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Cohomology of Deligne-Lusztig varieties for groups of type A

Olivier Dudas^{*†}

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Abstract

We study the cohomology of parabolic Deligne-Lusztig varieties associated to unipotent blocks of $\mathrm{GL}_n(q)$. We show that the geometric version of Broué's conjecture over $\overline{\mathbb{Q}}_\ell$, together with Craven's formula, holds for any unipotent block whenever it holds for the principal Φ_1 -block.

Introduction

Let \mathbf{G} be a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$ with an \mathbb{F}_q -structure associated to a Frobenius endomorphism F . Let ℓ be a prime number different from p and b be a unipotent ℓ -block of $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$. When ℓ is large, the defect group of b is abelian, and the geometric version of Broué's conjecture predicts that the cohomology of some Deligne-Lusztig variety should induce a derived equivalence between b and its Brauer correspondent [2].

When the centraliser of the defect group of b is a torus, then in [4] Broué and Michel identified which specific class of Deligne-Lusztig varieties should be considered. They correspond to good d -regular elements or equivalently to d -roots of $\pi = \mathbf{w}_0^2$ in the Braid monoid. In a recent work [9], Digne and Michel introduced the notion of *d-periodic element* to generalise this to the parabolic setting. If b is a unipotent Φ_d -block, then it is to be expected that there exists a d -periodic element (\mathbf{I}, \mathbf{w}) such that the corresponding parabolic Deligne-Lusztig variety $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}F)$ is a good candidate for inducing the derived equivalence predicted by Broué's conjecture. Furthermore, Chuang and Rouquier conjectured in [6] that this equivalence is perverse, with a perversity function that has recently been conjectured by Craven in [7]. Surprisingly, it can be expressed by a function C_d depending only on the generic degrees of the corresponding characters.

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If we restrict our attention to the characteristic zero then we obtain a conjectural explicit description of the unipotent part of the cohomology of $\tilde{X}(\mathbf{I}, \mathbf{w}F)$. The fundamental property that we derive from Broué's conjecture is that the cohomology groups of $X(\mathbf{I}, \mathbf{w}F)$ are mutually disjoint. More precisely, it can be formulated as follows:

Conjecture 1. *Any d -cuspidal pair is conjugate to a pair (\mathbf{I}, \mathbf{w}) where (\mathbf{I}, \mathbf{w}) is a d -periodic element. Moreover, if \mathcal{F}_χ is the corresponding \mathbb{Q}_ℓ -local system on $X(\mathbf{I}, \mathbf{w}F)$, then (\mathbf{I}, \mathbf{w}) can be chosen such that*

- (i) *The $\overline{\mathbb{Q}_\ell} \mathbf{G}^F$ -modules $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$ are mutually disjoint.*
- (ii) *If ρ is a irreducible unipotent constituent of $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$ then ρ lies in the Φ_d -block associated to χ and $i = C_d(\deg \rho / \deg \chi)$.*

In addition, the endomorphism algebra $\text{End}_{\mathbf{G}^F}(H^\bullet(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi))$ should be endowed with a natural structure of d -cyclotomic Hecke algebra. Let us note the following important consequence of this property: the eigenvalue of any sufficiently divisible power F^m of F on ρ should be $q^{m(\alpha_\rho + A_\rho - \alpha_\chi - A_\chi)/d}$.

The choice of a specific d -periodic element in this conjecture is not very relevant: it is conjectured that any other d -periodic element $(\mathbf{I}, \mathbf{w}')$ can be obtained from (\mathbf{I}, \mathbf{w}) by cyclic shifts, so that the cohomology of the corresponding varieties are isomorphic. This has already been proven when $\mathbf{I} = \emptyset$ and F acts trivially on W (see [9, Remark 7.4]).

When $d = 1$, the unipotent blocks correspond to the usual Harish-Chandra series. In particular, when (\mathbf{G}, F) has type A , there is a unique unipotent block and it contains all the unipotent characters. The purpose of this paper is to show that from the cohomology of $X(\pi)$ one can actually deduce all the other interesting cases (see Corollary 3.2):

Theorem. *For groups of type A , Conjecture 1 holds whenever it holds for $d = 1$, that is for $X(\pi)$.*

Let us emphasize that Conjecture 1 is known to be true only in a very small number of cases, namely when $d = h$ is the Coxeter number by Lusztig [12], for groups of rank 2 by Digne, Michel and Rouquier [10] and when $d = n$ for A_n and $d = 4$ for D_4 by Digne and Michel [8]. Therefore this theorem represents a very important step towards a proof of the geometric version of Broué's conjecture.

Even though this result depends on the conjectural description of $X(\pi)$, one can give an effective proof of Conjecture 1 for principal Φ_d -blocks when $d > (n + 1)/2$. In that case the defect group is cyclic, and the modular representation theory of the block is fully understood. We will address this problem in a subsequent paper, where we will compute the cohomology of $\overline{\mathbb{Z}_\ell}$ of the corresponding Deligne-Lusztig variety.

To give a flavour of the proof of the main theorem, recall that for groups of type A , we have by [3] a combinatorial description of the Deligne-Lusztig

induction associated to $\tilde{X}(\mathbf{I}, \mathbf{w}F)$. In terms of partitions, it corresponds to adding a certain number of d -hooks. By transitivity, one can decompose $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ by means of simpler varieties $\tilde{X}_{n,d}$, each of which corresponds to adding a single d -hook to a partition. Now, using the methods developed in [11] one can compute the cohomology of (some quotient of) $\tilde{X}_{n,d}$ in terms of $\tilde{X}_{n-1,d}$ and $\tilde{X}_{n-1,d-1}$ (see Theorem 2.1), providing an inductive argument to tackle Conjecture 1.

Note finally that Theorem 2.1 can be generalised to many other situations in type B , C and D . However, two main problems arise: firstly, the limit case is either $X(w_0)$ or $X(\pi)$ and does not contain all the unipotent characters. Secondly, the methods in [11] work obviously for non-cuspidal unipotent characters only. We can obtain partial results on the principal series in that situation, which we believe are too coarse to be mentioned in this paper.

1 Parabolic varieties in type A

Throughout this paper, \mathbf{G} will denote any connected reductive algebraic group of type A_n over $\mathbb{F} = \overline{\mathbb{F}}_p$. We will consider a Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ defining a standard \mathbb{F}_q -structure on \mathbf{G} . Since we will be interested in unipotent characters only, we will not make any specific choice for (\mathbf{G}, F) in its isogeny class. If \mathbf{H} is any F -stable subgroup of \mathbf{G} we will denote by $H = \mathbf{H}^F$ the associated finite group.

