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# Cohomology of Deligne-Lusztig varieties for groups of type $A$

Olivier Dudas<sup>\*†</sup>

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## Abstract

We study the cohomology of parabolic Deligne-Lusztig varieties associated to unipotent blocks of  $\mathrm{GL}_n(q)$ . We show that the geometric version of Broué's conjecture over  $\overline{\mathbb{Q}}_\ell$ , together with Craven's formula, holds for any unipotent block whenever it holds for the principal  $\Phi_1$ -block.

## Introduction

Let  $\mathbf{G}$  be a connected reductive algebraic group over  $\mathbb{F} = \overline{\mathbb{F}}_p$  with an  $\mathbb{F}_q$ -structure associated to a Frobenius endomorphism  $F$ . Let  $\ell$  be a prime number different from  $p$  and  $b$  be a unipotent  $\ell$ -block of  $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ . When  $\ell$  is large, the defect group of  $b$  is abelian, and the geometric version of Broué's conjecture predicts that the cohomology of some Deligne-Lusztig variety should induce a derived equivalence between  $b$  and its Brauer correspondent [2].

When the centraliser of the defect group of  $b$  is a torus, then in [4] Broué and Michel identified which specific class of Deligne-Lusztig varieties should be considered. They correspond to good  $d$ -regular elements or equivalently to  $d$ -roots of  $\pi = \mathbf{w}_0^2$  in the Braid monoid. In a recent work [9], Digne and Michel introduced the notion of *d-periodic element* to generalise this to the parabolic setting. If  $b$  is a unipotent  $\Phi_d$ -block, then it is to be expected that there exists a  $d$ -periodic element  $(\mathbf{I}, \mathbf{w})$  such that the corresponding parabolic Deligne-Lusztig variety  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}F)$  is a good candidate for inducing the derived equivalence predicted by Broué's conjecture. Furthermore, Chuang and Rouquier conjectured in [6] that this equivalence is perverse, with a perversity function that has recently been conjectured by Craven in [7]. Surprisingly, it can be expressed by a function  $C_d$  depending only on the generic degrees of the corresponding characters.

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If we restrict our attention to the characteristic zero then we obtain a conjectural explicit description of the unipotent part of the cohomology of  $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ . The fundamental property that we derive from Broué's conjecture is that the cohomology groups of  $X(\mathbf{I}, \mathbf{w}F)$  are mutually disjoint. More precisely, it can be formulated as follows:

**Conjecture 1.** *Any  $d$ -cuspidal pair is conjugate to a pair  $(\mathbf{I}, \mathbf{w})$  where  $(\mathbf{I}, \mathbf{w})$  is a  $d$ -periodic element. Moreover, if  $\mathcal{F}_\chi$  is the corresponding  $\overline{\mathbb{Q}}_\ell$ -local system on  $X(\mathbf{I}, \mathbf{w}F)$ , then  $(\mathbf{I}, \mathbf{w})$  can be chosen such that*

- (i) *The  $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules  $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$  are mutually disjoint.*
- (ii) *If  $\rho$  is a irreducible unipotent constituent of  $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$  then  $\rho$  lies in the  $\Phi_d$ -block associated to  $\chi$  and  $i = C_d(\deg \rho / \deg \chi)$ .*

In addition, the endomorphism algebra  $\text{End}_{\mathbf{G}^F}(H^\bullet(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi))$  should be endowed with a natural structure of  $d$ -cyclotomic Hecke algebra. Let us note the following important consequence of this property: the eigenvalue of any sufficiently divisible power  $F^m$  of  $F$  on  $\rho$  should be  $q^{m(\alpha_\rho + A_\rho - \alpha_\chi - A_\chi)/d}$ .

The choice of a specific  $d$ -periodic element in this conjecture is not very relevant: it is conjectured that any other  $d$ -periodic element  $(\mathbf{I}, \mathbf{w}')$  can be obtained from  $(\mathbf{I}, \mathbf{w})$  by cyclic shifts, so that the cohomology of the corresponding varieties are isomorphic. This has already been proven when  $\mathbf{I} = \emptyset$  and  $F$  acts trivially on  $W$  (see [9, Remark 7.4]).

When  $d = 1$ , the unipotent blocks correspond to the usual Harish-Chandra series. In particular, when  $(\mathbf{G}, F)$  has type  $A$ , there is a unique unipotent block and it contains all the unipotent characters. The purpose of this paper is to show that from the cohomology of  $X(\pi)$  one can actually deduce all the other interesting cases (see Corollary 3.2):

**Theorem.** *For groups of type  $A$ , Conjecture 1 holds whenever it holds for  $d = 1$ , that is for  $X(\pi)$ .*

Let us emphasize that Conjecture 1 is known to be true only in a very small number of cases, namely when  $d = h$  is the Coxeter number by Lusztig [12], for groups of rank 2 by Digne, Michel and Rouquier [10] and when  $d = n$  for  $A_n$  and  $d = 4$  for  $D_4$  by Digne and Michel [8]. Therefore this theorem represents a very important step towards a proof of the geometric version of Broué's conjecture.

Even though this result depends on the conjectural description of  $X(\pi)$ , one can give an effective proof of Conjecture 1 for principal  $\Phi_d$ -blocks when  $d > (n + 1)/2$ . In that case the defect group is cyclic, and the modular representation theory of the block is fully understood. We will address this problem in a subsequent paper, where we will compute the cohomology of  $\overline{\mathbb{Z}}_\ell$  of the corresponding Deligne-Lusztig variety.

To give a flavour of the proof of the main theorem, recall that for groups of type  $A$ , we have by [3] a combinatorial description of the Deligne-Lusztig

induction associated to  $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ . In terms of partitions, it corresponds to adding a certain number of  $d$ -hooks. By transitivity, one can decompose  $\tilde{X}(\mathbf{I}, \mathbf{w}F)$  by means of simpler varieties  $\tilde{X}_{n,d}$ , each of which corresponds to adding a single  $d$ -hook to a partition. Now, using the methods developed in [11] one can compute the cohomology of (some quotient of)  $\tilde{X}_{n,d}$  in terms of  $\tilde{X}_{n-1,d}$  and  $\tilde{X}_{n-1,d-1}$  (see Theorem 2.1), providing an inductive argument to tackle Conjecture 1.

Note finally that Theorem 2.1 can be generalised to many other situations in type  $B$ ,  $C$  and  $D$ . However, two main problems arise: firstly, the limit case is either  $X(w_0)$  or  $X(\pi)$  and does not contain all the unipotent characters. Secondly, the methods in [11] work obviously for non-cuspidal unipotent characters only. We can obtain partial results on the principal series in that situation, which we believe are too coarse to be mentioned in this paper.

## 1 Parabolic varieties in type A

Throughout this paper,  $\mathbf{G}$  will denote any connected reductive algebraic group of type  $A_n$  over  $\mathbb{F} = \overline{\mathbb{F}}_p$ . We will consider a Frobenius endomorphism  $F : \mathbf{G} \rightarrow \mathbf{G}$  defining a standard  $\mathbb{F}_q$ -structure on  $\mathbf{G}$ . Since we will be interested in unipotent characters only, we will not make any specific choice for  $(\mathbf{G}, F)$  in its isogeny class. If  $\mathbf{H}$  is any  $F$ -stable subgroup of  $\mathbf{G}$  we will denote by  $H = \mathbf{H}^F$  the associated finite group.

