



HAL
open science

First order Two-Scale Particle-in-Cell numerical method for Vlasov equation

Emmanuel Frénod, Michael Gutnic, Sever Adrian Hirstoaga

► **To cite this version:**

Emmanuel Frénod, Michael Gutnic, Sever Adrian Hirstoaga. First order Two-Scale Particle-in-Cell numerical method for Vlasov equation. 2012. hal-00654928v1

HAL Id: hal-00654928

<https://hal.science/hal-00654928v1>

Preprint submitted on 23 Dec 2011 (v1), last revised 12 Jan 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FIRST ORDER TWO-SCALE PIC NUMERICAL SIMULATION FOR VLASOV EQUATION

EMMANUEL FRÉNOT^{1,2}, MICHAËL GUTNIC² AND SEVER HIRSTOAGA²

Abstract. The aim of this work is to simulate the long time evolution of the Vlasov solution f^ε for an electric field $E^\varepsilon = E_0 + \varepsilon E_1$ in two regimes $\varepsilon \rightarrow 0$ and $\varepsilon \sim 1$. To this purpose, we use the Two-Scale Convergence theory to determine an first order approximation $F + \varepsilon F_1$ of f^ε , then a PIC method to build an algorithm to obtain a numerical approximation of $F + \varepsilon F_1$.

Résumé. On cherche à simuler l'évolution en temps long de la solution f^ε de l'équation de Vlasov pour un champ électrique $E^\varepsilon = E_0 + \varepsilon E_1$ dans les deux régimes donnés par $\varepsilon \rightarrow 0$ et $\varepsilon \sim 1$. À cet effet, on utilise la théorie de la Convergence Deux Échelles pour obtenir une approximation d'ordre 1 $F + \varepsilon F_1$ de f^ε , puis une méthode PIC pour construire l'algorithme d'approximation numérique de $F + \varepsilon F_1$.

CONTENTS

1.	Introduction	1
2.	On Two-Scale Convergence statements	2
2.1.	Zero order approximation	4
2.2.	First order approximation	4
3.	Numerical algorithm	10
	References	12

1. INTRODUCTION

In this paper, we are interested in the solution f^ε to the following Vlasov problem set in a bi-dimensional phase space

$$\begin{cases} \frac{\partial f^\varepsilon}{\partial t} + \frac{v}{\varepsilon} \frac{\partial f^\varepsilon}{\partial r} + \left(E^\varepsilon - \frac{r}{\varepsilon} \right) \frac{\partial f^\varepsilon}{\partial v} = 0, \\ f^\varepsilon(t = 0, r, v) = f_0(r, v). \end{cases} \quad (1.1)$$

¹ Université Européenne de Bretagne, Lab-STICC (UMR CNRS 3192), Université de Bretagne-Sud, Centre Yves Coppens, Campus de Tohannic, F-56017, Vannes

² INRIA CALVI Project & IRMA (UMR 7501), Université de Strasbourg, 7 rue René Descartes, F-67084 Strasbourg Cedex, France

where $f^\varepsilon : (t, r, v) \mapsto f^\varepsilon(t, r, v)$, for $t \in [0, T]$, $r \in \mathbb{R}^+$ and $v \in \mathbb{R}$. In (1.1), let

$$E^\varepsilon(t, r) = E_0\left(t, \frac{t}{\varepsilon}, r\right) + \varepsilon E_1\left(t, \frac{t}{\varepsilon}, r\right), \quad (1.2)$$

where $E_{i=0,1} : (t, \tau, r) \mapsto E_{i=0,1}(t, \tau, r)$ are regular and 2π -periodics with respect to τ . Our purpose is to simulate the long time evolution of f^ε in two regimes $\varepsilon \rightarrow 0$ and $\varepsilon \sim 1$.

In Section 2, we use Two-Scale Convergence theory to determine a first-order approximation of f^ε , that is $f^\varepsilon(t, r, v) \approx F(t, t/\varepsilon, r, v) + \varepsilon F_1(t, t/\varepsilon, r, v)$ where $F : (t, \tau, r, v) \mapsto F(t, \tau, r, v)$ and $F_1 : (t, \tau, r, v) \mapsto F_1(t, \tau, r, v)$ are 2π -periodic functions with respect to $\tau \in \mathbb{R}$.

We first show in Section 2.1 that, for the zero order term, there exists a function $G : (t, q, u) \mapsto G(t, q, u)$ solution of an initial boundary condition partial differential problem such that $F(t, \tau, r, v) = G(t, \mathcal{R}^\tau(r, v))$, where \mathcal{R}^τ is the 2D-rotation of angle τ . In Section 2.2, we handle first order term and show that there exists two functions $G_1 : (t, q, u) \mapsto G_1(t, q, u)$ and $W : (t, \tau, q, u) \mapsto W(t, \tau, q, u)$ such that $F_1(t, \tau, r, v) = G_1(t, \mathcal{R}^\tau(r, v)) + W(t, \tau, \mathcal{R}^\tau(r, v))$ where W can be computed from $\nabla_{q,u} G$ and G_1 is yet the solution of an initial boundary condition partial differential problem.

