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# A NEW DISCOURSE ON UNDECIDABILITY AND PROOF SYSTEMS.

RAJAH IYER

ABSTRACT. We delve into the fundamental concept of undecidability. In particular, with an understanding of the inherent undecidability of statements within PA (Peano Arithmetic), our investigation aims to illuminate the extent to which statements can be conclusively proven in ZFC. This examination encompasses even Diophantine equations, leading us to argue that the decidability of these equations exists entirely beyond the boundaries of ZFC. To achieve this, we will critically assess certain facets of incompleteness, as they have evolved since the ground breaking contributions of Gödel, Cohen, and Yuri.M. We will extend the argument to the theory of primes and demonstrate the existence of principles that mechanistically illustrate the 'knowability' of the distribution of primes. Furthermore, we will address the Riemann Zeta Function by proving that the non-trivial zeros of the function are likely to exist on the critical strip  $x = \frac{1}{2}$ . This will be accomplished through the introduction and demonstration of a novel convergence test that we propose in this work.

## 1. INTRODUCTION

Statements within any formal theory share several fundamental elements. They pertain to or discuss a foundational set of objects, often numbers, and describe how these objects undergo 'transformations.' These transformations, more precisely mappings, are governed by operations defined on these objects. The representation of each object enables a distinct mapping, serving as the 'alphabet' of the language in modern logic.

Among the various possible mappings, there exists a core set of fundamental mappings upon which all others are constructed. This mechanical nature of the system is akin to a finite set of mechanical principles forming the basis of a mechanical system. Consequently, it should be feasible to describe certain mechanical features linguistically.

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*Key words and phrases.* Diophantine Equations, Primes, Proveability in Second Order Arithmetic.

Consider the mechanical properties of an engine, where pressing the accelerator at a specific angle may limit the resulting velocity. This claim can be examined by delving into the internal mechanisms of the engine. The axle's rotation leads to the piston's movement, regulated by fuel combustion. This knowledge, coupled with physical principles, suffices to substantiate the claim.

To articulate the reasons behind the behaviour, language can be mapped to the associated mechanisms. For instance, the phrase 'revolution of piston' can be mapped to the concept 'piston moves up and down.' The 'how' linked to 'somehow' can be further mapped, if known, to a sequence of words, and so forth. However, if the language lacks the capacity to express specific mechanical properties, an explanation for the behaviour becomes elusive.

Our research suggests that the provability of a mathematical claim is akin to the informative capacity of an associated language. Specifically, a series of representations (R) reflecting information (I) restricts observations and deductions made on R by I. This set (R, I) forms a language (L) capable of revealing deductions possible on I. Thus, the linguistic capacity is intricately tied to the linked information.

Following the modern definition of proof systems, the language resembles theorems constructed upon a base set of axioms, which could be limited in their exposed properties. Gauss, Bolyai, Beltrami, and others confronted challenges stemming from such limitations. Their investigations revealed the inability to prove the parallel postulate from the first four [1]. This highlights the potential existence of other axioms, and Russell's struggles further underscored this notion.

Our unique perspective centres on language and its connection to information. In essence, if a finite and informative collection of mechanical components can be understood, then a language corresponding to the exposed information can always be constructed. No restrictions impede creating representations based on exposed information.

These investigations hold particular relevance within the formal arithmetic system. Our goal is to explore the extent to which the arithmetic system reveals information on which such languages can be built.

## PRELIMINARIES

First-order arithmetic, also known as Peano arithmetic or first-order number theory, is a formal system within first-order logic that aims to capture the properties of the natural numbers and basic arithmetic operations. In first-order arithmetic, formulas are constructed using

symbols and variables according to certain rules. The components of a formula in first-order arithmetic include:

**Variables:** Variables are placeholders for elements in the domain of discourse. In first-order arithmetic, variables typically represent natural numbers. They are usually denoted by lowercase letters, such as  $x, y, z, \dots$  and so on.

**Logical Symbols:** These symbols are used to express logical connectives and quantifiers. The main logical symbols are:

$\neg$ : Negation (not)  $\wedge$ : Conjunction (and)  $\vee$ : Disjunction (or)  $\Rightarrow$ : Implication (if...then)  $\iff$ : Biconditional (if and only if)  $\forall$ : Universal quantifier (for all)  $\exists$ : Existential quantifier (there exists) **Arithmetic Symbols:** These symbols are used to represent arithmetic operations and relations. The main arithmetic symbols are:

$+$ : Addition  $\times$ : Multiplication  $=$ : Equality  $<$ : Less than **Constants:** Constants are specific elements of the domain of discourse. In first-order arithmetic, constants often represent specific natural numbers, such as 0 or 1.

**Function Symbols:** Function symbols represent operations that take one or more arguments and produce a result. In first-order arithmetic, common function symbols might represent addition, multiplication, or successor functions.

**Predicate Symbols:** Predicate symbols represent relations between elements. In first-order arithmetic, common predicate symbols might represent equality or order relations.

A formula in first-order arithmetic is constructed by combining these components according to the rules of first-order logic. The formulas can be used to express properties of natural numbers, such as the commutativity of addition or the existence of additive inverses.

### 1.1. SOME REFRESHER POINTS.

**Lemma 1.1.** (*Zorns Lemma*) *If  $P$  is a poset in which every well-ordered subset has an upper bound, and if  $x$  is any element of  $P$ , then  $P$  has a maximal element greater than or equal to  $x$ . That is, there is a maximal element which is comparable to  $x$ .*

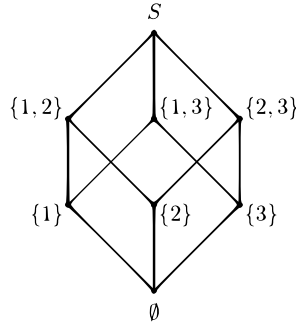
*Proof.* For an outline of proofs assuming the axiom of choice, suppose the lemma is false. then there exists a poset, with  $T_i \subset P$  for totally ordered subsets  $T_i$ . Given this, we have that anyone specific  $T_p$  has an upper bound  $b \in T_p$ , with every such bound having a 'larger' element  $b_{T_p}$ .  $\square$

The word *larger* is where the complications begin. There are many schools of thought and arguments surrounding this matter. The difficulty arises when considering that in order to say that something is larger than another there needs to be an inherent meaning in the word larger when applied to an arbitrary collection of objects. In the previous statement, numbers are no exception, as there is no reason for  $4 > 3$ , except that we define it inductively to be so. We are as such *choosing* for this to be the case, and hence the need for the Axiom-of-Choice.

To define a function  $f_b$  that effects ordering, we need to employ this axiom. For a *class* of ordinals, we may thus say  $a_w = f(a_v | v < w)$ . Continuing with the outline of the formal proof, we have that the class of all ordinals has by AC a choice-function, and as such,  $T_i$  is well ordered. This is because the class of all ordinals has more ordinals than any one single set, implying a maximal element in  $T_p$ . It almost requires maturity to be able to speak of a *class* in set theory, and as such we encourage the unfamiliar reader to take some time in understanding the thinking surrounding classes in set theory.

We additionally encourage the interested reader to see [1][2]. For an additional overview of how the concepts of ordinals and cardinals began, see[3].

**Definition 1.1.** (*Hasse diagram*) A Hasse diagram, is a diagram of the following form in which a line segment joins two nodes  $x$  and  $y$  if  $x < y$  and there is no  $z$  for which  $x < z < y$ . In such a case we say  $y$  covers  $x$ .



If  $S = \{1, 2, 3\}$ , then  $\mathcal{P}(S)$  is as above.

**Definition 1.2.** (*Well-Ordering*) requires that every non-empty subset of  $S$  has a least element in this ordering.

Direct products satisfy the property that, given maps  $\alpha : S \rightarrow A$  and  $\beta : S \rightarrow B$ , there exists a unique map  $S \rightarrow A \times B$  given by  $(\alpha(s), \beta(s))$

**Definition 1.3.** (*Choice Function*)

$f$  is a choice function for  $X$  if and only if it belongs to the direct product of  $X$ .

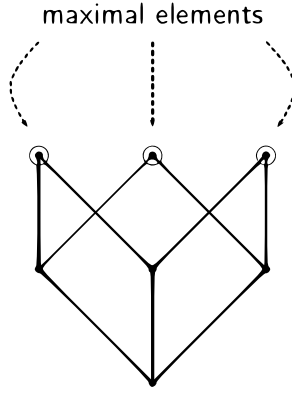
**Definition 1.4.** (*Total Order*) Formally, a binary relation  $\leq$  is a total order on a set  $X$  if the following statements hold for all  $a, b$  and  $c$  in  $X$ :

If  $a \leq b$  and  $b \leq a$  then  $a = b$  (antisymmetry);

If  $a \leq b$  and  $b \leq c$  then  $a \leq c$  (transitivity);

$a \leq b$  or  $b \leq a$  (connex property).

**Definition 1.5.** (*Stronger*) A maximal element of a subset  $S \in \mathbb{P}$  is an element  $s_i \in S$  that is not stronger than any other element in  $S$ .



**Definition 1.6.** Let  $D(x_1, \dots, x_m)$  be a polynomial in  $n$  variables with integer coefficients which admits only integer values for  $x_1, \dots, x_m$ . The equation  $D(x_1, \dots, x_m) = 0$  is called Diophantine.

More generally, we usually speak of a Diophantine equation as accepting two sets of inputs. That is,  $D(a_1, \dots, a_n, x_1, \dots, x_m)$  takes as input an  $n$ -tuple, called its parameters, and an  $m$ -tuple, called its unknowns. (Intuitively, we immediately recognize that  $n \leq m + 1$ .)

**Definition 1.7.** (*Diophantine Representation*) Let  $S$  be a subset of all  $n$ -tuples of integers and consider an arbitrary Diophantine equation. If for every  $a \in S$  there exists an  $m$ -tuple  $(x_1, \dots, x_m)$  such that  $D(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ , and the converse is also true, then the set  $S$  is Diophantine. The dimension of the set is  $m$ . Similarly, we consider  $D$  to be a Diophantine representation of  $S$ .

Limited to the natural numbers.

**Lemma 1.2.** (*Sum of Squares*) Every natural number can be decomposed as the sum of squares of four integers, not necessarily unique. (This is known as Lagrange's four-square theorem.)