The Weyl group  $W$  of  $\mathbf{G}$  is the symmetric group  $\mathfrak{S}_n$  and its Braid monoid  $B^+$  is the usual Artin monoid. It is generated by a set  $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$  corresponding to simple reflections  $s_1, \dots, s_n$  of  $W$ . Following [9], we define for  $1 \leq d \leq n+1$

$$\mathbf{v}_d = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_{n - \lfloor \frac{d}{2} \rfloor} \mathbf{s}_n \mathbf{s}_{n-1} \cdots \mathbf{s}_{\lfloor \frac{d+1}{2} \rfloor}$$

and  $\mathbf{J}_d = \{\mathbf{s}_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{d}{2} \rfloor\} \subset \mathbf{S}$ .

We are interested in computing the cohomology of the variety

$$X_{n,d} = X(\mathbf{J}_d, \mathbf{v}_d F)$$

with coefficients in any unipotent local system. Note that for  $d > 1$ , the element  $\mathbf{v}_d$  is reduced so that we can work with the variety  $X(\mathbf{J}_d, v_d F)$ . By [9, Lemma 11.7 and 11.8], the pair  $(\mathbf{J}_d, \mathbf{v}_d)$  is  $d$ -periodic so that it makes sense to study the cohomology of  $X_{n,d}$ . Recall from [9] that a  $d$ -periodic element is any pair  $(\mathbf{I}, \mathbf{b})$  with  $\mathbf{I} \subset \mathbf{S}$  and  $\mathbf{b} \in B^+$  such that  $\mathbf{b}F(\mathbf{b}) \cdots F^{d-1}(\mathbf{b}) = \pi/\pi_I$  where  $\pi = \mathbf{w}_0^2$  is the generator of the pure Braid group. It has been shown in [9] that this forces  $\mathbf{b}F$  to normalise  $\mathbf{I}$ . Note that when  $d \leq (n+1)/2$ ,  $\mathbf{v}_d$  is not *maximal* in the sense that it is not extendable to a  $d$ th root of  $\pi/\pi_I$  for a proper subset  $\mathbf{I}$  of  $\mathbf{J}_d$ . However, it can still be used to associate to any unipotent block a "good" parabolic Deligne-Lusztig variety. Before making any precise statement, we shall briefly recall the combinatorial objects that we will use.

**1.1.  $\Phi_d$ -blocks of  $G$ .** The unipotent characters of  $G$  are labeled by the partitions of  $n+1$ . If  $\lambda$  is such a partition, we will denote by  $\chi_\lambda$  the corresponding character,

with the convention that  $\chi_{(1,1,\dots,1)} = \text{St}_G$  is the Steinberg character of  $G$ . We shall also fix a representation  $V_\lambda$  over  $\overline{\mathbb{Q}}_\ell$  of character  $\chi_\lambda$ . For  $1 \leq d \leq n+1$ , the pair  $(\mathbf{L}_{J_d}, v_d F)$  represents a  $d$ -Levi subgroup of  $\mathbf{G}$ . From [3], we know how to express the  $d$ -Harish-Chandra induction in terms of combinatorics of partitions. To fix the notation, let  $\mu$  be a partition of  $n+1-d$  and  $X = \{x_1 < x_2 < \dots < x_s\}$  be a  $\beta$ -set associated to  $\mu$ . We may and we will assume that  $X$  is big enough, so that it contains  $\{0, 1, \dots, d-1\}$ . Let  $X'$  be the subset of  $X$  defined by  $X' = \{x \in X \mid x+d \notin X\}$ . It represents the possible  $d$ -hooks that can be added to  $\mu$ . For  $x \in X'$  we will denote by  $\mu * x$  the partition of  $n+1$  which has  $(X \setminus \{x\}) \cup \{x+d\}$  as a  $\beta$ -set.

We fix an  $F$ -stable Tits homomorphism  $t : B^+ \rightarrow N_{\mathbf{G}}(\mathbf{T})$ . By [9] the variety  $X_{n,d}$  has an étale covering  $\tilde{X}_{n,d} = \tilde{X}(\mathbf{J}_d, \mathbf{v}_d F)$  with Galois group  $\mathbf{L}_{J_d}^{t(\mathbf{v}_d)F}$ . Since  $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$  is a split group of type  $A_{n-d}$ , the partition  $\mu$  defines a unipotent local system  $\mathcal{F}_\mu$  on  $X_{n,d}$  such that  $\mathbf{H}_c^*(X_{n,d}, \mathcal{F}_\mu)$  and  $\mathbf{H}_c^*(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$  are isomorphic. Then we deduce from [3, Section 3.4] that there exist signs  $\varepsilon_x = \pm 1$  such that the  $d$ -Harish-Chandra induction of  $\chi_\mu$  is given by

$$\mathbf{R}_{\mathbf{L}_{J_d}}^{\mathbf{G}}(\chi_\mu) = \sum (-1)^i \mathbf{H}_c^i(X_{n,d}, \mathcal{F}_\mu) = \sum_{x \in X'} \varepsilon_x \chi_{\mu * x}.$$

In particular, the  $d$ -Harish-Chandra restriction of  $\chi_\lambda \in \text{Irr } G$  is non-zero until we reach the  $d$ -core  $\nu$  of  $\lambda$ , which corresponds to a  $d$ -cuspidal character  $\chi_\nu$ . The unipotent characters in the  $\Phi_d$ -block of  $G$  containing  $\chi_\lambda$  are all the characters that can be obtained by successive  $d$ -inductions from  $\chi_\nu$ . They correspond to partitions of  $n+1$  that have  $\nu$  as a  $d$ -core.

**1.2. A parabolic variety associated to a  $\Phi_d$ -block.** The cohomology of the variety  $X_{n,d}$  induces to a minimal  $d$ -induction since there is no  $d$ -split Levi between  $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$  and  $(\mathbf{G}, F)$ . By transitivity, one can form a Deligne-Lusztig variety  $\tilde{X}(\mathbf{I}, \mathbf{w})$  associated to the  $d$ -cuspidal character  $\chi_\nu$ . Let  $n+1-ad$  be the size of  $\nu$  and consider for  $i = 1, \dots, a$  the pairs  $(\mathbf{J}_d^{(i)}, \mathbf{v}_d^{(i)})$  where  $(\mathbf{J}_d^{(0)}, \mathbf{v}_d^{(0)}) = (\mathbf{J}_d, \mathbf{v}_d)$  and  $(\mathbf{J}_d^{(i+1)}, \mathbf{v}_d^{(i+1)})$  is the analogue of the pair  $(\mathbf{J}_d, \mathbf{v}_d)$  for the split group  $(\mathbf{L}_{J_d^{(i)}}, t(\mathbf{v}_d^{(i)} \dots \mathbf{v}_d^{(1)})F)$  of type  $A_{n-id}$ . Then one can readily check that the pair  $(\mathbf{I}, \mathbf{w}) = (\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} \dots \mathbf{v}_d^{(1)})$  is  $d$ -periodic.

By [9, Proposition 8.26] the associated Deligne-Lusztig variety  $\tilde{X}(\mathbf{I}, \mathbf{w}F)$  is isomorphic to the following amalgamated product

$$\tilde{X}(\mathbf{J}_d^{(1)}, \mathbf{v}_d^{(1)}F) \times_{\mathbf{L}_{J_d^{(1)}}^{t(\mathbf{v}_d^{(1)})F}} \dots \times_{\mathbf{L}_{J_d^{(a-1)}}^{t(\mathbf{v}_d^{(a-1)} \dots \mathbf{v}_d^{(1)})F}} \tilde{X}_{\mathbf{L}_{J_d^{(a-1)}}}(\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} t(\mathbf{v}_d^{(a-1)} \dots \mathbf{v}_d^{(1)})F).$$

Now each variety in this decomposition corresponds to a variety  $\tilde{X}_{n-id,d}$  for some  $i = 0, \dots, a-1$ . Since the cohomology of the latter with coefficients in a unipotent local system depends only on the isogeny class of the group (here, the split type  $A_{n-id}$ ) we obtain

$$\mathbf{R}\Gamma_c(\tilde{X}(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\nu) \simeq \mathbf{R}\Gamma_c(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_\ell) \otimes_{A_{n-d}} \dots \otimes_{A_{n-(a-1)d}} \mathbf{R}\Gamma_c(\tilde{X}_{n-(a-1)d,d}, \mathcal{F}_\nu). \quad (1.1)$$

Consequently, if we believe that the vanishing property in Conjecture 1 holds for the cohomology of  $X(\mathbf{I}, \mathbf{w})$ , then for any partition  $\mu$  of  $n + 1 - d$ , the graded  $G$ -module  $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$  should be multiplicity-free.