In Section 3, we introduce a PIC based approximation to build a numerical algorithm which will allow us to solve numerically the problems satisfied by G and G_1 as well as to compute a numerical approximation of W .

2. ON TWO-SCALE CONVERGENCE STATEMENTS

The concept of Two-Scale Convergence was introduced at the end of the 80's by Nguetseng [5,6]. In 1992, Allaire gave a very understandable proof of this result [1]. Then several authors used this theory to build numerical methods called Two-Scale Numerical Methods (e.g. Frénod and Sonnendrücker [3], Frénod, Salvarani and Sonnendrücker [2], or Mouton [4]). For more details about Two-Scale Convergence theory, we refer to these authors and their references.

Let us here only recall the definition of the Two-Scale Convergence.

Definition 2.1. A function $f^\varepsilon : (t, r, v) \in [0, T] \times \mathbb{R}^2 \mapsto f^\varepsilon(t, r, v)$ Two-Scale converges to a function $F : (t, \tau, r, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2 \mapsto F(t, \tau, r, v)$ 2π -periodic in τ as $\varepsilon \rightarrow 0$ if:

$$\int_0^T \int_{\mathbb{R}^2} f^\varepsilon(t, r, v) [\phi]^\varepsilon(t, r, v) dr dv dt = \int_0^T \int_0^{2\pi} \int_{\mathbb{R}^2} F(t, \tau, r, v) \phi(t, \tau, r, v) dr dv d\tau dt, \quad (2.1)$$

for all $\phi \in \mathcal{C}^0$ where $\mathcal{C}^i := C_c^i([0, T]; C_{\#}^i(\mathbb{R}; C_c^i(\mathbb{R}^2)))$ for $i = 0, 1$.

Remark 2.1 (Notations). In the above definition and in the following of the paper, we introduce the following general notations:

- (i) Subscript # in space definition stands for 2π -periodicity in τ .
- (ii) Bracket $[\cdot]^\varepsilon$ stands for $[\phi]^\varepsilon(t, r, v) := \phi(t, t/\varepsilon, r, v)$ for any function ϕ depending of the four variables $(t, \tau, r, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2$.

Moreover, in the whole section we will use the 2D-rotation of angle τ of a two-dimensionnal vector.

Definition 2.2. For all $\tau \in [0, 2\pi]$ and all $(r, v) \in \mathbb{R}^2$, we define the 2D-rotation \mathcal{R}^τ of angle τ by:

$$\mathcal{R}^\tau(r, v) = \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} r \cos \tau - v \sin \tau \\ r \sin \tau + v \cos \tau \end{pmatrix}. \quad (2.2)$$

Its inverse is then given by $\mathcal{R}^{-\tau}$ for which the first component is denoted by $\mathcal{R}_r^{-\tau}$.

Remark 2.2. Thanks to the above definition, we have that :

$$\mathcal{R}^\tau(r, v) \cdot \begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} r \\ v \end{pmatrix} \cdot \mathcal{R}^{-\tau}(q, u). \quad (2.3)$$

Theorem 2.1. Let $(f^\varepsilon)_\varepsilon$ be a sequence of solution to Vlasov problem (1.1) with the electric field E^ε given by (1.2). Therefore there exists two functions $F : (t, \tau, r, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2 \mapsto F(t, \tau, r, v)$ and $F_1 : (t, \tau, r, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2 \mapsto F(t, \tau, r, v)$ 2π -periodic with respect to $\tau \in \mathbb{R}$ such that

$$f^\varepsilon(t, r, v) \approx F\left(t, \frac{t}{\varepsilon}, r, v\right) + \varepsilon F_1\left(t, \frac{t}{\varepsilon}, r, v\right), \quad (2.4)$$

Moreover, regarding the zero order term, there exists a function $G : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto G(t, q, u)$ such that

$$F(t, \tau, r, v) = G(t, \mathcal{R}^\tau(r, v)), \quad (2.5)$$

where \mathcal{R}^τ is the rotation define in Definition 2.2 and G is solution to the problem

$$\begin{cases} \frac{\partial G}{\partial t}(t, q, u) + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) d\tau \cdot \nabla_{q,u} G(t, q, u) = 0 \\ G(t=0, q, u) = \frac{1}{2\pi} f_0(q, u), \end{cases} \quad (2.6)$$

