*$, $y \in X^*$ and $r \in \mathcal{R}$ be such that

$$y_n \rightarrow y \quad \text{and} \quad f_x(y_n) \xrightarrow{\mathcal{R}} r.$$

We infer from condition (1) that for any $k \in \mathbb{N}$

$$f_x(y_n)(k) \xrightarrow{n \rightarrow \infty} r(k).$$

For any $k \in \mathbb{N}$

$$f_x(y_n)(k) = \langle F_k(T)x; y_n \rangle \xrightarrow{n \rightarrow \infty} \langle F_k(T)x; y \rangle = f_x(y)(k).$$

Thus $f_x(y) = r$ and, using the closed graph theorem, f_x is bounded. In particular, for any $x \in X$

$$(3.1) \quad \sup \left\{ \left\| (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \right\|_{\mathcal{R}} ; \|y\| \leq 1 \right\} < \infty.$$

For any $y \in X^*$ let

$$\begin{aligned} g_y : X &\rightarrow \mathcal{R} \\ x &\mapsto f_x(y). \end{aligned}$$

Then, in the same way as for f_x , we can prove that, for every y , the map g_y is bounded. Using (3.1) we obtain that

$$\sup_{\|y\| \leq 1} \|g_y(x)\| < \infty,$$

for an arbitrary $x \in X$. Using the Banach-Steinhaus theorem we infer that $\{g_y\}_{\|y\| \leq 1}$ is a bounded set. Thus

$$\sup \left\{ \left\| (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \right\|_{\mathcal{R}} ; \|y\| \leq 1, \|x\| \leq 1 \right\} < \infty,$$

which completes the proof. \square

Proof of Theorem 3.2. Lemma 3.3 implies that

$$C := \sup \left\{ \left\| (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \right\|_{\mathcal{R}} ; \|y\| \leq 1, \|x\| \leq 1 \right\} < \infty.$$

Thus for any $\varphi \in \mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$, using Proposition 3.1

$$\begin{aligned} \left\| \sum_{n \geq 0} \varphi_n F_n(T) \right\| &= \sup \left\{ \left| \sum_{n \geq 0} \varphi_n \langle F_n(T)x; y \rangle \right| ; \|x\| \leq 1, \|y\| \leq 1 \right\} \\ &\leq \sup \left\{ \left\| (\langle F_n(T)x; y \rangle)_{n \in \mathbb{N}} \right\|_{\mathcal{R}} ; \|x\| \leq 1, \|y\| \leq 1 \right\} \|\varphi\|_{\mathcal{R}^*} \\ &\leq C \|\varphi\|_{\mathcal{R}^*}. \end{aligned}$$

If $\zeta \in \sigma(T)$, we have

$$\left| \sum_{n \geq 0} \varphi_n F_n(\zeta) \right| \leq r \left(\sum_{n \geq 0} \varphi_n F_n(T) \right) \leq \left\| \sum_{n \geq 0} \varphi_n F_n(T) \right\| \leq C \|\varphi\|_{\mathcal{R}^*}.$$

Hence for any $\zeta \in \sigma(T)$, the map V_ζ is bounded in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$.

This gives us

$$\partial K \cap \sigma(T) = \emptyset.$$

□

Example (1). Let $1 \leq p < \infty$ and (ω_n) be a bounded sequence of positive numbers such that

$$|\omega_n|^{1/n} \xrightarrow{n \rightarrow \infty} 1 \quad \text{and} \quad \sum_{n=0}^{\infty} \omega_n = \infty.$$

We suppose that K is such that for any $\zeta \in \partial K$,

$$\sum_{n=0}^{\infty} \omega_n |F_n(\zeta)|^p = \infty.$$

Then the space

$$l^p(\omega) := \left\{ (a_n); \sum_{n=0}^{\infty} \omega_n |a_n|^p < \infty \right\}$$

with the norm:

$$\|(a_n)\|_{l^p(\omega)} = \left(\sum_{n=0}^{\infty} \omega_n |a_n|^p \right)^{1/p}.$$

satisfies conditions (1) to (4).

Proof. It is easy to check that conditions (1) to (3) are satisfied. Let us prove that the condition (4) is also satisfied.

Let $\zeta \in \partial K$. We argue by contradiction. Suppose that V_ζ is bounded. Then V_ζ has a continuous extension to $(l^p(\omega))^*$, that is to say that $V_\zeta \in l^p(\omega)^{**}$. In the case of $p \in (1; +\infty)$ we get $V_\zeta \in l^p(\omega)$, which gives a contradiction. Thus V_ζ is unbounded. The same holds true for $p = 1$. □

We deduce the following result from Theorems 3.2 and 2.2.