**1.3. Craven's formula in type A.** Conjecturally, the unipotent character  $\chi_\lambda$  is a constituent of only one cohomology group of  $R\Gamma(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\nu)$ . Craven proposed in [7] a formula which gives the degree of this cohomology group in terms of  $d$  and the generic degree of  $\chi_\lambda$  and  $\chi_\mu$ . More precisely, he considers a function  $C_d$  on some set of enhanced cyclotomic polynomials and conjectured that

$$\langle \chi_\lambda; H^i(X(\mathbf{I}, \mathbf{w}), \mathcal{F}_\nu) \rangle_G \neq 0 \iff i = C_d(\deg \chi_\lambda) - C_d(\deg \chi_\nu). \quad (1.2)$$

Let us recall the definition of  $C_d$ : assume that  $P \in \mathbb{Q}[x]$  is a polynomial such that the non-zero roots  $z_1, \dots, z_m$  (written with multiplicity) of  $P$  are all roots of unity. Let us denote by  $d^\circ(P)$  the degree of  $P$  and by  $v(P)$  its valuation, that is the degree of  $x^{d^\circ(P)}P(x^{-1})$ . Then Craven's function  $C_d$  is defined by

$$C_d(P) = \frac{1}{d} (d^\circ(P) + v(P)) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < 2\pi/d\} - \frac{1}{2} \#\{i = 1, \dots, m \mid z_i = 1\}.$$

Here, the argument  $\text{Arg } z$  of a non-zero complex number  $z$  is taken in  $[0; 2\pi)$ . More generally, if  $\zeta = \exp(2i\pi k/d)$  is a primitive  $d$ -root of unity, one can define a function  $C_\zeta$  by replacing  $d$  by  $d/k$  and 1.2 should hold for  $d$ th roots of  $(\pi/\pi_I)^k$ . Note also that Craven's function is additive: it satisfies  $C_\zeta(PQ) = C_\zeta(P) + C_\zeta(Q)$ .

For groups of type A, the degree  $\deg \chi_\lambda$  of the unipotent character  $\chi_\lambda$  is explicitly known (see for example [5, Section 13]). It is a polynomial in  $q$  of degree  $A_\lambda$  and valuation  $a_\lambda$  and no factors of the form  $(q - 1)$  can appear. In particular, Craven's function can be written

$$C_\zeta(\deg \chi_\lambda) = \frac{2\pi}{\text{Arg } \zeta} (a_\lambda + A_\lambda) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < \text{Arg } \zeta\}$$

where  $z_1, \dots, z_m$  are the roots with multiplicity of the polynomial  $\deg \chi_\lambda$ . Note that with this description it is already not obvious that the rational number on the right-hand side of 1.2 is actually an integer.

Since  $C_d$  is additive, formula 1.2 together with the quasi-isomorphism 1.1 suggests that the partition  $\nu$  should not be necessarily a  $d$ -core. In the case of an elementary  $d$ -induction (when  $a = 1$ ) we can write everything explicitly using [7, Proposition 9.1]; the second equality follows from an easy calculation:

**Lemma 1.3.** *Let  $\mu$  be a partition with corresponding  $\beta$ -set  $X$  that we assume to be large enough. For  $x \in X'$ , we have*

$$C_d(\deg \chi_{\mu^{*x}}) - C_d(\deg \chi_\mu) = 2(n + 1 - d - x + \#\{y \in X \mid y < x\}) + \#\{y \in X \mid x < y < x + d\}$$

and

$$a_{\mu^{*x}} + A_{\mu^{*x}} - a_\mu - A_\mu = d(n - d + s - x).$$

These integers give conjecturally the degree of the cohomology group of  $X_{n,d}$  in which  $\chi_{\mu^{*x}}$  will appear, as well as the corresponding eigenvalue of the Frobenius. Since we will work with the cohomology with compact support, we shall rather work with the integers

$$\pi_d(X, x) = 2(n + x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x + d\}$$

and

$$\gamma_d(X, x) = n + 1 + x - s.$$

They are readily deduced from the previous ones by taking into account the dimension of  $X_{n,d}$  (which is equal to  $\ell(v_d) = 2n + 1 - d$ ). Now Conjecture 1 can be deduced from the following:

**Conjecture 1.4.** *Let  $n \geq 1$  be a positive integer and  $1 \leq d \leq n + 1$ . Let  $\mu$  be a partition of  $n - d + 1$  and  $X$  be its  $\beta$ -set, assumed to be large enough. Then*

$$\mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu) \simeq \bigoplus_{x \in X'} V_{\mu * x}[-\pi_d(X, x)] \otimes \overline{\mathbb{Q}}_\ell(\gamma_d(X, x))$$

as a complex of  $G \times \langle F \rangle$ -modules.

The purpose of this paper is to prove that this conjecture holds for any  $d$  whenever it holds for  $d = 1$ . As a byproduct, we shall deduce the cohomology of parabolic Deligne-Lusztig varieties associated to any unipotent block from the knowledge of the cohomology of  $X(\mathbf{w}_0^2)$ .

## 2 Decomposition of the quotient

The group  $(\mathbf{L}_{J_d}, \dot{v}_d F)$  is split of type  $A_{n-d}$ , therefore the unipotent representations of the corresponding finite group are labelled by partitions  $\mu$  of  $n - d + 1$ . To such a partition one can associate a unipotent  $\overline{\mathbb{Q}}_\ell$ -local system  $\mathcal{F}_\mu$  on  $X_{n,d}$ . From [3] we know that the irreducible constituents of the virtual character  $\sum (-1)^i \mathrm{H}_c^i(X_{n,d}, \mathcal{F}_\mu)$  correspond to the partitions of  $n + 1$  obtained by adding a  $d$ -hook to  $\mu$ . The restriction to  $\mathfrak{S}_n$  of the corresponding irreducible representation corresponds to a partition obtained by

- either restricting the hook (usually in two different ways),
- or restricting  $\mu$ .

The main result of this section gives a geometric interpretation of this phenomenon.

**Theorem 2.1.** *Assume that  $d \geq 2$ . Let  $I = \{s_j \mid 1 \leq j \leq n - 1\}$ . Let  $\mu$  be a partition of  $n - d + 1$  and  $\{\mu^{(j)}\}$  be the set of partitions of  $n - d$  obtained by restricting  $\mu$ . Then there is a distinguished triangle in  $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$*

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

**Remark 2.2.** From [1, Proposition 1.1] we can deduce that the cohomology of a Deligne-Lusztig variety with coefficients in a unipotent local system depends only on the type of  $(\mathbf{G}, F)$ . Therefore there is no ambiguity in the statement of the theorem.