Regarding the first order term, there exists two functions $G_1 : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto G_1(t, q, u)$ and $W : (t, \tau, q, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2 \mapsto W(t, \tau, q, u)$ such that

$$F_1(t, \tau, r, v) = G_1(t, \mathcal{R}^\tau(r, v)) + W(t, \tau, \mathcal{R}^\tau(r, v)), \quad (2.7)$$

where on the one hand, W is given by

$$\begin{aligned} W(t, \tau, q, u) &= \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\ &\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u), \end{aligned} \quad (2.8)$$

and on the other hand, G_1 is solution to the problem

$$\begin{cases} \frac{\partial G_1}{\partial t}(t, q, u) + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) d\tau \cdot \nabla_{q,u} G_1(t, q, u) = \\ \frac{1}{2\pi} \int_0^{2\pi} \left[\int_0^\tau \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right. \\ \quad \left. - \frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right] d\tau \cdot \nabla_{q,u} G(t, q, u) + \\ \left[\frac{\pi}{4\pi^2} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\ \quad \left. - \frac{1}{2\pi} \int_0^{2\pi} \left(\nabla_{q,u} \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right) d\tau \right] \\ \quad \cdot \nabla_{q,u} G(t, q, u) d\tau. \\ G_1(t=0, q, u) = 0. \end{cases} \quad (2.9)$$

Remark 2.3. *Note that the expression of W in (2.8) as well as problems (2.6) and (2.9) do not depend on $1/\varepsilon$ -frequency oscillations any more. The proof of Theorem 2.2 is given in the two following sections.*

2.1. Zero order approximation

Here, we briefly recall arguments developed in [2]. Let us consider the Vlasov equation for f^ε in problem (1.1). First, we multiply by f^ε and integrate over $r \in \mathbb{R}^+$ and $v \in \mathbb{R}$ to obtain:

$$\frac{1}{2} \frac{d \left(\|f^\varepsilon(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}^2 \right)}{dt} = 0, \quad (2.10)$$

which leads to the following estimate:

$$\|f^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C. \quad (2.11)$$

Then it is known that $(f^\varepsilon)_\varepsilon$ Two-Scale converges to some $F : (t, \tau, r, v) \mapsto F(t, \tau, r, v)$ with $F \in L^\infty \left([0, T]; L^\infty_{\#}(\mathbb{R}, L^2(\mathbb{R}^2)) \right)$. Now, we multiply Vlasov equation in (1.1) by $[\phi]^\varepsilon$ with $\phi \in \mathcal{C}^1$, integrate over $t \in [0, T]$ and $(r, v) \in \mathbb{R}^2$ and then integrate by parts. Thus we show that F satisfies in a weak sense the following equation

$$\frac{\partial F^\varepsilon}{\partial \tau} + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r, v} F = 0, \quad (2.12)$$

where we note that ε is not involved anymore. From (2.12), we deduce that F is constant along the characteristics associated to this advection equation, and deduce from Theorem 4.2 in [2] that there exists a function $G : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto G(t, q, u)$ such that (2.5) is satisfied with G is solution to problem (2.6) of Theorem 2.1.

Remark 2.4. *Let $G : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto G(t, q, u)$ be a given function and define the family $(F^\varepsilon)_{\varepsilon > 0}$ by*

$$F^\varepsilon(t, r, v) = G(t, \mathcal{R}^{t/\varepsilon}(r, v)). \quad (2.13)$$

Then

- (i) $\nabla_{r, v} F^\varepsilon(t, r, v) = \mathcal{R}^{-t/\varepsilon}(\nabla_{q, u} G(t, \mathcal{R}^{t/\varepsilon}(r, v)))$.
- (ii) $\frac{\partial F^\varepsilon}{\partial t}(t, r, v) + \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} F^\varepsilon(t, r, v) = \frac{\partial G}{\partial t}(t, \mathcal{R}^{t/\varepsilon}(r, v))$.

2.2. First order approximation

We consider the zero order Two-Scale approximation F of f^ε introduced in section 2.1. Let us define $[F]^\varepsilon$ as in Remark 2.1(ii). Then thanks to Remark 2.4(ii)

$$\frac{\partial [F]^\varepsilon}{\partial t}(t, r, v) + \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r, v} [F]^\varepsilon(t, r, v) = \frac{\partial G}{\partial t}(t, \mathcal{R}^{t/\varepsilon}(r, v)). \quad (2.14)$$

For the sake of simplicity in computations below, we introduce the notation $E^\varepsilon = E_0(t, t/\varepsilon, r) + \varepsilon E_1(t, t/\varepsilon, r)$ and omit the dependency of $[F]^\varepsilon$ and its partial derivatives in (t, r, v) . First, we add the term $E^\varepsilon \partial [F]^\varepsilon / \partial v$ on both sides of equation (2.14) to get (2.15). Then we rewrite the right-hand-side to show up the term $\nabla_{r, v} [F]^\varepsilon$ and get (2.16). Using Remark 2.4(i) then Remark 2.2, we

obtain respectively (2.17) then (2.18).

