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# Some infinite divisibility properties of the reciprocal of planar Brownian motion exit time from a cone

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## Abstract

With the help of the Gauss-Laplace transform for the exit time from a cone of planar Brownian motion, we obtain some infinite divisibility properties for the reciprocal of this exit time.

**Key words:** Bougerol's identity, infinite divisibility, Chebyshev polynomials, Lévy measure, Thorin measure, generalized Gamma convolution (GGC).

**MSC Classification (2010):** 60J65, 60E07.

## 1 Introduction

Let  $(Z_t = X_t + iY_t, t \geq 0)$  denote a standard planar Brownian motion<sup>§</sup>, starting from  $x_0 + i0, x_0 > 0$ , where  $(X_t, t \geq 0)$  and  $(Y_t, t \geq 0)$  are two independent linear Brownian motions, starting respectively from  $x_0$  and 0.

It is well known [ItMK65] that, since  $x_0 \neq 0$ ,  $(Z_t, t \geq 0)$  does not visit a.s. the point 0 but keeps winding around 0 infinitely often. Hence, the continuous winding process  $\theta_t = \text{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$  is well defined. Using a scaling argument, we may assume  $x_0 = 1$ , without loss of generality, since, with obvious notation:

$$\left( Z_t^{(x_0)}, t \geq 0 \right) \stackrel{(law)}{=} \left( x_0 Z_{(t/x_0^2)}^{(1)}, t \geq 0 \right). \quad (1)$$

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<sup>§</sup>When we write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

From now on, we shall take  $x_0 = 1$ .

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t} = \int_0^t \frac{ds}{|Z_s|^2}, \quad (2)$$

where  $(\beta_u + i\gamma_u, u \geq 0)$  is another planar Brownian motion starting from  $\log 1 + i0 = 0$  (for further study of the Bessel clock  $H$ , see [Yor80]).

We may rewrite (2) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}. \quad (3)$$

One now easily obtains that the two  $\sigma$ -fields  $\sigma\{|Z_t|, t \geq 0\}$  and  $\sigma\{\beta_u, u \geq 0\}$  are identical, whereas  $(\gamma_u, u \geq 0)$  is independent from  $(|Z_t|, t \geq 0)$ .

Bougerol's celebrated identity in law ([Bou83, ADY97] and [Yor01] (p. 200)), which says that:

$$\text{for fixed } t, \quad \sinh(\beta_t) \stackrel{(law)}{=} \delta_{A_t(\beta)} \quad (4)$$

where  $(\beta_u, u \geq 0)$  is 1-dimensional BM,  $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$  and  $(\delta_v, v \geq 0)$  is another BM, independent of  $(\beta_u, u \geq 0)$ , will also be used. We define the random times  $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$ , and  $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$ , ( $c > 0$ ). From the skew-product representation (3) of planar Brownian motion, we obtain [ReY99]:

$$A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} ds \exp(2\beta_s) = H_u^{-1} \Big|_{u=T_c^{|\gamma|}} = T_c^{|\theta|}. \quad (5)$$

Then, Bougerol's identity (4) for the random time  $T_c^{|\theta|}$  yields the following [Vak11, VaY11]:

**Proposition 1.1** *The distribution of  $T_c^{|\theta|}$  is characterized by:*

$$E \left[ \sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp \left( -\frac{x}{2T_c^{|\theta|}} \right) \right] = \frac{1}{\sqrt{1+x}} \varphi_m(x), \quad (6)$$

for every  $x \geq 0$ , with  $m = \frac{\pi}{2c}$ , and

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad \text{with } G_{\pm}(x) = \sqrt{1+x} \pm \sqrt{x}. \quad (7)$$

**Comment and Terminology:**

If  $S > 0$  a.s. is independent from a Brownian motion  $(\delta_u, u \geq 0)$ , we call the density of  $\delta_S$ , which is:

$$E \left[ \frac{1}{\sqrt{2\pi S}} \exp \left( -\frac{x^2}{2S} \right) \right] \quad (8)$$

the Gauss-Laplace transform of  $S$  (see e.g. [ChY03] ex.4.16, or [BPY01]). Thus, formula (6) expresses - up to simple changes - the Gauss-Laplace transform of  $T_c^{|\theta|}$ .