We will use the results in [11] to compute the quotient of  $\tilde{X}(J_d, v_d F)$  by the finite group  $U_I$ . Recall that  $X_{n,d} = X(J_d, v_d F)$  can be decomposed into locally closed  $P_I$ -subvarieties  $X_x$ , where  $x$  is a  $I$ -reduced- $J_d$  element of  $W$ . In our situation, at most two pieces will appear:

**Lemma 2.3.** *Assume that  $2 \leq d \leq n$ . The variety  $X_x$  is non-empty if and only if  $x$  is one of the two following elements:*

- (1)  $x_0 = s_n s_{n-1} \cdots s_1$
- (2)  $x_1 = s_n s_{n-1} \cdots s_{n - \lfloor \frac{d}{2} \rfloor + 1}$ .

*Proof.* For simplicity, we shall denote  $a = \lfloor \frac{d+1}{2} \rfloor$  and  $b = n - \lfloor \frac{d}{2} \rfloor$  so that  $w = v_d = s_1 \cdots s_b s_n \cdots s_a$  and  $J = J_d = \{a+1, \dots, b\}$ . If  $x$  is a  $I$ -reduced- $J$  element, then  $x = s_n s_{n-1} \cdots s_i$  with  $b+1 \leq i \leq n+1$  or  $1 \leq i \leq a$ . Recall from [11] that the variety  $X_x$  is non-empty if and only if there exists  $y = y_1 \cdots y_r \in W_J$  and an  $x$ -distinguished subexpression  $\gamma$  of  $yw$  such that the products of the elements of  $\gamma$  lies in  $(W_I)^x$ . We first observe that for  $i \notin \{1, n+1\}$  we have

$$(W_I)^x = \langle s_1, \dots, s_{i-2}, s_i s_{i-1} s_i, s_{i+1}, \dots, s_n \rangle.$$

Now, since  $x$  is reduced- $J$ , the subexpression  $\gamma$  is the concatenation of  $(y_1, \dots, y_r)$  and an  $xy$ -distinguished subexpression  $\tilde{\gamma}$  of  $w$ . If  $i > b+1$  or  $2 \leq i \leq a$ , the group  $W_J$  is included in  $(W_I)^x$ . Therefore the product of the elements of  $\tilde{\gamma}$  must lie in  $(W_I)^x$ . We shall distinguish two cases:

**Case (1).** We assume that  $i > b+1$ . In that case  $x$  commutes with any element of  $W_J$ , so that  $\tilde{\gamma}$  is an  $yx$ -distinguished subexpression of  $w$ . Then

- if  $x$  is trivial (that is if  $i = n+1$ ), then any  $y$ -distinguished subexpression of  $w$  contains necessarily  $s_n$  and hence cannot produce any element of  $(W_I)^x = W_I$ ;
- if  $x$  is non-trivial then  $i \leq n$ , and a subexpression of  $w$  lies in  $(W_I)^x$  if and only if it does not contain  $s_i$  or  $s_{i-1}$ . However, such a subexpression will never be  $yx$ -distinguished since for all  $v$  in  $W_I$  we have  $v x s_{i-1} > v x$ .

We deduce that the variety  $X_x$  is empty in this case.

**Case (2).** We assume that  $2 \leq i \leq a$ . The subexpression  $\tilde{\gamma}$  is  $^x yx$ -distinguished. Since  $i \leq a$ , we have  $^x W_J = W_{a, \dots, b-1}$ . For  $j < i-1$ , we have  $x s_j = s_j x$ ; moreover,  $x s_{i-1}$  is  $I$ -reduced, so that  $\tilde{\gamma}$  should start with  $(s_1, \dots, s_{i-1})$ . In that case, the product of the elements of  $\tilde{\gamma}$  cannot belong to  $(W_I)^x$ . Indeed, a subexpression of  $s_{i-1} s_i \cdots s_b s_n \cdots s_a$  starting with  $s_{i-1}$  will never give an element of  $(W_I)^x$ , the only non-trivial situation being the case  $a = i$ :

- with the notation in [10, Section 2.1.2] we have  $s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a = \underline{s_{a+1} \cdots s_b s_n \cdots s_{a+1} s_{a-1} s_a}$  and neither  $s_{a-1}$  nor  $s_{a-1} s_a$  belongs to  $(W_I)^x$ ;
- $s_{a-1} s_a s_{a+1} \cdots s_b s_n \cdots s_a = (s_{a-1} s_a s_{a-1}) s_{a-1} s_{a+1} \cdots s_b s_n \cdots s_a$  and we are back to the previous case.

This forces the variety  $X_x$  to be empty. □

*Proof of the Theorem.* From the previous lemma we deduce that  $\tilde{X}(J_d, v_d F)$  decomposes as a disjoint union  $\tilde{X}(J_d, v_d F) = \tilde{X}_{x_0} \cup \tilde{X}_{x_1}$  with  $\tilde{X}_{x_0}$  being open. Using [11] we shall now determine the cohomology of the quotient of each of these varieties by  $U_I$ . Throughout the proof, we will denote  $\tilde{I} = \{s_2, \dots, s_n\} \subset S$  the conjugate of  $I$  by  $w_0$ .

When  $x = x_0 = w_I w_0$  and  $d > 2$  we are in the situation of [11, Proposition 3.4]. Indeed,  $v_d = s_1 w'$  with  $w' \in W_{2, \dots, n}$  and  $s_1$  commutes with  $W_{J_d} \subset W_{3, \dots, n-1}$  so that we obtain

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}/N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(K_{x_0}, \dot{v}F)/{}^{x_0}N', \overline{\mathbb{Q}}_\ell)$$

with  $v = {}^{x_0}w'$  and  $K_{x_0} = I \cap {}^{x_0}\Phi_{J_d} = {}^{x_0}J_d$ . For simplicity, we shall rather consider the conjugate by  $x_0$  of the right-hand side

Recall that  $N$  and  $N'$  are normal subgroup of  $\mathbf{L}_{J_d}$  and are both contained in  $\mathbf{T}$ . In particular, any unipotent character of  $\mathbf{L}_{J_d}^{wF}$  (resp. of  $\mathbf{L}_{J_d}^{w'F}$ ) is trivial on  $N$  (resp.  $N'$ ). Consequently, for any unipotent character  $\chi$  of  $\mathbf{L}_{J_d}^{wF}$  we obtain the following quasi-isomorphism of complexes of  $L_I \times \langle F \rangle$ -modules:

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}, \overline{\mathbb{Q}}_\ell)_\chi \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F), \overline{\mathbb{Q}}_\ell)_{x_0 \chi}.$$

Finally, we observe that the varieties  $\tilde{X}_{\mathbf{L}_I}({}^{x_0}J_d, \dot{v}F)$  and  $\tilde{X}_{n-1, d-1}$  have the same cohomology with coefficients in any unipotent local system. Indeed, if we denote  $(s_1, \dots, s_{n-1})$  by  $(t_1, \dots, t_{n-1})$  if  $d$  is odd or by  $(t_{n-1}, \dots, t_1)$  if  $d$  is even, then we have

$$v = t_1 t_2 \cdots t_{n-1 - \lfloor \frac{d-1}{2} \rfloor} t_{n-1} t_{n-2} \cdots t_{\lfloor \frac{d}{2} \rfloor}$$

which corresponds to the element  $v_{d-1}$  in the Weyl group  $W_I = \langle t_1, \dots, t_{n-1} \rangle$  of type  $A_{n-1}$ .