The Weyl group W of \mathbf{G} is the symmetric group \mathfrak{S}_n and its Braid monoid B^+ is the usual Artin monoid. It is generated by a set $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ corresponding to simple reflections s_1, \dots, s_n of W . Following [9], we define for $1 \leq d \leq n+1$

$$\mathbf{v}_d = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_{n - \lfloor \frac{d}{2} \rfloor} \mathbf{s}_n \mathbf{s}_{n-1} \cdots \mathbf{s}_{\lfloor \frac{d+1}{2} \rfloor}$$

and $\mathbf{J}_d = \{\mathbf{s}_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{d}{2} \rfloor\} \subset \mathbf{S}$.

We are interested in computing the cohomology of the variety

$$X_{n,d} = X(\mathbf{J}_d, \mathbf{v}_d F)$$

with coefficients in any unipotent local system. Note that for $d > 1$, the element \mathbf{v}_d is reduced so that we can work with the variety $X(\mathbf{J}_d, v_d F)$. By [9, Lemma 11.7 and 11.8], the pair $(\mathbf{J}_d, \mathbf{v}_d)$ is d -periodic so that it makes sense to study the cohomology of $X_{n,d}$. Recall from [9] that a d -periodic element is any pair (\mathbf{I}, \mathbf{b}) with $\mathbf{I} \subset \mathbf{S}$ and $\mathbf{b} \in B^+$ such that $\mathbf{b}F(\mathbf{b}) \cdots F^{d-1}(\mathbf{b}) = \pi/\pi_I$ where $\pi = \mathbf{w}_0^2$ is the generator of the pure Braid group. It has been shown in [9] that this forces $\mathbf{b}F$ to normalise \mathbf{I} . Note that when $d \leq (n+1)/2$, \mathbf{v}_d is not *maximal* in the sense that it is not extendable to a d th root of π/π_I for a proper subset \mathbf{I} of \mathbf{J}_d . However, it can still be used to associate to any unipotent block a "good" parabolic Deligne-Lusztig variety. Before making any precise statement, we shall briefly recall the combinatorial objects that we will use.

1.1. Φ_d -blocks of G . The unipotent characters of G are labeled by the partitions of $n+1$. If λ is such a partition, we will denote by χ_λ the corresponding character,

with the convention that $\chi_{(1,1,\dots,1)} = \text{St}_G$ is the Steinberg character of G . We shall also fix a representation V_λ over $\overline{\mathbb{Q}}_\ell$ of character χ_λ . For $1 \leq d \leq n+1$, the pair $(\mathbf{L}_{J_d}, v_d F)$ represents a d -Levi subgroup of \mathbf{G} . From [3], we know how to express the d -Harish-Chandra induction in terms of combinatorics of partitions. To fix the notation, let μ be a partition of $n+1-d$ and $X = \{x_1 < x_2 < \dots < x_s\}$ be a β -set associated to μ . We may and we will assume that X is big enough, so that it contains $\{0, 1, \dots, d-1\}$. Let X' be the subset of X defined by $X' = \{x \in X \mid x+d \notin X\}$. It represents the possible d -hooks that can be added to μ . For $x \in X'$ we will denote by $\mu * x$ the partition of $n+1$ which has $(X \setminus \{x\}) \cup \{x+d\}$ as a β -set.

We fix an F -stable Tits homomorphism $t : B^+ \rightarrow N_{\mathbf{G}}(\mathbf{T})$. By [9] the variety $X_{n,d}$ has an étale covering $\tilde{X}_{n,d} = \tilde{X}(\mathbf{J}_d, \mathbf{v}_d F)$ with Galois group $\mathbf{L}_{J_d}^{t(\mathbf{v}_d)F}$. Since $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$ is a split group of type A_{n-d} , the partition μ defines a unipotent local system \mathcal{F}_μ on $X_{n,d}$ such that $\mathbf{H}_c^*(X_{n,d}, \mathcal{F}_\mu)$ and $\mathbf{H}_c^*(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$ are isomorphic. Then we deduce from [3, Section 3.4] that there exist signs $\varepsilon_x = \pm 1$ such that the d -Harish-Chandra induction of χ_μ is given by

$$\mathbf{R}_{\mathbf{L}_{J_d}}^{\mathbf{G}}(\chi_\mu) = \sum (-1)^i \mathbf{H}_c^i(X_{n,d}, \mathcal{F}_\mu) = \sum_{x \in X'} \varepsilon_x \chi_{\mu * x}.$$

In particular, the d -Harish-Chandra restriction of $\chi_\lambda \in \text{Irr} G$ is non-zero until we reach the d -core ν of λ , which corresponds to a d -cuspidal character χ_ν . The unipotent characters in the Φ_d -block of G containing χ_λ are all the characters that can be obtained by successive d -inductions from χ_ν . They correspond to partitions of $n+1$ that have ν as a d -core.

1.2. A parabolic variety associated to a Φ_d -block. The cohomology of the variety $X_{n,d}$ induces to a minimal d -induction since there is no d -split Levi between $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$ and (\mathbf{G}, F) . By transitivity, one can form a Deligne-Lusztig variety $\tilde{X}(\mathbf{I}, \mathbf{w})$ associated to the d -cuspidal character χ_ν . Let $n+1-ad$ be the size of ν and consider for $i = 1, \dots, a$ the pairs $(\mathbf{J}_d^{(i)}, \mathbf{v}_d^{(i)})$ where $(\mathbf{J}_d^{(0)}, \mathbf{v}_d^{(0)}) = (\mathbf{J}_d, \mathbf{v}_d)$ and $(\mathbf{J}_d^{(i+1)}, \mathbf{v}_d^{(i+1)})$ is the analogue of the pair $(\mathbf{J}_d, \mathbf{v}_d)$ for the split group $(\mathbf{L}_{J_d^{(i)}}, t(\mathbf{v}_d^{(i)} \dots \mathbf{v}_d^{(1)})F)$ of type A_{n-id} . Then one can readily check that the pair $(\mathbf{I}, \mathbf{w}) = (\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} \dots \mathbf{v}_d^{(1)})$ is d -periodic.

