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Integrability properties and Limit Theorems for the exit time from a cone of planar Brownian motion

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Abstract
We obtain some integrability properties and some limit Theorems for the exit time from a cone of a planar Brownian motion, and we check that our computations are correct via Bougerol’s identity.

Key words: Bougerol’s identity, planar Brownian motion, skew-product representation, exit time from a cone.

MSC Classification (2010): 60J65, 60F05.

1 Introduction
We consider a standard planar Brownian motion§ (Zt = Xt + iYt, t ≥ 0), starting from x0 + i0, x0 > 0, where (Xt, t ≥ 0) and (Yt, t ≥ 0) are two independent linear Brownian motions, starting respectively from x0 and 0.
As is well known [ItMK65], since x0 ≠ 0, (Zt, t ≥ 0) does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process θt = Im(ʃ0t dtZs), t ≥ 0 is well defined. A scaling argument shows that we may assume x0 = 1, without loss of generality, since, with obvious notation:

\[ \left( Z_{t}^{(x_{0})}, t \geq 0 \right)^{\text{(law)}} = \left( x_{0}Z_{t/(x_{0})}^{(1)}, t \geq 0 \right). \]

§When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

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Thus, from now on, we shall take $x_0 = 1$.
Furthermore, there is the skew product representation:

$$\log |Z_t| + i \theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i \gamma_u) \bigg|_{u=H_t-f_0^t \frac{ds}{|Z_s|^2}},$$  \tag{2}$$

where $(\beta_u + i \gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i \theta = 0$.
Thus, the Bessel clock $H$ plays a key role in many aspects of the study of the winding number process $(\theta_t, t \geq 0)$ (see e.g. [Yor80]).

Rewriting (2) as:

$$\log |Z_t| = \beta H_t; \quad \theta_t = \gamma H_t,$$ \tag{3}$$

we easily obtain that the two $\sigma$-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

We shall also use Bougerol’s celebrated identity in law [Bou83, ADY97] and [Yor01] (p. 200), which may be written as:

$$\text{for fixed } t, \quad \sinh(\beta t) \overset{(\text{law})}{=} \hat{\beta} A_\beta t \quad \Rightarrow \tag{4}$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2 \beta s)$ and $(\hat{\beta}_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$.

For the random times $T_{\theta|c} \equiv \inf\{t : |\theta_t| = c\}$, and $T_{\gamma|c} \equiv \inf\{t : |\gamma_t| = c\}$, $(c > 0)$ by using the skew-product representation (3) of planar Brownian motion [ReY99], we obtain:

$$T_{\theta|c} = A_{T_{\gamma|c}}(\beta) \equiv \int_0^{T_{\gamma|c}} ds \exp(2 \beta s) = H_u^{-1} \bigg|_{u=T_{\gamma|c}}.$$ \tag{5}$$

Moreover, it has been recently shown that, Bougerol’s identity applied with the random time $T_{\theta|c}$ instead of $t$ in (4) yields the following [Vak11]:

**Proposition 1.1** The distribution of $T_{\theta|c}$ is characterized by its Gauss-Laplace transform:

$$E \left[ \frac{2c^2}{\pi T_{\theta|c}} \exp \left(-\frac{x}{2T_{\theta|c}}\right) \right] = \frac{1}{\sqrt{1+x}} \varphi_m(x),$$ \tag{6}$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and:

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad G_{\pm}(x) = \sqrt{1+x} \pm \sqrt{x}.$$ \tag{7}$$