*Proof.* The proof is given in Niven and Zuckerman (1960).  $\square$

**Theorem 1.1.** *(Existence of Solutions to a Diophantine Equation)*  
*The problem of determining the existence or nonexistence of solutions to a Diophantine equation which accepts natural numbers is reducible to the problem of determining the existence or nonexistence of solutions to a Diophantine equation which accepts integer values. The opposite is also true.*

*Proof.* Since  $\mathbb{N} \subset \mathbb{Z}$ , the second part is trivially true. To prove the first part, consider a Diophantine equation  $D(x_1, \dots, x_m) = 0$ , accepting only natural numbers for  $x_1, \dots, x_m$ . Write  $x_n = y_{n,1}^2 + y_{n,2}^2 + y_{n,3}^2 + y_{n,4}^2$ , for every  $n$  where each  $y$  is an integer; from Lagrange's four-square theorem, this is guaranteed to be possible. Substituting into  $D$ , we obtain a Diophantine equation in  $4m$  variables, all of which are integers. Diophantine sets and relations behave very well with respect to logical operations, which we can see in the following theorem.  $\square$

**Theorem 1.2.** *(Union of two Diophantine sets)*  
*The union of two Diophantine sets of the same dimension is Diophantine. The intersection of two Diophantine sets, of same or different dimension, is Diophantine.*

**Definition 1.8.** *(Recursive)* A subset  $S$  of  $\mathbb{N}$  is called recursive if there exists an algorithm which accepts a natural number  $n$  and is guaranteed to terminate after a finite amount of time, after which it correctly outputs the truth value of the statement  $n \in S$ .

**Definition 1.9.** *(Recursively Enumerable)* A subset  $S$  of  $\mathbb{N}$  is called recursively enumerable if there exists an algorithm which accepts a natural number  $n$  and, if  $n \in S$ , is guaranteed to terminate in a finite amount of time and confirm that  $n \in S$ .

It is easy to see from the above that all recursive sets are recursively enumerable. However, the fact that there exist recursively enumerable sets that are not recursive is nontrivial. The difference between the conditions is clear; recursively enumerable sets are not required to terminate if the input is not in the solution set  $S$ . However, it is not immediately clear that the set of recursively enumerable sets that are not also recursive is nonempty. Thankfully, we have the following theorem:

**Theorem 1.3.** *(Simple Set)*  
*A simple set is a set that is co-infinite and recursively enumerable but*

*also such that every infinite subset of its complement is not recursively enumerable. Simple sets are not recursive.*

*Proof.* Given in Soare (1987). □

We may reformulate a different definition of recursively enumerable in order to make the resolution of Hilbert’s Tenth Problem a little easier.

**Definition 1.10.** (*Recursively Enumerable Amended*)

*A set  $S$  is recursively enumerable if there exists an algorithm that enumerates  $S$ . It is not immediately clear that this definition is equivalent to the previously stated one. However, it is straightforward to prove.*

**Theorem 1.4.** (*Equivalence of Definitions*) *The two given definitions of recursive enumerability are equivalent.*

*Proof.* Take a set  $S \subset N$ . First suppose that there exists an algorithm that is guaranteed to terminate on inputs contained in  $S$  and run infinitely for inputs not contained in  $S$ . Let this algorithm be denoted  $A(n)$  where  $n$  is its input. To show that there exists an algorithm for enumerating the members of  $S$ , consider the following construction: Run  $A(0)$  for one time step, then run  $A(0), A(1)$  for one time step, then run  $A(0), A(1), A(2)$  for one time step, and so on. This described algorithm will “eventually” reach arbitrarily large timesteps for  $A(n)$  given any choice of  $n$ , and so for all  $n \in S$  it is guaranteed to confirm that  $n \in S$  in a finite span of time. Modify  $A(n)$  to print  $n$  if it halts, and we have the desired enumeration. Conversely we shall suppose that there exists an algorithm that enumerates  $S$ ; call it  $A$ . To construct an algorithm  $B(n)$  that halts only if  $n \in S$ , simply run  $A$  and halt if  $n$  is printed. □

## 2. PROVEABLE STATEMENTS IN PA: MODERN THEORY

Gödel, famous for the independence results, specifically the Gödel’s Incompleteness Theorems which demonstrate the existence of undecidable propositions in Peano Arithmetic (PA), is a famous result in mathematical logic. At a high level, Gödel’s proof begins by representing the syntax and semantics of PA within the system itself. This encoding allows the system to reason about its own statements and proofs. Gödel constructs a method to encode formulas and proofs of PA as numbers. This encoding enables the system to manipulate and reason about its own syntactic objects. Gödel assigns unique numbers (Gödel



numbers) to formulas and proofs in PA. This encoding is recursive and captures the structure of formulas and proofs. Gödel uses a diagonalization argument (Diagonal argument as is more conventional), to construct a formula that asserts its own unprovability within PA. This formula is referred to as the Gödel sentence or the diagonal lemma. By establishing the unprovability of the Gödel sentence within PA, Gödel demonstrates that there exists a true statement that is not provable in the system. This shows the incompleteness of PA. Gödel's proof shows that any consistent formal system that can encode arithmetic, such as PA, will have undecidable propositions. These undecidable propositions cannot be proven or disproven within the system itself. See for instance [34], and [35].

To provide an outline of this fascinating result, we take from *Smullyan*: Let  $X$  be some encoded-expression, then the following is possible:

Let  $P$  stand for printable,

$N$  norm of,

and ! not.

$P(X) \rightarrow True$  if  $X$  is 'printable'.

$P(N(X)) \rightarrow True$  if  $N(X)$  is 'printable'.

$!P(X) \rightarrow True$  if  $X$  is NOT 'printable'.

$!P(N(X)) \rightarrow True$  if  $N(X)$  is NOT 'printable'.

Given that 'the machine' never prints false sentences:

The sentence  $PN(!PN(X))$  is true if the norm of  $(!PN(X))$  is printable, as  $PN(..)$  means 'Printable, Norm of that which lies within  $(..)$ '. But this means that if we place  $!PN$  within, the statement then translates to 'Printable, Norm of that which lies within (Norm of this not Printable(X))'. This either means that: the sentence is true and not printable, or it is printable and not true. The latter violates our hypothesis that the machine is only capable of printing true statements. The significance of this is that all systems 'morphic' to the above in a manner of setting up statements, then Gödel's argument is made. The infinitely more significant result is that Arithmetic is one such formal system.

These results were devastating to the mathematical community as a whole, as how then does one work for many years on a problem all the while unsure of whether it is even proveable? Which questions, how does one even go about knowing such a thing? Amidst this unrest, it was commonplace for mathematicians to speculate that many outstanding problems fell into this category. One such major problem is

the Continuum Hypothesis (CH). Gödel along with Cohen undertook the task of showing demonstrating that at the very least, CH is undecidable from the axioms of ZFC alone.

Gödel's proof of the independence of CH builds upon this earlier work on incompleteness theorems. His ideas were highly inspired by the work of Cantor, as was the case for Turing as well. This was a truly revolutionary period of mathematical enlightenment. Gödel established that within any consistent formal system that is sufficiently 'powerful' to express arithmetic, there are true statements that cannot be proven within that system. Gödel used a technique called the constructible universe, denoted by  $L$ , which is a particular model of set theory. In this model, sets are constructed in a step-by-step fashion using a hierarchy of stages. Gödel then introduced a hierarchy of sets called the constructible hierarchy. Each stage of this hierarchy represents a level of the cumulative hierarchy of sets, and it is constructed based on the previous stages. A notion of constructible sets is then defined within his constructible hierarchy. These sets are built using formulas of set theory, and each constructible set is associated with a particular formula. The reflection principle ensures that if a statement is true at one stage of the constructible hierarchy, then it continues to be true at later stages. Gödel then showed that within the constructible universe  $L$ , the continuum hypothesis holds. In other words, within  $L$ , it is true that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers. Finally, Gödel constructs a different model of set theory (referred to as the "Gödel model") in which CH is false. This model is obtained by considering a larger universe of sets that extends beyond  $L$  and introducing certain additional sets that violate CH.

In 1963, Paul Cohen presented his ground breaking proof that the Continuum Hypothesis is independent of the standard axioms of set theory. This meant that the hypothesis cannot be proved or disproved within the existing framework of set theory. The results of Cohen were inspired by those of Gödel and Cantor.

The independence of the Continuum Hypothesis had a profound impact on the field of set theory and the understanding of mathematical infinity. It highlighted the inherent complexity and richness of infinite set theory and paved the way for further investigations into different cardinalities and the structure of the continuum. For a detailed account of independence results see for instance, [15], [16]. We here give a small account of the work done by Cohen on the Continuum Hypothesis.

First, let us define some notation. For any countable ordinal  $\alpha$ , let  $2^\alpha$  denote the set of all functions from  $\alpha$  to 2, and let  $2^{<\alpha}$  denote the set of all finite functions from  $\alpha$  to 2. We can think of  $2^{<\alpha}$  as the set of "partial" functions from  $\alpha$  to 2, i.e., functions that are only defined on a finite initial segment of  $\alpha$ . We order  $2^{<\alpha}$  by extension, so  $p \leq q$  means that  $p$  extends  $q$ , i.e.,  $p$  is a stronger condition than  $q$ . We say that  $p$  and  $q$  are compatible (written  $p \vdash q$ ) if there exists  $r$  such that  $r \leq p$  and  $r \leq q$ .

Now, let  $V$  be a model of ZFC, and let  $G$  be a generic filter over  $V$  for the forcing notion  $(2^{<\alpha}, \leq)$ . We say that  $G$  is a Cohen generic filter if it has the following two properties:

$G$  is downward-closed: if  $p \in G$  and  $p \leq q$ , then  $q \in G$ .  $G$  intersects every maximal antichain in  $2^{<\alpha}$ , i.e., every collection  $A$  of pairwise incompatible elements of  $2^{<\alpha}$  has a common extension in  $G$ . Note that property (2) implies that  $G$  is maximal with respect to the ordering  $\leq$ . In other words, if  $p \notin G$ , then there exists a  $q$  such that  $q \leq p$  and  $q$  is incompatible with every element of  $G$ .

We can now define the Cohen generic extension  $V[G]$  of  $V$ . The universe  $V[G]$  consists of all sets that can be constructed using elements of  $V$  and elements of  $G$ . Specifically, for each name  $\tau$  in  $V$ , we define its interpretation  $\tau^G$  in  $V[G]$  as follows:

If  $\tau$  is a ground set, then  $\tau^G = \tau$ . If  $\tau$  is a name for an element of  $2^{<\omega}$ , then  $\tau^G$  is the function in  $2^\omega$  defined by  $\tau^G(n) = 1$  if and only if  $m < n : \tau(m) = 1 \in G$ . The key fact about Cohen forcing is that it adds a new subset of  $\omega$  to  $V$ . Specifically, the set  $n \in \omega : \tau^G(n) = 1$  is a new subset of  $\omega$  that is not in  $V$ . This new subset has the property that it is not constructible from any set in  $V$ . In particular, it is not constructible from any countable sequence of sets in  $V$ .