By [9, Proposition 8.26] the associated Deligne-Lusztig variety $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ is isomorphic to the following amalgamated product

$$\tilde{X}(\mathbf{J}_d^{(1)}, \mathbf{v}_d^{(1)}F) \times_{\mathbf{L}_{J_d^{(1)}}^{t(\mathbf{v}_d^{(1)})F}} \dots \times_{\mathbf{L}_{J_d^{(a-1)}}^{t(\mathbf{v}_d^{(a-1)} \dots \mathbf{v}_d^{(1)})F}} \tilde{X}_{\mathbf{L}_{J_d^{(a-1)}}}(\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} t(\mathbf{v}_d^{(a-1)} \dots \mathbf{v}_d^{(1)})F).$$

Now each variety in this decomposition corresponds to a variety $\tilde{X}_{n-id,d}$ for some $i = 0, \dots, a-1$. Since the cohomology of the latter with coefficients in a unipotent local system depends only on the isogeny class of the group (here, the split type A_{n-id}) we obtain

$$\mathbf{R}\Gamma_c(\tilde{X}(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\nu) \simeq \mathbf{R}\Gamma_c(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_\ell) \otimes_{A_{n-d}} \dots \otimes_{A_{n-(a-1)d}} \mathbf{R}\Gamma_c(X_{n-(a-1)d,d}, \mathcal{F}_\nu). \quad (1.1)$$

Consequently, if we believe that the vanishing property in Conjecture 1 holds for the cohomology of $X(\mathbf{I}, \mathbf{w})$, then for any partition μ of $n + 1 - d$, the graded G -module $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ should be multiplicity-free.

1.3. Craven's formula in type A. Conjecturally, the unipotent character χ_λ is a constituent of only one cohomology group of $R\Gamma(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\nu)$. Craven proposed in [7] a formula which gives the degree of this cohomology group in terms of d and the generic degree of χ_λ and χ_μ . More precisely, he considers a function C_d on some set of enhanced cyclotomic polynomials and conjectured that

$$\langle \chi_\lambda; H^i(X(\mathbf{I}, \mathbf{w}), \mathcal{F}_\nu) \rangle_G \neq 0 \iff i = C_d(\deg \chi_\lambda) - C_d(\deg \chi_\nu). \quad (1.2)$$

Let us recall the definition of C_d : assume that $P \in \mathbb{Q}[x]$ is a polynomial such that the non-zero roots z_1, \dots, z_m (written with multiplicity) of P are all roots of unity. Let us denote by $d^\circ(P)$ the degree of P and by $v(P)$ its valuation, that is the degree of $x^{d^\circ(P)}P(x^{-1})$. Then Craven's function C_d is defined by

$$C_d(P) = \frac{1}{d} (d^\circ(P) + v(P)) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < 2\pi/d\} - \frac{1}{2} \#\{i = 1, \dots, m \mid z_i = 1\}.$$

Here, the argument $\text{Arg } z$ of a non-zero complex number z is taken in $[0; 2\pi)$. More generally, if $\zeta = \exp(2i\pi k/d)$ is a primitive d -root of unity, one can define a function C_ζ by replacing d by d/k and 1.2 should hold for d th roots of $(\pi/\pi_I)^k$. Note also that Craven's function is additive: it satisfies $C_\zeta(PQ) = C_\zeta(P) + C_\zeta(Q)$.

For groups of type A, the degree $\deg \chi_\lambda$ of the unipotent character χ_λ is explicitly known (see for example [5, Section 13]). It is a polynomial in q of degree A_λ and valuation a_λ and no factors of the form $(q - 1)$ can appear. In particular, Craven's function can be written

$$C_\zeta(\deg \chi_\lambda) = \frac{2\pi}{\text{Arg } \zeta} (a_\lambda + A_\lambda) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < \text{Arg } \zeta\}$$

where z_1, \dots, z_m are the roots with multiplicity of the polynomial $\deg \chi_\lambda$. Note that with this description it is already not obvious that the rational number on the right-hand side of 1.2 is actually an integer.

Since C_d is additive, formula 1.2 together with the quasi-isomorphism 1.1 suggests that the partition ν should not be necessarily a d -core. In the case of an elementary d -induction (when $a = 1$) we can write everything explicitly using [7, Proposition 9.1]; the second equality follows from an easy calculation:

Lemma 1.3. *Let μ be a partition with corresponding β -set X that we assume to be large enough. For $x \in X'$, we have*

$$C_d(\deg \chi_{\mu^{*x}}) - C_d(\deg \chi_\mu) = 2(n + 1 - d - x + \#\{y \in X \mid y < x\}) + \#\{y \in X \mid x < y < x + d\}$$

and

$$a_{\mu^{*x}} + A_{\mu^{*x}} - a_\mu - A_\mu = d(n - d + s - x).$$

These integers give conjecturally the degree of the cohomology group of $X_{n,d}$ in which $\chi_{\mu^{*x}}$ will appear, as well as the corresponding eigenvalue of the Frobenius. Since we will work with the cohomology with compact support, we shall rather work with the integers

$$\pi_d(X, x) = 2(n + x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x + d\}$$

and

$$\gamma_d(X, x) = n + 1 + x - s.$$

They are readily deduced from the previous ones by taking into account the dimension of $X_{n,d}$ (which is equal to $\ell(v_d) = 2n + 1 - d$). Now Conjecture 1 can be deduced from the following:

Conjecture 1.4. *Let $n \geq 1$ be a positive integer and $1 \leq d \leq n + 1$. Let μ be a partition of $n - d + 1$ and X be its β -set, assumed to be large enough. Then*

$$\mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu) \simeq \bigoplus_{x \in X'} V_{\mu * x}[-\pi_d(X, x)] \otimes \overline{\mathbb{Q}}_\ell(\gamma_d(X, x))$$

as a complex of $G \times \langle F \rangle$ -modules.

The purpose of this paper is to prove that this conjecture holds for any d whenever it holds for $d = 1$. As a byproduct, we shall deduce the cohomology of parabolic Deligne-Lusztig varieties associated to any unipotent block from the knowledge of the cohomology of $X(\mathfrak{w}_0^2)$.

2 Decomposition of the quotient

The group $(\mathbf{L}_{J_d}, \dot{v}_d F)$ is split of type A_{n-d} , therefore the unipotent representations of the corresponding finite group are labelled by partitions μ of $n - d + 1$. To such a partition one can associate a unipotent $\overline{\mathbb{Q}}_\ell$ -local system \mathcal{F}_μ on $X_{n,d}$. From [3] we know that the irreducible constituents of the virtual character $\sum (-1)^i \mathrm{H}_c^i(X_{n,d}, \mathcal{F}_\mu)$ correspond to the partitions of $n + 1$ obtained by adding a d -hook to μ . The restriction to \mathfrak{S}_n of the corresponding irreducible representation corresponds to a partition obtained by

- either restricting the hook (usually in two different ways),
- or restricting μ .