The remainder of this article is organized as follows: in Section 2 we study some integrability properties for the exit times from a cone; more precisely we obtain some new results concerning the negative moments of $T_{\theta|c}$ and of $T_{\theta} \equiv \inf\{t : \theta_t = c\}$. In Section 3 we state and prove some limit Theorems for these random times for $c \to 0$ and for $c \to \infty$ followed by several generalizations (for extensions of these works to more general planar processes, see e.g. [DoV12]). We use these results in order to obtain (see Remark 3.4) a
new simple non-computational proof of Spitzer’s celebrated asymptotic Theorem [Spi58], which states that:

\[ \frac{2}{\log t} \theta_t \xrightarrow{\text{law}} C_1, \]

with \( C_1 \) denoting a standard Cauchy variable (for other proofs, see also e.g. [Wil74, Dur82, MeY82, BeW94, Yor97, Vak11]). Finally, in Section 4 we use the Gauss-Laplace transform (6) which is equivalent to Bougerol’s identity (4) in order to check our results.

2 Integrability Properties

Concerning the moments of \( T_{c|\theta} \), we have the following (a more extended discussion is found in e.g. [MaY05]):

**Theorem 2.1** For every \( c > 0 \), \( T_{c|\theta} \) enjoys the following integrability properties:

(i) for \( p > 0 \), \( E \left[ \left( T_{c|\theta} \right)^p \right] < \infty \), if and only if \( p < \frac{\pi}{4c} \).

(ii) for any \( p < 0 \), \( E \left[ \left( T_{c|\theta} \right)^p \right] < \infty \).

**Corollary 2.2** For \( 0 < c < d \), the random times \( T_{\theta,d,c} \equiv \inf \{ t : \theta_t \notin (-d,c) \} \), \( T_{c|\theta} \) and \( T_{\theta,c} \) satisfy the inequality:

\[ T_{\theta,c} \geq T_{\theta,d,c} \geq T_{c|\theta}. \]

Thus, their negative moments satisfy:

\[ \text{for } p > 0, \ E \left[ \frac{1}{(T_{c|\theta})^p} \right] \leq E \left[ \frac{1}{(T_{\theta,d,c})^p} \right] \leq E \left[ \frac{1}{(T_{c|\theta})^p} \right] < \infty. \]

**Proofs of Theorem 2.1 and of Corollary 2.2**

(i) The original proof is given by Spitzer [Spi58], followed later by many authors [Wil74, Bur77, MeY82, Dur82, Yor85]. See also [ReY99] Ex. 2.21/page 196.

(ii) In order to obtain this result, we might use the representation \( T_{c|\theta} = A_{\theta|c} \) together with a recurrence formula for the negative moments of \( A_t \) [Duf00], Theorem 4.2, p. 417 (in fact, Dufresne also considers \( A_\mu = \int_0^t ds \exp(2\beta_s + 2\mu s) \), but we only need to take \( \mu = 0 \) for our purpose, and we note \( A_t \equiv A_t^{(0)} \) [Vakth11]. However, we can also obtain this result by simply remarking that the RHS of the Gauss-Laplace transform (6) in Proposition 1.1 is an infinitely differentiable function in 0 (see also [VaY11]), thus:

\[ E \left[ \frac{1}{(T_{c|\theta})^p} \right] < \infty, \ \text{for every } p > 0. \]

Now, Corollary 2.2 follows immediately from Theorem 2.1 (ii).
3 Limit Theorems for $T^{|\theta|}_c$

3.1 Limit Theorems for $T^{|\theta|}_c$, as $c \to 0$ and $c \to \infty$

The skew-product representation of planar Brownian motion allows to prove the three following asymptotic results for $T^{|\theta|}_c$.

**Proposition 3.1** a) For $c \to 0$, we have:

$$\frac{1}{c^2} T^{|\theta|}_c \xrightarrow[c \to 0]{(law)} T_1^{|\gamma|}. \quad (12)$$

b) For $c \to \infty$, we have:

$$\frac{1}{c} \log \left( T^{|\theta|}_c \right) \xrightarrow[c \to \infty]{(law)} 2|\beta| T_1^{|\gamma|}. \quad (13)$$

c) For $\varepsilon \to 0$, we have:

$$\frac{1}{\varepsilon^2} \left( T^{|\theta|}_{c+\varepsilon} - T^{|\theta|}_c \right) \xrightarrow[\varepsilon \to 0]{(law)} \exp \left( 2\beta T_1^{|\gamma|} \right) T_1^{'\gamma}, \quad (14)$$

where $\gamma'$ stands for a real Brownian motion, independent from $\gamma$, and $T_1^{'\gamma} = \inf \{ t : \gamma'_t = 1 \}$

