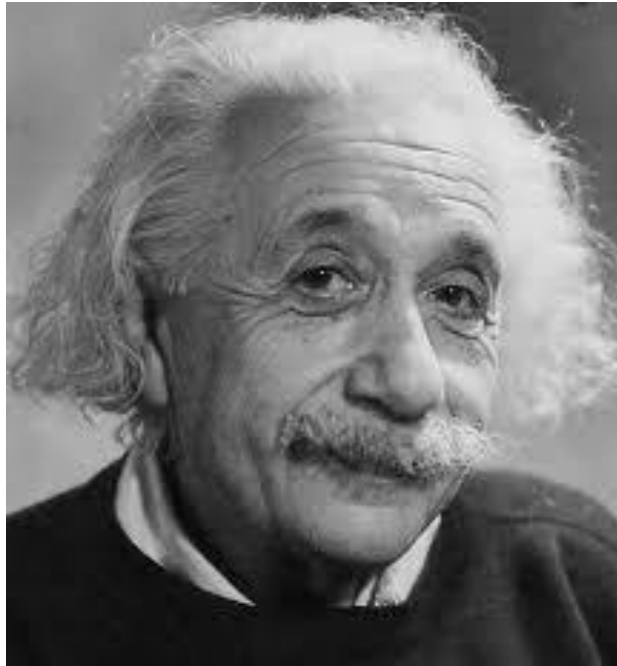
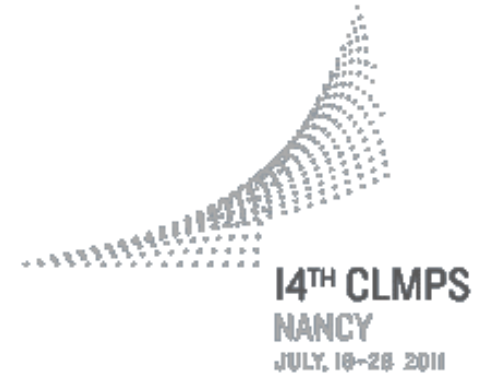


From the hole argument (A. Einstein) to the ball of clay argument (H. Weyl)



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Aim of this paper

Context: Hermann Weyl's *Space-Time-Matter* (1918-...)

Albert Einstein's General Theory of Relativity (GTR) is a brilliant confirmation of Bernhard Riemann's epistemological ideas on geometry

That involves a redistribution of the roles of mathematics and physics in the scientific constitution of the space concept.

Weyl's **Ball of Clay Argument (BCA)** 1918-...
(*Space-Time-Matter*, §12, 1918-1919-1921-1923)

Einstein's **Hole Argument (HA)** 1913-...
(Einstein&Grossman, "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation")

Aim of this paper

Two different points of view on BCA:

- 1) **A comparative approach** will highlight some features of GTR-like theories (→Mach's principle)
- 2) **An intrinsic approach** will illustrate that the BCA provides the key to understand the unity of Weyl's epistemology about space

Einstein's presentation and the modern presentation of the hole argument

1907-1915 Einstein's hypotheses to find gravitational equations:

- 1) Mach's principle
- 2) General principle of relativity → Covariant theory

Gravitational equations $G_{\mu\nu}(x) = T_{\mu\nu}(x)$ (*)

with $G_{\mu\nu} = F(g_{\mu\nu}, \partial g_{\mu\nu}/\partial x, \partial^2 g_{\mu\nu}/\partial x^2)$

Einstein's presentation and the modern presentation of the hole argument

Indeterminism (violation of Mach's principle) :

For the same $T_{ab}(x)$, we can find *two different solutions* $g_{ab}(x)$ and $g'_{ab}(x)$ that both verify (*)

Proof (Hole argument)

Additional Hypothesis: a region void of matter (Hole)

We take an (active) diffeomorphism ϕ that differ from identity only outside the hole

Then we take: $g' = \phi^* g$ (pulled-back metric)

Einstein's presentation and the modern presentation of the hole argument

Modern presentation of the hole argument :

John Stachel/John Earman/John Norton

- They minimize the importance of Mach's principle
- They minimize the importance of the hole itself

What is important for them:

- Before we put any field on, space-time as a purely topological manifold is not already a *physical* entity
- Two solutions (T, g) and (T', g') differing only by an active diffeomorphism are two *mathematical* expressions of the same *physical* situation

Einstein's presentation and the modern presentation of the hole argument

“Modern” hole argument:

Manifold \leftrightarrow Stress-energy tensor + Metrical tensor

Epistemological problem about the frontier between mathematics and physics in the new relativistic framework.