The main result of this section gives a geometric interpretation of this phenomenon.

Theorem 2.1. *Assume that $d \geq 2$. Let $I = \{s_j \mid 1 \leq j \leq n - 1\}$. Let μ be a partition of $n - d + 1$ and $\{\mu^{(j)}\}$ be the set of partitions of $n - d$ obtained by restricting μ . Then there is a distinguished triangle in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$*

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

Remark 2.2. From [1, Proposition 1.1] we can deduce that the cohomology of a Deligne-Lusztig variety with coefficients in a unipotent local system depends only on the type of (\mathbf{G}, F) . Therefore there is no ambiguity in the statement of the theorem.

We will use the results in [11] to compute the quotient of $\tilde{X}(J_d, v_d F)$ by the finite group U_I . Recall that $X_{n,d} = X(J_d, v_d F)$ can be decomposed into locally closed P_I -subvarieties X_x , where x is a I -reduced- J_d element of W . In our situation, at most two pieces will appear:

Lemma 2.3. *Assume that $2 \leq d \leq n$. The variety X_x is non-empty if and only if x is one of the two following elements:*

- (1) $x_0 = s_n s_{n-1} \cdots s_1$
- (2) $x_1 = s_n s_{n-1} \cdots s_{n-\lfloor \frac{d}{2} \rfloor + 1}$.

Proof. For simplicity, we shall denote $a = \lfloor \frac{d+1}{2} \rfloor$ and $b = n - \lfloor \frac{d}{2} \rfloor$ so that $w = v_d = s_1 \cdots s_b s_n \cdots s_a$ and $J = J_d = \{a+1, \dots, b\}$. If x is a I -reduced- J element, then $x = s_n s_{n-1} \cdots s_i$ with $b+1 \leq i \leq n+1$ or $1 \leq i \leq a$. Recall from [11] that the variety X_x is non-empty if and only if there exists $y = y_1 \cdots y_r \in W_J$ and an x -distinguished subexpression γ of yw such that the products of the elements of γ lies in $(W_I)^x$. We first observe that for $i \notin \{1, n+1\}$ we have

$$(W_I)^x = \langle s_1, \dots, s_{i-2}, s_i s_{i-1} s_i, s_{i+1}, \dots, s_n \rangle.$$

Now, since x is reduced- J , the subexpression γ is the concatenation of (y_1, \dots, y_r) and an xy -distinguished subexpression $\tilde{\gamma}$ of w . If $i > b+1$ or $2 \leq i \leq a$, the group W_J is included in $(W_I)^x$. Therefore the product of the elements of $\tilde{\gamma}$ must lie in $(W_I)^x$. We shall distinguish two cases:

Case (1). We assume that $i > b+1$. In that case x commutes with any element of W_J , so that $\tilde{\gamma}$ is an yx -distinguished subexpression of w . Then

- if x is trivial (that is if $i = n+1$), then any y -distinguished subexpression of w contains necessarily s_n and hence cannot produce any element of $(W_I)^x = W_I$;
- if x is non-trivial then $i \leq n$, and a subexpression of w lies in $(W_I)^x$ if and only if it does not contain s_i or s_{i-1} . However, such a subexpression will never be yx -distinguished since for all v in W_I we have $v x s_{i-1} > v x$.

We deduce that the variety X_x is empty in this case.

Case (2). We assume that $2 \leq i \leq a$. The subexpression $\tilde{\gamma}$ is $^x yx$ -distinguished. Since $i \leq a$, we have $^x W_J = W_{a, \dots, b-1}$. For $j < i-1$, we have $x s_j = s_j x$; moreover, $x s_{i-1}$ is I -reduced, so that $\tilde{\gamma}$ should start with (s_1, \dots, s_{i-1}) . In that case, the product of the elements of $\tilde{\gamma}$ cannot belong to $(W_I)^x$. Indeed, a subexpression of $s_{i-1} s_i \cdots s_b s_n \cdots s_a$ starting with s_{i-1} will never give an element of $(W_I)^x$, the only non-trivial situation being the case $a = i$:

- with the notation in [10, Section 2.1.2] we have $\underline{s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a} = \underline{s_{a+1} \cdots s_b s_n \cdots s_{a+1} s_{a-1} s_a}$ and neither s_{a-1} nor $s_{a-1} s_a$ belongs to $(W_I)^x$;
- $\underline{s_{a-1} s_a s_{a+1} \cdots s_b s_n \cdots s_a} = (s_{a-1} s_a s_{a-1}) \underline{s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a}$ and we are back to the previous case.

This forces the variety X_x to be empty. □

Proof of the Theorem. From the previous lemma we deduce that $\tilde{X}(J_d, v_d F)$ decomposes as a disjoint union $\tilde{X}(J_d, v_d F) = \tilde{X}_{x_0} \cup \tilde{X}_{x_1}$ with \tilde{X}_{x_0} being open. Using [11] we shall now determine the cohomology of the quotient of each of these varieties by U_I . Throughout the proof, we will denote $\tilde{I} = \{s_2, \dots, s_n\} \subset S$ the conjugate of I by w_0 .

When $x = x_0 = w_I w_0$ and $d > 2$ we are in the situation of [11, Proposition 3.4]. Indeed, $v_d = s_1 w'$ with $w' \in W_{2, \dots, n}$ and s_1 commutes with $W_{J_d} \subset W_{3, \dots, n-1}$ so that we obtain

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}/N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(K_{x_0}, \dot{v}F)^{/x_0} N', \overline{\mathbb{Q}}_\ell)$$

with $v = {}^{x_0}w'$ and $K_{x_0} = I \cap {}^{x_0}\Phi_{J_d} = {}^{x_0}J_d$. For simplicity, we shall rather consider the conjugate by x_0 of the right-hand side

Recall that N and N' are normal subgroup of \mathbf{L}_{J_d} and are both contained in \mathbf{T} . In particular, any unipotent character of $\mathbf{L}_{J_d}^{wF}$ (resp. of $\mathbf{L}_{J_d}^{w'F}$) is trivial on N (resp. N'). Consequently, for any unipotent character χ of $\mathbf{L}_{J_d}^{wF}$ we obtain the following quasi-isomorphism of complexes of $L_I \times \langle F \rangle$ -modules:

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}, \overline{\mathbb{Q}}_\ell)_\chi \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F), \overline{\mathbb{Q}}_\ell)_{x_0 \chi}.$$

Finally, we observe that the varieties $\tilde{X}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F)$ and $\tilde{X}_{n-1, d-1}$ have the same cohomology with coefficients in any unipotent local system. Indeed, if we denote (s_1, \dots, s_{n-1}) by (t_1, \dots, t_{n-1}) if d is odd or by (t_{n-1}, \dots, t_1) if d is even, then we have

$$v = t_1 t_2 \cdots t_{n-1 - \lfloor \frac{d-1}{2} \rfloor} t_{n-1} t_{n-2} \cdots t_{\lfloor \frac{d}{2} \rfloor}$$

which corresponds to the element v_{d-1} in the Weyl group $W_I = \langle t_1, \dots, t_{n-1} \rangle$ of type A_{n-1} .

