Hardy spaces, commutators of singular integral operators related to Schrödinger operators and applications

Luong Dang Ky

HARDY SPACES, COMMUTATORS OF SINGULAR INTEGRAL OPERATORS RELATED TO SCHRÖDINGER OPERATORS AND APPLICATIONS

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Abstract. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative function, $V \neq 0$, and belongs to the reserve Hölder class $RH_{d/2}$. The purpose of this paper is three-fold. First, we prove a version of the classical theorem of Jones and Journé on weak*-convergence in $H^1_L(\mathbb{R}^d)$. Secondly, we give a bilinear decomposition for the product space $H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$. Finally, we study the commutators $[b, T]$ for T belongs to a class $\mathcal{K}_L$ of sublinear operators containing almost all fundamental operators related to L. More precisely, when $T \in \mathcal{K}_L$, we prove that there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|, \tag{1}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ which does not depend on T. In the particular case of the Riesz transforms $R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$, and $b \in BMO(\mathbb{R}^d)$, we prove that the commutators $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$ iff $b \in BMO^{\log}_L(\mathbb{R}^d)$– a new space of BMO type, which coincides with the space $LMO(\mathbb{R}^d)$ when $L = -\Delta + 1$. Furthermore,

$$\|b\|_{BMO^{\log}_L} \approx \|b\|_{BMO} + \sum_{j=1}^{d} \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

The subbilinear decomposition (1) explains why almost commutators of the fundamental operators are of weak type (H^1_L, L^1), and when a commutator $[b, T]$ is of strong type (H^1_L, L^1).

Contents

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1. Introduction

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, with V is a nonnegative function, $V \neq 0$, and belongs to the class $RH_{d/2}$. Here RH_q is the class of functions satisfying the reverse Hölder inequality of order $q > 1$. In the recent years, there is an increasing interest on the study of the problems of harmonic analysis associated with these operators. In particular, Fefferman [18], Shen [39] and Zhong [46] obtained some basic results on L, including certain estimates of the fundamental solutions of L and the boundedness of Riesz transforms $\nabla L^{-1/2}$ on Lebesgue spaces $L^p(\mathbb{R}^d)$ for some $p \in (1, \infty)$. In [17], Dziubański and Zienkiewicz considered the Hardy space $H^1_L(\mathbb{R}^d)$ defined via the maximal function $\mathcal{M}_L$ (see Section 2) related to the semigroup $T_t = e^{-tL}, t > 0$, and characterized it in terms of atomic decomposition and in terms of the Riesz transforms $\nabla L^{-1/2}$. Then, Dziubański et al. [16] introduced a BMO-type space $BMO_L(\mathbb{R}^d)$ associated with L and established the duality between $H^1_L(\mathbb{R}^d)$ and $BMO_L(\mathbb{R}^d)$. Later, Deng el al. [15] introduced and developed new function spaces of VMO type associated with some operators which have a bounded holomorphic functional calculus on $L^2(\mathbb{R}^d)$. More precisely, in the particular case of the Schrödinger operator L, their space $\widetilde{VMO}_L(\mathbb{R}^d)$ is the subspace of $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO_L(\mathbb{R}^d)$ such that

$\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^d, r\le t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{t\to \infty}\left(\sup_{x\in\mathbb{R}^d, r\ge t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_3(f) = \lim_{t\to \infty}\left(\sup_{B(x,r)\subset(B(0,t))^c}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right).$$

Also, the authors in [15] showed that $H^1_L(\mathbb{R}^d)$ is just the dual of $\widetilde{VMO}_L(\mathbb{R}^d)$. This fact allows us to study the weak*-convergence in the setting of $H^1_L(\mathbb{R}^d)$. Motivated by this, thanks to some ideas from [14] we introduce the space $VMO_L(\mathbb{R}^d)$ as the closure of $C^\infty_c(\mathbb{R}^d)$, the space of C^∞-functions with compact support, in $BMO_L(\mathbb{R}^d)$. We then prove that $VMO_L(\mathbb{R}^d)$ coincides with $\widetilde{VMO}_L(\mathbb{R}^d)$ and establish a version of the Jones-Journé theorem for $H^1_L(\mathbb{R}^d)$. To do this, we introduce and study the discrete Riesz transforms $\widetilde{R}_j$ (see Section 3).