Albert Einstein original problem:

Manifold + Stress-energy tensor \leftrightarrow Metrical tensor

Ontological problem about the nature of the relations between metrics and mass (\rightarrow Mach)

Hole Argument (Einstein)

Topological background: (M, g)

Matter: $T_{\mu\nu}$

Mach's principle:

$$G_{\mu\nu}(x) = k * T_{\mu\nu}(x)$$

$$G_{\mu\nu} = F(g_{\mu\nu}, \partial g_{\mu\nu} / \partial x, \partial^2 g_{\mu\nu} / \partial x^2)$$

3 steps:

1) $(T_{\mu\nu}(x), g_{\mu\nu}(x))$

2) $(T'_{\mu\nu}(x'), g'_{\mu\nu}(x')) \quad (T'_{\mu\nu} = T_{\mu\nu})$

3) $(T_{\mu\nu}(x), g'_{\mu\nu}(x))$

Ball of Clay Argument (Weyl)

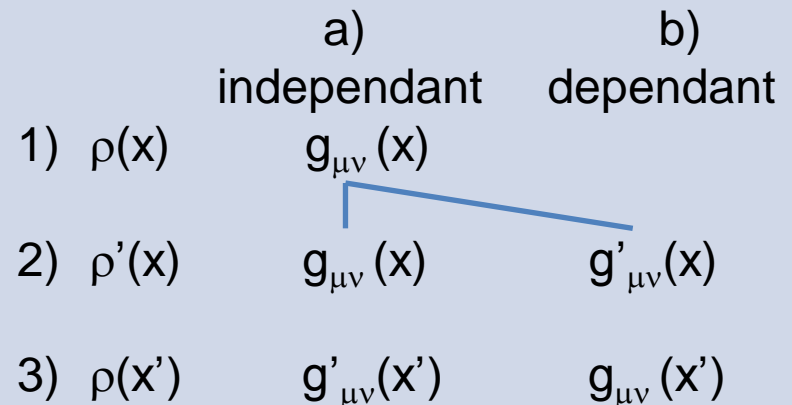
Topological background: R^4

Matter: ρ

Mach's principle:

$$g_{\mu\nu}(x) = F(x, \rho)$$

3 steps and 2 possibilities :



The ball of Clay Paradox

“The simple fact that I can squeeze a ball of modeling clay with my hands into any regular shape totally different from a sphere would seem to reduce Riemann’s view to absurdity.”

Space-Time-Matter, §12

A new kind of paradox of motion like Zeno’s

- 1) If the inner properties of matter can be entirely determined by fields whose signification need not any metric (like scalar fields)
- 2) And if these inner non-metrical properties of matter entirely determine the metric

Then no motion is possible in the universe (without changing the inner properties of matter)

The ball of Clay Paradox

◦ *What does this paradox tell us about Mach's principle?*

1) Mach's principle has to be expressed in a Local-Dynamic (not a Global-Static) form. We have to take time into account in order to express correctly Mach's principle.

2) Mach's principle cannot link a totally non-metrical concept of matter with the metric. Rather, it must link matter, *as it is distributed in the current metrical web (metrical distribution)*, with the evolution of the metric. Matter does not determine the metric but the *evolution* of the metric.