When $x = x_0$ and $d = 2$, we can write $v_2 = w w'$ with $w = s_n s_{n-1} \cdots s_2$ and $w' = s_1 s_2 \cdots s_n = s_1 w''$ so that $\tilde{X}_{n,2} \simeq \tilde{X}(\{s_2, \dots, s_{n-1}\}, \mathbf{w} w' F)$. Moreover, via this isomorphism we have

$$\tilde{X}_{x_0} \simeq \bigcup_{y \in W} \tilde{X}_{(x_0, y)}.$$

We claim that $\tilde{X}_{(x_0, y)}$ is empty unless $y \in W_I w_0 W_{J'_2}$ where $J'_2 = J_2^w = \{s_3, s_4, \dots, s_n\}$. The piece $\tilde{X}_{(x_0, y)}$ consists of pairs $(p x_0 \mathbf{P}_{J_2}, p' y \mathbf{P}_{J'_2})$ with $p, p' \in \mathbf{P}_I$ such that $p^{-1} p' \in x_0 \mathbf{P}_{J_2} w \mathbf{P}_{J'_2} y^{-1}$ and $p'^{-1} p \in y \mathbf{P}_{J'_2} w' \mathbf{P}_{J_2} x_0^{-1}$. In particular, if $\tilde{X}_{(x_0, y)}$ is non-empty then the double coset $\mathbf{P}_I y \mathbf{P}_{J'_2}$ has a non-trivial intersection with $x_0 \mathbf{B} w$. But $x_0 = w_I w_0$ is reduced- \tilde{I} so that $\ell(x_0 w) = \ell(x_0) + \ell(w)$ and $x_0 \mathbf{B} w \subset \mathbf{P}_I w_0 \mathbf{P}_{J'_2}$. This forces y to lie in $W_I w_0 W_{J'_2}$. Note that $w_0(J'_2) \subset I$ so that the minimal element in this coset is x_0 and we have $\tilde{X}_{x_0} \simeq \tilde{X}_{(x_0, x_0)}$.

Now s_1 commutes with J'_2 and we can apply [11, Proposition 3.4] to obtain, after conjugation by x_0 :

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}/N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(\mathbf{K}_{x_0}, \mathbf{v} w' F)^{/x_0} N', \overline{\mathbb{Q}}_\ell).$$

with $K_{x_0} = {}^{x_0}J_2$, $v = {}^{x_0}w$ and $v' = {}^{x_0}w''$. If we denote (s_1, \dots, s_{n-1}) by (t_{n-1}, \dots, t_1) we obtain ${}^{x_0}J_2 = \{t_2, \dots, t_{n-1}\}$ and $\mathbf{vv}' = \mathbf{t}_1 \cdots \mathbf{t}_{n-1} \mathbf{t}_{n-1} \cdots \mathbf{t}_1$ so that the pair $(\mathbf{K}_{x_0}, \mathbf{vv}')$ corresponds to $(\mathbf{J}_1, \mathbf{v}_1)$ in the Weyl group W_I of type A_{n-1} . As before, N and N' do not play any role if we consider the unipotent part of the previous quasi-isomorphism.

When $x = x_1$ we use [11, Proposition 3.2]: the conjugate of v_d by x_1 is

$$v = x_1 v_d x_1^{-1} = s_1 s_2 \cdots s_{n-1-\lfloor \frac{d}{2} \rfloor} s_{n-1} s_{n-2} \cdots s_{\lfloor \frac{d+1}{2} \rfloor}$$

which corresponds exactly to the element v_d in W_I . We can therefore identify the cohomology of the varieties $\mathbf{X}_{\mathbf{L}_I}(K_{x_1}, vF)$ and $\mathbf{X}_{n-1,d}$ with coefficients in any unipotent local system (see Remark 2.2). The group $\mathbf{P}_I \cap {}^{x_1}\mathbf{L}_{J_d}$ is a vF -stable parabolic subgroup of ${}^{x_1}\mathbf{L}_{J_d}$ and $\mathbf{L}_{K_{x_1}} = \mathbf{L}_I \cap {}^{x_1}\mathbf{L}_{J_d}$ is a stable Levi complement. Therefore it makes sense to consider the Harish-Chandra restriction ${}^*\mathbf{R}_{\mathbf{K}}^J \chi$ of any unipotent character χ of $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$. From [11, Proposition 3.2] (see also [11, Remark 3.12]) we deduce the following quasi-isomorphism

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_1}, \overline{\mathbb{Q}}_\ell)_\chi[2](-1) \simeq \mathrm{R}\Gamma_c(\tilde{\mathbf{X}}_{\mathbf{L}_I}(K_{x_1}, vF), \overline{\mathbb{Q}}_\ell)_{* \mathbf{R}_{\mathbf{K}}^J \chi}.$$

Let μ be a partition of $n - d + 1$. The cohomology of the variety $\mathbf{X}_{n,d}$ with coefficients in the local system \mathcal{F}_μ is given by

$$\mathrm{R}\Gamma_c(\mathbf{X}_{n,d}, \mathcal{F}_\mu) \simeq \mathrm{R}\Gamma_c(\tilde{\mathbf{X}}(\mathbf{J}_d, \mathbf{v}_d F), \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$$

where χ_μ is the unipotent character of $\mathbf{L}_{J_d}^{\dot{v}F}$ corresponding to the partition μ . Since $(\mathbf{L}_{K_{x_1}}, vF)$ is a split group of type A_{n-d-1} , the Harish-Chandra restriction of χ_μ from $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$ is the sum of the χ_{μ_i} 's where the μ_i 's are the partitions of $n - d$ obtained by restricting μ . With with description, we get the following isomorphisms in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_0}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathbf{X}_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$$

and $\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_1}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c(\mathbf{X}_{n-1,d}, \bigoplus \mathcal{F}_{\mu_i})[-2](1)$.