**Proof of Proposition 3.1:**

We rely upon (5) for the three proofs. By using the scaling property of BM, we obtain:

$$T^{|\theta|}_c = A_{T_1^{|\gamma|}}(\beta) \xrightarrow[c \to \infty]{(law)} A_u(\beta) \bigg|_{u=c^2 T_1^{|\gamma|}}$$

thus:

$$\frac{1}{c^2} T^{|\theta|}_c \xrightarrow[c \to \infty]{(law)} \int_0^{T_1^{|\gamma|}} dv \exp \left( 2c\beta_v \right). \quad (15)$$

a) For $c \to 0$, the RHS of (15) converges to $T_1^{|\gamma|}$, thus we obtain part a) of the Proposition.

b) For $c \to \infty$, taking logarithms on both sides of (15) and dividing by $c$, on the LHS we obtain $\frac{1}{c} \log \left( T^{|\theta|}_c \right) - \frac{2}{c} \log c$ and on the RHS:

$$\frac{1}{c} \log \left( \int_0^{T_1^{|\gamma|}} dv \exp \left( 2c\beta_v \right) \right) = \log \left( \int_0^{T_1^{|\gamma|}} dv \exp \left( 2c\beta_v \right) \right)^{1/c} \quad ,$$

which, from the classical Laplace argument: $\|f\|_p \overset{p \to \infty}{\longrightarrow} \|f\|_\infty$, converges for $c \to \infty$, towards:

$$2 \sup_{v \leq T_1^{|\gamma|}} (\beta_v) \xrightarrow{(law)} 2|\beta| T_1^{|\gamma|}.$$
This proves part b) of the Proposition.

c) $T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|} = \int_{T_c^{|\gamma|}}^{T_{c+\varepsilon}^{|\gamma|}} du \ \exp (2\beta_u) = \int_0^{T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}} dv \ \exp (2\beta_{T_c^{|\gamma|}}) \ \exp \left( 2 \left( \beta_v - \beta_{T_c^{|\gamma|}} \right) \right) = \exp \left( 2\beta_{T_c^{|\gamma|}} \right) \int_0^{T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}} dv \ \exp (2B_v), \quad (16)$

where $\left( B_s \equiv \beta_s + T_c^{|\gamma|} - \beta_{T_c^{|\gamma|}}, s \geq 0 \right)$ is a BM independent of $T_c^{|\gamma|}$.

We study now $\tilde{T}_{c+\varepsilon}^{|\gamma|} = T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|}$, the first hitting time of the level $c+\varepsilon$ from $|\gamma|$, starting from $c$. Thus, we define: $\rho_u \equiv |\gamma_u|$, starting also from $c$. Thus, $\rho_u = c + \delta_u + L_u$, where $(\delta_s, s \geq 0)$ is a BM and $(L_s, s \geq 0)$ is the local time of $\rho$ at $0$. Thus:

$$\tilde{T}_{c,c+\varepsilon}^{|\gamma|} = \inf \{ u \geq 0 : \rho_u = c + \varepsilon \} \equiv \inf \{ u \geq 0 : \delta_u + L_u = \varepsilon \} \equiv \varepsilon^2 \inf \left\{ v \geq 0 : \frac{1}{\varepsilon^2} \delta_{\varepsilon v} + \frac{1}{\varepsilon} L_{\varepsilon v^2} = 1 \right\}. \quad (17)$$