We also recall several notions which will be used throughout the following text:

a) A stochastic process  $\zeta = (\zeta_t, t \geq 0)$  is called a *Lévy process* if  $\zeta_0 = 0$  a.s., it has stationary and independent increments and it is almost surely right continuous with left limits. A Lévy process which is increasing is called a *subordinator*.

b) Following e.g. [ReY99], a probability measure  $\pi$  on  $\mathbb{R}$  (resp. a real-valued random variable with law  $\pi$ ) is said to be *infinitely divisible* if, for any  $n \geq 1$ , there is a probability measure  $\pi_n$  such that  $\pi = \pi_n^{*n}$  (resp. if  $\zeta_1, \dots, \zeta_n$  are  $n$  i.i.d. random variables,  $\zeta \stackrel{(law)}{=} \zeta_1 + \dots + \zeta_n$ ). For instance, Gaussian, Poisson and Cauchy variables are infinitely divisible.

It is well-known that (e.g. [Ber96]),  $\pi$  is infinitely divisible if and only if, its Fourier transform  $\hat{\pi}$  is equal to  $\exp(\psi)$ , with:

$$\psi(u) = ibu - \frac{\sigma^2 u^2}{2} + \int \left( e^{iux} - 1 - \frac{iux}{1+x^2} \right) \nu(dx),$$

where  $b \in \mathbb{R}$ ,  $\sigma^2 \geq 0$  and  $\nu$  is a Radon measure on  $\mathbb{R} \setminus \{0\}$  such that:

$$\int \frac{x^2}{1+x^2} \nu(dx) < \infty.$$

This expression of  $\hat{\pi}$  is known as the *Lévy-Khintchine formula* and the measure  $\nu$  as the *Lévy measure*.

c) Following [Bon92] (p.29) and [JRY08], a positive random variable  $\Gamma$  is a *generalized Gamma convolution* (GGC) if there exists a positive Radon measure  $\mu$  on  $]0, \infty[$  such that:

$$E[e^{-\lambda\Gamma}] = \exp\left(-\int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} \int_0^\infty e^{-xz} \mu(dz)\right) \quad (9)$$

$$= \exp\left(-\int_0^\infty \log\left(1 + \frac{\lambda}{z}\right) \mu(dz)\right), \quad (10)$$

with:

$$\int_{]0,1]} |\log x| \mu(dx) \quad \text{and} \quad \int_{]1,\infty[} \frac{\mu(dx)}{x} < \infty. \quad (11)$$

We remark that (10) follows immediately from (9) using the elementary Frullani formula (see e.g. [Leb72], p.6). The measure  $\mu$  is called *Thorin's measure* associated with  $\Gamma$ .

We return now to the case of planar Brownian motion and the exit times from a cone. Below, we state and prove the following:

**Proposition 1.2** *For every integer  $m$ , the function  $x \rightarrow \varphi_m(x)$ , is the Laplace transform of an infinitely divisible random variable  $K$ ; more specifically, the following decompositions hold:*

- for  $m = 2n + 1$ ,

$$K = \frac{\mathcal{N}^2}{2} + \sum_{k=1}^n a_k \mathbf{e}_k, \quad a_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right)}; \quad k = 1, 2, \dots, n, \quad (12)$$

- for  $m = 2n$ ,

$$K = \sum_{k=1}^n b_k \mathbf{e}_k, \quad b_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n}\right)}; \quad k = 1, 2, \dots, n, \quad (13)$$

where  $\mathcal{N}$  is a centered, reduced Gaussian variable and  $\mathbf{e}_k$ ,  $k \leq n$  are  $n$  independent exponential variables, with expectation 1.

Looking at formula (6), it is also natural to consider:

$$\tilde{\varphi}_m(x) \equiv \frac{1}{\sqrt{1+x}} \varphi_m(x). \quad (14)$$

We note that:

$$\tilde{K} \equiv \frac{\mathcal{N}^2}{2} + K, \quad (15)$$

admits the RHS of (6) as its Laplace transform. Hence,

- for  $m = 2n + 1$ ,

$$\tilde{K} \stackrel{(law)}{=} \mathbf{e}_0 + \sum_{k=1}^n a_k \mathbf{e}_k, \quad (16)$$

- for  $m = 2n$ ,

$$\tilde{K} \stackrel{(law)}{=} \frac{\mathcal{N}^2}{2} + \sum_{k=1}^n b_k \mathbf{e}_k, \quad (17)$$

with obvious notation.