We conclude using the distinguished triangle associated to the decomposition $\tilde{\mathbf{X}}_{n,d} = \tilde{\mathbf{X}}_{x_0} \cup \tilde{\mathbf{X}}_{x_1}$ in which $\tilde{\mathbf{X}}_{x_0}$ is open. \square

3 Cohomology over $\overline{\mathbb{Q}}_\ell$

We have just seen how to relate the Harish-Chandra restriction of the cohomology of $\mathbf{X}_{n,d}$ to the cohomology of smaller parabolic Deligne-Lusztig varieties. We shall now explain how this strategy provides an inductive method for a thorough determination of the cohomology of $\mathbf{X}_{n,d}$ with coefficients in any unipotent local system. The main result in this section gives an inductive strategy towards a proof of Conjecture 1:

Theorem 3.1. *Let $n \geq 1$ and $2 \leq d \leq n$. If Conjecture 1.4 holds for $(n, d+1)$, $(n-1, d-1)$ and $(n-1, d)$ then it holds for (n, d) .*

Note that we already know from [12] that Conjecture 1.4 holds in the Coxeter case, corresponding to $(n, n+1)$. Therefore $d = 1$ is the only limit case. But $\pi = \mathbf{w}_0^2$ is a maximal 1-periodic element in the sense of [9] and in that specific case, a general conjecture for the cohomology has been already formulated in [4]: a unipotent character χ_λ can appear in $H_c^i(X(\pi))$ for $i = 4\nu_G - 2A_\lambda$ only, where ν_G is the number of positive roots. An important consequence of Theorem 3.1 is that knowing the cohomology of $X(\pi)$ is sufficient for determining all the other interesting cases:

Corollary 3.2. *For groups of type A, Conjecture 1 holds for any $d \geq 1$ as soon as it holds for $d = 1$.*

Proof. Assume that Conjecture 1 holds for $d = 1$, that is for the variety $X(\pi)$. Let $I = J_1 = \{s_2, \dots, s_n\}$ and $\mathbf{b} = \mathbf{v}_1 = \mathbf{s}_1 \cdots \mathbf{s}_n \mathbf{s}_n \cdots \mathbf{s}_1$. By [9, Proposition 8.26] we have

$$\mathrm{R}\Gamma_c(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\tilde{X}(\mathbf{I}, \mathbf{b}F), \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}_I^{t(\mathbf{b})F}} \mathrm{R}\Gamma_c(X(\pi_I), \overline{\mathbb{Q}}_\ell).$$

Since the cohomology of $X(\pi_I)$ contains all the unipotent characters of $\mathbf{L}_I^{t(\mathbf{b})F}$, we deduce that for any partition μ of n , the groups $H_c^i(X_{n,1}, \mathcal{F}_\mu)$ are submodules of the cohomology groups of $X(\pi)$. Consequently, they are disjoint as soon as Conjecture 1 holds for $X(\pi)$. Since we have assumed that it holds also for $X(\pi_I)$ we have actually

$$H_c^i(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\mu \vdash n} H_c^{i-4\nu_{\mathbf{L}_I}+2A_\mu}(X_{n,1}, \mathcal{F}_\mu)(2\nu_{\mathbf{L}_I} - a_\mu - A_\mu) \quad (3.3)$$

as a $G \times \langle F \rangle$ -module. Now, the alternating sum of the cohomology groups of $X_{n,1}$ represents the Deligne-Lusztig induction from $\mathbf{L}_I^{t(\mathbf{b})F} \simeq L_I$ to G . Therefore a character χ_λ appear in $H_c^i(X_{n,1}, \mathcal{F}_\mu)$ if and only if μ is the restriction of λ , or equivalently, if λ is obtained from μ by adding a 1-hook. This, together with 3.3 and Lemma 1.3 proves that Conjecture 1.4 holds for $X_{n,1}$, and therefore for any variety $X_{n,d}$ by 3.1. We use 1.1 to conclude. \square

Proof of the Theorem. Let X be a β -set associated the partition μ of $n-d+1$. We can always assume that it contains $\{0, 1, \dots, d-1\}$. The partitions $\mu^{(j)}$'s of $n-d$ which are obtained by restricting μ can be associated to the following β -set:

$$X^{(j)} = \{x_1^{(j)} < \dots < x_s^{(j)}\} \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_j - 1 & \text{if } i = j; \\ x_i & \text{otherwise.} \end{cases}$$

Let $I = \{s_1, \dots, s_{n-1}\}$. By Theorem 2.1, the Harish-Chandra restriction of the cohomology of $X_{n,d}$ can be fitted into the following distinguished triangle:

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

Now, if we assume that Conjecture 1.4 holds for both $(n-1, d-1)$ and $(n-1, d)$, the complexes on the left and right-hand side are completely determined. Let us examine the different eigenvalues of F that can appear:

- (a) on $\mathcal{C} = \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$, the eigenvalues of F are q^{n+x-s} and $q^{n+1+x-s}$ with $x \in X$ such that $x+d-1 \notin X$. The character of the corresponding eigenspace is χ_λ where λ is the partition of n obtained by adding to μ a $(d-1)$ -hook represented by x ;
- (b) on $\mathcal{D}^{(j)} = \mathrm{R}\Gamma_c(X_{n-1,d}, \mathcal{F}_{\mu^{(j)}})[-2](1)$, the eigenvalues of F are $q^{n+1+x-s}$ where $x \in X^{(j)}$ is such that $x+d \notin X^{(j)}$. The character of the corresponding eigenspace is χ_λ where λ is the partition of n obtained by adding to $\mu^{(j)}$ a d -hook represented by x .

We shall now determine $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ by studying each eigenspace of F separately. For x a positive integer, we can separate the following cases:

Case (1). Assume first that $x \in X$ and $x+d \notin X$. Let $\lambda = \mu * x$ be the partition of $n+1$ obtained by adding to μ a d -hook from x . We want to prove that the $q^{n+1-s+x}$ -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is non-zero in degree $\pi_d(X, x)$ only and that its character is the Harish-Chandra restriction of χ_λ .