$$\begin{aligned} \frac{\partial [F]^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} F^\varepsilon + E^\varepsilon \frac{\partial [F]^\varepsilon}{\partial v} &= \\ &= E^\varepsilon \frac{\partial [F]^\varepsilon}{\partial v} + \frac{\partial G}{\partial t} \left(t, \mathcal{R}^{t/\varepsilon}(r, v) \right), \end{aligned} \quad (2.15)$$

$$= \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot \nabla_{r,v} F^\varepsilon + \frac{\partial G}{\partial t} \left(t, \mathcal{R}^{t/\varepsilon}(r, v) \right), \quad (2.16)$$

$$= \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot \mathcal{R}^{-t/\varepsilon} \left(\nabla_{q,u} G(t, \mathcal{R}^{t/\varepsilon}(r, v)) \right) + \frac{\partial G}{\partial t} \left(t, \mathcal{R}^{t/\varepsilon}(r, v) \right), \quad (2.17)$$

$$= \mathcal{R}^{t/\varepsilon}(0, E^\varepsilon) \cdot \nabla_{q,u} G(t, \mathcal{R}^{t/\varepsilon}(r, v)) + \frac{\partial G}{\partial t} \left(t, \mathcal{R}^{t/\varepsilon}(r, v) \right). \quad (2.18)$$

To handle the right-hand-side of (2.18), we introduce the function Υ^ε such that

$$\Upsilon^\varepsilon(t, \tau, r, v) := \mathcal{R}^\tau(0, E^\varepsilon) \cdot \nabla_{q,u} G(t, \mathcal{R}^\tau(r, v)) + \frac{\partial G}{\partial t} \left(t, \mathcal{R}^\tau(r, v) \right). \quad (2.19)$$

Then defining $[\Upsilon^\varepsilon]^\varepsilon$ as in Remark 2.1(ii), we can subtract (2.18) from (1.1) and multiply the result by $1/\varepsilon$ to obtain

$$\frac{\partial}{\partial t} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} \right) + \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} \right) + E^\varepsilon \frac{\partial}{\partial v} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} \right) = -\frac{1}{\varepsilon} [\Upsilon^\varepsilon]^\varepsilon. \quad (2.20)$$

Now, let a function $W^\varepsilon : (t, \tau, q, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^2 \mapsto W^\varepsilon(t, \tau, q, u)$ such that

$$\frac{\partial W^\varepsilon}{\partial \tau} (t, \tau, \mathcal{R}^\tau(r, v)) = -\Upsilon^\varepsilon(t, \tau, r, v). \quad (2.21)$$

On the one hand, if we substitute Υ^ε by its expression in (2.19) and then consider equation (2.21) in variables $(q, u) = \mathcal{R}^\tau(r, v)$, we obtain

$$\frac{\partial W^\varepsilon}{\partial \tau} (t, \tau, q, u) = -\mathcal{R}^\tau(0, E^\varepsilon(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} G(t, q, u) - \frac{\partial G}{\partial t} (t, q, u). \quad (2.22)$$

We substitute the time derivative of G on the right-hand-side of (2.22) by its expression in (2.6). Integrating the result over the second variable from 0 to τ gives

$$\begin{aligned} W^\varepsilon(t, \tau, q, u) &= \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\ &\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) \\ &\quad - \varepsilon \int_0^\tau \mathcal{R}^\sigma(0, E^1(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \cdot \nabla_{q,u} G(t, q, u). \end{aligned} \quad (2.23)$$

Let ε goes to 0 in (2.23), we obtain the limit W of W^ε define by (2.8) in Theorem 2.1.

On the other hand, let us introduce $\widetilde{W}^\varepsilon$ define by $\widetilde{W}^\varepsilon(t, \tau, r, v) := W^\varepsilon(t, \tau, \mathcal{R}^\tau(r, v))$. Therefore, (2.21) is equivalent to

$$\frac{\partial \widetilde{W}^\varepsilon}{\partial \tau} (t, \tau, r, v) + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} \widetilde{W}^\varepsilon(t, \tau, r, v) = -\Upsilon^\varepsilon(t, \tau, r, v). \quad (2.24)$$

Using Remark 2.1(ii), we note that $\partial_t[\widetilde{W}^\varepsilon]^\varepsilon = [\partial_t \widetilde{W}^\varepsilon]^\varepsilon + [\partial_\tau \widetilde{W}^\varepsilon]^\varepsilon/\varepsilon$ and $\partial_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon = [\partial_{r,v} \widetilde{W}^\varepsilon]^\varepsilon$ so that considering (2.24) in $\tau = t/\varepsilon$ and multiply the result by $1/\varepsilon$ leads to

$$\frac{\partial[\widetilde{W}^\varepsilon]^\varepsilon}{\partial t} - \left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t} \right]^\varepsilon + \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon = -\frac{1}{\varepsilon}[\Upsilon^\varepsilon]^\varepsilon. \quad (2.25)$$