In Section 2 we first illustrate Proposition 1.2 for  $m = 1$  and  $m = 2$ ; we may also verify equation (6) by using the laws of  $T_c^{|\theta|}$ , for  $c = \pi/2$  and  $c = \pi/4$ , which are well known [ReY99].

In Section 3, we prove Proposition 1.2, where the Chebyshev polynomials play an essential role, we calculate the Lévy measure in the Lévy-Khintchine representation of  $\varphi_m$  and we obtain the following asymptotic result:

**Proposition 1.3** *With  $c$  denoting a positive constant, the distribution of  $T_{c\varepsilon}^{|\theta|}$ , for every  $x \geq 0$ , follows the asymptotics:*

$$\left( E \left[ \sqrt{\frac{2(c\varepsilon)^2}{\pi T_{c\varepsilon}^{|\theta|}}} \exp\left(-\frac{x}{2T_{c\varepsilon}^{|\theta|}}\right) \right] \right)^{1/\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{(\sqrt{x} + \sqrt{1+x})^{\pi/2c}}, \quad (18)$$

which, from [JRY08], is the Laplace transform of a subordinator  $(\Gamma_t(\mathbb{G}_{1/2}), t \geq 0)$  with Thorin measure that of the arc sine law, taken at  $t = \pi/2c$ .

Finally, we state a conjecture concerning the case where  $m$  is not necessarily an integer.

## 2 Examples

**2.1**  $\boxed{m = 1 \Rightarrow c = \frac{\pi}{2}}$

Then:

$$\tilde{\varphi}_1(x) = \frac{1}{1+x}, \quad (19)$$

is the Laplace transform of an exponential variable  $\mathbf{e}_1$ .

Indeed, with  $(Z_t = X_t + iY_t = |Z_t| \exp(i\theta_t), t \geq 0)$  a planar BM starting from  $(1, 0)$ ,  $T_{\pi/2}^{|\theta|} = \inf\{t : X_t = 0\} = \inf\{t : X_t^0 = 1\}$ ,

with  $(X_t^0, t \geq 0)$  denoting another one-dimensional BM starting from 0. Formula (6) states that:

$$E \left[ \sqrt{\frac{2}{\pi T_{\pi/2}^{|\theta|}}} \exp\left(-\frac{x}{2T_{\pi/2}^{|\theta|}}\right) \right] = \frac{1}{1+x}. \quad (20)$$

However, we know that:  $T_{\pi/2}^{|\theta|} \stackrel{(law)}{=} \frac{1}{N^2}$ ,  $N \sim \mathcal{N}(0, 1)$ .

The LHS of the previous equality (20) gives:

$$E \left[ \sqrt{\frac{2}{\pi}} |N| \exp\left(-\frac{x}{2} N^2\right) \right] = \int_0^\infty dy y e^{-\frac{x+1}{2} y^2} = \frac{1}{1+x}, \quad (21)$$

thus, we have verified directly that (20) holds.

**2.2**  $\boxed{m = 2 \Rightarrow c = \frac{\pi}{4}}$

Similarly,

$$\tilde{\varphi}_2(x) = \frac{1}{\sqrt{1+x}} \frac{1}{1+2x}, \quad (22)$$

is the Laplace transform of the variable  $\frac{N^2}{2} + 2\mathbf{e}_1$ .

Again, this can be shown directly; indeed, with obvious notation:

$$\begin{aligned} T_{\pi/4}^{|\theta|} &= \inf\{t : X_t + Y_t = 0, \text{ or } X_t - Y_t = 0\} \\ &= \inf\left\{t : \frac{X_t^0 + Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ or } \frac{X_t^0 - Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}\right\} \\ &= T_{1/\sqrt{2}} \wedge \tilde{T}_{1/\sqrt{2}} \stackrel{(law)}{=} \frac{1}{2} (T \wedge \tilde{T}). \end{aligned}$$

Hence, formula (6) now writes, in this particular case:

$$E \left[ \sqrt{\frac{\pi}{4(T \wedge \tilde{T})}} \exp\left(-\frac{x}{T \wedge \tilde{T}}\right) \right] = \frac{1}{\sqrt{1+x}} \frac{1}{1+2x}. \quad (23)$$

This is easily proven, using:  $T \stackrel{(law)}{=} \frac{1}{N^2}$ ,  $\tilde{T} \stackrel{(law)}{=} \frac{1}{\tilde{N}^2}$ , which yields:

$$\begin{aligned} E \left[ \left( |N| \vee |\tilde{N}| \right) \exp \left( -x \left( N^2 \vee \tilde{N}^2 \right) \right) \right] &= 2E \left[ |N| \exp \left( -xN^2 \right) 1_{(|N| \geq |\tilde{N}|)} \right] \\ &= C \int_0^\infty du u e^{-xu^2} e^{-\frac{u^2}{2}} \int_0^u dy e^{-\frac{y^2}{2}}. \end{aligned}$$

Fubini's theorem now implies that (23) holds.