To see why this is the case, suppose for contradiction that there exists a sequence  $(S_n : n < \omega)$  of sets in  $V$  such that  $n \in \omega : \tau^G(n) = 1 = \bigcup_{n < \omega} S_n$ . Then each  $S_n$  is constructible from a countable sequence of

sets in  $V$ , say  $(T_{n,m} : m < \omega)$ . Since  $V$  is a model of ZFC, there exists a formula  $\phi(x)$  such that for each  $n$ , the set  $m < \omega : T_{n,m} \in x$  is the  $n$ -th element of the sequence  $S_n$  if  $\phi(x)$  is true, and the empty set otherwise. Since the sequence  $(S_n : n < \omega)$  is not in  $V$ , there exists a Cohen condition  $p$  such that  $p$  forces  $\neg\phi(G)$ . But this contradicts the fact that  $G$  intersects every maximal antichain in  $2^{<\omega}$  (See 1.1).

Finally, it is a well-known result that the addition of a Cohen subset of  $\omega$  to  $V$  is independent of ZFC (See [18], [19], [20]).

One way to visualize the Cohen forcing notion is to imagine a binary tree whose nodes correspond to partial functions from  $\omega$  to 2. The root of the tree corresponds to the empty function  $\{\}$ , and the children of a node corresponding to a partial function  $f$  are obtained by extending  $f$  with a new pair  $(n, b)$ , where  $n$  is a natural number not already in  $\text{dom}(f)$  and  $b$  is either 0 or 1. The nodes are formed in a manner that have a chain for each ordinal in the base model.

At each level of the tree, we have a finite number of choices to make, corresponding to the possible values of the next unused natural number and the next bit in the binary representation of the function. At the limit levels of the tree, we have a branch for each possible function from  $\omega$  to 2. The partial order on  $P$  is defined by saying that a node corresponding to a partial function  $f$  is less than or equal to a node corresponding to a partial function  $g$  if and only if  $g$  extends  $f$ , that is,  $\text{dom}(f)$  is a subset of  $\text{dom}(g)$  and  $g(x) = f(x)$  for all  $x$  in  $\text{dom}(f)$ . A generic filter for the Cohen forcing notion can be thought of as a path through the tree that includes all the branches that correspond to a condition in the filter. Intuitively, a generic filter "chooses" one branch from each level of the tree in a way that is consistent with the ordering relation.

In this way, the Cohen forcing notion allows us to construct a model of set theory in which we have "chosen" a particular path through the tree, corresponding to a random subset of  $\omega$  that cannot be constructed from the usual axioms of set theory.

The beauty of the *forcing* technique is that, should the models constructed, behave transitively, then the model is one of ZFC. However since we can find two models that model ZFC, one supporting CH and one supporting  $\neg\text{CH}$ , one has that CH is undecidable from ZFC alone.

**Forcing**, as a method in set theory, allows us to construct mathematical models (forcing extensions) in which certain statements are either true or false. However, forcing cannot definitively resolve the Continuum Hypothesis (CH) because it does not provide a conclusive answer as to whether CH is true or false in the standard set-theoretic universe.

The main reason forcing cannot settle CH is that it does not add any new information about the truth value of CH in the original set-theoretic universe. Instead, forcing allows us to construct additional models of set theory, called forcing extensions, in which we have more freedom to manipulate certain properties and values.

When applying forcing to the Continuum Hypothesis, we can construct forcing extensions in which CH is true and others in which CH is false. This shows that CH is independent of the standard axioms of set theory because both possibilities can be consistently realized.

In other words, forcing demonstrates that there are models of set theory in which CH is true and models in which CH is false. This independence result implies that CH cannot be settled within the confines of the standard axioms of set theory alone. It indicates that additional axioms or principles beyond the standard ones are needed to establish the truth or falsity of CH (Again, see [15], [16]).

According to Gödel's incompleteness theorems, the statement that Peano Arithmetic (PA) is consistent, in its guise as a number-theoretic statement (given the technique of Gödel numbering), cannot be derived in PA itself. But PA can be strengthened by adding this consistency statement or by stronger axioms. In particular, axioms partially expressing the soundness of PA can be added. These are known as reflection principles. An example of a reflection principle for PA would be the set of sentences  $Bew_{PA}(\ulcorner \phi \urcorner) \rightarrow \phi$  where  $\phi$  is a formula of the language of arithmetic,  $\ulcorner \phi \urcorner$  a name<sup>1</sup> for  $\phi$  and  $Bew_{PA}(x)$  is the standard provability predicate for PA (*Bew* was introduced by Gödel and is short for the German word 'beweisbar', that is, 'provable').

The process of adding reflection principles can be iterated: one can add, for example, a reflection principle R for PA to PA; this results in a new theory PA+R. Then one adds the reflection principle for the system PA+R to the theory PA+R. This process can be continued into the transfinite (see Feferman 1962 and Franzén 2004).

The reflection principles express at least partially the soundness of the system. The most natural and full expression of the soundness of a system involves the truth predicate and is known as the Global Reflection Principle (see Kreisel and Lévy 1968). The Global Reflection Principle for a formal system S states that all sentences provable in S are true:

$\forall x (Bew_S(x) \rightarrow Tx)$ ,  $Bew_S(x)$  expresses here provability of sentences in the system S (there are considerable problems with defining  $Bew_S(x)$ ). The truth predicate has to satisfy certain principles; otherwise the global reflection principle would be facile.

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<sup>1</sup>The name of a mathematical object is a word or phrase in math English used to identify an object. A name is a special kind of description – a one-word description.

The relation of truth theories and (iterated) reflection principles also became prominent in the discussion of truth-theoretic deflationism (see Tennant 2002 and the follow-up discussion).

### 3. A SET OF PROVEABLE STATEMENTS

In this section, we intend to discuss the tangible symbols that form an alphabet within a language and their capacity to store information and 'state' data. To accomplish this, we present foundational definitions that are relevant to this undertaking, forming a comprehensive framework aligned with this endeavour. Ultimately, our contention rests on demonstrating that modifying the definitions of  $+$ ,  $=$ ,  $<$  in ways that remain consistent with ZFC (Zermelo-Fraenkel set theory) has a discernible impact on the truth values of statements. Consequently, it becomes evident that the influence of ZFC on the truth evaluation of such statements is non-existent.

We will be working within the framework of modern first order logic and the ZFC axioms, where here, instead of classical numbers, the variables constitute place holders for elements within the set  $\mathbb{E}$  defined shortly. Formulas here, operate over the alphabet, which consists of symbols representing variables, logical connectives, arithmetic operations, constants, function symbols, and predicate symbols consistent with modern logic. Formulas are constructed using these symbols according to the rules we will define in this section, which will include definitions for the arithmetic and relational symbols  $\{+, <, =\}$ .

In modern First-Order-Logic, the alphabet includes:

**Variables:** These symbols represent place holders for natural numbers and are typically denoted by lower-case letters, such as  $x, y, z$ .

**Logical Symbols:** These symbols represent logical connectives and quantifiers. Examples include  $\neg$  (negation),  $\wedge$  (conjunction),  $\vee$  (disjunction),  $\Rightarrow$  (implication),  $\Leftrightarrow$  (biconditional),  $\forall$  (universal quantifier), and  $\exists$  (existential quantifier).

**Arithmetic and Relational Symbols:** These symbols represent arithmetic operations and relations. Common arithmetic symbols include  $+$  (addition),  $\times$  (multiplication),  $=$  (equality), and  $<$ : (less than).

**Constants:** Constants represent specific elements in the domain of discourse, such as natural numbers. For example, 0 and 1 might be constants, however these derive here from  $\mathbb{E}$ .

**Function Symbols:** Function symbols represent operations that take arguments and produce a result. In PA, they might represent addition,

multiplication, successor functions, etc.

**Predicate Symbols:** Predicate symbols represent relations between elements. For instance, equality or order relations.

**Definition 3.1.** (*Mechanical-set element*) A mechanical-set element  $e$  is an element of the set  $\mathbb{E} := \mathbb{B} \times \mathbb{N} | \mathbb{B} := \{0, 1\}$ .

**Definition 3.2.** (*Pos, Val*) For an element  $e \in \mathbb{E}$ ,  $Pos(\mathbb{E}) = n \in \mathbb{N}$  and  $Val(\mathbb{E}) = b \in \mathbb{B}$ , with  $b$  being the number assigned to  $e$  at position  $n$ .

**Definition 3.3.** ( $\{+\}$ )

$$\mathbb{E} + \mathbb{E} := \begin{cases} r = \{(1, n_\omega + 1) | n_\omega \in \mathbb{N}\} & \text{if } (Pos(v) = Pos(e)) \wedge (Val(v) = Val(e) = 1) \\ r = \{(1, n_\omega) | n_\omega \in \mathbb{N}\} & \text{if } (Pos(v) = Pos(e)) \wedge (Val(v) = 0, Val(e) = 1) \end{cases}$$

**Definition 3.4.** (*Number-Object*) A single number object  $\mathcal{O}$  symbol, is the union of :  $\bigcup e_i \in \mathbb{E}$ . The set of number-objects is denoted by  $\overline{\mathcal{O}}$

A total order on  $\overline{\mathcal{O}}$ , is defined via the finite ordering algorithm see [4].

**Definition 3.5.** (*Knowable*)

Given a set of number objects  $\mathcal{O}_i = (\overline{Pos_i}, \overline{Val_i})$ , the predicate distribution of  $\mathcal{O}_j := (\overline{Pos_j}, \overline{Val_j})$  given  $f$  the function responsible for the predicate, is knowable if a well-defined continuous (nothing missing) sequence of steps  $\Pi$  involving  $(\mathbb{E}, \{+\})$  exists, that forms the function  $f : \mathcal{O}_i \mapsto \mathcal{O}_j$ , written  $Know(\mathcal{O}_j | f)$ .

**Definition 3.6.** (*Equivalence*) Considering a collection of numerical entities (number objects), denoted as  $\mathcal{O}_i = (\overline{Pos_i}, \overline{Val_i})$  and  $\mathcal{O}_j = (\overline{Pos_j}, \overline{Val_j})$ , the notation  $=$  is employed to signify the uniformity of both  $Pos$  and  $Val$  across  $\mathcal{O}_i$  and  $\mathcal{O}_j$  for all relevant instances of  $i$  and  $j$ .