Theorem 3.4. *Let $1 \leq p < \infty$ and let (ω_n) be a bounded sequence of positive numbers such that $|\omega_n|^{1/n} \xrightarrow{n \rightarrow \infty} 1$. Suppose either that ∂K is a C^p curve with angles and*

$$\sum_{n=0}^{\infty} \omega_n = \infty,$$

or that for any $\zeta \in \partial K$,

$$\sum_{n=0}^{\infty} \omega_n |F_n(\zeta)|^p = \infty.$$

Let X be a Banach space. Let $T \in B(X)$ be such that for every $(x, y) \in X \times X^$ we have*

$$\sum_{n=0}^{\infty} \omega_n |\langle F_n(T)x; y \rangle|^p < \infty.$$

Then $\sigma(T) \subset G$.

Example (2). For $p \in [1; +\infty]$, let $H^p(\mathbb{D})$ be the Hardy space on the unit disk. We suppose that for no $\zeta \in \partial K$, we have $\lim_n |F_n(\zeta)| = 0$. Then the space

$$h_p := \left\{ (x_n); f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\mathbb{D}) \right\},$$

with the norm

$$\|(x_n)\| = \left\| \sum_{n=0}^{\infty} x_n z^n \right\|_{H^p(\mathbb{D})}$$

satisfies conditions (1) to (4).

Proof. It is easy to check that conditions (1) to (3) are satisfied. Let us prove that the condition (4) is also satisfied.

Suppose on the contrary that condition (4) does not hold. Then there exists $\zeta \in \partial K$ such that V_ζ is bounded in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$. Let $a_n = F_n(\zeta)$ for every n .

According to the Hahn-Banach theorem, there exists a bounded extension of V_ζ to $(h^p)^*$ still denoted V_ζ . Hence there exists $M \geq 0$ such that for every $(x_n^*) \in (h^p)^*$,

$$|V_\zeta(x_n^*)| \leq M \|(x_n^*)\|.$$

But $\|f(r \cdot)\|_{H^p} \leq \|f\|_{H^p}$ for any $r \in (0; 1)$ and every $f \in H^p$. Thus, for every $(x_n^*) \in (h^p)^*$,

$$\begin{aligned} \|(r^n x_n^*)\| &= \sup\{|\langle r^n x_n^*; x_n \rangle|; \|(x_n)\|_{h^p} \leq 1\} \\ &= \sup\{|\langle x_n^*; r^n x_n \rangle|; \|(x_n)\|_{h^p} \leq 1\} \\ &\leq \|(x_n^*)\|. \end{aligned}$$

So, for every $r \in (0; 1)$ and for every $(x_n^*) \in (h^p)^*$,

$$\left| \sum_{n=0}^{\infty} x_n^* r^n a_n \right| \leq M \| (x_n^*) \|.$$

Consequently, $\| (r^n a_n) \| \leq M$ for each $r \in (0; 1)$, and thus (a_n) is in h^p . We have $f(e^{i\theta}) = \sum a_n e^{in\theta} \in H^p(\mathbb{T})$ and $a_n \xrightarrow{n \rightarrow \infty} 0$. This contradiction completes the proof. \square

Definition ([5]). For $p \in [1; \infty)$, a function f analytic in $\mathbb{C} \setminus K$ is said to be of class $E^p(\mathbb{C} \setminus K)$ if there exists a sequence of rectifiable Jordan curves C_1, C_2, \dots in $\mathbb{C} \setminus K$ such that C_n eventually surrounds each compact subset of $\mathbb{C} \setminus K$, and such that

$$\sup_n \int_{C_n} |f(z)|^p |dz| < \infty.$$

Theorem 3.5 ([5] Theorem 10.1). *A function f is of class $E^p(\mathbb{C} \setminus K)$ if and only if,*

$$\sup_{r \in (1; 2)} \int_{|\phi(z)|=r} |f(z)|^p |dz| < \infty.$$

We deduce the following result which extends Mlak's result from [9].

Theorem 3.6. *We suppose either that ∂K is a C^p curve with angles or that for no $\zeta \in \partial K$, we have $\lim_n |F_n(\zeta)| = 0$. Let $T \in B(X)$ be such that $\sigma(T) \subset K$.*

If for any $x \in X$ and $x^ \in X^*$ we have*

$$z \mapsto \langle (zI - T)^{-1} x; x^* \rangle \in E^1(\mathbb{C} \setminus K),$$

then $\sigma(T) \subset \text{int}(K)$.