Weyl's ball of clay argument, intrinsic approach

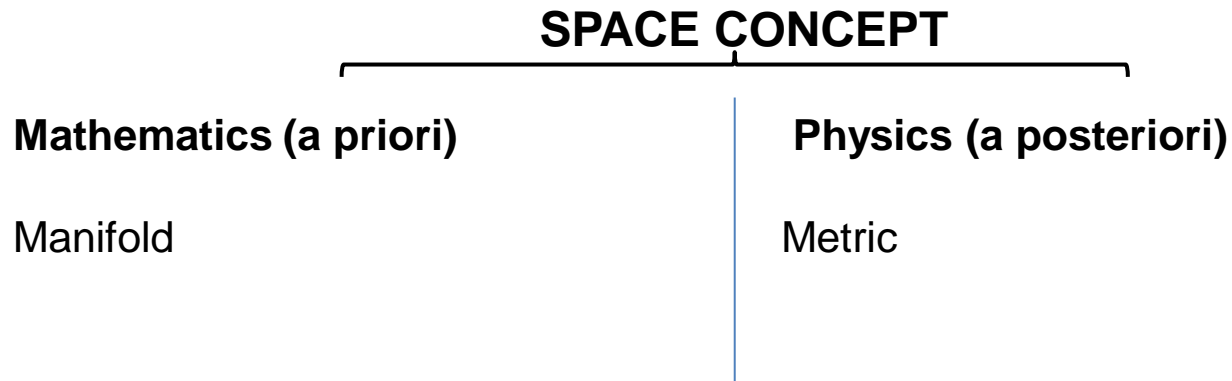
Weyl's problem:

In the context of a Riemannian metrics whose coefficients are determined by matter, can we continue to think about geometry as an a priori mathematical knowledge about a space characterized by its homogeneity?

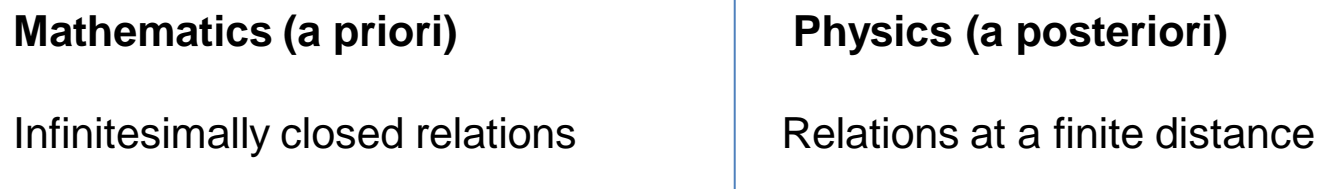
Weyl's ball of clay argument, intrinsic approach

Yes. We can think about the homogeneity of space in a Riemannian context if we consider the frontier between mathematics and physics, inside the space concept, like this:

Weyl's first answer (inside the Ball of Clay Argument):



Weyl's second answer (idea of *Nahegeometrie*):



Conclusion

To conclude, we have first highlighted some important features of Mach's principle, by a comparative interpretation of the Ball of Clay argument with the hole argument.

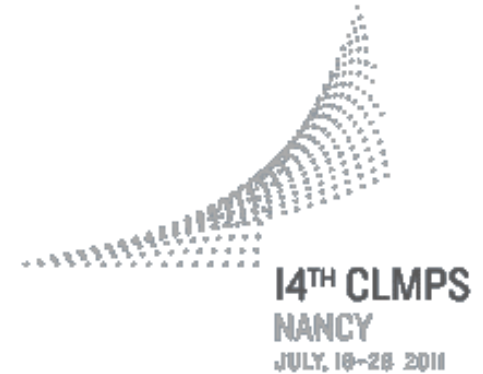
Then, we turned to an intrinsic interpretation of the text of the Ball of Clay Argument. And we saw that it can be considered as the place, in *Space-Time-Matter*, where Hermann Weyl puts down the problem of the conciliation of Riemannian geometry with the idea of the homogeneity of space.

That idea of homogeneity was very important for Hermann Weyl, because he defended some kind of idealism of space. Therefore, the issue presented in the Ball of Clay Argument is the key to have a global understanding of the great geometrical works of this author. This is the birth of the epistemological problem that has guided Hermann Weyl to his Nahegeometrie, which is an epistemological schema where GTR became coherent with a kind of transcendental idealism, reduced to infinitesimal relations.

What can we see now?