**Definition 3.7.** (*Writeable*)

If names exist for each step in  $\Pi$ , we then say that the reason for  $f : \mathcal{O}_i \mapsto \mathcal{O}_j$  is Writeable, written  $Write(f)$ .

**Definition 3.8.** (*Predictable*) If there exists a finite set of names expressing the sequence  $f_0, \dots, f_k, \dots$  of choices (infinite or otherwise)  $f_i : \mathcal{O}_i \mapsto_f \mathcal{O}_j$  in finite alphabets, we say that  $\bigcup_{\forall i} f_i$  is Predictable. Where

'sufficient' well-formed names exist that enable the comparison  $Pred(L.H.S(\phi), R.H.S(\phi))$ ,

for a formula  $\phi(\mathbb{E}, +, =)$  in this canonical model over some domain set  $A \subseteq \mathbb{E}$ , we say that  $\phi$  is Predictably-Writeable, written<sup>2</sup>  $PredW(\bigcup_{\forall i} f_i, \phi_{ij})$ .

This does not include statements involving primes, as these are not yet 'written', i.e., no formula exists this far for its exact positional distribution over  $\mathbb{N}$ , we are making no claim that it is not writeable, we are of the opinion that it is.<sup>3</sup>

Arithmetic statements involving formulas  $\phi_{ij}(\mathbb{E}, \{+, =\})$  are in essence comparisons of distributions of  $(\overline{Pos_{ij}}, \overline{Val_{ij}})$ , that are coincident on both sides of  $\{=\}$  for some domain value set, post operation of the formulas  $\phi_{ij}$ , progressively or otherwise. As a result, **Truth** (involving the  $=, <$  relations alone, and we limit ourselves in this manner) can be defined for a given formula where post evaluation, both sides share the same  $Pred(\overline{Pos_{ij}}, \overline{Val_{ij}})$  distribution across  $=$ , within a specified domain. The validity of distributions in relation to  $<$  hinges on the specific definitions used in determining whether the distributions of PV and PV' hold consistent with the relation definition. These and other such statements is further **provable** when  $PredW(\overline{Pos_{ij}}, \overline{Val_{ij}})$ , so as to express the comparison in terms of the same base language  $\mathcal{L}$ . When such a scenario is achievable, the presence of an indisputable deterministic finite algorithm  $c_{fin}$  for determining truth is inevitable.

### partial proof

*Proof.* For any statement  $\phi$  involving the alphabets and relational symbols  $(\mathbb{E}, \{+, =\})$ , we have that  $K(Pred(L.H.S(\phi_i), R.H.S(\phi_j)))$ , this is a result of the binomial theorem, which gives us a  $\Pi$  and  $f_{Bnom}$  for all arithmetic statements  $(\Phi)$  in this space (proven more definitively later on in this paper). As a result, such statements involving the amalgamation of  $(\mathbb{E}, \{+, =\})$  for a language  $\mathcal{L}$  associated with this theory  $T$ , are provable. and is therefore writeable, implying  $PredW(\overline{Pos_{ij}}, \overline{Val_{ij}})$ .  $\square$

Of course, we do not expect our audience to take this as evidence alone and as such we will get back to this in a later section covering this aspect in more detail.

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<sup>2</sup>Inductive statements are an example.

<sup>3</sup>N.B. If we did not *know* we would not be able to perform any arithmetic operation successfully.



#### 4. PSEUDO-FORCING AND GENERAL FIRST ORDER STATEMENTS

Based on our proposed framework in the previous chapter, we will investigate within this section whether a false statement  $\phi_F$  in the model  $M := \{\overline{\mathbf{O}}, \{+, =, <\}\}$ , implies a break in the axiomatic set ZFC. We will undertake this task by hypothetically altering the definitions of  $\{+, <\}$  (assuming a) and noting the impact of the truth value of the statements possible and observe the logical impact of such a thing on the axioms of ZFC, i.e., does this contradict an axiom in ZFC. We note importantly that changing the definitions of  $\{+, <\}$ , impacts only the collective distributions of the PV set associated with each  $o_i \in M$ . More precisely, our primary focus is on identifying instances where statements lead to an axiomatic contradiction during a modification of the definitions  $\{+, <\}$ . Modifying the definitions of  $+, <$  can render previously true statements (i.e., before changes) false, without necessarily contradicting ZFC.

**A1) Axiom of extensionality:** *Two sets are equal if and only if they have the same members.*

$$\forall x \forall y [\forall z (z \in x \iff z \in y) \implies x = y].$$

(Speaks to equality. general consensus "basically to help us prove that sets are equal..."). We have defined equality in our structure, and it does not violate A1). For  $E_1 := \bigcup_{\forall i} e_i, E_2 := \bigcup_{\forall j} e_j | \forall i, j \exists e_i = e_j \implies E_1 = E_2$ . The truth of a statement (where  $Pos$  and  $Val$  coincide on both sides of an equality relation  $' = '$ ) within the space  $(\mathbb{E}, +, =, <)$  is solely affected by modifications in the definitions  $\delta Def$  of  $+, <$ . Consequently, the variation  $\Phi_{true} \rightarrow \Phi_{false}(\delta Def\{\overline{+}, \overline{<}\})$  is exclusively driven by alterations in the distributions of  $(\overline{Pos}, \overline{Val})$  across the equality relation  $' = '$ . However, the co-incidence of these values, when they do in fact coincide, satisfies the equality relation  $' = '$ , and this outcome is solely influenced by changes in the definitions of  $+, <$ . Nevertheless, in the event of coincidence, equality is clearly upheld.

**A2) Axiom of regularity:** *In every set  $x$  which is not empty, there is a member  $y$  which does not have any members shared with  $x$ .*

$$\forall x (x \neq \emptyset \implies \exists y (y \in x \wedge y \cap x = \emptyset))$$

This (along with the Axiom of Pairing) implies, for example, that no set is an element of itself and that every set has an ordinal rank. The

intent here as just described is to establish in a deep and general sense, rank and order of set elements. As  $M$ , is well defined, this principle holds. Furthermore, this holds true regardless of a well defined change in definitions to  $\{+, <\}$

**A3) Axiom schema of specification:** *If we have a subset  $x$  and a property  $\phi$ , then the collection of all the members of  $x$  with the property  $\phi$  is a set.*

$$\forall z \forall w_1, \forall w_2, \dots, \forall w_n \exists y \forall x [x \in y \iff ((x \in z) \wedge \phi(\overline{W}))]$$

Here again, truth is always independent of this Axiom, as one is clearly able to attribute well defined conditions to the set  $PV := \{Pos, Val\}$ , and have it remain a set further. This holds true regardless of a well defined change in definitions to  $\{+, <\}$

**A4) Axiom of pairing:** *If we have a set of sets  $I$ , then we have a set  $U$  which is the union of those sets, namely all the members of  $I$  are subsets of  $U$ , and  $U$  is the smallest possible set with this property.*

$$\forall x \forall y \exists z ((x \in z) \wedge (y \in z))$$

$PV$ , is a well defined set as part of the collective model  $M$ , as such these principles are upheld here further. This holds true regardless of a well defined change in definitions to  $\{+, <\}$

**A5) Axiom schema of replacement:** *If we can describe a function whose domain is a set, then its image is a set.*

$$\forall A \forall w_1 \forall w_2 \dots \forall w_n [\forall x (x \in A \Rightarrow \exists! y \phi) \Rightarrow \exists B \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \phi))].$$

In other words, if the relation  $\phi$  represents a definable function  $f$ ,  $A$  represents its domain, and  $f(x)$  is a set for every  $x \in A$ , then the range of  $f$  is a subset of some set  $B$ . The form stated here, in which  $B$  may be larger than strictly necessary, is sometimes called the axiom schema of collection.

Modifications to  $+$ ,  $<$ , so long as they are well defined, do not infringe upon this principle. In particular, sets of the format  $\cup\{pos, val\}$  adhere to this principle. When this set constitutes a domain, it can support a well-defined function on this domain.

**A6) Axiom of infinity:** *There exists an infinite set.* The set  $PV$  is by definition infinitely extensible, the concept of inductive infinity is upheld here. Furthermore alterations to the definitions of  $\{+, <\}$ , so

long as they are well defined, does not violate this principle.

**A7) Axiom of empty set:** *There exists a set which has no members.*

The set  $PV$  can by definition hold no values, i.e., can be attributed to  $0 \in \mathbb{R}$  for both  $pos, val$ . As such, the principle of the existence of a null set is upheld here. Furthermore alterations to  $\{+, <\}$ , so long as they are well defined (as we have), i.e., in an manner not contradicting this principle, does not violate this principle.

**A8) Axiom of power set:** *If we have a set, then the collection of all of its subsets is another set.*

By definition a set  $z$  is a subset of a set  $x$  if and only if every element of  $z$  is also an element of  $x$  :

$$(z \subseteq x) \Leftrightarrow (\forall q(q \in z \Rightarrow q \in x)).$$

The Axiom of Power Set states that for any set  $x$ , there is a set  $y$  that contains every subset of  $x$  :

$$\forall x \exists y \forall z [z \subseteq x \Rightarrow z \in y].$$

The axiom schema of specification is then used to define the power set  $\mathcal{P}(x)$  as the subset of such a  $y$  containing the subsets of  $x$  exactly:

$$P(x) = \{z \in y : z \subseteq x\}.$$

Alterations to  $\{+, <\}$  does not violate this principle. Specifically, sets of the form  $\cup\{pos, val\}$  can have an associated power-set. In order to support this further, alterations to  $\{+, <\}$ , so long as they are well defined and defined as we have, i.e., in an manner not contradicting this principle, this does not violate this principle. This is evident with  $M$  effecting ordering along with the capacity for a well defined power-set of this set  $PV$ .

A1)  $\rightarrow$  A8) remains upheld in our model and holds no relevance to the truth of statements within this theory. Specifically, we hold that truth and fallacy as a whole is completely outside the scope of the axioms of ZFC From our perspective, the definitions of  $+, <$  embody fundamental truths at an atomic level, intricately linked to the structure  $(\mathbb{E}, (+, =, Pos, Val))$ . These truths encompass associativity, commutativity, and transitivity. We will now endeavour to formalize these ideas.

Suppose we were to modify the relation  $+$  on  $\mathbb{E}$  while preserving associativity, commutativity, and transitivity. This alteration would directly influence the distribution of  $(L.H.S(\phi), R.H.S(\phi))$ , the  $(Pos, Val)$

set tied to  $\phi(\mathbb{E})$ , which is employed to establish truth through comparison. Similarly, modifying the order relation  $<$  would impact the order of elements within  $\mathbb{E}$  and subsequently affect the truth of statements involving  $<$ . Notably, our theory  $(\mathcal{T}, <)$  satisfies well ordering.