Proof. We get from example (2) and Theorem 3.2 that it is sufficient to prove that for any $x \in X$ and $x^* \in X^*$ we have

$$\langle F_n(T)x; x^* \rangle_n \in h^1.$$

From lemma 2.4 we get that for any $r > 1$,

$$\begin{aligned} \int_{|\phi(z)|=r} |\langle (zI - T)^{-1} x; x^* \rangle| |dz| &= \int_{z \in r\mathbb{T}} |\langle (\psi(z)I - T)^{-1} x; x^* \rangle| |\psi'(z)| |dz| \\ &= \int_{z \in r\mathbb{T}} \left| \sum_{n=0}^{\infty} z^{-n-1} \langle F_n(T)x; x^* \rangle \right| |dz| \\ &= \int_{z \in r^{-1}\mathbb{T}} \left| \sum_{n=0}^{\infty} z^n \langle F_n(T)x; x^* \rangle \right| r |dz|. \end{aligned}$$

This proves that

$$z \mapsto \left| \sum_{n=0}^{\infty} z^n \langle F_n(T)x; x^* \rangle \right| \in H^1(\mathbb{D}).$$

\square

4. ADMISSIBLE BANACH FUNCTION SPACES

In this section, we consider K a simply connected compact set with a non empty interior G .

Definition. A Banach space $E \subset \mathbb{C}^{\mathbb{N}}$ is said to be an *admissible Banach function space over \mathbb{N}* if it satisfies the following properties:

- if $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$ and $(b_n) \in E$, then $(a_n) \in E$ and $\|(a_n)\|_E \leq \|(b_n)\|_E$;
- for all $k \in \mathbb{N}$ there exists a sequence $(a_n) \in E$ such that $a_k \neq 0$.

Proposition 4.1. *Let E be an admissible Banach function space over \mathbb{N} . Then for any finite set $F \subset \mathbb{N}$, the characteristic function of F , denoted χ_F , belongs to E .*

Proof. Let $k \in \mathbb{N}$. There exists a sequence $(a_n) \in E$ such that $a_k \neq 0$ by the second axiom of admissibility. Then $|\chi_{\{k\}}(n)| \leq |a_k|^{-1}|a_n|$ for all $n \in \mathbb{N}$. It follows then from the first axiom of admissibility that $\chi_{\{k\}}$ belongs to E . By taking finite sums, we get that the characteristic functions of finite sets belong to E . \square

The main result of this section is the following one, generalising a result from [16].

Theorem 4.2. *Let X be a Banach space, $T \in B(X)$ and E an admissible Banach function space over \mathbb{N} such that for any $\zeta \in \partial K$*

$$\|(F_n(\zeta)\chi_{\{0,\dots,N\}}(n))_n\|_E \xrightarrow{N \rightarrow \infty} \infty.$$

If T is of weak type E , then

$$\sigma(T) \subset G.$$

Lemma 4.3. *Let $T \in B(X)$ be a linear operator acting on the Banach space X . Suppose that $\sigma(T) \not\subset G$. Then there exists $\lambda \notin G$ such that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$ there exist $x_N \in X$ and $x_N^* \in X^*$ satisfying $\|x_N\| = 1$, $\|x_N^*\| = 1$ and*

$$|\langle F_n(T)x_N; x_N^* \rangle| > |F_n(\lambda)| - \varepsilon, \quad n = 0, 1, \dots, N-1.$$

Proof. Let $\lambda \in \partial\sigma(T) \setminus G$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$. As $\lambda \in \partial\sigma(T)$, we have that λ belongs to $\sigma_{app}(T)$, the approximate point spectrum of T . Hence, there exists a sequence (y_n) in X such that $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|Ty_n - \lambda y_n\| \xrightarrow{n \rightarrow \infty} 0.$$

Thus, for each $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \|F_k(T)y_n - F_k(\lambda)y_n\| = 0.$$

Let n_1 be such that

$$\|F_k(T)y_{n_1} - F_k(\lambda)y_{n_1}\| < \varepsilon, \quad \forall k = 0, 1, \dots, N-1.$$

Let $x_N = y_{n_1}$ and $x_N^* \in X^*$ be such that $\|x_N^*\| = 1$ and $|\langle x_N^*; x_N \rangle| = 1$. Then for $k \in \{0, 1, \dots, N-1\}$, we have

$$\begin{aligned} |\langle F_k(T)x_N; x_N^* \rangle| &\geq |\langle F_k(\lambda)x_N; x_N^* \rangle| - |\langle F_k(T)x_N - F_k(\lambda)x_N; x_N^* \rangle| \\ &> |F_k(\lambda)| - \varepsilon. \end{aligned}$$