When  $x = x_0$  and  $d = 2$ , we can write  $v_2 = w w'$  with  $w = s_n s_{n-1} \cdots s_2$  and  $w' = s_1 s_2 \cdots s_n = s_1 w''$  so that  $\tilde{X}_{n,2} \simeq \tilde{X}(\{s_2, \dots, s_{n-1}\}, \mathbf{w}w'F)$ . Moreover, via this isomorphism we have

$$\tilde{X}_{x_0} \simeq \bigcup_{y \in W} \tilde{X}_{(x_0, y)}.$$

We claim that  $\tilde{X}_{(x_0, y)}$  is empty unless  $y \in W_I w_0 W_{J'_2}$  where  $J'_2 = J_2^w = \{s_3, s_4, \dots, s_n\}$ . The piece  $\tilde{X}_{(x_0, y)}$  consists of pairs  $(p x_0 \mathbf{P}_{J_2}, p' y \mathbf{P}_{J'_2})$  with  $p, p' \in \mathbf{P}_I$  such that  $p^{-1} p' \in x_0 \mathbf{P}_{J_2} w \mathbf{P}_{J'_2} y^{-1}$  and  $p'^{-1} p \in y \mathbf{P}_{J'_2} w' \mathbf{P}_{J_2} x_0^{-1}$ . In particular, if  $\tilde{X}_{(x_0, y)}$  is non-empty then the double coset  $\mathbf{P}_I y \mathbf{P}_{J'_2}$  has a non-trivial intersection with  $x_0 \mathbf{B}w$ . But  $x_0 = w_I w_0$  is reduced- $\tilde{I}$  so that  $\ell(x_0 w) = \ell(x_0) + \ell(w)$  and  $x_0 \mathbf{B}w \subset \mathbf{P}_I w_0 \mathbf{P}_{J'_2}$ . This forces  $y$  to lie in  $W_I w_0 W_{J'_2}$ . Note that  $w_0(J'_2) \subset I$  so that the minimal element in this coset is  $x_0$  and we have  $\tilde{X}_{x_0} \simeq \tilde{X}_{(x_0, x_0)}$ .

Now  $s_1$  commutes with  $J'_2$  and we can apply [11, Proposition 3.4] to obtain, after conjugation by  $x_0$ :

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{x_0}/N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(\mathbf{K}_{x_0}, \mathbf{v}v'F)/{}^{x_0}N', \overline{\mathbb{Q}}_\ell).$$

with  $K_{x_0} = {}^{x_0}J_2$ ,  $v = {}^{x_0}w$  and  $v' = {}^{x_0}w''$ . If we denote  $(s_1, \dots, s_{n-1})$  by  $(t_{n-1}, \dots, t_1)$  we obtain  ${}^{x_0}J_2 = \{t_2, \dots, t_{n-1}\}$  and  $\mathbf{vv}' = \mathbf{t}_1 \cdots \mathbf{t}_{n-1} \mathbf{t}_{n-1} \cdots \mathbf{t}_1$  so that the pair  $(\mathbf{K}_{x_0}, \mathbf{vv}')$  corresponds to  $(\mathbf{J}_1, \mathbf{v}_1)$  in the Weyl group  $W_I$  of type  $A_{n-1}$ . As before,  $N$  and  $N'$  do not play any role if we consider the unipotent part of the previous quasi-isomorphism.

When  $x = x_1$  we use [11, Proposition 3.2]: the conjugate of  $v_d$  by  $x_1$  is

$$v = x_1 v_d x_1^{-1} = s_1 s_2 \cdots s_{n-1-\lfloor \frac{d}{2} \rfloor} s_{n-1} s_{n-2} \cdots s_{\lfloor \frac{d+1}{2} \rfloor}$$

which corresponds exactly to the element  $v_d$  in  $W_I$ . We can therefore identify the cohomology of the varieties  $\mathbf{X}_{\mathbf{L}_I}(K_{x_1}, vF)$  and  $\mathbf{X}_{n-1,d}$  with coefficients in any unipotent local system (see Remark 2.2). The group  $\mathbf{P}_I \cap {}^{x_1}\mathbf{L}_{J_d}$  is a  $vF$ -stable parabolic subgroup of  ${}^{x_1}\mathbf{L}_{J_d}$  and  $\mathbf{L}_{K_{x_1}} = \mathbf{L}_I \cap {}^{x_1}\mathbf{L}_{J_d}$  is a stable Levi complement. Therefore it makes sense to consider the Harish-Chandra restriction  ${}^*\mathbf{R}_{\mathbf{K}}^J \chi$  of any unipotent character  $\chi$  of  $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$  to  $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$ . From [11, Proposition 3.2] (see also [11, Remark 3.12]) we deduce the following quasi-isomorphism

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_1}, \overline{\mathbb{Q}}_\ell)_\chi[2](-1) \simeq \mathrm{R}\Gamma_c(\tilde{\mathbf{X}}_{\mathbf{L}_I}(K_{x_1}, vF), \overline{\mathbb{Q}}_\ell)_{* \mathbf{R}_{\mathbf{K}}^J \chi}.$$

Let  $\mu$  be a partition of  $n - d + 1$ . The cohomology of the variety  $\mathbf{X}_{n,d}$  with coefficients in the local system  $\mathcal{F}_\mu$  is given by

$$\mathrm{R}\Gamma_c(\mathbf{X}_{n,d}, \mathcal{F}_\mu) \simeq \mathrm{R}\Gamma_c(\tilde{\mathbf{X}}(\mathbf{J}_d, \mathbf{v}_d F), \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$$

where  $\chi_\mu$  is the unipotent character of  $\mathbf{L}_{J_d}^{\dot{v}F}$  corresponding to the partition  $\mu$ . Since  $(\mathbf{L}_{K_{x_1}}, vF)$  is a split group of type  $A_{n-d-1}$ , the Harish-Chandra restriction of  $\chi_\mu$  from  $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{x_1}\mathbf{L}_{J_d})^{\dot{v}F}$  to  $\mathbf{L}_{K_{x_1}}^{\dot{v}F}$  is the sum of the  $\chi_{\mu_i}$ 's where the  $\mu_i$ 's are the partitions of  $n - d$  obtained by restricting  $\mu$ . With with description, we get the following isomorphisms in  $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_0}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \mathbf{X}_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$$

and  $\mathrm{R}\Gamma_c(U_I \backslash \tilde{\mathbf{X}}_{x_1}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} \simeq \mathrm{R}\Gamma_c(\mathbf{X}_{n-1,d}, \bigoplus \mathcal{F}_{\mu_i})[-2](1)$ .

We conclude using the distinguished triangle associated to the decomposition  $\tilde{\mathbf{X}}_{n,d} = \tilde{\mathbf{X}}_{x_0} \cup \tilde{\mathbf{X}}_{x_1}$  in which  $\tilde{\mathbf{X}}_{x_0}$  is open.  $\square$

### 3 Cohomology over $\overline{\mathbb{Q}}_\ell$

We have just seen how to relate the Harish-Chandra restriction of the cohomology of  $\mathbf{X}_{n,d}$  to the cohomology of smaller parabolic Deligne-Lusztig varieties. We shall now explain how this strategy provides an inductive method for a thorough determination of the cohomology of  $\mathbf{X}_{n,d}$  with coefficients in any unipotent local system. The main result in this section gives an inductive strategy towards a proof of Conjecture 1:

**Theorem 3.1.** *Let  $n \geq 1$  and  $2 \leq d \leq n$ . If Conjecture 1.4 holds for  $(n, d + 1)$ ,  $(n - 1, d - 1)$  and  $(n - 1, d)$  then it holds for  $(n, d)$ .*