**Remark 2.1** *In a first draft, we continued looking at the cases:  $m = 3, 4, 5, 6, \dots$ , in a direct manner. But, these studies are now superseded by the general discussion in Section 3.*

## 2.3 A "small" generalization

As we just wrote in Remark 2.1, before finding the proof of Proposition 1.2 (see below, Subsection 3.1), we kept developing examples for larger values of  $m$ , and in particular, we encountered quantities of the form:

$$\frac{1}{\mathcal{P}_{u,v}(x)}, \text{ with } \mathcal{P}_{u,v}(x) = 1 + ux + vx^2. \quad (24)$$

These quantities turn out to be the Laplace transforms of variables of the form  $ae + be'$ , with  $a, b > 0$  constants and  $e, e'$  two independent exponential variables. In this Subsection, we characterize the polynomials  $\mathcal{P}_{u,v}(x)$  such that this is so.

**Lemma 2.2** a) *A necessary and sufficient condition for  $1/\mathcal{P}_{u,v}$  to be the Laplace transform of the law of  $ae + be'$ , is:*

$$u, v > 0 \quad \text{and} \quad \Delta \equiv u^2 - 4v \geq 0. \quad (25)$$

b) *Then, we obtain:*

$$a = \frac{u - \sqrt{\Delta}}{2} \quad ; \quad b = \frac{u + \sqrt{\Delta}}{2}. \quad (26)$$

### Proof of Lemma 2.2

i)  $1/\mathcal{P}_{u,v}$  is the Laplace transform of  $ae + be'$ , then:

$$\mathcal{P}_{u,v}(x) = (1 + ax)(1 + bx).$$

Both  $u = a + b$  and  $v = ab$  are positive.

Moreover,  $\mathcal{P}_{u,v}$  admits two real roots, thus  $\Delta \equiv u^2 - 4v \geq 0$ ; i.e.: (25) is satisfied.

ii) Conversely, if the two conditions (25) are satisfied, then the 2 roots of the polynomial are  $-1/a$  and  $-1/b$ . Hence,  $\mathcal{P}_{u,v}(x) = C(1 + ax)(1 + bx)$ , where  $C$  is a constant. However, from the definition of  $\mathcal{P}_{u,v}$  (24), we have:  $\mathcal{P}_{u,v}(0) = 1$ , hence  $C = 1$ . Thus,  $1/\mathcal{P}_{u,v}$  is the Laplace transform of  $ae + be'$ .

iii) To show b), we note that:

$$\left\{ -\frac{1}{a}, -\frac{1}{b} \right\} = \left\{ \frac{-u - \sqrt{\Delta}}{2v}, \frac{-u + \sqrt{\Delta}}{2v} \right\}.$$

as well as:  $(u - \sqrt{\Delta})(u + \sqrt{\Delta}) = 4v$ , which finishes the proof of the second part of the Lemma. ■

## 3 A discussion of Proposition 1.2 in terms of the Chebyshev polynomials

### 3.1 Proof of Proposition 1.2

a) Assuming, to begin with, the validity of our Proposition 1.2, for any integer  $m$ , the function  $\varphi_m$  should admit the following representation:

$$\varphi_m(x) = \frac{1}{D_m(x)}, \quad (27)$$

where

- for  $m = 2n + 1$ ,  $D_m(x) = \sqrt{1+x}P_n(x)$ , with  $P_n(x) = \prod_{k=1}^n (1 + a_k x)$ ,
- for  $m = 2n$ ,  $D_m(x) = Q_n(x)$ , with  $Q_n(x) = \prod_{k=1}^n (1 + b_k x)$ .

In particular,  $P_n$  and  $Q_n$  are polynomials of degree  $n$ , each of which has its  $n$  zeros, that is  $(-1/a_k; k = 1, 2, \dots, n)$ , resp.  $(-1/b_k; k = 1, 2, \dots, n)$ , on the negative axis  $\mathbb{R}_-$ .