□

Proof of Theorem 4.2. From Lemma 3.3 we know that there exists $M \geq 0$ such that for any $x \in X$, $x^* \in X^*$, with $\|x\| = 1$ and $\|x^*\| = 1$, we have

$$\left\| (\langle F_n(T)x; x^* \rangle)_{n \in \mathbb{N}} \right\|_E \leq M.$$

Suppose that $\sigma(T) \not\subseteq G$. Then, using Lemma 4.3, there exist $\lambda \notin G$ such that for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $x_N \in X$ and $x_N^* \in X^*$ satisfying $\|x_N\| \leq 1$, $\|x_N^*\| \leq 1$ and

$$|\langle F_n(T)x_N; x_N^* \rangle| > |F_n(\lambda)| - \varepsilon, \quad n = 0, 1, \dots, N-1.$$

Therefore, $|\langle F_n(T)x_N; x_N^* \rangle| \geq |F_n(\lambda)|\chi_{\{0, \dots, N-1\}} - \varepsilon\chi_{\{0, \dots, N-1\}}(n)$ for all n . Thus

$$\left\| (\langle F_n(T)x_N; x_N^* \rangle)_{n \in \mathbb{N}} \right\|_E \geq \|(F_n(\zeta)\chi_{\{0, \dots, N\}}(n))_n\|_E - \varepsilon\|\chi_{\{0, \dots, N-1\}}\|_E.$$

We obtain that for any $\varepsilon > 0$ and $N \in \mathbb{N}$

$$\|(F_n(\zeta)\chi_{\{0, \dots, N\}}(n))_n\|_E - \varepsilon\|\chi_{\{0, \dots, N-1\}}\|_E \leq M.$$

Hence, for every $N \in \mathbb{N}$ we have

$$\|(F_n(\zeta)\chi_{\{0, \dots, N\}}(n))_n\|_E \leq M,$$

which gives the desired contradiction if $\lambda \in \partial K$. If $\lambda \notin K$, then

$$|F_n(\lambda)|^{1/n} \xrightarrow{n \rightarrow \infty} R > 1.$$

But we know that for any $\lambda \in \partial K$, $\limsup_{n \rightarrow \infty} |F_n(\lambda)|^{1/n} \leq 1$, and

$$\|(F_n(\zeta)\chi_{\{0, \dots, N\}}(n))_n\|_E \xrightarrow{n \rightarrow \infty} \infty.$$

Thus, if $\lambda \notin K$ we get from the first axiom of admissibility that

$$\|(F_n(\zeta)\chi_{\{0, \dots, N\}}(n))_n\|_E \xrightarrow{N \rightarrow \infty} \infty,$$

which gives a contradiction. □

Remark. It is easy to verify that $l^p(\omega)$ (cf. Example 1 in Section 3) is an admissible Banach function space over \mathbb{N} and the condition

$$\|(F_n(\zeta)\chi_{\{0, \dots, N-1\}}(n))\|_{l^p(\omega)} \xrightarrow{N \rightarrow \infty} \infty$$

is equivalent to

$$\sum_{n=0}^{\infty} \omega_n |F_n(\zeta)|^p = \infty.$$

Theorem 4.4. *Let $1 \leq p < \infty$ and let (ω_n) be a sequence of non negative numbers. We suppose either that ∂K is a C^p curve with angles and*

$$\sum_{n=0}^{\infty} \omega_n = \infty,$$

or that for any $\zeta \in \partial K$,

$$\sum_{n=0}^{\infty} \omega_n |F_n(\zeta)|^p = \infty.$$

Let X be a Banach space. Let $T \in B(X)$ be such that for every $(x, x^) \in X \times X^*$ we have*

$$\sum_{n=0}^{\infty} \omega_n |\langle F_n(T)x; x^* \rangle|^p < \infty.$$

Then $\sigma(T) \subset G$.

Remark. Contrarily to Theorem 3.4 we do not need to suppose here that

$$|\omega_n|^{1/n} \xrightarrow{n \rightarrow \infty} 1.$$

Proof of Theorem 4.4. We have that

$$\lim_{|z| \rightarrow \infty} |\phi(z)| = \infty.$$

Therefore, there exists a real $R > 0$ such that

$$\sigma(T) \cap \{z : \phi(z) \geq R\} = \emptyset.$$

Thus, we get from Theorem 2.5 that

$$\limsup_{n \rightarrow \infty} \|F_n(T)\|^{1/n} < R.$$

Let $(\tilde{\omega}_n)$ be the positive sequence defined by

$$\tilde{\omega}_n = \begin{cases} \omega_n & \text{if } \omega_n \neq 0, \\ R^{-n} & \text{if } \omega_n = 0. \end{cases}$$

For every $(x, y) \in X \times X^*$ we have

$$\sum_{n=0}^{\infty} \tilde{\omega}_n |\langle F_n(T)x; y \rangle|^p < \infty.$$

It is then sufficient to apply Theorem 4.2 with the space $l^p(\tilde{\omega})$ to conclude. \square

We give now other applications of Theorem 4.2.