From Skorokhod’s Lemma [ReY99]:

$$L_u = \sup_{y \leq u} \left( (c - \delta_y) \vee 0 \right)$$

we deduce:

$$\frac{1}{\varepsilon} L_{\varepsilon v^2} = \sup_{y \leq \varepsilon v^2} \left( (c - \delta_y) \vee 0 \right) \overset{\sigma \leq \varepsilon}{=} \sup_{\sigma \leq \varepsilon} \left( (c - \varepsilon^2 \delta_{\varepsilon^2}) \vee 0 \right) = 0. \quad (18)$$

Hence, with $\gamma'$ denoting a new BM independent from $\gamma$, (16) writes:

$$T_{c+\varepsilon}^{|\gamma|} - T_c^{|\gamma|} = \exp \left( 2\beta_{T_c^{|\gamma|}} \right) \int_0^{\varepsilon^2 T_{1'}} dv \ \exp (2B_v). \quad (19)$$

Thus, dividing both sides of (19) by $\varepsilon^2$ and making $\varepsilon \to 0$, we obtain part c) of the Proposition. ■

**Remark 3.2** The asymptotic result c) in Proposition 3.1 may also be obtained in a straightforward manner from (16) by analytic computations. Indeed, using the Laplace transform of the first hitting time of a fixed level by the absolute value of a linear Brownian motion $E \left[ e^{-\frac{\lambda^2}{2} T_{c}^{|\gamma|}} \right] = \frac{1}{\cosh(\lambda c)}$ (see e.g. Proposition 3.7, p 71 in Revuz and Yor [ReY99]), we have that for $0 < c < b$, and $\lambda \geq 0$:

$$E \left[ e^{-\frac{\lambda^2}{2} (T_{b}^{|\gamma|} - T_{c}^{|\gamma|})} \right] = \frac{\cosh(\lambda c)}{\cosh(\lambda b)} \quad (20)$$

Using now $b = c + \varepsilon$, for every $\varepsilon > 0$, the latter equals:

$$\frac{\cosh(\lambda c)}{\cosh \left( \frac{\lambda}{2} (c + \varepsilon) \right)} \xrightarrow{\varepsilon \to 0} e^{-\lambda}.$$

The result follows now by remarking that $e^{-\lambda}$ is the Laplace transform (for the argument $\lambda^2/2$) of the first hitting time of 1 by a linear Brownian motion $\gamma'$, independent from $\gamma$. 5
3.2 Generalizations

Obviously we can obtain several variants of Proposition 3.1, by studying $T_{\theta}^{\theta - \beta c, \alpha c}$, $0 < a, b \leq \infty$, for $c \to 0$ or $c \to \infty$, and $a, b$ fixed. We define $T^{-\gamma}_{-d, c} \equiv \inf\{t : \gamma t \notin (-d, c)\}$ and we have:

- $\frac{1}{c^2} T_{\theta}^{-\beta c, \alpha c} \xrightarrow{\text{law}} T_{-\beta, a}^{-\gamma}$

- $\frac{1}{c} \log (T_{\theta}^{-\beta c, \alpha c}) \xrightarrow{\text{law}} 2|\beta|T_{-\beta, a}^{-\gamma}.$

In particular, we can take $b = \infty$, hence:

**Corollary 3.3**

a) For $c \to 0$, we have:

$$\frac{1}{c^2} T_{\theta}^{-\beta c, \alpha c} \xrightarrow{\text{law}} T_{-\beta, a}^{-\gamma}. \quad (21)$$

b) For $c \to \infty$, we have:

$$\frac{1}{c} \log (T_{\theta}^{-\beta c, \alpha c}) \xrightarrow{\text{law}} 2|\beta|T_{-\beta, a}^{-\gamma} = 2|C_a|, \quad (22)$$

where $(C_a, a \geq 0)$ is a standard Cauchy process.