Products of functions in H^1 and BMO have been considered by Bonami et al. in [6]. Such products in general are not integrable. However, following [6], they make sense as distributions, and can be written as the sum of an integrable function and a function in a weighted Hardy-Orlicz space. To be more precise, for $f \in H^1(\mathbb{R}^d)$ and $g \in BMO(\mathbb{R}^d)$, we define the product (in the distribution sense) $f \times g$ as the distribution whose action on the Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$\langle f \times g, \phi\rangle := \langle \phi g, f\rangle,$$

where the second bracket stands for the duality bracket between $H^1(\mathbb{R}^d)$ and its dual $BMO(\mathbb{R}^d)$. Then, it was shown in [6] that for each $f \in H^1(\mathbb{R}^d)$, there are two bounded linear operators $L_f : BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $H_f : BMO(\mathbb{R}^d) \to H^\Phi(\mathbb{R}^d, d\mu)$ such that for every $g \in BMO(\mathbb{R}^d)$,

$$f \times g = L_f(g) + H_f(g).$$

Here $H^\Phi(\mathbb{R}^d, d\mu)$ is the weighted Hardy-Orlicz space related to the Orlicz function $\Phi(t) = \frac{t}{\log(e+t)}$ and the weight $d\mu(x) = (\log(e + |x|))^{-1}dx$. To be more precise, $H^\Phi(\mathbb{R}^d, d\mu)$ consists of all distributions f such that for some $\lambda > 0$,

$$\int\limits_{\mathbb{R}^d}\frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right)}\frac{dx}{\log(e+|x|)} < \infty$$

with the Luxemburg norm

$$\|f\|_{H^\Phi_\mu}=\inf\left\{\lambda>0:\int\limits_{\mathbb{R}^d}\frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right)}\frac{dx}{\log(e+|x|)}\le 1\right\}.$$

Here and in what follows the grand maximal operator $\mathfrak{M}$ is defined by

$$\mathfrak{M}f(x)=\sup_{\phi\in\mathcal{A}}\sup_{|y-x|<t}t^{-d}|f*\phi(t^{-1}\cdot)(y)|$$

with $\mathcal{A}=\{\phi\in\mathcal{S}(\mathbb{R}^d):|\phi(x)|+|\nabla\phi(x)|\le(1+|x|^2)^{-(d+1)}\}$. Unfortunately, as the classical spaces $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$, the pointwise product $b.f$ of functions $b\in BMO_L(\mathbb{R}^d)$ and $f\in H^1_L(\mathbb{R}^d)$ need not be integrable. Similarly to the classical case in [6], Li and Peng showed in [32] that they can make in the sense of distributions. Furthermore, for each $f\in H^1_L(\mathbb{R}^d)$, there are two bounded linear operators $L_f: BMO_L(\mathbb{R}^d)\to L^1(\mathbb{R}^d)$ and $H_f: BMO_L(\mathbb{R}^d)\to H^\Phi_L(\mathbb{R}^d,d\mu)$ such that for every $g\in BMO_L(\mathbb{R}^d)$,

$$f\times g=L_f(g)+H_f(g). \tag{1.1}$$

Here $H^\Phi_L(\mathbb{R}^d,d\mu)$ is defined as $H^\Phi(\mathbb{R}^d,d\mu)$ with the grand maximal operator $\mathfrak{M}$ replaced by the maximal operator $\mathcal{M}_L$. Motivated by [6], [32] and some recent results of Bonami et al. [4], in this paper, we prove that there are two bounded bilinear operators $S_L: H^1_L(\mathbb{R}^d)\times BMO_L(\mathbb{R}^d)\to L^1(\mathbb{R}^d)$ and $T_L: H^1_L(\mathbb{R}^d)\times BMO_L(\mathbb{R}^d)\to H^{\log}(\mathbb{R}^d)$ such that for every $(f,g)\in H^1_L(\mathbb{R}^d)\times BMO_L(\mathbb{R}^d)$,

$$f\times g=S_L(f,g)+T_L(f,g). \tag{1.2}$$

Here $H^{\log}(\mathbb{R}^d)$ is a new kind of Hardy-Orlicz space consisting of all distributions f such that for some $\lambda>0$,

$$\int\limits_{\mathbb{R}^d}\frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right)+\log(e+|x|)}dx<\infty$$

with the Luxemburg norm

$$\|f\|_{H^{\log}}=\inf\left\{\lambda>0:\int\limits_{\mathbb{R}^d}\frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\log\left(e+\frac{\mathfrak{M}f(x)}{\lambda}\right)+\log(e+|x|)}dx\le 1\right\}.$$