With these arguments in mind, we venture to assert that the validity or falsehood of all arithmetic statements is independent of the axioms of ZFC. Furthermore, the provability is confined solely to the *names* accessible to  $\mathcal{L}$ .

This assertion is substantiated through the equation:

$$(4.1) \quad Bew(\exists \mathcal{E}_R | \mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E})$$

We have defined a choice function  $f_{<}$  that influences transitivity from its very foundation. Consequently, Axiom A2) emerges as a 'property' inherent to the structure  $(\mathbb{E}, +, <, =)$  (based on definition), signifying a consequence and intrinsic characteristic:  $f_{<} \implies OrderProperty \implies A2)$ . This harmonizes with Zorn's principle, which asserts  $Chooseability \implies Order \implies Transitivity$ .

In succession, we posit that Axioms serve as fundamental properties 'Prop' of the defined space  $S$ . The truth or falsity of first-order arithmetic statements is contingent upon the *choice*  $Def_{<, +}$  applied to the set  $\mathbb{E}$ , where  $S := (f_{<, +}, \mathbb{E})$ . Thus,  $Prop(S) \implies Axioms$ , and the definitions of  $f_{<, +}$  impact the truth of statements involving  $<, +$ , as long as these definitions uphold the concept of order. However, should modifications be introduced without preserving the concept of order, the choice of axioms from the Axiomatic-Set containing  $\delta S$  would be affected. This reasoning extends to the truth or falsity of statements concerning the entire structure  $S$ . For instance, consider a simple structure  $(\mathbb{P}, f) | f : \mathbb{P} \rightarrow (0)$ , where an axiom asserts that  $Image(f)$  comprises a single element and hence lacks order within the image alone.

**4.1. Extraction and provability**  $Bew(\exists \mathcal{E}_R | \mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E})$ . Given a set of variables  $t_i \in T_i$  (what these are variables of, is covered later on and will be of radical fields and extensions where appropriate), here  $T_i$  denotes the set of 'form' associated with the variable-elements  $\cup t_i := \{*_1, *_2, \dots, *_n\}$  forming a term:

**Definition 4.1.** (*Radical*) We define a minimal-radical as  $T_i := \{n, *_1^{e_1}, *_2^{e_2}, \dots, *_n^{e_n} | n \in \mathbb{N}, e_i \in \mathbb{Z}\}$ , where  $e_i \in \mathbb{Z}_-, \mathbb{Q}$ , and arbitrary variables  $*_i$ , we say  $T_i$  is a Radical (short).

**Definition 4.2.** (*Expression*) An expression is defined to be  $\mathcal{E} := \{\cup_{\forall i}(T_i, \pm n_i)\}$ ,  $n_i \in \mathbb{N}$ .

**Definition 4.3.** (*Equivalent*) We say that  $T, T'$  are equivalent, written:  $T \equiv' T'$  and is defined to mean:  $\forall t_i^{e_i} \in T, t_i'^{e_i} \in T', \exists i, j | t_i = t_j, e_i = e_j$  and  $\kappa(T) = \kappa(T')$ . The binary operations  $\{\times\}, \{+\}$  are only defined, when  $T \equiv' T'$ .

**Definition 4.4.** ( $\{\times\}$  Term) Given two terms  $\mathcal{T}_a, \mathcal{T}_b$ , a binary operation  $\{\times\}$  on the terms is defined as :  $\forall e_i \in T_i, \forall b_i \in T_i, \mathcal{T} \times \mathcal{T}' := \{n_1 \times n_2, *_1^{e_1+b_1}, *_2^{e_2+b_2}, \dots, *_n^{e_n+b_n} | n_1, n_2 \in \mathbb{N}, e_i, b_i \in \mathbb{Z}\}$ .

**Definition 4.5.** ( $\{+\}$  Term) A binary operation  $\{+\}$  on the terms is defined as :  $n_1 \in T, n_2 \in T, \mathcal{T} + \mathcal{T}' := \{n_1 + n_2, *_1^{e_1}, *_2^{e_2}, \dots, *_n^{e_n} | n_1, n_2 \in \mathbb{N}, e_i, b_i \in \mathbb{Z}\}$ .

All of the sets are more naturally akin to spaces rather than forming a group of some sort.

**Definition 4.6.** ( $\{+\}$  Expression) The binary relation  $\{+\}$  on two expressions  $\mathcal{E}, \mathcal{E}'$ , is defined as  $\mathcal{E} + \mathcal{E}' := \cup_{\forall (T_i \equiv T_j)} \{T_i + T_j | T_i \in \mathcal{E}, T_j \in \mathcal{E}'\}$

**Definition 4.7.** (*Expression Equality*)  $\mathcal{E}, \mathcal{E}'$  are equal when  $\forall T_i \in \mathcal{E}, T_i' \in \mathcal{E}', T_i \equiv' T_i'$  and  $m = n, m \in T_i, n \in T_i' | T_i \equiv' T_i', T_i \in \mathcal{E}, T_i' \in \mathcal{E}'$ .

**Definition 4.8.** (*Substitution*) A substitution of a Radical expression  $\mathcal{E}_R$  for a variable in some term/expression  $T[t_i], E[t_i]$ , is the term/expression  $T[t_i = \mathcal{E}_R], \mathcal{E}[t_i = \mathcal{E}_R]$ .

**Definition 4.9.** (*Resolved expression*) A resolved expression or term, is considered resolved when every equivalent pair of terms are  $\{+\}$ -paired, written  $Res(T), Res(\mathcal{E})$ .

#### 4.2. APPLICATION OF THIS PROPOSED THEORY TOWARD ESTABLISHING: PROVEABILITY OF ALL FINITE DIOPHANTINE EQUATIONS.

We are primarily concerned with the theory that governs the conditions under which  $\mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E}$  holds true, and when occasionally these require more stringent conditions. In our context, our focus lies on expressions of the form:  $\cup^n C_{r,i}^{n-r} \cdot_j^r$  and when these are equivalent.

In a broader scope, considering terms  $T, \mathcal{E}$ , and  $J$  as defined earlier, a generalized Diophantine equation takes the form:  $P(x, y, \dots) = z^n$ , which can be easily reconfigured as:  $T + (\mathcal{E}) = Jz^n$ .

Representing all variables in terms of a single variable, say  $x$ , takes on the form (where each subsequent variable takes on the form):  $x + m, x + n, x + o, \dots$

When asserting, for instance, that  $x^3 + Q[x + m] = (x + o)^3$ , where  $Q$  is a variable quantity with  $x + m$  as an argument, we recognize that  $Q$  can be any desired quantity based on  $x + m$ . Suppose that in this specific case, that it is required that  $Q[x + m] = (x + m)^3$ , what inquiry does this truly prompt, as a radical substitution for  $m$ , that renders the two quantities  $(+3x^2o + 3xo^2 + o^3 = (x + m)^3)$  equal, and resolves to  $m \in \mathbb{N}$ ? Essentially, any quantity larger than  $x$  for  $Q$ , must take the form  $x^3 + 3x^2o + 3xo^2 + o^3$ . When the left-hand side (L.H.S.) of the equation already contains an  $x^3$  term, it is required in addition that,  $Q$  must be of the form  $f = +3x^2o + 3xo^2 + o^3$  for the two to be similar and equatable. In other words, a viable substitution for  $o, m, f[m \text{ or } o = \mathcal{E}]$ , is required to construct  $Q[x + m]$  that results in the desired form of  $(x + m)^3 + 3x^2o + 3xo^2 + o^3 = (x + m)^3$ , i.e., it is required that the form associated with  $f$  is morph-able to  $(x + m)^3$  for some  $o \text{ or } m \in \mathcal{R} \mid +3x^2o + 3xo^2 + o^3 = (x + m)^3$ . Failure to achieve such substitution, in a manner where  $\text{Res}(o \text{ or } m) \in \mathbb{N}$ , renders the equation unsolvable. In its most comprehensive sense, this is the underlying question: When does  $\mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E}$  hold true?

Let's consider the example  $x^n + y^n = z^n$ . Through modest algebraic manipulation, it becomes apparent that for each  $n > 2 \in \mathbb{N}$ , the following forms  $F_i$  must expand into a complete n-th volume:

$$\begin{aligned} & 3x^2o + 3o^2x + o^3 \\ & 4x^3o + 6o^2x^2 + 3o^3x + o^4 \\ & \dots \end{aligned}$$

With some mental manoeuvring, we see that having two or more terms in a substitution for  $m$  prevents the morphing into an n-th form. Hence, we confine ourselves to a single term. Our aim is to substitute  $m$  in such a way that  $F_n = (x + m)^n$ . Therefore, our objective is to make substitutions for  $m$  such that  $(x + m)^n = F_n$ , and in a manner that results in forms of the type  $F_i$  for some  $m \in \mathbb{N} \mid m \in \mathcal{R}$ , and this substitution must take an initial radical form for logical reasons. This also implies that  $m \in \mathbb{N}$ , because we only permit substitutions that result in no remaining radicals once all terms are resolved. Thus, what remains instead of  $f[m = E = p^{\frac{n}{m}}]$  is  $p^n \mid p, n \in \mathbb{N}$ . Substituting a radical for  $m$ , should produce the desired form. With the present terms, we must extract some  $m^n$  via substitution for  $Q[m]$  in  $(Eq =$

$x^n + Q = z^n$ ) without disturbing a potential isomorphism, in order for the equation to equate in  $Eq$ .

It becomes evident that for each  $n$ , one term exists post-substitution  $f[m = E]$  that introduces an additional  $m^n$  term. However, there is always one term per  $m$  that jeopardizes the potential for an isomorphism to persist. This is due to the fact that, for all  $n$ , the substitution's requirements are contradictory – they need to behave differently for distinct terms to avoid being left with additional unnecessary terms. Employing multiple substitutions yields  $n-1$  terms with different coefficients, one for each term in the expansion, which can't be resolved to  $F_n$  unless they are equatable via substitution, which in this instance clearly can't be achieved.

Ultimately, there are  $x_n$  terms, along with  $n$  associated forms, and  $n +$  (Range from 0 to the highest power within the equation), potential extractions possible with a limit on the radical in the extraction process, equivalent, in essence, to the highest power associated with the term implying the existence of a  $c_{fin}$  alternative to  $c_\infty$ . This in addition, likely explains why the Fermat equation remains un-equatable. For these reasons, we assert that proofs exist for all finite Diophantine equations, specifically a  $c_{fin}$  alternative exists, in these circumstances, as an alternative to one equivalent to  $c_\infty$ , for all finite Diophantine Equations.