Note that we already know from [12] that Conjecture 1.4 holds in the Coxeter case, corresponding to  $(n, n + 1)$ . Therefore  $d = 1$  is the only limit case. But  $\pi = \mathbf{w}_0^2$  is a maximal 1-periodic element in the sense of [9] and in that specific case, a general conjecture for the cohomology has been already formulated in [4]: a unipotent character  $\chi_\lambda$  can appear in  $H_c^i(X(\pi))$  for  $i = 4\nu_G - 2A_\lambda$  only, where  $\nu_G$  is the number of positive roots. An important consequence of Theorem 3.1 is that knowing the cohomology of  $X(\pi)$  is sufficient for determining all the other interesting cases:

**Corollary 3.2.** *For groups of type A, Conjecture 1 holds for any  $d \geq 1$  as soon as it holds for  $d = 1$ .*

*Proof.* Assume that Conjecture 1 holds for  $d = 1$ , that is for the variety  $X(\pi)$ . Let  $I = J_1 = \{s_2, \dots, s_n\}$  and  $\mathbf{b} = \mathbf{v}_1 = \mathbf{s}_1 \cdots \mathbf{s}_n \mathbf{s}_n \cdots \mathbf{s}_1$ . By [9, Proposition 8.26] we have

$$\mathrm{R}\Gamma_c(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\tilde{X}(\mathbf{I}, \mathbf{b}F), \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}_I^{t(\mathbf{b})F}} \mathrm{R}\Gamma_c(X(\pi_I), \overline{\mathbb{Q}}_\ell).$$

Since the cohomology of  $X(\pi_I)$  contains all the unipotent characters of  $\mathbf{L}_I^{t(\mathbf{b})F}$ , we deduce that for any partition  $\mu$  of  $n$ , the groups  $H_c^i(X_{n,1}, \mathcal{F}_\mu)$  are submodules of the cohomology groups of  $X(\pi)$ . Consequently, they are disjoint as soon as Conjecture 1 holds for  $X(\pi)$ . Since we have assumed that it holds also for  $X(\pi_I)$  we have actually

$$H_c^i(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\mu \vdash n} H_c^{i-4\nu_{\mathbf{L}_I}+2A_\mu}(X_{n,1}, \mathcal{F}_\mu)(2\nu_{\mathbf{L}_I} - a_\mu - A_\mu) \quad (3.3)$$

as a  $G \times \langle F \rangle$ -module. Now, the alternating sum of the cohomology groups of  $X_{n,1}$  represents the Deligne-Lusztig induction from  $\mathbf{L}_I^{t(\mathbf{b})F} \simeq L_I$  to  $G$ . Therefore a character  $\chi_\lambda$  appear in  $H_c^i(X_{n,1}, \mathcal{F}_\mu)$  if and only if  $\mu$  is the restriction of  $\lambda$ , or equivalently, if  $\lambda$  is obtained from  $\mu$  by adding a 1-hook. This, together with 3.3 and Lemma 1.3 proves that Conjecture 1.4 holds for  $X_{n,1}$ , and therefore for any variety  $X_{n,d}$  by 3.1. We use 1.1 to conclude.  $\square$

*Proof of the Theorem.* Let  $X$  be a  $\beta$ -set associated the partition  $\mu$  of  $n - d + 1$ . We can always assume that it contains  $\{0, 1, \dots, d - 1\}$ . The partitions  $\mu^{(j)}$ 's of  $n - d$  which are obtained by restricting  $\mu$  can be associated to the following  $\beta$ -set:

$$X^{(j)} = \{x_1^{(j)} < \dots < x_s^{(j)}\} \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_j - 1 & \text{if } i = j; \\ x_i & \text{otherwise.} \end{cases}$$

Let  $I = \{s_1, \dots, s_{n-1}\}$ . By Theorem 2.1, the Harish-Chandra restriction of the cohomology of  $X_{n,d}$  can be fitted into the following distinguished triangle:

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

Now, if we assume that Conjecture 1.4 holds for both  $(n - 1, d - 1)$  and  $(n - 1, d)$ , the complexes on the left and right-hand side are completely determined. Let us examine the different eigenvalues of  $F$  that can appear:

- (a) on  $\mathcal{C} = \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ , the eigenvalues of  $F$  are  $q^{n+x-s}$  and  $q^{n+1+x-s}$  with  $x \in X$  such that  $x+d-1 \notin X$ . The character of the corresponding eigenspace is  $\chi_\lambda$  where  $\lambda$  is the partition of  $n$  obtained by adding to  $\mu$  a  $(d-1)$ -hook represented by  $x$ ;
- (b) on  $\mathcal{D}^{(j)} = \mathrm{R}\Gamma_c(X_{n-1,d}, \mathcal{F}_{\mu^{(j)}})[-2](1)$ , the eigenvalues of  $F$  are  $q^{n+1+x-s}$  where  $x \in X^{(j)}$  is such that  $x+d \notin X^{(j)}$ . The character of the corresponding eigenspace is  $\chi_\lambda$  where  $\lambda$  is the partition of  $n$  obtained by adding to  $\mu^{(j)}$  a  $d$ -hook represented by  $x$ .

We shall now determine  $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$  by studying each eigenspace of  $F$  separately. For  $x$  a positive integer, we can separate the following cases:

**Case (1).** Assume first that  $x \in X$  and  $x+d \notin X$ . Let  $\lambda = \mu * x$  be the partition of  $n+1$  obtained by adding to  $\mu$  a  $d$ -hook from  $x$ . We want to prove that the  $q^{n+1-s+x}$ -eigenspace of  $F$  on  $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$  is non-zero in degree  $\pi_d(X, x)$  only and that its character is the Harish-Chandra restriction of  $\chi_\lambda$ .

By (a), the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $\mathcal{C}$  will produce non-zero representations in the following two cases:

- if  $x+d-1 \notin X$ , then one obtains a character associated to the  $\beta$ -set  $(X \setminus \{x\}) \cup \{x+d-1\}$  and it is concentrated in degree

$$\begin{aligned} 2 + \pi_{d-1}(X, x) &= 2 + 2(n-1+x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d-1\} \\ &= 2(n+x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d-1\} \\ &= \pi_d(X, x). \end{aligned}$$

- if  $x+1 \in X$ , then the corresponding  $\beta$ -set is  $(X \setminus \{x+1\}) \cup \{x+d\}$  and the associated character appears in degree  $1 + \pi_{d-1}(X, x+1)$  only. But we have

$$\begin{aligned} \pi_{d-1}(X, x+1) &= 2(n+x - \#\{y \in X \mid y < x+1\}) - \#\{y \in X \mid x+1 < y < x+d\} \\ &= 2(n+x-1 - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x+d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

On the other hand, the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $\mathcal{D}^{(j)}$  is non-zero if and only if  $x \in X^{(j)}$  and  $x+d \notin X^{(j)}$ . This happens if and only if  $x$  and  $x+d+1$  are different from  $x_j$ . In that case, the  $\beta$ -set corresponding to the character of the eigenspace will be  $(X^{(j)} \setminus \{x\}) \cup \{x+d\}$ . Furthermore, the degree in which this character will appear is  $2 + \pi_d(X^{(j)}, x)$ , which is clearly equal to  $\pi_d(X, x)$  in that case.

Now, the  $\beta$ -set  $Y = (X \setminus \{x\}) \cup \{x+d\}$  is associated to the partition  $\lambda = \mu * x$ . As mentioned in the beginning of Section 2, the restriction of  $\lambda$  is obtained by restricting the hook (usually in two different ways) or by restricting  $\mu$ . In the framework of  $\beta$ -sets, it corresponds to decreasing specific elements of  $Y$ :

- if  $x+d-1 \notin X$ , one can replace  $x+d$  by  $x+d-1$  in  $Y$  and we obtain the  $\beta$ -set  $(X \setminus \{x\}) \cup \{x+d-1\}$ ;

- if  $x + 1 \in X$ , one can replace  $x + 1$  by  $x$  in  $Y$  and we obtain the  $\beta$ -set  $(X \setminus \{x + 1\}) \cup \{x + d\}$ ;
- if  $x_j \in X$  is different from  $x$  or from  $x + d + 1$ , and if  $x_j - 1 \notin X$ , then one can replace  $x_j$  by  $x_j - 1$  in  $Y$  to obtain the  $\beta$ -set  $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$ .