It is not difficult, from the explicit expression of  $D_m(x) = \frac{1}{2}((G_+(x))^m + (G_-(x))^m)$ , to find the polynomials  $P_n$  and  $Q_n$ . They are given by the formulas:

$$\begin{cases} P_n(x) = \sum_{k=0}^n C_{2n+1}^{2k+1} (1+x)^k x^{n-k}, \\ Q_n(x) = \sum_{k=0}^n C_{2n}^{2k} (1+x)^k x^{n-k}. \end{cases} \quad (28)$$

In order to prove Proposition 1.2, we shall make use of Chebyshev's polynomials of the first kind (see e.g. [Riv90] ex.1.1.1 p.5 or [KVA02] ex.25, p.195):

$$\begin{aligned} T_m(y) &\equiv \frac{(y + \sqrt{y^2 - 1})^m + (y - \sqrt{y^2 - 1})^m}{2} \\ &\equiv \begin{cases} \cos(m \arg \cos(y)), & y \in [-1, 1] \\ \cosh(m \arg \cosh(y)), & y \geq 1 \\ (-1)^m \cosh(m \arg \cosh(-y)), & y \leq -1. \end{cases} \end{aligned} \quad (29)$$

b) We now start the proof of Proposition 1.2 in earnest. First, we remark that:

$$\varphi_m(x) = \frac{1}{T_m(\sqrt{1+x})}, \quad (30)$$



hence:

$$D_m(x) = T_m\left(\sqrt{1+x}\right), \quad (31)$$

with  $x \geq -1$ , thus we are interested only in the positive zeros of  $T_m$ , and we study separately the cases  $m$  odd and  $m$  even.

$$\boxed{m = 2n + 1}$$

$$D_{2n+1}(y) \equiv \sqrt{1+y}P_n(y) = T_{2n+1}\left(\sqrt{1+y}\right)$$

and the zeros of  $T_{2n+1}$  are:  $x_k = \cos\left(\frac{\pi 2k-1}{2 2n+1}\right)$ ,  $k = 1, 2, \dots, (2n+1)$ . However,  $x_k$  is positive if and only if  $k = 1, 2, \dots, n$ , thus:

$$y_k = x_k^2 - 1 = \cos^2\left(\frac{\pi 2k-1}{2 2n+1}\right) - 1 = -\sin^2\left(\frac{\pi 2k-1}{2 2n+1}\right); k = 1, 2, \dots, n.$$

Finally:

$$a_k = \frac{1}{\sin^2\left(\frac{\pi 2k-1}{2 2n+1}\right)}; k = 1, 2, \dots, n, \quad (32)$$

and

$$P_n(x) = \prod_{k=1}^n \left(1 + \frac{x}{\sin^2\left(\frac{\pi 2k-1}{2 2n+1}\right)}\right). \quad (33)$$

$\boxed{m = 2n}$  Similarly, we obtain:

$$b_k = \frac{1}{\sin^2\left(\frac{\pi 2k-1}{2 2n}\right)}; k = 1, 2, \dots, n, \quad (34)$$

and

$$Q_n(x) = \prod_{k=1}^n \left(1 + \frac{x}{\sin^2\left(\frac{\pi 2k-1}{2 2n}\right)}\right). \quad (35)$$

■

### 3.2 Search for the Lévy measure of $\varphi_m$ and proof of Proposition 1.3

We have proved that  $\varphi_m$  is infinitely divisible. In this Subsection, we shall calculate its Lévy measure. For this purpose, we shall make use of the following (recall that  $\mathbf{e}_k$ ,  $k \leq n$  are  $n$  independent exponential variables, with expectation 1):

**Lemma 3.1** *With  $(c_k, k = 1, 2, \dots, n)$  denoting a sequence of positive constants,  $\prod_{k=1}^n \frac{1}{(1+c_k x)}$  is the Laplace transform of  $\sum_{k=1}^n c_k \mathbf{e}_k$ , which is an infinitely divisible random variable with Lévy measure:*

$$\frac{dz}{z} \sum_{k=1}^n e^{-z/c_k}.$$

### Proof of Lemma 3.1

Using the elementary Frullani formula (see e.g. [Leb72], p.6), we have:

$$\begin{aligned} \prod_{k=1}^n \frac{1}{(1+c_k x)} &= \exp \left\{ - \sum_{k=1}^n \log(1+c_k x) \right\} = \exp \left\{ - \sum_{k=1}^n \int_0^\infty \frac{dy}{y} e^{-y} (1-e^{-c_k x y}) \right\} \\ &\stackrel{z=c_k y}{=} \exp \left\{ - \sum_{k=1}^n \int_0^\infty \frac{dz}{z} e^{-z/c_k} (1-e^{-xz}) \right\}, \end{aligned}$$

which finishes the proof. ■

We return now to the proof of Proposition 1.3 and we study separately the cases  $m$  odd and  $m$  even and we apply Lemma 3.1 with  $c_k = a_k$  and  $c_k = b_k$  respectively.