- More about Weyl's different solutions of his Ball of Clay Paradox
- More about what we learned about Mach's principle by comparing Einstein's HA with Weyl's BCA
- I can simply answer your questions

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For more details:

“Modelling a ball of clay: *How Hermann Weyl modified Albert Einstein's famous Hole Argument*”

(submitted to *Studies in History and Philosophy of Modern Physics*)

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Mach's Principle: Global-Static VS Local-Dynamic

Global-Static-Mach Principle:

The repartition of mass in the Universe at a given moment determines the metric (everywhere) at the same moment.

Local-Dynamic-Mach's Principle:

The mass at one event of space-time determines only the *curvature* of the metric *at that point*. Thus, the link between mass and metric is given by *differential equations*. As a consequence, the global determination of the metric by the masses has to be solved as a Cauchy problem with boundary conditions.

Weyl's solution to his Ball of Clay Paradox

1) Cosmological solution:

“a reorganization of all the masses in Universe would be necessary in order to make the distorted shape [of the ball of clay] appear spherical to an observer from any point of view.”

(solution of 1918, erased from the third edition 1919)

2) Microscopic solution:

“Riemann is right, a deformation of the inner atomistic structure of the clay, which is entirely different from that which I can effect with my hands, would be necessary”

In fact the inner properties of matter ρ have a metrical meaning
(example: density of charge)

Weyl's solution to his Ball of Clay Paradox

3) A third solution:

Weyl admits that his simplified hypotheses hide an important part of the topic. A satisfying answer to the paradox needs to have more subtle hypotheses. In particular:

“we now recognize that these ideas could give rise to a valid theory only after *time* had been added as a fourth dimension to the three-space dimensions in the manner set forth in the so-called special theory of relativity”

(→Weyl understand that Mach's principle have to be expressed on a Local-Dynamic way?)

“The essential point is that a piece of space has no visual form at all, but that this form depends on the material content occupying this world, and that by an appropriate rearrangement of the mode of occupation I can give it any visual form.”

(→the metric is not a part of the space concept anymore)

The Ball of Clay Argument: Reconstruction

The BCA claims that in a world where the inner properties of matter could be reduced to scalar fields $\rho(x)$, and where we have a strong RMP, then we cannot have any motion at all, unless the properties of matter change.

The proof is simple and is formally close to the HA. Indeed, the difficulty was to make explicit the form of RMP that is assumed by Weyl. This we have achieved in the body of our text.

Strong-Mach's principle: The metric field $g_{\mu\nu}$ is a function univocally determined by the inner properties of matter: $g_{\mu\nu}(x) = F(\rho, x)$ (this equation should be independent of the chosen coordinates frame)

The Ball of Clay Argument: Reconstruction

Proof of the BCA: We chose a frame of coordinates. The respective positions of each part of matter at the initial time will be described by all the values of the metric $g_{\mu\nu}$ on the numerical space representing the universe.

Now we suppose that something has moved in our universe but that everywhere matter has kept the same inner properties.

The functions ρ_b and ρ_a will represent matter in our frame of coordinates, *before* and *after* the motion respectively.

Each part of matter that was at the point of coordinates (x_b) has been moved to the new point (x_a) . But, we will have $\rho_a(x_a) = \rho_b(x_b)$ because the inner properties of the object have not changed.

The Ball of Clay Argument: Reconstruction

Now, we change our coordinates frame. The coordinates (x_a) refers now to the point that the coordinates (x_b) had before. In these new coordinates, after the motion, each part of matter is represented by exactly the same numerical values as the same part of matter before the motion in the first frame of coordinates.

Now, the function ρ'_a will represent matter *after* the move in the *new* coordinates frame. It is easy to see that the function ρ'_a and ρ_b are literally the same mathematical functions.

Thus, by the strong Mach's principle, we know that the functions $g'_{\mu\nu}$ that represent the metric *after* the motion, in the *new* frame of coordinates, will be exactly the same as the functions $g_{\mu\nu}$, which represented the metric *before* the motion in the *first* coordinate frame:

$$g'_{\mu\nu}(x_a) = F(\rho'_a, x_a) = F(\rho_b, x_b) = g_{\mu\nu}(x_b)$$

So, the respective metrical positions of each part of matter have not changed at all. A motion (without a change of the inner properties of matter) is impossible in such a world. Ultimately, we cannot understand the simple possibility of modeling a ball of clay by hand.