We urge our audience to consider these ideas.

Interestingly, the range of the parameters, number of terms  $N_t$  and range of the exponents  $e_R$  in the expression being substituted are finite for Diophantine Equation with regards to testing its validity combinatorially. This is easily evident from the principles governing the method of establishing the validity of a claim, as described previously.

This tells us in Gödel's terms

$$(4.2) \quad Bew(\exists \mathcal{E}_R | \mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E})$$

This interestingly enough tells us that all arithmetic statements involving equality are proveable by virtue of  $PredW_{Fin}(\mathcal{E}[t_i = \mathcal{E}_R] \equiv' \mathcal{E})$ , for finite  $\mathcal{E}$ .

This necessitates further investigations into Hilbert's 10th problem.

## 5. HILBERT'S TENTH PROBLEM

Yuri Matiyasevich's proof of Hilbert's Tenth Problem, often referred to as Matiyasevich's theorem, is a groundbreaking result in number theory and logic. It demonstrates that there is no algorithmic procedure to determine whether a given Diophantine equation has integer

solutions. Diophantine equations are polynomial equations with integer coefficients, and the problem of deciding whether such equations have solutions is known as Hilbert's Tenth Problem.

Matiyasevich's proof is based on the concept of "Diophantine sets" and employs techniques from both number theory and formal logic. Here's an overview of the key steps in his proof:

**Definition 5.1.** (*Diophantine Equations and Sets*) A Diophantine equation is a polynomial equation with integer coefficients that seeks integer solutions. For example,  $x^2 + y^2 = z^2$  is a Diophantine equation. A Diophantine set is a set of natural numbers defined by a Diophantine equation.

For example, the set of all Pythagorean triples (integer solutions to  $x^2 + y^2 = z^2$ ) is a Diophantine set.

**Definition 5.2.** (*Universal Diophantine Equations*) A universal Diophantine equation is one that can encode any Diophantine set. In other words, it can simulate the characteristic function of any Diophantine set. Matiyasevich focused on a specific universal Diophantine equation called  $U(n, x, y)$ , which encodes whether the Diophantine equation with number  $n$  has a solution for the given values of  $x$  and  $y$ .

**Machines and Halting Problem:** Matiyasevich established a connection between Diophantine equations and Turing machines. Each universal Diophantine equation  $U(n, x, y)$  corresponds to a Turing machine that halts if and only if the Diophantine equation  $n$  has a solution for the given  $x$  and  $y$ .

**Undecidability of the Halting Problem:**

Turing's famous result shows that the halting problem (determining whether a given Turing machine halts on a given input) is undecidable. This means there is no algorithm that can determine whether an arbitrary Turing machine halts.

**Reduction to the Halting Problem:**

Matiyasevich demonstrated that if there were an algorithm to decide the solvability of universal Diophantine equations (i.e., to determine whether  $U(n, x, y) = 1$ ), it would lead to an algorithm to solve the halting problem. This reduction is achieved by constructing a Diophantine equation  $D(n, x)$  that simulates the behavior of a given Turing machine  $T$  on input  $x$ . The equation  $D(n, x)$  has a solution if and only if  $T$  halts on  $x$ .

**Gödel's Second Incompleteness Theorem:**

Matiyasevich also used concepts from Gödel's second incompleteness



theorem. He constructed a Diophantine equation that encodes a statement equivalent to the consistency of a formal system (such as Peano arithmetic). By Gödel's result, no consistent formal system can prove its own consistency. Therefore, the Diophantine equation encodes a statement that is undecidable within that formal system.

### **Implications for Hilbert's Tenth Problem:**

Combining the undecidability of the halting problem with the reduction from universal Diophantine equations, Matiyasevich concluded that there can be no algorithm to decide the solvability of Diophantine equations in general. This implies that Hilbert's Tenth Problem, which seeks an algorithm to determine the solvability of Diophantine equations, is undecidable. (See [37])

The halting problem is a fundamental concept in computer science and mathematics, specifically in the field of computability theory. It addresses the question of whether an algorithm (or program) can determine whether another algorithm, when given a particular input, will eventually halt (terminate) or continue running indefinitely (loop).

Formally, the halting problem can be stated as follows: Given a description of an algorithm (in the form of a computer program) and an input for that algorithm, is there an algorithm that can determine whether the given algorithm will halt when executed with the given input?

In simpler terms, the halting problem seeks to determine if there exists a general method (an algorithm) that can predict whether any arbitrary program will eventually stop running or continue running forever when provided a specific input.

Alan Turing, one of the pioneers of computer science, proved that the halting problem is undecidable, meaning that there is no such algorithm that can solve the halting problem for all possible programs and inputs. Turing's proof relies on a clever technique involving a hypothetical algorithm that leads to a contradiction, known as a diagonalization argument.

The undecidability of the halting problem has profound implications. It demonstrates that there are inherent limits to what algorithms can achieve in terms of predicting the behavior of other algorithms. It also highlights the existence of problems that are fundamentally unsolvable by computation. These ideas are nowhere under challenge in our discourse.

We are instead concerned with when algorithms can be re-factored into an halt-able condition, i.e., one that is specific (this is key). An

example would be loop 5-times, i.e, a variable holds state with each iteration, which is 'equate-able', and when at 5 for instance the process halts. This is due in part to the 'electricity' measure at which one is able to halt. Such a thing can be set. This is daily routine, and falls within scope of *halt-able* algorithms, as an example finite ones (algorithms).

### Overview of the Proof:

Assume there exists an algorithm H10 that can solve Hilbert's Tenth Problem. Given a Diophantine equation  $D$ ,  $H10(D)$  outputs yes if  $D$  has integer solutions, and no otherwise. Construct a new program  $C$  that operates as follows: Given a program  $P$  and input  $I$ ,  $C$  encodes  $P$  as a Diophantine equation and converts  $I$  into a solution to  $D$ . If  $H10(D)$  outputs yes,  $C$  halts. If  $H10(D)$  outputs no,  $C$  goes into an infinite loop. Consider the case where  $H10(D)$  is applied to the equation  $D$  that represents  $C$  itself and a specific input  $I$  that is a solution to  $D$ . If  $H10(D)$  outputs yes,  $C$  halts, leading to a contradiction because it should have gone into an infinite loop. If  $H10(D)$  outputs no,  $C$  goes into an infinite loop, which contradicts the output of  $H10(D)$ . This proof parallels the diagonalization argument used in the halting problem proof and shows that Hilbert's Tenth Problem is also uncomputable. In precise mathematical terms:

**Theorem 5.1.** *A set is Diophantine if and only if it is recursively enumerable.*

*Proof.* The existence of recursively enumerable sets that are not recursive immediately resolves Hilbert's Tenth Problem, because it implies the existence of a Diophantine set that is not recursive. To see this, consider the following reasoning: Let  $S$  be a Diophantine set and let  $D(a, x_1, \dots, x_m)$  be its Diophantine representation. By definition,  $a \in S$  if and only if there exists  $a \in N$  such that  $D(a_1, \dots, a_n, x) = 0$  has a solution  $m$ -tuple  $(x_1, \dots, x_m)$ . Suppose also that there does exist an algorithm capable of deciding the solvability of arbitrary Diophantine equations. This algorithm would be capable of deciding, in a finite amount of time, whether or not  $a \in S$ . As such, this would mean that every Diophantine set is recursive. However, the MRDP theorem asserts that every set is Diophantine if and only if it is recursively enumerable, so this implies that all recursively enumerable sets are also recursive, which is untrue. The contradiction yields a negative answer to Hilbert's Tenth Problem.  $\square$

However, to solidify Yuri's arguments, the following further formalization is necessary:

**Definition 5.3.** *A set is exponential Diophantine if its Diophantine representation is a polynomial which allows exponentiation.*

**Theorem 5.2.** *Every recursively enumerable set is exponential Diophantine. Applying the same argument as before, we may see that: Corollary 1. There exists no algorithm for the determination of solvability of arbitrary exponential Diophantine equations.*

*Proof.* By contradiction. Identical to proof for Diophantine equations from the full MRDP theorem.

Perhaps a little more interestingly, they note the following corollary:

### Corollary

There exists an algorithm which will accept a particular axiomatization of number theory and output an exponential Diophantine equation which has no solution, but cannot be proved to be unsolvable from the given axiomatization.  $\square$

Compare to the following statement of Gödel's first incompleteness theorem:

**Theorem 5.3.** *In any sufficiently strong system of arithmetic, there exists a statement that is true, but cannot be proven to be true in that system.*

From this "incomplete" version of the MRDP theorem alone, we already see a remarkable connection between Gödelian incompleteness" and number theory.

In both cases, the proofs rely on the notion of self-reference, where a program or equation refers to itself, leading to logical contradictions that demonstrate the impossibility of a general algorithmic solution. These results highlight the inherent limitations of computation and have significant implications for the boundaries of what can be computed algorithmically. However, in all cases, there is no assertion of the existence of an alternative algorithm, perhaps finite in nature capable of ascertaining truth of the equality of a statement for instance.

Fundamentally, if we cannot precisely determine the point at which an algorithm is designed to terminate, it will continue indefinitely. Consequently, the challenge lies in devising a halting program, not a multitude, but one that specifically addresses this particular issue. Fortunately, we have identified such a halting condition that involves a finite set of parameters and a finite set of inputs (denoted as 'I'). This condition ensures a resolution for all potential solutions to finite Diophantine equations.

Our focus is shifting from employing algorithms that may never halt to those that ensure termination. This transition is denoted as  $I \rightarrow c \rightarrow D$  progressing to  $I \rightarrow c_{fin} \rightarrow D$ . Our arguments provide a rationale for transitioning between different types of arguments or algorithms.

By establishing clear criteria for termination, we make the problem solvable. This approach follows an inductive technique as opposed to an iterative one. While both methods are logically valid means of achieving results, the inductive approach offers distinct advantages.

It is essential to avoid assuming that the failure of one logical approach, represented by  $c$ , implies the futility of an alternative approach,  $c'$ . This concept constitutes a profound insight.

An example of this would be: One can check infinitely by iterating over  $n$  in  $N$  over  $2n$  to see if every  $2n$  is even, or we have an alternate means of knowing so. This is the crux of our argument that such further investigations are necessary in the way of completeness. There is nothing in the theory, both within the scope of Gödel's incompleteness theorems or the work of Yuri, suggestive of the non existence of  $c_{fin}$ , or other alternate algorithms capable of reducing  $c \rightarrow c_{fin}$ .

Similar reasoning, aligning with the purpose of this article, can be applied to the truth of statements. Their provability is contingent upon the chosen techniques, namely theory and language.