$m = 2n + 1$  Lemma 3.1 yields that,  $\prod_{k=1}^n \frac{1}{(1+a_k x)}$  is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_+(dz) = \frac{dz}{z} \sum_{k=1}^n e^{-z/a_k}. \quad (36)$$

Moreover:

$$\frac{1}{\left(\prod_{k=1}^n (1+a_k x)\right)^{1/n}} = \exp \left\{ - \int_0^\infty \frac{dz}{z} (1-e^{-xz}) \frac{1}{n} \sum_{k=1}^n \exp \left\{ -\frac{z}{a_k} \right\} \right\}, \quad (37)$$

and  $\frac{1}{\left(\prod_{k=1}^n (1+a_k x)\right)^{1/n}}$ , for  $n \rightarrow \infty$ , converges to the Laplace transform of a variable which is a generalized Gamma convolution (GGC) with Thorin measure density:

$$\begin{aligned} \mu_+(z) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \exp \left\{ -\frac{z}{a_k} \right\} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \exp \left\{ -z \sin^2 \left( \frac{\pi 2k-1}{2 2n+1} \right) \right\} \\ &= \int_0^1 du \exp \left\{ -z \sin^2 \left( \frac{\pi}{2} u \right) \right\} \stackrel{v=\frac{\pi}{2}u}{=} \frac{2}{\pi} \int_0^{\pi/2} dv \exp \left\{ -z \sin^2(v) \right\} \\ &\stackrel{h=\sin^2 v}{=} \frac{1}{\pi} \int_0^1 \frac{dh}{\sqrt{h(1-h)}} e^{-hz}, \end{aligned} \quad (38)$$

which, following the notation in [JRY08], is the Laplace transform of the variable  $\mathbb{G}_{1/2}$  which is arc sine distributed on  $[0, 1]$ .

$m = 2n$  Lemma 3.1 yields that,  $\prod_{k=1}^n \frac{1}{(1+b_k x)}$  is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_-(dz) = \frac{dz}{z} \sum_{k=1}^n e^{-z/b_k} = \frac{dz}{z} \sum_{k=1}^n \exp \left\{ -z \sin^2 \left( \frac{\pi 2k-1}{2 2n} \right) \right\}. \quad (39)$$

Moreover  $\frac{1}{\left(\prod_{k=1}^n (1+b_k x)\right)^{1/n}}$ , for  $n \rightarrow \infty$ , converges to the Laplace transform of a GGC with Thorin measure density:

$$\mu_-(z) = \mu_+(z). \quad (40)$$

We now express the above results in terms of the Laplace transforms  $\varphi_m$  and  $\tilde{\varphi}_m$ . Using the following result from [JRY08], p.390, formula (193):

$$\begin{aligned} E [\exp (-x\Gamma_t (\mathbb{G}_{1/2}))] &= \exp \left\{ -t \int_0^\infty \frac{dz}{z} (1 - e^{-xz}) E [\exp (-z\mathbb{G}_{1/2})] \right\} \\ &= \frac{1}{(\sqrt{1+x} + \sqrt{x})^{2t}} \end{aligned} \quad (41)$$

with  $2t = m = \frac{\pi}{2c\varepsilon}$ , with  $c$  a positive constant, together with (38) and (40), we obtain (18). ■

**Remark:** The natural question that arises now is whether the results of Proposition 1.2 could be generalized for every  $m > 0$  (not necessarily an integer), in other words whether  $\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}$  is the Laplace transform of a generalized Gamma convolution (GGC, see [Bon92] or [JRY08]), that is:

$$\varphi_m(x) = E [e^{-x\Gamma_m}], \quad (42)$$

with

$$\Gamma_m \stackrel{(law)}{=} \int_0^\infty f_m(s) d\gamma_s, \quad (43)$$

where  $f_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $\gamma_s$  is a gamma process.

This conjecture will be investigated in a sequel of this article in [Vak12].

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