This shows that the character of the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$  is the Harish-Chandra restriction of  $\chi_{\mu * x}$ .

**Lemma 3.4.** *Let  $\lambda$  be a partition of  $n+1$ , with  $n \geq 3$  and let  $\chi$  be a (non-necessarily irreducible) unipotent character of  $G$ . Then the Harish-Chandra restriction of  $\chi$  and  $\chi_\lambda$  are equal if and only if  $\chi = \chi_\lambda$ .*

*Proof of the Lemma.* Assume that there exists a partition  $\nu = \{\nu_1 \leq \nu_2 \leq \dots \leq \nu_r\}$  of  $n+1$  with  $\nu_1 \neq 0$  such that the difference between the Harish-Chandra restriction of  $\chi_\lambda$  and  $\chi_\nu$  is still a unipotent character. This means that in the Young diagram of  $\nu$ , any box that can be removed can be replaced to form the Young diagram of  $\lambda$ . If  $\nu \neq \lambda$ , this is possible only if  $\nu_1 = \nu_2 = \dots = \nu_r$ .

Let  $\chi$  be a unipotent character of  $G$  which has the same Harish-Chandra restriction as  $\chi_\lambda$ . If  $\chi \neq \chi_\lambda$ , we deduce from the previous argument that all the irreducible constituents of  $\chi$  are of the form  $\chi_\nu$  with  $\nu = (a, a, \dots, a)$ . This can happen if and only if  $n = 2$  and  $\lambda = (1, 2)$ .  $\square$

When  $n \geq 3$ , we deduce that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)$  is actually  $\chi_{\mu * x}$ . If  $n = 2$ , then the only ambiguity concerns  $\chi_{\mu * x}$  when  $\mu * x = (1, 2)$ . In that case, the  $q^{n+1+x-s}$ -eigenspace can be either  $\chi_{\mu * x}$  or  $1_G + \text{St}_G$ . But by [9, Corollary 8.28], the trivial character and the Steinberg character cannot occur in the same cohomology group of  $X_{n,d}$  as soon as the dimension of this variety is non-zero.

**Case (2).** Assume now that  $x \notin X$ . The  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $\mathcal{D}^{(j)}$  is non-zero if and only if  $x \in X^{(j)}$  and  $x + d \notin X^{(j)}$ . Since  $x \notin X$ , this forces  $x = x_j^{(j)} = x_j - 1$  and  $x_j + d - 1 \notin X$ . In that case, its character corresponds to a partition with  $\beta$ -set  $(X \setminus \{x + 1\}) \cup \{x + d\}$  and it appears in degree  $2 + \pi_d(X^{(j)}, x)$  only. On the other hand, the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $\mathcal{C}$  is non-zero if and only if  $x + 1 \in X$  and  $x + 1 + d - 1 = x + d \notin X$ . By (a), the character of this eigenspace corresponds to a partition with  $\beta$ -set  $(X \setminus \{x + 1\}) \cup \{x + d\}$ . Furthermore, it appears in degree  $1 + \pi_{d-1}(X, x + 1)$  only. Note that in that case we have

$$\begin{aligned} \pi_{d-1}(X, x + 1) &= 2(n + x - \#\{y \in X \mid y < x + 1\}) - \#\{y \in X \mid x + 1 < y < x + d\} \\ &= 2(n + x - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) + 1 \end{aligned}$$

$$\begin{aligned} \text{and } \pi_d(X^{(j)}, x) &= 2(n - 1 + x - \#\{y \in X^{(j)} \mid y < x\}) - \#\{y \in X^{(j)} \mid x < y < x + d\} \\ &= 2(n - 1 + x - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

We deduce that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$  is isotypic and concentrated in two consecutive degrees. However, there are only a few unipotent characters that can have an isotypic Harish-Chandra restriction: they correspond to partitions of the form  $(a, a, \dots, a)$ . Among them we can find the Steinberg character  $\text{St}_G$  (with  $a = 1$ ) and the trivial character  $1_G$  (with  $a = n + 1$ ). But by [9, Corollary 8.28] they have respective eigenvalues 1 and  $q^{2n+1-d}$ . Let us write the  $\beta$ -set of  $\mu$  as  $X = \{0, 1, \dots, k-1, \mu_1+k, \mu_2+k+1, \dots, \mu_r+s-1\}$  with  $k \geq d$ . Since  $x \notin X$ , one must have  $k-1 < x < \mu_r+s-1$  and hence

$$d-1 \leq n+k-s < n+1+x-s < n+1+\mu_r-1 \leq 2n+1-d. \quad (3.5)$$

We deduce that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)$  is either zero, or consists of two copies of the character  $\chi_\lambda$  in two consecutive degrees, where  $\lambda = (a, a, \dots, a)$  with  $1 < a < n+1$ . We shall actually prove that it is always zero, but before that we need to study the last case.

**Remark 3.6.** The Harish-Chandra restriction of  $\chi_\lambda$  corresponds to the partition  $(a-1, a, \dots, a)$ . Therefore if the associated character appears in the cohomology of  $\mathcal{C}$  then the  $\beta$ -set  $(X \setminus \{x+1\}) \cup \{x+d\}$  must correspond to the partition  $(a-1, a, \dots, a)$ . This gives a rather strong condition on  $X$ : we will have either

$$X = \{0, 1, \dots, k-1, x+1, b, b+2, b+3, \dots, \widehat{x+d}, \dots, b+r\}$$

with  $b+2 \leq x+d \leq b+r$ , or

$$X = \{0, 1, \dots, k-1, x+1, x+d+2, x+d+3, \dots, x+d+r\}.$$

**Case (3).** Finally, assume that  $x \in X$  and  $x+d \in X$ . The  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $\mathcal{D}^{(j)}$  is non-zero if and only if  $x \in X^{(j)}$  and  $x+d \notin X^{(j)}$ . Since  $x+d \in X$ , this forces  $x+d = x_j$  (and therefore  $x_j - 1 \notin X$ ). In that case, the character of the eigenspace corresponds to a partition with  $\beta$ -set  $(X^{(j)} \setminus \{x\}) \cup \{x+d\} = (X \setminus \{x\}) \cup \{x+d-1\}$ . On  $\mathcal{C}$ , the Frobenius has a non-zero  $q^{n+1+x-s}$ -eigenspace if and only if  $x+d-1 \notin X$  and its character is again associated to the  $\beta$ -set  $(X \setminus \{x\}) \cup \{x+d-1\}$ . This ensures that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)^{U_I}$  is isotypic. Using  $x+d-1$  instead of  $x$  in the inequalities 3.5 yields  $0 < n+1+x-s < 2n+2-2d$  and therefore the previous argument applies. We deduce that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)$  is again either zero or consists of two copies of the character  $\chi_\lambda$  in two consecutive degrees, namely  $\pi_d(X, x) - 1$  and  $\pi_d(X, x)$ , where  $\lambda = (a, a, \dots, a)$  and  $1 < a < n+1$ .