Based on the argument presented, one can associate a tangible condition, namely an "electrical" equivalence, to determine the solvability of a Diophantine equation. This condition can be quantified by assessing a small number of combinatorial "extractions" (as we've illustrated) that have the potential to yield this outcome. In other words, when an extraction aligns with an anticipated electronically-defined outcome, we observe a point of termination. Drawing from extensive experience in implementing such algorithms over the years, we can confidently affirm the viability of this approach.

## 6. SOME INVESTIGATIONS INTO THE DISTRIBUTION OF PRIMES

The origins and details of series analysis can be found in almost any comprehensive treatise on Real Analysis. One of the primary concerns is determining whether a given series is convergent or not. To address this, a set of well-known tests and analytical techniques are available, which are covered in most undergraduate courses in mathematical analysis.

The field of infinite sum analysis is vast, and over the centuries, various techniques and tests have been formulated. However, our understanding is primarily limited to the 'well-behaved' subset of such series, particularly the monotone ones.

The origins of sequences and series in mathematical analysis can be traced back to ancient civilizations, but the systematic study and formalization of these concepts began to take shape in the works of ancient Greek mathematicians.

The ancient Greeks were among the first to explore the properties of sequences and series. The Pythagoreans, around the 5th century BCE, were interested in the properties of number sequences and made several important discoveries related to triangular numbers and perfect numbers.

Archimedes (circa 287-212 BCE), a renowned Greek mathematician, made significant contributions to the understanding of series. He is credited with calculating the areas of various geometric shapes by using infinite series. One of his most famous results is the approximation of the value of  $\pi$  using the method of exhaustion, which involved inscribing and circumscribing polygons to a circle and using series to find increasingly accurate approximations.

In the 12th century, the Indian mathematician Bhaskara II (1114-1185 CE) developed techniques to find the sum of certain arithmetic and geometric series. His work influenced later mathematicians and contributed to the understanding of series convergence.

In the 17th century, Sir Isaac Newton and Gottfried Wilhelm Leibniz independently developed calculus, which revolutionized the study of sequences and series. Newton introduced the concept of infinite series and developed methods to manipulate them, such as the binomial series. Leibniz, on the other hand, worked on power series expansions and developed methods for integrating and differentiating functions represented as series.

The 18th and 19th centuries saw significant advancements in mathematical analysis, and sequences and series played a central role in these developments. Mathematicians like Euler, Gauss, and Cauchy made important contributions to the theory of sequences and series, studying their convergence, divergence, and summability.

In the 20th century, the study of sequences and series continued to evolve, with mathematicians like Hardy, Ramanujan, and Kolmogorov making noteworthy contributions to the field. The concept of divergent series, Cesàro summation, and other methods for handling divergent series were explored.

Today, sequences and series remain fundamental topics in mathematical analysis, with applications in various branches of mathematics and its applications in physics, engineering, computer science, and many other fields.

For a well rounded introduction and development, see for instance [38], [39].

### 6.1. INVESTIGATION INTO PRIMES: TESTS FOR CON-

**VERGENCE.** An infinite sum, denoted as  $\sum_{\forall j} a_j$ , is a concise representation of the sum of terms:  $a_0 + a_1 + \dots + a_n + \dots$ . For instance, the harmonic series is represented as  $\sum_{\forall n} \frac{1}{n}$ . The harmonic series is an example of a well-behaved monotone series. One of the tests that can be employed to establish its divergence is the Integral Test.

#### **Definition 6.1.** (*Integral Test*)

*For a continuous function  $f$  defined over  $[N, \infty)$  that is monotone-decreasing;  $\sum_{n=N}^{\infty} f(n)$  converges to a real number if and only if the improper integral  $\int_N^{\infty} f(x)dx$  exists.*

Going back to the harmonic series and applying the integral test we obtain:  $\int_1^{\infty} \frac{1}{x} dx = [\ln(x)]_1^{\infty}$  which is clearly divergent.

For non-monotone, oscillating functions, the task of ascertaining whether

the associated series converges or not can be difficult to seemingly impossible.

Taking for instance the alternating series test:

**Definition 6.2.** (*Alternating Series Test*) If for all  $n$ ,  $a_n$  is positive, non-increasing (i.e.  $0 < a_{n+1} \leq a_n$ ), and approaching zero, then the alternating series  $\sum_{n=1}^{\infty} (-1)^n a_n$  and  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$  both converge, see for instance [42].

This test explicitly requires that  $a_n$  be monotone convergent, so one can't for instance, establish convergence of:  $\sum_{n=1}^{\infty} \frac{\sin(x)}{x}$  using this test alone. However we are saved by the **Squeeze-Theorem** which can be used to establish convergence by means of the following argument:  $\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ , since the series associated with either ends of the inequality are both absolutely convergent, it follows for:  $\sum_{n=1}^{\infty} \frac{\sin(x)}{x}$  as well. A comprehensive list of all such tests can be found in a variety of undergraduate texts, see for instance [40], [41], [43], [44] and of course [42].

**6.2. SOME PROPOSITIONS ON ESTABLISHING CONVERGENCE OF NON-MONOTONE SERIES.** The following proposal, though seemingly straightforward, originated from a subtle yet persistent intuition that there exists an integral expression closely linked to converging series, similar to the integral test, however applicable to both monotone and non-monotone series.

**Proposition 6.1.** *Given a single valued function  $f$  continuous and differentiable over  $[0, \infty)$  with  $\lim_{x \rightarrow \infty} f(x) = 0$ , the series  $\sum_{n=1}^{\infty} f_n$  is convergent if the integral :  $\lim_{h \rightarrow \infty} \int_1^h x f'(x) dx$  exists.*

**Proposition 6.2.**  $\sum_{n=1}^{\infty} f_n = \lim_{h \rightarrow \infty} \int_1^h x f'(x) dx + \mathcal{O}$  where  $\mathcal{O} < |f_1|$ .

*Proof.*

$$(6.1) \quad \int_c^\infty f(x)dx = - \int_\infty^c f(x)dx$$

From the above, we have that:

$$(6.2) \quad - \sum_1^\infty f_n = \int_1^\infty f'(x)dx + \int_2^\infty f'(x)dx + .. + \int_k^\infty f'(x)dx + ..$$

It is easy to see that over each interval  $[1,2]$ ,  $[2,3]$  etc.

The integrals may be re-written as:  $1 \int_1^2 f'(x)dx + 2 \int_2^3 f'(x)dx + ..$

and so on, as there is a repetition of the area in proportion with each integral value over which the original sum is evaluated. Given that this is the case, one may re-write the above in approximation by use of the

formula:  $\int_1^\infty x f'(x)dx$ .

In the way of establishing part two of the proposition, we note that the difference between:  $1 \int_1^\infty f'(x)dx + 2 \int_2^\infty f'(x)dx + ..$  (an upper-

bound) and:  $0 \int_1^\infty f'(x)dx + 1 \int_2^\infty f'(x)dx + ..$  (a lower-bound) is

simply  $\int_1^\infty f'(x)dx$ .

There is a lot of narration left to further clarify the above and is unfortunately the most difficult portion; to convince an audience of.

To felicitate this, we note trivially that:  $\lim_{h \rightarrow \infty} - \int_c^h f'(x)dx =$

$(-1)\lim_{h \rightarrow \infty} [f(h)]_c^h = f(c)$ . Further to this we have that:  $\sum_1^\infty f_n =$

$f(1) + f(2) + .. + f(n) + ..$  Which can be expressed as the sum of the entries in the last column of the *matrix* that follows:



(6.3)

$$\sum_1^{\infty} f_n = \begin{bmatrix} \int_1^2 f'(x)dx & \int_2^3 f'(x)dx & \int_3^4 f'(x)dx & . & \int_n^{n+1} f'(x)dx & \dots | & \int_1^{\infty} f'(x)dx \\ 0 & \int_2^3 f'(x)dx & \int_3^4 f'(x)dx & . & \int_n^{n+1} f'(x)dx & \dots | & \int_2^{\infty} f'(x)dx \\ 0 & 0 & \int_3^4 f'(x)dx & . & \int_n^{n+1} f'(x)dx & \dots | & \int_3^{\infty} f'(x)dx \\ 0 & 0 & 0 & . & \int_n^{n+1} f'(x)dx & \dots | & \int_n^{\infty} f'(x)dx \end{bmatrix}$$

Resulting in the approximation:  $\int_1^{\infty} x f'(x)dx$ . □

An example of (6.1) in practice can be achieved by proving the divergence of the harmonic series.

Taking  $\sum \frac{1}{n}$ ; with the obvious substitutions, we have  $\int x f'(x) = \int -x(\frac{1}{x^2})dx$  which becomes:  $\int -\frac{1}{x}dx$ , the integral of which forms:  $-\ln(x) + C$  which clearly diverges as:  $x \rightarrow \infty$ .

Another simple example is the divergence of  $\sum \frac{1}{\sqrt{n}}$ . Again with substitution;  $\int x f'(x)dx = \int -\frac{1}{2}x(x^{-\frac{3}{2}})dx = -\sqrt{x} + C$  which again clearly diverges as:  $x \rightarrow \infty$ .

**6.3. THE RIEMANN ZETA FUNCTION.** Its natural to provide a basic introduction to the origins of the zeta function. The Riemann Zeta function is named after the German mathematician Bernhard Riemann, who made significant contributions to its study. The origins of the Riemann Zeta function can be traced back to his work on understanding the distribution of prime numbers and investigating the properties of the complex number domain.

In 1859, Riemann published a ground breaking paper titled "On the Number of Primes Less Than a Given Magnitude," in which he introduced the Riemann Zeta function and its connection to the distribution of prime numbers (See [46]). In this paper, he defines the Riemann Zeta function (now baring his name) as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where  $s$  is a complex number with real part greater than 1. The Riemann Zeta function is initially defined for this region of complex numbers and is analytically continued to other regions.

Riemann's work on the zeta function was part of his broader study of the distribution of prime numbers and the nature of their distribution along the complex plane. He hypothesized that the nontrivial zeros of the Riemann Zeta function, known as the "Riemann Hypothesis," are all located on the critical line with real part  $1/2$ .

Riemann's pioneering work on the Riemann Zeta function laid the foundation for modern number theory and complex analysis. It provided deep insights into the behaviour of prime numbers and opened up new avenues for research in analytic number theory and the study of complex functions.