Now, we subtract (2.25) from (2.20) and subtract from the both sides of the result the term $E^\varepsilon \partial_v[\widetilde{W}^\varepsilon]^\varepsilon$ to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) &+ \frac{1}{\varepsilon} \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) \\ &+ E^\varepsilon \frac{\partial}{\partial v} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) \\ &= - \left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t} \right]^\varepsilon - E^\varepsilon \frac{\partial[\widetilde{W}^\varepsilon]^\varepsilon}{\partial v}. \end{aligned} \quad (2.26)$$

We want to obtain an a priori estimate in order to prove convergence result.. Therefore we multiply (2.26) by $\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon$ and integrate by part. This leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right\|_{L^2(\mathbb{R}^2)}^2(t) \\ \leq \left\| \left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t} \right]^\varepsilon + \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot \nabla_{r,v}[\widetilde{W}^\varepsilon]^\varepsilon \right\|_{L^2(\mathbb{R}^2)}(t) \left\| \frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right\|_{L^2(\mathbb{R}^2)}(t). \end{aligned} \quad (2.27)$$

If the first factor on the right-hand-side of (2.27) is bounded, by Gromwall Lemma, we obtain the following estimate

$$\left\| \frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} \leq C. \quad (2.28)$$

From this estimate, we deduce the following Two-Scale convergence result

$$\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \longrightarrow F_1 - \widetilde{W}, \quad (2.29)$$

where $\widetilde{W} = W(t, \tau, \mathcal{R}^\tau(r, v))$.

Now, we want to compute F_1 . Returning to (2.26), we multiply by $\phi \in \mathcal{C}^1$ (see Definition 2.1), integrate over $t \in [0, T)$ and $(r, v) \in \mathbb{R}^2$ and then integrate by parts. We obtain

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) \left(\left[\frac{\partial \phi}{\partial t} \right]^\varepsilon + E^\varepsilon \left[\frac{\partial \phi}{\partial v} \right]^\varepsilon \right) dr dv dt \\ & - \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^2} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) \left(\left[\frac{\partial \phi}{\partial \tau} \right]^\varepsilon + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot [\nabla_{r,v} \phi]^\varepsilon \right) dr dv dt \\ & = - \int_0^T \int_{\mathbb{R}^2} \left(\left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t} \right]^\varepsilon - \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot [\nabla_{r,v} \widetilde{W}^\varepsilon]^\varepsilon \right) [\phi]^\varepsilon dr dv dt. \end{aligned} \quad (2.30)$$

Multiply (2.30) by ε and pass to the limit using the Two-Scale convergence (2.29), we obtain

$$\frac{\partial}{\partial \tau} (F_1 - \widetilde{W}) + \begin{pmatrix} v \\ -r \end{pmatrix} \cdot \nabla_{r,v} (F_1 - \widetilde{W}) = 0, \quad (2.31)$$

in a weak sense. Thus $F_1 - \widetilde{W}$ is constant along the characteristics so that there exists a function $G_1 : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto G_1(t, q, u)$ such that

$$F_1(t, \tau, r, v) - \widetilde{W}(t, \tau, r, v) = G_1(t, \mathcal{R}^\tau(r, v)), \quad (2.32)$$

and therefore, thanks to the definition of \widetilde{W} , we obtain the decomposition (2.7) of F_1 in Theorem 2.1. Now, let $\gamma : (t, q, u) \in [0, T] \times \mathbb{R}^2 \mapsto \gamma(t, q, u)$ a function in $C_c^1([0, T]; C_c^1(\mathbb{R}^2))$ such that

$$\phi(t, \tau, r, v) = \gamma(t, \mathcal{R}^\tau(r, v)). \quad (2.33)$$

We consider (2.30) for this function ϕ . Then the second term on the left hand side cancels and we obtain

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} \left(\frac{f^\varepsilon - [F]^\varepsilon}{\varepsilon} - [\widetilde{W}^\varepsilon]^\varepsilon \right) \left(\left[\frac{\partial \phi}{\partial t} \right]^\varepsilon + E^\varepsilon \left[\frac{\partial \phi}{\partial v} \right]^\varepsilon \right) dr dv dt \\ &= - \int_0^T \int_{\mathbb{R}^2} \left(\left[\frac{\partial \widetilde{W}^\varepsilon}{\partial t} \right]^\varepsilon - \begin{pmatrix} 0 \\ E^\varepsilon \end{pmatrix} \cdot [\nabla_{r,v} \widetilde{W}^\varepsilon]^\varepsilon \right) [\phi]^\varepsilon dr dv dt. \end{aligned} \quad (2.34)$$