It is easy to see that $H^{\log}(\mathbb{R}^d)\subset H^\Phi(\mathbb{R}^d,d\mu)$ with continuous embedding. Moreover, similarly to the inclusion $H^1(\mathbb{R}^d)\subset H^1_L(\mathbb{R}^d)$, in a forecoming paper, using the atomic decompositions, we also obtain that $H^\Phi(\mathbb{R}^d,d\mu)\subset H^\Phi_L(\mathbb{R}^d,d\mu)$ with continuous embedding. Compared with the main result in [32] (see [32], Theorem 1), our results make an essential improvement in two directions. The first one consists in proving that the space $H^\Phi_L(\mathbb{R}^d,d\mu)$ can be replaced by a smaller space $H^{\log}(\mathbb{R}^d)$. Secondly, we give the bilinear decomposition (1.2) for the product space $H^1_L(\mathbb{R}^d)\times BMO_L(\mathbb{R}^d)$ instead of the linear decomposition (1.1) depending on $f\in H^1_L(\mathbb{R}^d)$, which was

conjectured by Bonami, Iwaniec, Jones and Zinsmeister (see [6], Conjecture 1.7) for the classical case. It allows to study the commutator regularity properties which are of increasing interest in this setting.

Given a function b locally integrable on $\mathbb{R}^d$, and a (classical) Calderón-Zygmund operator T, we consider the linear commutator $[b,T]$ defined for smooth, compactly supported functions f by

$$[b,T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberg and Weiss (see [12]), states that the commutator $[b,T]$ is continuous on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, when $b \in BMO(\mathbb{R}^d)$. Unlike the theory of (classical) Calderón-Zygmund operators, the proof of this result does not rely on a weak type $(1,1)$ estimate for $[b,T]$. Instead, an endpoint theory was provided for this operator. A general overview about these facts can be found in the recent paper of Ky [31]. In the present paper, we consider commutators of singular integral operators T related to the Schrödinger operator L, where T is in the class $\mathcal{K}_L$ of all sublinear operators T, bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ satisfying that there are $q \in (1,\infty]$, $\varepsilon > 0$ such that

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for all BMO-function b, generalized (H^1_L, q, ε)-atom (see Section 2) a related to the ball B. Here b_B denotes the average of b on B, and $C > 0$ is a constant independent of b, a. This class $\mathcal{K}_L$ contains almost all fundamental operators (we refer the reader to [31] for the classical case $L = -\Delta$) related to the Schrödinger operator L: Schrödinger-Calderón-Zygmund operators, maximal type operators, L-square operators, etc... (see Section 6). Let $R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$, be the Riesz transforms associated with L. Although Schrödinger-Calderón-Zygmund operators (related to L) map $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ (see Section 6), it was observed in [33] that, when $b \in BMO(\mathbb{R}^d)$, the commutators $[b, R_j]$ do not map, in general, $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Formally, when $L = -\Delta$, Schrödinger-Calderón-Zygmund operators are just the classical Calderón-Zygmund operators. Similarly to the classical case, it is natural to ask for subspaces of $H^1_L(\mathbb{R}^d)$ such that the commutators $[b, R_j]$ map continuously these spaces into $L^1(\mathbb{R}^d)$. In Section 7, we give such subspaces. Now, two natural questions arise.

Question 1. *For $b \in BMO(\mathbb{R}^d)$. Can one find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ of $H^1_L(\mathbb{R}^d)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$?*

Remark that the Riesz transforms R_j are just, in general, Schrödinger-Calderón-Zygmund operators when $V \in RH_d$. In this paper, we consider all potentials V which belong to the reserve Hölder class $RH_{d/2}$.

Question 2. *Can one find all functions b in $BMO(\mathbb{R}^d)$ such that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$?*

Let X be a Banach space. We say that an operator $T : X \to L^1(\mathbb{R}^d)$ is a sublinear operator if for all $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$, we have

$$|T(\alpha f + \beta g)(x)| \leq |\alpha||Tf(x)| + |\beta||Tg(x)|.$$

Obviously, a linear operator $T : X \to L^1(\mathbb{R}^d)$ is a sublinear operator. We also say that a operator $\mathfrak{T} : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a subbilinear operator if for every $(f, g) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, the operators $\mathfrak{T}(f, \cdot) : BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $\mathfrak{T}(\cdot, g) : H^1_L(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ are sublinear operators.