By (a), the $q^{n+1+x-s}$ -eigenspace of F on \mathcal{C} will produce non-zero representations in the following two cases:

- if $x+d-1 \notin X$, then one obtains a character associated to the β -set $(X \setminus \{x\}) \cup \{x+d-1\}$ and it is concentrated in degree

$$\begin{aligned} 2 + \pi_{d-1}(X, x) &= 2 + 2(n-1+x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d-1\} \\ &= 2(n+x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d-1\} \\ &= \pi_d(X, x). \end{aligned}$$

- if $x+1 \in X$, then the corresponding β -set is $(X \setminus \{x+1\}) \cup \{x+d\}$ and the associated character appears in degree $1 + \pi_{d-1}(X, x+1)$ only. But we have

$$\begin{aligned} \pi_{d-1}(X, x+1) &= 2(n+x - \#\{y \in X \mid y < x+1\}) - \#\{y \in X \mid x+1 < y < x+d\} \\ &= 2(n+x-1 - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x+d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

On the other hand, the $q^{n+1+x-s}$ -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x+d \notin X^{(j)}$. This happens if and only if x and $x+d+1$ are different from x_j . In that case, the β -set corresponding to the character of the eigenspace will be $(X^{(j)} \setminus \{x\}) \cup \{x+d\}$. Furthermore, the degree in which this character will appear is $2 + \pi_d(X^{(j)}, x)$, which is clearly equal to $\pi_d(X, x)$ in that case.

Now, the β -set $Y = (X \setminus \{x\}) \cup \{x+d\}$ is associated to the partition $\lambda = \mu * x$. As mentioned in the beginning of Section 2, the restriction of λ is obtained by restricting the hook (usually in two different ways) or by restricting μ . In the framework of β -sets, it corresponds to decreasing specific elements of Y :

- if $x+d-1 \notin X$, one can replace $x+d$ by $x+d-1$ in Y and we obtain the β -set $(X \setminus \{x\}) \cup \{x+d-1\}$;

- if $x + 1 \in X$, one can replace $x + 1$ by x in Y and we obtain the β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$;
- if $x_j \in X$ is different from x or from $x + d + 1$, and if $x_j - 1 \notin X$, then one can replace x_j by $x_j - 1$ in Y to obtain the β -set $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$.

This shows that the character of the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is the Harish-Chandra restriction of $\chi_{\mu * x}$.

Lemma 3.4. *Let λ be a partition of $n + 1$, with $n \geq 3$ and let χ be a (non-necessarily irreducible) unipotent character of G . Then the Harish-Chandra restriction of χ and χ_λ are equal if and only if $\chi = \chi_\lambda$.*

Proof of the Lemma. Assume that there exists a partition $\nu = \{\nu_1 \leq \nu_2 \leq \dots \leq \nu_r\}$ of $n + 1$ with $\nu_1 \neq 0$ such that the difference between the Harish-Chandra restriction of χ_λ and χ_ν is still a unipotent character. This means that in the Young diagram of ν , any box that can be removed can be replaced to form the Young diagram of λ . If $\nu \neq \lambda$, this is possible only if $\nu_1 = \nu_2 = \dots = \nu_r$.

Let χ be a unipotent character of G which has the same Harish-Chandra restriction as χ_λ . If $\chi \neq \chi_\lambda$, we deduce from the previous argument that all the irreducible constituents of χ are of the form χ_ν with $\nu = (a, a, \dots, a)$. This can happen if and only if $n = 2$ and $\lambda = (1, 2)$. \square

When $n \geq 3$, we deduce that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)$ is actually $\chi_{\mu * x}$. If $n = 2$, then the only ambiguity concerns $\chi_{\mu * x}$ when $\mu * x = (1, 2)$. In that case, the $q^{n+1+x-s}$ -eigenspace can be either $\chi_{\mu * x}$ or $1_G + \text{St}_G$. But by [9, Corollary 8.28], the trivial character and the Steinberg character cannot occur in the same cohomology group of $X_{n,d}$ as soon as the dimension of this variety is non-zero.

Case (2). Assume now that $x \notin X$. The $q^{n+1+x-s}$ -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. Since $x \notin X$, this forces $x = x_j^{(j)} = x_j - 1$ and $x_j + d - 1 \notin X$. In that case, its character corresponds to a partition with β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$ and it appears in degree $2 + \pi_d(X^{(j)}, x)$ only. On the other hand, the $q^{n+1+x-s}$ -eigenspace of F on \mathcal{C} is non-zero if and only if $x + 1 \in X$ and $x + 1 + d - 1 = x + d \notin X$. By (a), the character of this eigenspace corresponds to a partition with β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$. Furthermore, it appears in degree $1 + \pi_{d-1}(X, x + 1)$ only. Note that in that case we have

$$\begin{aligned} \pi_{d-1}(X, x + 1) &= 2(n + x - \#\{y \in X \mid y < x + 1\}) - \#\{y \in X \mid x + 1 < y < x + d\} \\ &= 2(n + x - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) + 1 \end{aligned}$$

$$\begin{aligned} \text{and } \pi_d(X^{(j)}, x) &= 2(n - 1 + x - \#\{y \in X^{(j)} \mid y < x\}) - \#\{y \in X^{(j)} \mid x < y < x + d\} \\ &= 2(n - 1 + x - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

We deduce that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is isotypic and concentrated in two consecutive degrees. However, there are only a few unipotent characters that can have an isotypic Harish-Chandra restriction: they correspond to partitions of the form (a, a, \dots, a) . Among them we can find the Steinberg character St_G (with $a = 1$) and the trivial character 1_G (with $a = n + 1$). But by [9, Corollary 8.28] they have respective eigenvalues 1 and q^{2n+1-d} . Let us write the β -set of μ as $X = \{0, 1, \dots, k-1, \mu_1+k, \mu_2+k+1, \dots, \mu_r+s-1\}$ with $k \geq d$. Since $x \notin X$, one must have $k-1 < x < \mu_r+s-1$ and hence

$$d-1 \leq n+k-s < n+1+x-s < n+1+\mu_r-1 \leq 2n+1-d. \quad (3.5)$$

We deduce that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)$ is either zero, or consists of two copies of the character χ_λ in two consecutive degrees, where $\lambda = (a, a, \dots, a)$ with $1 < a < n+1$. We shall actually prove that it is always zero, but before that we need to study the last case.