Since Riemann's time, the Riemann Zeta function has become one of the most important and extensively studied functions in mathematics. It has connections to various areas of mathematics and physics, including number theory, analytic number theory, algebraic geometry, quantum mechanics, and more. The Riemann Zeta function remains a subject of active research and fascination for mathematicians and scientists.

$$(6.4) \quad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

One of the most significant results on the zeta function  $\zeta(s)$  due to Hardy is that there are an infinite number of zeros on the critical strip  $x = \frac{1}{2}$  [49].

The outline of this fantastic result follows with  $\Xi(t)$  having real zeros for zeros of  $\zeta$ .

$$(6.5) \quad \Xi(t) := \xi\left(\frac{1}{2} + it\right) = -\frac{1}{2} \left(t^2 + \frac{1}{4}\right) \pi^{-\frac{1}{4} - \frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \zeta\left(\frac{1}{2} + it\right)$$

$$(6.6) \quad \xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} s(s-1) \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s)$$

Using a result by Ramanujan on integrals involving  $\Xi(t)$ , we have:

$$(6.7) \quad \int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} \cos(xt) dt = \frac{\pi}{2} \left( e^{\frac{x}{2}} - 2e^{-\frac{x}{2}} \psi(e^{-2x}) \right)$$

where  $\psi(s) := \sum_{n=1}^\infty e^{-n^2 \pi s}$  is the theta function. Now setting  $x := -i\alpha$ , we get:

$$(6.8) \quad \lim_{\alpha \rightarrow \frac{\pi}{4}} \int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh(\alpha t) dt = \frac{(-1)^n \pi \cos\left(\frac{\pi}{8}\right)}{4^n}$$

The following integral for  $0 \leq \alpha \leq \frac{\pi}{4}$  is uniformly convergent with respect to  $\alpha$ :

$$(6.9) \quad \int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} t^{2n} \cosh\left(\frac{\pi t}{4}\right) dt = \frac{(-1)^n \pi \cos\left(\frac{\pi}{8}\right)}{4^n}$$

With signs alternating infinitely often in the resulting expression, along the left-hand side having the same sign for sufficiently large values of  $n$ , we can infer that  $\zeta\left(\frac{1}{2} + it\right)$  has an infinite number of zeros on the critical strip [50].

#### 6.4. CALCULATIONS ON THE RIEMANN ZETA FUNCTION.

The objective from this point onwards is to build upon our proposal, enabling us to approximate  $\sum_1^\infty \frac{1}{n^s}$  with a high level of accuracy (within  $f(1)$ ) and determine when the approximate sum is minimal. We begin with the trivial expression:

$$(6.10) \quad n^{-x-iy} = n^{-x} [\cos(-y \ln(n)) + i \sin(-y \ln(n))]$$

The above follows trivially from the relation:  $n^{-iy} = e^{-iy \ln(n)} = \cos(-y \ln(n)) + i \sin(-y \ln(n))$ .

*Note:* At this critical juncture, there is a significant potential for confusion. The complex value in the integrand remains fixed and serves merely as a parameter. The summation, in reality, extends over the range of real-valued  $n$ . The influence of this parameter results in the presence of the quantity  $i \times \sin(-y \ln(n))$  in the second term. Consequently, we can treat these terms independently, with the understanding that  $i$  will be multiplied by the outcome of the second term.

Given the trivial relationship:  $n(-x)n^{-x-1} = (-x)n^{-x}$ , our objective

is to derive an expression for  $n\zeta'(s)$ . To achieve this, we evaluate the derivative with respect to  $n$  and then multiply the resulting expression by  $n$ .

(6.11)

$$n^{-x}(-x)[\text{Cos}(-y\ln(n))+i\text{Sin}(-y\ln(n))]+n^{1-x}(\frac{-y}{n})[-\text{Sin}(-y\ln(n))+i\text{Cos}(-y\ln(n))]$$

To utilize (6.2), we must integrate the respective terms in order to find

$$\int_1^\infty n\zeta'(s)dn$$

Let  $u = -y\ln(n)$ . We then have that:  $du = \frac{-y}{n}dn$  and  $e^{-u} = n^y$ ;  $e^{\frac{-u}{y}} = n$  and  $e^{\frac{-u(1-x)}{y}} = n^{-x}$ , with new bounds  $[0, \infty]$ .

Finally, given the previous; the expression for the real portion of the integrand follows as:

$$(6.12) \quad \frac{x}{y} \int e^{\frac{u(x-1)}{y}} \text{Cos}(u)du - \int e^{\frac{-u(1-x)}{y}} \text{Sin}(u)du$$

Substituting  $\frac{(x-1)}{y}$  with  $a$

$$(6.13) \quad \frac{x}{y} \int e^{au} \text{Cos}(u)du - \int e^{au} \text{Sin}(u)du$$

in the imaginary case :

$$(6.14) \quad \frac{x}{y} \int e^{au} \text{Sin}(u)du + \int e^{au} \text{Cos}(u)du$$

Using integral tables to integrate the above general form,(See for instance [48], along with the web reference: [47]), we have for the real part:

$$(6.15) \quad \frac{x}{y} \left[ \frac{e^{\frac{u(x-1)}{y}}}{(\frac{(x-1)}{y})^2 + 1} \right] \left( \frac{(x-1)}{y} \text{Cos}(u) + \text{Sin}(u) \right) - \left[ \frac{e^{\frac{u(x-1)}{y}}}{(\frac{(x-1)}{y})^2 + 1} \right] \left( \frac{(x-1)}{y} \text{Sin}(u) - \text{Cos}(u) \right)$$

In the imaginary case:

$$(6.16) \quad \frac{x}{y} \left[ \frac{e^{\frac{u(x-1)}{y}}}{(\frac{(x-1)}{y})^2 + 1} \right] \left( \frac{(x-1)}{y} \text{Sin}(u) + \text{Cos}(u) \right) + \left[ \frac{e^{\frac{u(x-1)}{y}}}{(\frac{(x-1)}{y})^2 + 1} \right] \left( \frac{(x-1)}{y} \text{Cos}(u) - \text{Sin}(u) \right)$$

When evaluated over the limits, and taking note that for  $x < 1$  and non-zero  $y$  the expression in both cases reduces to zero at  $u \rightarrow \infty$ . As

such, we have:

$$(6.17) \quad -\frac{x}{y} \left[ \frac{1}{\left(\frac{(x-1)}{y}\right)^2 + 1} \right] \left( \frac{(x-1)}{y} \right) + \left[ \frac{1}{\left(\frac{(x-1)}{y}\right)^2 + 1} \right] (-1)$$

$$(6.18) \quad -\frac{x}{y} \left[ \frac{1}{\left(\frac{(x-1)}{y}\right)^2 + 1} \right] (1) - \left[ \frac{1}{\left(\frac{(x-1)}{y}\right)^2 + 1} \right] \left( \frac{(x-1)}{y} \right)$$

$$(6.19) \quad -\left[ \frac{1}{\left(\frac{(x-1)}{y}\right)^2 + 1} \right] \left( \frac{x}{y} \left( \frac{(x-1)}{y} \right) + 1 \right)$$

$$(6.20) \quad -\frac{y^2}{(x-1)^2 + y^2} \left( \left( \frac{(2x-1)}{y} \right) \right)$$

Real:

$$(6.21) \quad -\frac{y^2}{(x-1)^2 + y^2} \left( \frac{x(x-1) + y^2}{y^2} \right)$$

Imaginary:

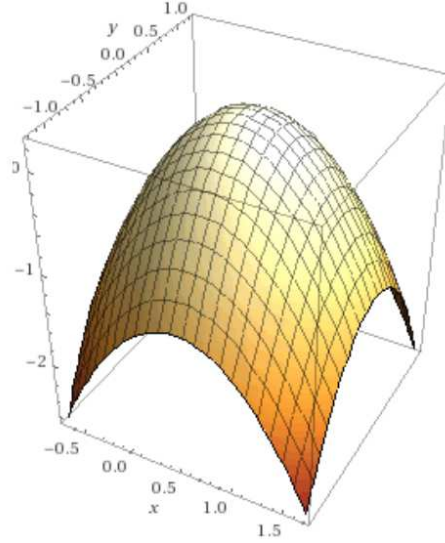
$$(6.22) \quad -\frac{y(2x-1)}{(x-1)^2 + y^2}$$

Naturally minimal at  $x = \frac{1}{2}$

Real:

$$(6.23) \quad -\frac{x(x-1) + y^2}{(x-1)^2 + y^2}$$

$$(6.24) \quad -\frac{x + \frac{y^2}{x-1}}{(x-1) + \frac{y^2}{x-1}}$$

FIGURE 1. Plot:  $-x^2 + x - y^2$ 

From (6.23), the roots follow as  $\frac{1 \pm \sqrt{1-4y^2}}{2}$ , indicating a range of zeros for  $x$  from  $-\frac{1}{2}$  to  $\frac{1}{2}$ , which includes  $x = \frac{1}{2}$  as one of the zeros. Given that the only requirement for the imaginary part is  $x = \frac{1}{2}$ , we observe intersecting minimization for both the real and imaginary components at  $x = \frac{1}{2}$ .

Of exceptional importance to note is that  $\zeta(1)$  possesses a real component and lacks a complex counterpart, meaning it exhibits no complex error; the error margin is zero. Additionally, the error, as we know, constitutes a real fraction of  $f(1)$  and hence cannot be complex. One can argue that, therefore, the minimum values for  $\zeta$  occur at  $x = 1/2$ .

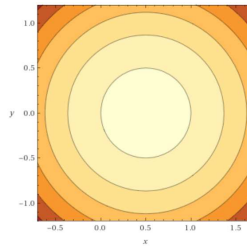


FIGURE 2. Plot: Contour

I humbly endeavour to express several thoughts here, many of which

may be deemed controversial at the very least. Nevertheless, I shall present them, as the realm of logic is confined solely by its own boundaries.

Every composite number  $np$  is a product of factors. Let's denote certain entities as the set of primes, encompassing 1, and 2 up to and less than some  $n = \sqrt{p} \in \mathbb{N}$ , inclusive of  $1_1, 1_2, \dots, 1_{\sqrt{p}}$ . It is evident that within

$$(6.25) \quad \binom{2\sqrt{P}}{\sqrt{p}} | \text{Number of Non-Primes} < \sqrt{p},$$

the count of factor expressions will yield a value smaller than  $Q = p - \sqrt{p}$ , providing us with a measure of the number of primes within an interval when subtracted from the quantity  $Q$ .

Expressed in  $\mathbb{R}_2$ , one gains greater control over the conditioning process in terms of precision. The underlying reason is profound and is also explored as part of a subsequent communication to experts. We have formed an intuition that all statements involving the count of primes in an equality predicate, along with Diophantine statements involving the relation  $=$ , are provable and can be verified in a finite, procedural manner.

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