Thanks to the Two-Scale convergence, we pass to the limit as $\varepsilon \rightarrow 0$ and integrate over $\tau \in [0, 2\pi]$ leads to

$$\begin{aligned} & \int_0^T \int_0^{2\pi} \int_{\mathbb{R}^2} G_1(t, \mathcal{R}^\tau(r, v)) \left(\frac{\partial \phi}{\partial t} + \begin{pmatrix} 0 \\ E^0 \end{pmatrix} \cdot \nabla_{r,v} \phi \right) dr dv d\tau dt \\ &= \int_0^T \int_0^{2\pi} \int_{\mathbb{R}^2} \left(\frac{\partial \widetilde{W}}{\partial t} - \begin{pmatrix} 0 \\ E^0 \end{pmatrix} \cdot \nabla_{r,v} \widetilde{W} \right) \phi dr dv d\tau dt. \end{aligned} \quad (2.35)$$

Then substituting ϕ with respect to γ and changing the variables (r, v) in $(q, u) = \mathcal{R}^\tau(r, v)$ that is $(r, v) = \mathcal{R}^{-\tau}(q, u)$ gives

$$\begin{aligned} & \int_0^T \int_0^{2\pi} \int_{\mathbb{R}^2} G_1(t, q, u) \left(\frac{\partial \gamma}{\partial t}(t, q, u) \right. \\ & \quad \left. + \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} \gamma(t, q, u) \right) dq du d\tau dt \\ &= \int_0^T \int_0^{2\pi} \int_{\mathbb{R}^2} \left(\frac{\partial W}{\partial t}(t, \tau, q, u) \right. \\ & \quad \left. + \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} W(t, \tau, q, u) \right) \gamma(t, q, u) dq du d\tau dt. \end{aligned} \quad (2.36)$$

We finally deduce that G_1 satisfies in a weak sense the following problem

$$\begin{cases} \frac{\partial G_1}{\partial t}(t, q, u) + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) d\tau \cdot \nabla_{q,u} G_1(t, q, u) = \\ - \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial W}{\partial t}(t, \tau, q, u) + \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} W(t, \tau, q, u) \right) d\tau \\ G_1(t=0, q, u) = 0. \end{cases} \quad (2.37)$$

To handle the right hand side of (2.37), we need to compute the partial derivatives of W . First from (2.8), we have that

$$\begin{aligned}
\frac{\partial W}{\partial t}(t, \tau, q, u) &= \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) \\
&+ \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \nabla_{q,u} \frac{\partial G}{\partial t}(t, q, u),
\end{aligned} \tag{2.38}$$

so that we can then replace $\partial_t G$ by its expression in (2.6) to get

$$\begin{aligned}
\frac{\partial W}{\partial t}(t, \tau, q, u) &= \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) \\
&\quad \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \\
&\quad \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \nabla_{q,u} G(t, q, u) \\
&\quad + H_{q,u} G(t, q, u) \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma,
\end{aligned} \tag{2.39}$$

where $H_{q,u}$ is the Hessian matrix with respect to variable q and u . Next, from (2.8), we deduce that

$$\begin{aligned}
\nabla_{q,u} W(t, \tau, q, u) &= \left[\frac{\tau}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. - \int_0^\tau \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) \\
&+ H_{q,u} G(t, q, u) \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right].
\end{aligned} \tag{2.40}$$

We consider the scalar product of (2.40) by vector $\mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u)))$, sum the result with (2.39) and then integrate over $\tau \in [0, 2\pi]$. Using Fubini's Theorem for the terms involving the

Hessian matrix of G which is symmetric yields

$$\begin{aligned}
 & \int_0^{2\pi} \left(\frac{\partial W}{\partial t}(t, \tau, q, u) + \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} W(t, \tau, q, u) \right) d\tau \\
 &= \int_0^{2\pi} \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right. \\
 &\quad \left. - \int_0^\tau \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) d\tau \\
 &+ \int_0^{2\pi} \left[\left[\frac{\tau}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \right. \\
 &\quad \left. \left. - \int_0^\tau \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \right. \\
 &\quad \left. - \frac{1}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
 &\quad \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
 &\quad \left. \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \right] \cdot \nabla_{q,u} G(t, q, u) d\tau \\
 &- \int_0^{2\pi} H_{q,u} G(t, q, u) \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
 &\quad \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \cdot \\
 &\quad \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
 &\quad \left. \left. - \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \right] d\tau.
 \end{aligned} \tag{2.41}$$