Definition. A function $\Phi : [0; \infty) \rightarrow [0; \infty)$ is a *Young function* if it is convex and such that

$$\frac{\Phi(x)}{x} \xrightarrow{x \rightarrow \infty} \infty \text{ and } \frac{\Phi(x)}{x} \xrightarrow{x \rightarrow 0} 0.$$

For any complex sequence (x_n) let

$$M_{\Phi}(x) = \sum_{n=0}^{\infty} \Phi(|x_n|).$$

The Orlicz space L^{Φ} is the space of all sequences (x_n) such that there exists a $k > 0$ such that $M_{\Phi}(kx) < \infty$.

Theorem 4.5. *Let Φ be a Young function, the Orlicz space L^{Φ} with the norm*

$$\|(x_n)\| = \inf\{k; M_{\Phi}(\frac{1}{k}x) \leq 1\}$$

is a Banach space and if $\Phi(t) > 0$ for all $t > 0$, then

$$\|\chi_{\{0, \dots, n-1\}}\|_{L^{\Phi}} \xrightarrow{n \rightarrow \infty} \infty.$$

Lemma 4.6 ([16]). *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be non-decreasing with $\varphi(t) > 0$ for all $t > 0$. Then there exists a Young function Φ such that the Orlicz space L^{Φ} contains every sequence (a_n) such that*

$$\sum_{n \in \mathbb{N}} \varphi(|a_n|) < \infty,$$

and satisfies

$$\|\chi_{\{0, \dots, n-1\}}\|_{L^{\Phi}} \xrightarrow{n \rightarrow \infty} \infty.$$

Proof. Replacing φ by a multiple of φ we may assume that $\varphi(1) = 1$. Let Φ be defined by

$$\Phi(t) = \int_0^t \varphi(s) ds.$$

Then Φ is a Young function and if (a_n) is such that

$$\sum_{n \in \mathbb{N}} \varphi(|a_n|) < \infty,$$

then the set of indices n such that $|a_n| > 1$ is finite and if $|a_n| \leq 1$, then $\Phi(|a_n|) \leq \varphi(|a_n|)$. Thus we get

$$\sum_{n \in \mathbb{N}} \Phi(|a_n|) < \infty,$$

so $(a_n) \in L^{\Phi}$.

And as Φ is strictly positive and increasing on $(0; \infty)$, we get

$$\|\chi_{\{0, \dots, n-1\}}\|_{L^{\Phi}} = \inf \left\{ k; \sum_{k=0}^{n-1} \Phi(1/k) \leq 1 \right\} \xrightarrow{n \rightarrow \infty} \infty.$$

□

As Orlicz spaces of sequences are admissible Banach function spaces over \mathbb{N} , we obtain the following result.

Theorem 4.7. *We suppose either that ∂K is a C^p curve with angles or that $(|F_n(\zeta)|)$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(X)$ and let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function with $\varphi(t) > 0$ for all $t > 0$. If for every $x \in X$ and $x^* \in X^*$*

$$\sum_{n=0}^{\infty} \varphi(|\langle F_n(T)x; x^* \rangle|) < \infty,$$

then

$$\sigma(T) \subset G.$$

Definition. The function φ is said to satisfy the Δ_2 -condition at 0 if there exist $\varepsilon > 0$ and $K > 0$ such that for any $x \in [0; \varepsilon]$, we have $\varphi(t/2) \geq K\varphi(t)$.

Theorem 4.8. *We suppose either that ∂K is a C^p curve with angles or that $(|F_n(\zeta)|)$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(X)$, let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying the Δ_2 -condition at 0 with $\varphi(t) > 0$ for all $t > 0$, and let (ω_n) be a sequence of non negative numbers such that*

$$\sum_{n=0}^{\infty} \omega_n \varphi(\omega_n) = \infty.$$

If for every $x \in X$ and $x^* \in X^*$

$$\sum_{n=0}^{\infty} \varphi(\omega_n |\langle F_n(T)x; x^* \rangle|) < \infty,$$