**Remark 3.4** *(Yet another proof of Spitzer’s Theorem)*

Taking $a = 1$, from Corollary 3.3(b), we can obtain yet another proof of Spitzer’s celebrated asymptotic Theorem stated in (8). Indeed, (22) can be equivalently stated as:

$$P \left( \log T_{\theta}^{-\beta c, \alpha c} \xrightarrow{\text{law}} T_{-\beta, a}^{-\gamma} \right) \xrightarrow{\text{law}} P \left( 2|C_1| < x \right). \quad (23)$$

Now, the LHS of (23) equals:

$$P \left( \log T_{\theta}^{-\beta c, \alpha c} \right) \xrightarrow{\text{law}} P \left( \sup_{u \leq \exp(cx)} \theta_u > c \right)$$

$$= P \left( |\theta_{\exp(cx)}| > c \right) = P \left( \left| \theta_t \right| > \frac{\log t}{x} \right), \quad (24)$$

with $t = \exp(cx)$. Thus, because $|C_1| \xrightarrow{\text{law}} |C_1|^{-1}$, (23) now writes:

$$P \left( \left| \theta_t \right| > \frac{\log t}{x} \right) \xrightarrow{\text{law}} P \left( |C_1| > \frac{2}{x} \right), \quad (25)$$

which yields precisely Spitzer’s Theorem (8).
3.3 Speed of convergence

We can easily improve upon Proposition 3.1 by studying the speed of convergence of the distribution of $\frac{1}{c^2}T_{c}^{(\theta)}$ towards that of $T_1^{[\gamma]}$, i.e.:

**Proposition 3.5** For any function $\varphi \in C^2$, with compact support,

$$\frac{1}{c^2} \left( E \left[ \varphi \left( \frac{1}{c^2} T_{c}^{(\theta)} \right) \right] - E \left[ \varphi \left( T_1^{[\gamma]} \right) \right] \right) \rightarrow_{c \to 0} E \left[ \varphi' \left( T_1^{[\gamma]} \right) \left( T_1^{[\gamma]} \right)^2 + \frac{2}{3} \varphi'' \left( T_1^{[\gamma]} \right) \left( T_1^{[\gamma]} \right)^3 \right].$$ (26)

**Proof of Proposition 3.5:**

We develop $\exp (2c\beta_v)$, up to the second order term, i.e.:

$$e^{2c\beta_v} = 1 + 2c\beta_v + 2c^2\beta_v^2 + \ldots .$$

More precisely, we develop up to the second order term, and we obtain:

$$E \left[ \varphi \left( \frac{1}{c^2} T_{c}^{(\theta)} \right) \right] = E \left[ \varphi \left( \int_0^{T_1^{[\gamma]}} dv \exp (2c\beta_v) \right) \right] = E \left[ \varphi \left( T_1^{[\gamma]} \right) + \varphi' \left( T_1^{[\gamma]} \right) \int_0^{T_1^{[\gamma]}} (2c\beta_v + 2c^2\beta_v^2) dv \right] + \frac{1}{2} E \left[ \varphi'' \left( T_1^{[\gamma]} \right) \left( \int_0^{T_1^{[\gamma]}} \beta_v dv \right)^2 \right] + c^2 o(c).$$

We then remark that $E \left[ \int_0^t \beta_v dv \right] = 0$, $E \left[ \int_0^t \beta_v^2 dv \right] = t^2/2$ and $E \left[ \left( \int_0^t \beta_v dv \right)^2 \right] = t^3/3$, thus we obtain (26).