Remark 3.6. The Harish-Chandra restriction of χ_λ corresponds to the partition $(a-1, a, \dots, a)$. Therefore if the associated character appears in the cohomology of \mathcal{C} then the β -set $(X \setminus \{x+1\}) \cup \{x+d\}$ must correspond to the partition $(a-1, a, \dots, a)$. This gives a rather strong condition on X : we will have either

$$X = \{0, 1, \dots, k-1, x+1, b, b+2, b+3, \dots, \widehat{x+d}, \dots, b+r\}$$

with $b+2 \leq x+d \leq b+r$, or

$$X = \{0, 1, \dots, k-1, x+1, x+d+2, x+d+3, \dots, x+d+r\}.$$

Case (3). Finally, assume that $x \in X$ and $x+d \in X$. The $q^{n+1+x-s}$ -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x+d \notin X^{(j)}$. Since $x+d \in X$, this forces $x+d = x_j$ (and therefore $x_j - 1 \notin X$). In that case, the character of the eigenspace corresponds to a partition with β -set $(X^{(j)} \setminus \{x\}) \cup \{x+d\} = (X \setminus \{x\}) \cup \{x+d-1\}$. On \mathcal{C} , the Frobenius has a non-zero $q^{n+1+x-s}$ -eigenspace if and only if $x+d-1 \notin X$ and its character is again associated to the β -set $(X \setminus \{x\}) \cup \{x+d-1\}$. This ensures that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is isotypic. Using $x+d-1$ instead of x in the inequalities 3.5 yields $0 < n+1+x-s < 2n+2-2d$ and therefore the previous argument applies. We deduce that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)$ is again either zero or consists of two copies of the character χ_λ in two consecutive degrees, namely $\pi_d(X, x) - 1$ and $\pi_d(X, x)$, where $\lambda = (a, a, \dots, a)$ and $1 < a < n+1$.

To conclude, we need to prove that the $q^{n+1+x-s}$ -eigenspaces of F are actually zero whenever $x \notin X$ or $x+d \in X$. Let us first summarize what we have proven so far:

- (1) if $x \in X$ and $x+d \notin X$ then the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)$ is $\chi_{\mu*x}$ and it appears in degree $\pi_d(X, x)$ only;
- (2) if $x \notin X$, the $q^{n+1+x-s}$ -eigenspace of F is zero unless $x+1 \in X$ and $x+d \notin X$. In that case, it may consist of two copies of χ_λ , one in degree $\pi_d(X, x) + 1$ and one in degree $\pi_d(X, x) + 2$, where $\lambda = (a, a, \dots, a)$ with $1 < a < n+1$.

Moreover, the β -set $(X \setminus \{x+1\}) \cup \{x+d\}$ must correspond to the partition $(a-1, a, \dots, a)$ (see Remark 3.6);

- (3) if $x \in X$ and $x+d \in X$, the $q^{n+1+x-s}$ -eigenspace of F is zero unless $x+d-1 \notin X$. In that case, it can only be χ_λ -isotypic with $\lambda = (a, a, \dots, a)$ and $1 < a < n+1$. Moreover, it is non-zero in degrees $\pi_d(X, x) - 1$ and $\pi_d(X, x)$ only, and $(X \setminus \{x\}) \cup \{x+d-1\}$ must be a β -set of the partition $(a-1, a, \dots, a)$.

Now, if we assume that Conjecture 1.4 holds for the variety $X_{n,d+1}$, then we can use the distinguished triangle

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

from Theorem 2.1 to prove that the eigenspaces of F on $H_c^*(X_{n,d}, \mathcal{F}_\mu)$ in cases (2) and (3) are indeed zero.

Assume that $x \notin X$ and that there exists $1 < a < n+1$ such that the character $\chi_\lambda = \chi_{(a, a, \dots, a)}$ appears twice in the $q^{n+1+x-s}$ -eigenspace of F on the cohomology of $X_{n,d}$ – that is in degrees $\pi_d(X, x) + 1$ and $\pi_d(X, x) + 2$. Then,

- if $x-1 \notin X$, the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ is χ_λ -isotypic by (2) (we have $x-1+1 \notin X$). Moreover, the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$ is zero since Conjecture 1.4 holds for the variety $X_{n,d+1}$. We deduce that the eigenspace on $H_c^*(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I}$ is χ_λ -isotypic, which is impossible since no unipotent character can have χ_λ as a Harish-Chandra restriction when $1 < a < n+1$.
- if $x-1 \in X$, then since $x-1+d+1 \notin X$, the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ and $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$ can be determined as in case (1). It corresponds to the Harish-Chandra restriction of the partition $\mu * (x-1)$ obtained from μ by adding a $(d+1)$ -hook from $x-1$. Furthermore, they will appear in degree $\pi_{d+1}(X, x-1)$ only, which is equal to

$$\begin{aligned} \pi_{d+1}(X, x-1) &= 2(n+x - \#\{y \in X \mid y < x-1\}) - \#\{y \in X \mid x-1 < y < x+d\} \\ &= 2(n+x+1 - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d\} \\ &= \pi_d(X, x) + 2. \end{aligned}$$

To these characters we have to add the contribution of χ_λ and possibly of an other character $\chi_{\lambda'}$ corresponding to $\lambda = (a', a', \dots, a')$ (when the case (3) applies to $x-1$). Now, we claim that neither χ_λ nor $\chi_{\lambda'}$ can appear in the $q^{n+1+x-s}$ -eigenspace of F on $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$. Indeed the assumptions on x force X (see Remark 3.6) to be either

$$X = \{0, 1, \dots, k-1, k+1, b, b+2, b+3, \dots, \widehat{k+d}, \dots, b+r\}$$

with $b+2 \leq k+d \leq b+r$, or

$$X = \{0, 1, \dots, k-1, k+1, k+d+2, k+d+3, \dots, k+d+r\}.$$

Therefore a β -set corresponding to the partition $\mu * (x-1)$ of $n+2$ is either

$$\{0, 1, \dots, k-2, k+1, b, b+2, b+3, \dots, b+r\}$$

with $b+2 \leq k+d \leq b+r$, or

$$\{0, 1, \dots, k-2, k+1, k+d, k+d+2, k+d+3, \dots, k+d+r\}.$$

We deduce that the restriction of $\mu^*(x-1)$ will never produce λ or λ' unless $r=2$, $b=k+2$ and $d=4$ in the first case, or $r=2$ and $d=2$ in the second case. In these very specific cases, we have either $X = \{0, \dots, k-1, k+1, k+2\}$, which corresponds to the partition $\mu = (1, 1)$ or $X = \{0, \dots, k-1, k+1, k+4\}$, which corresponds to $\mu = (1, 3)$. In this situation, we get $\lambda = (3, 3)$ and $\mu^*(x-1) = (2, 2, 3)$. But $(3, 3)$ cannot be obtained by restricting $\mu^*(x-1)$. This proves that the $q^{n+1+x-s}$ -eigenspace of F on $H_c^{\pi_d(X, x)+2}(\mathbb{X}_{n+1, d+1}, \mathcal{F}_\mu)^{U_I}$ is just χ_λ (plus possibly $\chi_{\lambda'}$), which is impossible by the properties of the Harish-Chandra restriction.

The same argument can be adapted to deal with the case (3), if we rather look at the $q^{n+2+x-s}$ -eigenspace and distinguish whether $x+1+d$ is an element of X or not. The details are left to the reader. \square

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