For both last factors on the right hand side of (2.41), we remark that the second one is the exact derivative of the first one with respect to τ . Since the Hessian matrix is symmetric and does not depend on τ , we can take it out of the integral. Thus, the remaining integrand is the exact derivative of a quadratic form with respect to τ , so that integrating over a period for variable τ gives 0. Moreover, we can perform an integration by parts in the first part of the second term on

the right hand side of (2.41). We obtain

$$\begin{aligned}
& \int_0^{2\pi} \left(\frac{\partial W}{\partial t}(t, \tau, q, u) + \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \cdot \nabla_{q,u} W(t, \tau, q, u) \right) d\tau \\
&= \int_0^{2\pi} \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right. \\
&\quad \left. - \int_0^\tau \mathcal{R}^\sigma \left(0, \frac{\partial E^0}{\partial t}(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u)) \right) d\sigma \right] \cdot \nabla_{q,u} G(t, q, u) d\tau \\
&- \int_0^{2\pi} \left[\left[\frac{1}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \right. \\
&\quad \left. \left. - \nabla_{q,u} \mathcal{R}^\tau(0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) \right] \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. - \frac{1}{2\pi} \int_0^{2\pi} \nabla_{q,u} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left[\frac{\tau}{2\pi} \int_0^{2\pi} \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right. \\
&\quad \left. \left. - \int_0^\tau \mathcal{R}^\sigma(0, E^0(t, \sigma, \mathcal{R}_r^{-\sigma}(q, u))) d\sigma \right] \right] \cdot \nabla_{q,u} G(t, q, u) d\tau. \tag{2.42}
\end{aligned}$$

In the second term on the right hand side of (2.42), we remark that first and last products cancel. Moreover, $\tau/2\pi$ is the only factor depending of τ in the third product, integration of which over $[0, 2\pi]$ gives π . Substituting the right-hand side of equation in problem (2.37) by its expression in (2.42) finally leads to problem (2.9)

3. NUMERICAL ALGORITHM

To build our numerical algorithm, we use a PIC method, that is dealing with particles rather than directly with the distribution function.

Theorem 2.1 tells that, provided that problem (2.6) is solved and space derivatives of G can be computed, then we can compute W from (2.8) and source term in problem (2.9). Then it remains to solve problem (2.9) to obtain G_1 . From G , G_1 and W , we finally get the Two-Scale first order approximation of f^ε thanks to (2.4), (2.5) and (2.7). Thus, the main steps of the algorithm are the following.

Mains steps of the algorithm

- (i) Compute a PIC approximation of G , that is push the macroparticles with respect to the advection operator associated to the partial differential equation in problem (2.6) as in [2].
- (ii) Compute a PIC approximation of the gradient $\nabla_{q,u} G$ from the PIC approximation of G .
- (iii) Compute a PIC approximation of W from equation (2.8).
- (iv) Compute a PIC approximation of the source term of the partial differential equation in problem (2.9), and then a PIC approximation of G_1 by solving problem (2.9) with the same advection operator than in problem (2.6).

To deal with the first step, we introduce the following PIC approximation of function G

$$G(t, q, u) = \sum_{k=1}^{N_p} \omega_k \delta(q = Q_k(t)) \delta(u = U_k(t)), \tag{3.1}$$

where δ is the Dirac mass, N_p is the number of macroparticles and $(Q_k(t), U_k(t))$ is a numerical approximation of the solution to the dynamical system associated with the advection operator in problem (2.6). More precisely, $(Q_k(t), U_k(t))$ is the position in phase space of macro-particle k which moves along a characteristic curve of the equation in (2.6). Hence our problem is reduced to compute the location of macro-particles at the next time step from their positions at the previous time step, as the solution of the following system

$$\frac{d}{dt} \begin{pmatrix} Q_k \\ U_k \end{pmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}^\tau (0, E^0(t, \tau, \mathcal{R}_r^{-\tau}(q, u))) d\tau. \quad (3.2)$$

The main difficulty in the solution of problem (3.2) lies in the fact that Q_k and U_k are coupled in each equation and that we need to approximate an integral term. A way to solve this problem is described in section 5.1 of [2].

Once the macro-particles are pushed along the characteristics, we can compute G at the next time step. Applying the *ad hoc* rotation, we then obtain a PIC approximation of F from (2.5). Finally, evaluating this approximation in $\tau = t/\varepsilon$ gives the zero order approximation of f^ε .

For the second step of the algorithm, we need to recover an approximation of the space derivatives of G thanks to the PIC approximation of G . Therefore, a regularization of approximation (3.1) is needed. To this purpose, we introduce a regular function γ^α with support included in the ball of radius α centered in 0, and such that the function $\gamma_k^\alpha : (q, u) \mapsto \gamma^\alpha(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2})$ has an integral over q and u with worth 1. Thus a regularization of (3.1) is given by

$$G(t, q, u) = \sum_{k=1}^{N_p} \omega_k \gamma^\alpha \left(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2} \right), \quad (3.3)$$

Thus we can compute numerical approximations of the space derivatives of G .