4 Checks via Bougerol’s identity

So far, we have not made use of Bougerol’s identity (4), which helps us to characterize the distribution of $T_{c}^{[\theta]}$ [Vak11]. In this Subsection, we verify that writing the Gauss-Laplace transform in (6) as:

$$E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 + x^2 T_{c}^{[\theta]}}} \exp \left( -\frac{xc^2}{2T_{c}^{[\theta]}} \right) \right] = \frac{1}{\sqrt{1 + x^2c^2}} \varphi_m(xc^2),$$ (27)

with $m = \pi/(2c)$, we find asymptotically for $c \to 0$ the Gauss-Laplace transform of $T_1^{[\gamma]}$. Indeed, from (27), for $c \to 0$, we obtain:

$$E \left[ \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{[\gamma]}}} \exp \left( -\frac{x}{2T_1^{[\gamma]}} \right) \right] = \lim_{c \to 0} \frac{2}{\sqrt{1 + xc^2 + \sqrt{xc^2}} \pi^{2c} \left( \sqrt{1 + xc^2} - \sqrt{xc^2} \right)^{2/2c}}.$$ (28)
Let us now study:

\[
\left(\sqrt{1+xc^2} + \sqrt{xc^2}\right)^{\pi/2c} = \exp\left(\frac{\pi}{2c} \log \left[1 + \left(\sqrt{1+xc^2} - 1\right) + \sqrt{xc^2}\right]\right) \\
\sim \exp\left(\frac{\pi}{2c} \left[c\sqrt{x} + \frac{xc^2}{2}\right]\right) \xrightarrow{c \to 0} \exp\left(\frac{\pi \sqrt{x}}{2}\right).
\]

A similar calculation finally gives:

\[
E\left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{\gamma}}} \exp\left(-\frac{x}{2T_1^{\gamma}}\right)\right] = \frac{1}{\cosh\left(\frac{\pi}{2}\sqrt{x}\right)}, \quad (29)
\]

a result which is in agreement with the law of \(\beta_{T_1^{\gamma}}\), whose density is:

\[
E\left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{\gamma}}} \exp\left(-\frac{y^2}{2T_1^{\gamma}}\right)\right] = \frac{1}{2 \cosh\left(\frac{\pi y}{2}\right)}. \quad (30)
\]

Indeed, the law of \(\beta_{T_1^{\gamma}}\) may be obtained from its characteristic function which is given by [ReY99], page 73:

\[
E\left[\exp(i\lambda \beta_{T_1^{\gamma}})\right] = \frac{1}{\cosh(\lambda c)}.
\]

It is well known that [Lev80, BiY87]:

\[
E\left[\exp(i\lambda \beta_{T_1^{\gamma}})\right] = \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi \lambda \frac{2}{\pi})} = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{2 \pi \cosh\left(\frac{\pi x}{2}\right)} dx = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{1}{2 \cosh\left(\frac{\pi x}{2c}\right)} dx . \quad (31)
\]

So, the density \(h_{-c,c}\) of \(\beta_{T_1^{\gamma}}\) is:

\[
h_{-c,c}(y) = \left(\frac{1}{2c}\right) \frac{1}{\cosh\left(\frac{\pi y}{2c}\right)} = \left(\frac{1}{c}\right) e^{\frac{\pi y}{2c}} + e^{-\frac{\pi y}{2c}},
\]

and for \(c = 1\), we obtain (30).

We recall from Remark 3.2 that (see also [PiY03], where further results concerning the infinitely divisible distributions generated by some Lévy processes associated with the hyperbolic functions cosh, sinh and tanh can also be found):

\[
E\left[\exp\left(-\frac{\lambda^2}{2} T_1^{\gamma}\right)\right] = \frac{1}{\cosh(\lambda c)} , \quad (32)
\]

thus, for \(c = 1\) and \(\lambda = \frac{\pi}{2} \sqrt{x}\), (29) now writes:

\[
E\left[\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T_1^{\gamma}}} \exp\left(-\frac{x}{2T_1^{\gamma}}\right)\right] = E\left[\exp\left(-\frac{x \pi^2}{8} T_1^{\gamma}\right)\right] , \quad (33)
\]
a result which gives a probabilistic proof of the reciprocal relation in [BPY01] (using the notation of this article, Table 1, p.442):

$$f_{C_1}(x) = \left(\frac{2}{\pi x}\right)^{3/2} f_{C_1} \left(\frac{4}{\pi^2 x}\right).$$

References