To anwser *Question 1* and *Question 2*, we study commutators of sublinear operators in $\mathcal{K}_L$. More precisely, when $T \in \mathcal{K}_L$ is a sublinear operator, we prove that there exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ so that for all $(f, b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \leq |[b, T](f)| \leq \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|, \tag{1.3}$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ which does not depend on T (see Section 5). The subbilinear decomposition (1.3) is strongly related to our previous results in [4, 31] on paraproduct and product on $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$. Also, it gives a general overview. Namely, *it explains why* almost commutators of the fundamental operators are of weak type (H^1_L, L^1), and when a commutator $[b, T]$ is of strong type (H^1_L, L^1).

Let b be a non-constant BMO-function, otherwise $[b, T] = 0$. We define the space $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ is the set of all f in $H^1_L(\mathbb{R}^d)$ such that $[b, \mathcal{M}_L](f)(x) = \mathcal{M}_L(b(x)f(\cdot) - b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^d)$, and the norm on $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ is defined by $\|f\|_{\mathcal{H}^1_{L,b}} = \|f\|_{H^1_L}\|b\|_{BMO} + \|[b, \mathcal{M}_L](f)\|_{L^1}$. Then, using the subbilinear decomposition (1.3), we prove that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Furthermore, $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ *is the largest space having this property* (see Theorem 5.3). This answers *Question 1*.

Recall that $BMO^{\log}_L(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO^{\log}_L} = \sup_{B(x,r)} \left(\log\Big(e + \frac{\rho(x)}{r}\Big) \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|dy \right) < \infty,$$

where $\rho(x) = \sup\{r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)dy \leq 1\}$. This space arises naturally in the study of characterizations of pointwise multipliers on $BMO_L(\mathbb{R}^d)$, see for example [3, 35]. Then, we also use the decomposition (1.3) to prove that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$ iff $b \in BMO^{\log}_L(\mathbb{R}^d)$ (see Theorem 5.4), which answers *Question 2*.

When T is linear and belongs to $\mathcal{K}_L$, we prove that there exists a bounded bilinear operators $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f, b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have the following bilinear decomposition

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)). \tag{1.4}$$

In the particular case of the Riesz transforms R_j, $j = 1, ..., d$, Gou et al. showed in [23] that the classical theorem of Coifman, Rochberg and Weiss still holds for $b \in BMO(\mathbb{R}^d)$. Recently, this result was extended by Bongioanni et al. [8], there they obtained that the classical theorem of Coifman, Rochberg and Weiss still holds for $b \in BMO_{L,\infty}(\mathbb{R}^d) = \cup_{\theta\geq 0} BMO_{L,\theta}(\mathbb{R}^d)$ (see Section 5) containing $BMO(\mathbb{R}^d)$ as a proper subset. Let R_j^*, $j = 1, ..., d$, be the adjoint operators of R_j. In [7], Bongioanni et al. established that the operators R_j^* are bounded on $BMO_L(\mathbb{R}^d)$, and thus from $L^\infty(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$. Later, they (see [8]) ask for a class of functions b such that the commutators $[b, R_j^*]$ are bounded from $L^\infty(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$, and such a class of functions $BMO^{\log}_{L,\infty}(\mathbb{R}^d) = \cup_{\theta\geq 0} BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ (see Section 5) was found. A natural question arises: can one replace the space $L^\infty(\mathbb{R}^d)$ by $BMO_L(\mathbb{R}^d)$?

Question 3. *Are the commutators $[b, R_j^*]$, $j = 1, ..., d$, bounded on $BMO_L(\mathbb{R}^d)$ whenever $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$?*

Motivated by this question, we study the Hardy estimates for the commutators of Riesz transforms $[b, R_j]$. Let us remind that in the setting of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, Janson, Peetre and Semmes showed in [27] that the commutator of the Hilbert transform $[b, H]$ is bounded on the Hardy space $H^1(\mathbb{T})$ whenever $b \in BMO^{\log}(\mathbb{T})$, with

$$\|b\|_{BMO^{\log}(\mathbb{T})} = \frac{1}{2\pi}\Big|\int_{\mathbb{T}} b(z)|dz|\Big| + \sup_I \frac{\log\frac{4}{|I|}}{|I|}\int_I \Big|b(\eta) - \frac{1}{|I|}\int_I b(z)|dz|\Big||d\eta| < \infty$$

where the supremum is taken over all arcs I of $\mathbb{T}$ and $|I|$ is the length of I. In the setting of Schrödinger operators L on $\mathbb{R}^d$, an interesting question is for which functions b the commutators of the Riesz transforms $[b, R_j]$ are bounded on $H^1_L(\mathbb{R}^d)$. Here, we give such a class of functions, however we do not know whether this class is the largest (see *Question 4*). More precisely, given $b \in BMO_{L,\infty}(\mathbb{R}^d)$, we prove that the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$ if and only if b belongs to $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ (see Theorem 5.6). Furthermore, when $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$, we have

$$\|b\|_{BMO^{\log}_{L,\theta}} \approx \|b\|_{BMO_{L,\theta}} + \sum_{j=1}^d \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