Now we can handle the third step of the algorithm. Indeed, in (2.8), we can compute an approximation of the right hand side which has the following shape

$$\sum_{k=1}^{N_p} \beta_k(t, \tau, q, u) (\gamma^\alpha)' \left(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2} \right). \quad (3.4)$$

In order to get W , we need to distribute the above approximation over the macroparticles. Let the pic approximation of W be given by

$$W(t, \tau, q, u) = \sum_{k=1}^{N_p} \tilde{\beta}_k(t, \tau) \delta(q = Q_k(t)) \delta(u = U_k(t)), \quad (3.5)$$

which can be regularized by

$$W(t, \tau, q, u) = \sum_{k=1}^{N_p} \tilde{\beta}_k(t, \tau) \gamma^\alpha \left(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2} \right). \quad (3.6)$$

Then we want that all the integrals of the sums in (3.4) and (3.6) over each subdomain of the phase space are almost equals. From the numerical point of view, we want to verify this proximity only for the support C_i of functions γ_i^α that is

$$\int_{C_i} \sum_{k=1}^{N_p} \tilde{\beta}_k(t, \tau) \gamma_k^\alpha(q, u) dq du = \int_{C_i} \sum_{k=1}^{N_p} \beta_k(t, \tau, q, u) \gamma_k^\alpha(q, u) dq du, \quad (3.7)$$

which writes again

$$\sum_{k=1}^{N_p} \tilde{\beta}_k(t, \tau) \int_{C_i} \gamma_k^\alpha(q, u) dq du = \int_{C_i} \sum_{k=1}^{N_p} \beta_k(t, \tau, q, u) \gamma_k^\alpha(q, u) dq du. \quad (3.8)$$

Therefore we can determine the coefficient $\tilde{\beta}_k$ of the PIC approximation of W up to the solution the linear system (3.8).

Finally, we consider the last step of the algorithm. We want to build a PIC approximation of G_1 from problem (2.9). We first note that in this problem, the advection operator is exactly the same than in problem (2.6). Therefore the macroparticles to approximate G_1 can be chosen as being localized exactly at the same phase space locations as the ones used to approximate G . Thus G_1 is approximated by

$$G_1(t, q, u) = \sum_{k=1}^{N_p} \omega_k^1(t) \delta(q - Q_k(t)) \delta(u - U_k(t)), \quad (3.9)$$

where the weights ω_k^1 now depend on time t , which allows to take into account the effect of the source term on the right-hand side of the equation in (2.9). Once more, we can regularize this approximation by

$$G_1(t, q, u) = \sum_{k=1}^{N_p} \omega_k^1(t) \gamma^\alpha \left(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2} \right). \quad (3.10)$$

Then we handle the right hand side of equation in problem (2.9) exactly as for W . Indeed, PIC approximation of this term has the following shape

$$\sum_{k=1}^{N_p} \eta_k(t, \tau, q, u) (\gamma^\alpha)' \left(\sqrt{(q - Q_k(t))^2 + (u - U_k(t))^2} \right). \quad (3.11)$$

Therefore in order to distribute this approximation on the macroparticles, we want to determine $\tilde{\eta}_k$ such that

$$\int_{C_i} \sum_{k=1}^{N_p} \tilde{\eta}_k(t, \tau) \gamma_k^\alpha(q, u) dq du = \int_{C_i} \sum_{k=1}^{N_p} \eta_k(t, \tau, q, u) \gamma_k^\alpha(q, u) dq du, \quad (3.12)$$

which writes again

$$\sum_{k=1}^{N_p} \tilde{\eta}_k(t, \tau) \int_{C_i} \gamma_k^\alpha(q, u) dq du = \int_{C_i} \sum_{k=1}^{N_p} \eta_k(t, \tau, q, u) \gamma_k^\alpha(q, u) dq du. \quad (3.13)$$

Therefore we can determine the coefficient $\tilde{\eta}_k$ up to the solution the linear system (3.13). Since G_1 is solution to problem (2.9), we finally have that

$$\frac{d\omega_k^1}{dt} = \tilde{\eta}_k. \quad (3.14)$$

REFERENCES

- [1] G. Allaire. Homogenization and Two-scale Convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.
- [2] E. Frénod, F. Salvarani, and E. Sonnendrücker. Long time simulation of a beam in a periodic focusing channel via a two-scale PIC-method. *Mathematical Models and Methods in Applied Sciences*, 19(2):175–197, 2009.

- [3] E. Frénod and E. Sonnendrücker. The Finite Larmor Radius Approximation. *SIAM J. Math. Anal.*, 32(6):1227–1247, 2001.
- [4] A. Mouton. Two-scale semi-lagrangian simulation of a charged particles beam in a periodic focusing channel. *Kinet. Relat. Models*, 2(2):251–274, 2009.
- [5] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20(3):608–623, 1989.
- [6] G. Nguetseng. Asymptotic analysis for a stiff variational problem arising in mechanics. *SIAM Journal on Mathematical Analysis*, 21(6):1394–1414, 1990.