then

$$\sigma(T) \subset G.$$

Proof. Adapting the proof of Theorem 4.4, we may suppose w_n to be positive for every n . Replacing if necessary ω_n by $\min(\omega_n, \varepsilon)$, we may also assume that $\omega_n \in (0; \varepsilon]$ for each n . Let Φ be as in Lemma 4.6 and let E be the space

$$E = \{(x_n); (\omega_n x_n) \in L^\Phi\},$$

with the norm

$$\|(x_n)\|_E = \|(\omega_n x_n)\|_{L^\Phi}.$$

The Δ_2 condition gives that for any $t \in (0; \varepsilon]$,

$$\Phi(t) = \int_0^t \varphi(s) ds \geq \frac{t}{2} \varphi(t/2) \geq \frac{Kt}{2} \varphi(t).$$

Let $k > 1$, and $m = E(\ln(k)/\ln(2)) + 1$. Then

$$\sum_{j=0}^{n-1} \Phi(\omega_j/k) \geq \frac{K^m}{2^m} \sum_{j=0}^{n-1} \omega_j \varphi(\omega_j).$$

This proves that

$$\|\chi_{\{0, \dots, n-1\}}\|_E \xrightarrow{n \rightarrow \infty} \infty$$

allowing the application of Theorem 4.2. □

Lemma 4.9. *Let φ be a positive function on $[0; 1)$ such that $\varphi(r)$ tends to infinity and $\sqrt{1-r}\varphi(r)$ tends to 0 as r tends to 1. Then the space E defined by*

$$E = \left\{ (x_n); \frac{1}{\varphi(r)} \|(r^n x_n)\|_2 \xrightarrow{r \rightarrow 1} 0 \right\},$$

with the norm

$$\|(x_n)\|_E = \sup_{r \in (0; 1)} \frac{1}{\varphi(r)} \|(r^n x_n)\|_2,$$

is a Banach function space over \mathbb{N} such that

$$\|\chi_{\{0, \dots, n-1\}}\|_E \xrightarrow{n \rightarrow \infty} \infty.$$

Proof. It is easy to see that E is an admissible Banach function space over \mathbb{N} . Furthermore, we have

$$\|\chi_{\{0, \dots, n-1\}}\|_E = \sup_{r \in (0; 1)} \frac{1}{\varphi(r)} \|(r^k \chi_{\{0, \dots, n-1\}}(k))_{k \in \mathbb{N}}\|_2,$$

and

$$\|(r^k \chi_{\{0, \dots, n-1\}}(k))\|_2^2 = \sum_{k=0}^{n-1} r^{2k} = \frac{1-r^{2n}}{1-r^2}.$$

Therefore,

$$\begin{aligned} \frac{1}{\varphi(r)} \|(r^k \chi_{\{0, \dots, n-1\}}(k))_{k \in \mathbb{N}}\|_2 &= \frac{1}{\varphi(r)} \frac{\sqrt{1-r^n}}{\sqrt{1-r}} \frac{\sqrt{1+r^n}}{\sqrt{1+r}} \\ &\geq \frac{1}{2} \frac{\sqrt{1-r^n}}{\varphi(r)\sqrt{1-r}}. \end{aligned}$$

Let $M > 0$. As $\sqrt{1-r}\varphi(r)$ tends to 0 as r tends to 1, there exists $r_0 < 1$ such that

$$\frac{1}{\varphi(r_0)\sqrt{1-r_0}} > 4M.$$

Let N be such that $\sqrt{1-r_0^N} > 1/2$. We get that for each $n \geq N$ we have

$$\|\chi_{\{0, \dots, n-1\}}\|_E \geq M.$$

The proof is complete. \square

Theorem 4.10. *We suppose either that ∂K is a C^p curve with angles or that $(|F_n(\zeta)|)$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(X)$. Suppose there exists a positive function φ on $[0; 1)$ such that*

$$\sqrt{1-r}\varphi(r) \xrightarrow{r \rightarrow 1} 0$$

and such that

$$\frac{1}{\varphi(r)} \|(r^n \langle F_n(T)x; x^* \rangle)_{n \in \mathbb{N}}\|_2 \xrightarrow{r \rightarrow 1} 0,$$

for every $x \in X$ and every $x^* \in X^*$. Then

$$\sigma(T) \subset G.$$

Proof. Replacing, if necessary, $\varphi(r)$ by $\max(\varphi(r); (1-r)^{-1/4})$, we may assume that $\varphi(r) \xrightarrow{r \rightarrow 1} \infty$. We can then apply Lemma 4.9 and Theorem 4.2 to obtain the inclusion $\sigma(T) \subset G$. \square

5. EXPLICIT ESTIMATES

5.1. General domains. We suppose in this section that Γ is an analytic Jordan curve. Let E be a Banach function space over \mathbb{N} containing $(r^n)_n$ for any $r \in (0; 1)$ and satisfying the condition

$$\|\chi_{\{0, \dots, n-1\}}\|_E \xrightarrow{n \rightarrow \infty} \infty.$$

Let $T \in B(X)$ be a weak type E operator.