To conclude, we need to prove that the  $q^{n+1+x-s}$ -eigenspaces of  $F$  are actually zero whenever  $x \notin X$  or  $x+d \in X$ . Let us first summarize what we have proven so far:

- (1) if  $x \in X$  and  $x+d \notin X$  then the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)$  is  $\chi_{\mu * x}$  and it appears in degree  $\pi_d(X, x)$  only;
- (2) if  $x \notin X$ , the  $q^{n+1+x-s}$ -eigenspace of  $F$  is zero unless  $x+1 \in X$  and  $x+d \notin X$ . In that case, it may consist of two copies of  $\chi_\lambda$ , one in degree  $\pi_d(X, x) + 1$  and one in degree  $\pi_d(X, x) + 2$ , where  $\lambda = (a, a, \dots, a)$  with  $1 < a < n+1$ .

Moreover, the  $\beta$ -set  $(X \setminus \{x+1\}) \cup \{x+d\}$  must correspond to the partition  $(a-1, a, \dots, a)$  (see Remark 3.6);

- (3) if  $x \in X$  and  $x+d \in X$ , the  $q^{n+1+x-s}$ -eigenspace of  $F$  is zero unless  $x+d-1 \notin X$ . In that case, it can only be  $\chi_\lambda$ -isotypic with  $\lambda = (a, a, \dots, a)$  and  $1 < a < n+1$ . Moreover, it is non-zero in degrees  $\pi_d(X, x) - 1$  and  $\pi_d(X, x)$  only, and  $(X \setminus \{x\}) \cup \{x+d-1\}$  must be a  $\beta$ -set of the partition  $(a-1, a, \dots, a)$ .

Now, if we assume that Conjecture 1.4 holds for the variety  $X_{n,d+1}$ , then we can use the distinguished triangle

$$\mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) \longrightarrow \mathrm{R}\Gamma_c(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I} \longrightarrow \mathrm{R}\Gamma_c(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow$$

from Theorem 2.1 to prove that the eigenspaces of  $F$  on  $H_c^*(X_{n,d}, \mathcal{F}_\mu)$  in cases (2) and (3) are indeed zero.

Assume that  $x \notin X$  and that there exists  $1 < a < n+1$  such that the character  $\chi_\lambda = \chi_{(a,a,\dots,a)}$  appears twice in the  $q^{n+1+x-s}$ -eigenspace of  $F$  on the cohomology of  $X_{n,d}$  – that is in degrees  $\pi_d(X, x) + 1$  and  $\pi_d(X, x) + 2$ . Then,

- if  $x-1 \notin X$ , the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$  is  $\chi_\lambda$ -isotypic by (2) (we have  $x-1+1 \notin X$ ). Moreover, the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$  is zero since Conjecture 1.4 holds for the variety  $X_{n,d+1}$ . We deduce that the eigenspace on  $H_c^*(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I}$  is  $\chi_\lambda$ -isotypic, which is impossible since no unipotent character can have  $\chi_\lambda$  as a Harish-Chandra restriction when  $1 < a < n+1$ .
- if  $x-1 \in X$ , then since  $x-1+d+1 \notin X$ , the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$  and  $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$  can be determined as in case (1). It corresponds to the Harish-Chandra restriction of the partition  $\mu * (x-1)$  obtained from  $\mu$  by adding a  $(d+1)$ -hook from  $x-1$ . Furthermore, they will appear in degree  $\pi_{d+1}(X, x-1)$  only, which is equal to

$$\begin{aligned} \pi_{d+1}(X, x-1) &= 2(n+x - \#\{y \in X \mid y < x-1\}) - \#\{y \in X \mid x-1 < y < x+d\} \\ &= 2(n+x+1 - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x+d\} \\ &= \pi_d(X, x) + 2. \end{aligned}$$

To these characters we have to add the contribution of  $\chi_\lambda$  and possibly of an other character  $\chi_{\lambda'}$  corresponding to  $\lambda = (a', a', \dots, a')$  (when the case (3) applies to  $x-1$ ). Now, we claim that neither  $\chi_\lambda$  nor  $\chi_{\lambda'}$  can appear in the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^*(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$ . Indeed the assumptions on  $x$  force  $X$  (see Remark 3.6) to be either

$$X = \{0, 1, \dots, k-1, k+1, b, b+2, b+3, \dots, \widehat{k+d}, \dots, b+r\}$$

with  $b+2 \leq k+d \leq b+r$ , or

$$X = \{0, 1, \dots, k-1, k+1, k+d+2, k+d+3, \dots, k+d+r\}.$$

Therefore a  $\beta$ -set corresponding to the partition  $\mu * (x-1)$  of  $n+2$  is either

$$\{0, 1, \dots, k-2, k+1, b, b+2, b+3, \dots, b+r\}$$

with  $b+2 \leq k+d \leq b+r$ , or

$$\{0, 1, \dots, k-2, k+1, k+d, k+d+2, k+d+3, \dots, k+d+r\}.$$

We deduce that the restriction of  $\mu^*(x-1)$  will never produce  $\lambda$  or  $\lambda'$  unless  $r=2$ ,  $b=k+2$  and  $d=4$  in the first case, or  $r=2$  and  $d=2$  in the second case. In these very specific cases, we have either  $X = \{0, \dots, k-1, k+1, k+2\}$ , which corresponds to the partition  $\mu = (1, 1)$  or  $X = \{0, \dots, k-1, k+1, k+4\}$ , which corresponds to  $\mu = (1, 3)$ . In this situation, we get  $\lambda = (3, 3)$  and  $\mu^*(x-1) = (2, 2, 3)$ . But  $(3, 3)$  cannot be obtained by restricting  $\mu^*(x-1)$ . This proves that the  $q^{n+1+x-s}$ -eigenspace of  $F$  on  $H_c^{\pi_d(X, x)+2}(\mathbb{X}_{n+1, d+1}, \mathcal{F}_\mu)^{U_I}$  is just  $\chi_\lambda$  (plus possibly  $\chi_{\lambda'}$ ), which is impossible by the properties of the Harish-Chandra restriction.

The same argument can be adapted to deal with the case (3), if we rather look at the  $q^{n+2+x-s}$ -eigenspace and distinguish whether  $x+1+d$  is an element of  $X$  or not. The details are left to the reader.  $\square$

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## References

- [1] C. Bonnafé and R. Rouquier. Coxeter orbits and modular representations. *Nagoya Math. J.*, 183:1–34, 2006.
- [2] M. Broué and G. Malle. Zyklotomische Heckealgebren. *Astérisque*, (212): 119–189, 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.
- [3] M. Broué, G. Malle, and J. Michel. Generic blocks of finite reductive groups. *Astérisque*, (212):7–92, 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.
- [4] M. Broué and J. Michel. Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées. In *Finite reductive groups (Luminy, 1994)*, volume 141 of *Progr. Math.*, pages 73–139.
- [5] R. W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

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- [6] J. Chuang and R. Rouquier. Calabi-Yau algebras and perverse Morita equivalences. *In preparation*.
- [7] D. Craven. On the cohomology of Deligne-Lusztig varieties, arXiv:math/1107.1871. *Preprint*, 2011.
- [8] F. Digne and J. Michel. Endomorphisms of Deligne-Lusztig varieties. *Nagoya Math. J.*, 183:35–103, 2006.
- [9] F. Digne and J. Michel. Parabolic Deligne-Lusztig varieties, arXiv:math/1110.4863. *preprint*, 2011.
- [10] F. Digne, J. Michel, and R. Rouquier. Cohomologie des variétés de Deligne-Lusztig. *Adv. Math.*, 209(2):749–822, 2007.
- [11] O. Dudas. Quotient of Deligne-Lusztig varieties, arXiv:math/1112.4942. *preprint*, 2011.
- [12] G. Lusztig. Coxeter orbits and eigenspaces of Frobenius. *Invent. Math.*, 38(2):101–159, 1976/77.