As we have supposed Γ to be analytic, we get from Theorem 2.3 that the map ψ has an analytic extension to $\overline{r_0 \mathbb{D}^c}$ for some $r_0 < 1$.

Notation. For any $r \in (r_0; 1)$, let Γ_r be the analytic Jordan curve defined by

$$\Gamma_r = \{\psi(z); |z| = r\}$$

and let G_r be the interior of Γ_r .

We know from Theorem 4.2 that the spectrum of T is included in G . Having in mind possible applications in numerical analysis, it is the aim of this section to give an estimate of the “shrinking radius” $r \in (r_0; 1)$ such $\sigma(T) \subset \overline{G_r}$. In the case of the disc, this was done by N. Nikolski in [12]. Our estimate of the “shrinking radius” is given in terms of the constant $C(T, E)$ defined by

$$C(T, E) = \sup \left\{ \left\| \left(\langle F_n(T)x; y \rangle \right)_{n \in \mathbb{N}} \right\|_E ; \|y\| \leq 1, \|x\| \leq 1 \right\}.$$

Notation. For $r \in (r_0; 1)$ we denote

$$p(r) = \inf_{r \leq |\phi(\zeta)| < 1} \|(F_n(\zeta))_{n \in \mathbb{N}}\|_E$$

and, for $t > 0$

$$p^{-1}(t) = \begin{cases} r_0 & \text{if } \{r \in (r_0; 1); p(r) \leq t\} = \emptyset, \\ \sup\{r \in (r_0; 1); p(r) \leq t\} & \text{otherwise.} \end{cases}$$

Lemma 5.1. *Let $r \in (r_0; 1)$. If $\sigma(T) \not\subset G_r$, then there exists $\lambda \in (G_r)^c$ such that for every $N \in \mathbb{N}$ and every $\varepsilon > 0$, there exist $x_N \in X$ and $x_N^* \in X^*$ satisfying $\|x_N\| = 1$, $\|x_N^*\| = 1$ such that*

$$|\langle x_N^*; F_k(T)x_N \rangle| > |F_k(\lambda)| - \varepsilon, \quad k = 0, 1, \dots, N-1.$$

Proof. As $\sigma(T) \not\subset G_r$, there exist $\lambda \in (G_r)^c \cap \partial\sigma(T)$. In particular λ belongs to the approximate point spectrum $\sigma_{app}(T)$ of T . Thus, for every $N \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $x_N \in X$ such that $\|x_N\| = 1$ and

$$\|F_k(T)x_N - F_k(\lambda)x_N\| < \varepsilon, \quad (k \in \{0, 1, \dots, N-1\}).$$

Let $x_N^* \in X^*$ be such that $\|x_N^*\| = 1$ and $|\langle x_N^*; x_N \rangle| = 1$. Then, for every $k \in \{0, 1, \dots, N-1\}$, we have

$$\begin{aligned} |\langle x_N^*; F_k(T)x_N \rangle| &\geq |\langle x_N^*; F_k(\lambda)x_N \rangle| - |\langle x_N^*; F_k(T)x_N - F_k(\lambda)x_N \rangle| \\ &\geq |F_k(\lambda)| - \varepsilon. \end{aligned}$$

□

We deduce the following result.

Theorem 5.2. *Let T be a weak type E operator for the compact K and set $r = p^{-1}(C(T, E))$. Then*

$$\sigma(T) \subset \overline{G_r}.$$

Proof. We argue by contradiction. Suppose that $\sigma(T) \not\subset \overline{G_r}$. Let λ be the complex number satisfying the properties described in Lemma 5.1. For every $N \in \mathbb{N}$ and every $\varepsilon > 0$, there exist $x_N \in X$ and $x_N^* \in X^*$ satisfying $\|x_N\| = 1$, $\|x_N^*\| = 1$ and

$$|\langle x_N^*; F_k(T)x_N \rangle| > |F_k(\lambda)| - \varepsilon, \quad (k \in \{0, 1, \dots, N-1\}).$$

By definition of $C(T, E)$, we have

$$C(T, E) \geq \|(\langle F_n(T)x_N; x_N^* \rangle)_n\|_E \geq \|((F_n(\lambda) - \varepsilon)\chi_{\{0,1,\dots,N-1\}}(n))_n\|_E.$$

Thus, for every $N \in \mathbb{N}$,

$$\|(F_0(\lambda), F_1(\lambda), \dots, F_{N-1}(\lambda), 0, \dots)_n\|_E \leq C(T, E).$$

But $F_n(\lambda) \sim \phi(\lambda)^n$, $n \rightarrow \infty$, and $|\phi(\lambda)| < 1$. Therefore $(F_n(\lambda))_n \in E$ and

$$\|(F_n(\lambda))_n\|_E \leq C(T, E).$$

Using $\lambda \notin \overline{G_r}$, and taking into account the definition of r as $r = p^{-1}(C(T, E))$, we have

$$\|(F_n(\lambda))_n\|_E > C(T, E).$$

This contradiction completes the proof. □

5.2. Case of an ellipse. In case of some concrete domains, the previously described quantities can be explicitly estimated. We present here the case of an ellipse.

Consider the compact $K_1 = [-1; 1]$. In this case the conformal map ψ is given by the formula

$$\psi(z) = \frac{1}{2} \left(z + \frac{1}{z} \right).$$

The contour lines of this function are the ellipses Γ_R with the foci -1 and 1, and semi-axes

$$a = \frac{1}{2} \left(R + \frac{1}{R} \right) \quad \text{and} \quad b = \frac{1}{2} \left(R - \frac{1}{R} \right).$$

Let K_R the convex hull of Γ_R .

Theorem 5.3. *Let $R > r > 1$ and $1 \leq p < \infty$. Let T be a weak type l^p operator for $K = K_R$. Suppose that*

$$C(T, l^p) < \left(\frac{1}{1 - (r/R)^p} \right)^{1/p} - \frac{2}{\pi(r + 1/r - 2)(1 - R^{-p})^{1/p}}.$$

Then

$$\sigma(T) \subset K_r.$$

Proof. Let $1 < t < \rho < R$. From the Cauchy formula in the Riemann sphere, we have for $n \in \mathbb{N}$ and $z \in \Gamma_\rho$,

$$\begin{aligned} |F_n(z) - \phi^n(z)| &= \left| \frac{1}{2i\pi} \int_{\Gamma_t} \frac{\phi^n(\zeta)}{\zeta - z} d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{\Gamma_t} \frac{|\phi^n(\zeta)|}{|\zeta - z|} |d\zeta| \\ &\leq \frac{(t/R)^n}{2\pi \operatorname{dist}(\Gamma_\rho, \Gamma_t)} \ell(\Gamma_t). \end{aligned}$$

Here $\operatorname{dist}(\Gamma_\rho, \Gamma_t)$ denotes the distance between the curves Γ_ρ and Γ_t , and $\ell(\Gamma_t)$ denotes the length of the curve Γ_t .

We have

$$\ell(\Gamma_t) \xrightarrow{t \rightarrow 1} 2 \quad \text{and} \quad \operatorname{dist}(\Gamma_t, \Gamma_\rho) \xrightarrow{t \rightarrow 1} \frac{1}{2} \left(\rho + \frac{1}{\rho} \right) - 1.$$

Hence, for every $n \in \mathbb{N}$,

$$\gamma_n(\rho) := \sup_{z \in \Gamma_\rho} |F_n(z) - \phi^n(z)| \leq \frac{2}{\pi R^n} \left(\rho + \frac{1}{\rho} - 2 \right)^{-1}.$$

Let $\zeta \in \Gamma_\rho$. Then

$$\begin{aligned} M(\zeta) &= \left\| \sum_{n=0}^{\infty} F_n(\zeta) z^{-n} \right\|_{l^p_A} = \left(\sum_{n=0}^{\infty} |F_n(\zeta)|^p \right)^{1/p} \\ &\geq \left(\sum_{n=0}^{\infty} \left((\rho/R)^n - \gamma_n(\rho) \right)^p \right)^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} p(r) &\geq \left(\sum_{n=0}^{\infty} \left((r/R)^n - \gamma_n(r) \right)^p \right)^{1/p} \\ &\geq \left(\sum_{n=0}^{\infty} (r/R)^{np} \right)^{1/p} - \frac{2}{\pi} \left(r + \frac{1}{r} - 2 \right)^{-1} \left(\sum_{n=0}^{\infty} R^{-np} \right)^{1/p} \\ &\geq \left(\frac{1}{1 - (r/R)^p} \right)^{1/p} - \frac{2}{\pi(r + 1/r - 2)(1 - R^{-p})^{1/p}}. \end{aligned}$$

The conclusion follows now from Theorem 5.2. \square

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