As a consequence, we obtain that if $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$, then the commutators $[b, R_j^*]$ are bounded on $BMO_L(\mathbb{R}^d)$, which gives a positive answer for *Question 3*. Moreover, for every $b \in BMO^{\log}_L(\mathbb{R}^d)$, we have

$$\|b\|_{BMO^{\log}_L} \approx \|b\|_{BMO} + \sum_{j=1}^d \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

We conclude the introduction by giving an *open question*:

Question 4. *Is it true that $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$ is the set of all functions b such that the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$?*

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. The symbol $f \approx g$ means that f is equivalent to g (i.e. $C^{-1}f \le g \le Cf$). In $\mathbb{R}^d$, we denote by $B = B(x, r)$ an open ball with center x and radius $r > 0$, and $tB(x, r) := B(x, tr)$ whenever $t > 0$. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^d \setminus E$. For a ball B and f a locally integrable function, we denote by f_B the average of f on B.

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2. Hardy spaces via generalized atoms

A nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int\limits_B V^q dx\Big)^{1/q} \le \frac{C}{|B|}\int\limits_B V dx$$

holds for every balls B in $\mathbb{R}^d$. By Hölder inequality we can get that $RH_{q_1} \subset RH_{q_2}$ if $q_1 \ge q_2 > 1$. For $q > 1$, it is well-known that $V \in RH_q$ implies $V \in RH_{q+\varepsilon}$ for some $\varepsilon > 0$ (see [21]). Moreover, $V(y)dy$ is a doubling measure, namely for any ball $B(x, r)$ we have

$$\int\limits_{B(x,2r)} V(y)dy \le C_0 \int\limits_{B(x,r)} V(y)dy. \tag{2.1}$$

In this paper, we *always assume that* $L = -\Delta + V$ is a Schrödinger operator on $\mathbb{R}^d$ with $0 \ne V$ belongs to the reverse Hölder class $RH_{d/2}$. We then define the auxiliary function ρ by

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int\limits_{B(x,r)} V(y)dy \le 1\Big\},$$

$x \in \mathbb{R}^d$, and for any $n \in \mathbb{Z}$,

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \le 2^{-n/2}\}.$$

Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$.

The following lemma is important and will be used often.

Lemma 2.1 (see [39], Lemma 1.4)**.** *There exist $C > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}.$$

Let $\{T_t\}_{t>0}$ be semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL}f(x) = \int\limits_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

Then the maximal operator is defined by

$$\mathcal{M}_L f(x) = \sup_{t>0} |T_t f(x)|.$$

We say that a function $f \in L^2(\mathbb{R}^d)$ belongs to the space $\mathbb{H}^1_L(\mathbb{R}^d)$ if

$$\|f\|_{\mathbb{H}^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty.$$

The space $H^1_L(\mathbb{R}^d)$ is then defined as the completion of $\mathbb{H}^1_L(\mathbb{R}^d)$ with respect to this norm.

Throughout the whole paper, we denote by $\mathcal{C}_L$ the constant

$$\mathcal{C}_L = 8.9^{k_0}C$$

where k_0 and C are defined as in Lemma 2.1.

Thank to the ideas from [11] and [13], we give here some variants of the definition of atoms for $H^1_L(\mathbb{R}^d)$ which are useful for our study.

Definition 2.1. *Let $1 < q \leq \infty$ and $\varepsilon > 0$.*

(1) Recall that a function a is called a classical (H^1, q)-atom related to the ball $B(x_0, r)$ if
   (a) supp $a \subset B(x_0, r)$,
   (b) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,
   (c) $\int_{\mathbb{R}^d} a(x)dx = 0$.

(2) A function a is called a (H^1_L, q)-atom related to the ball $B(x_0, r)$ if $r \leq \mathcal{C}_L\rho(x_0)$ and
   (a) supp $a \subset B(x_0, r)$,
   (b) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,
   (c) if $r \leq \frac{1}{\mathcal{C}_L}\rho(x_0)$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

(3) A function a is called a generalized (H^1_L, q, ε)-atom related to the ball $B(x_0, r)$ if
   (a) supp $a \subset B(x_0, r)$,
   (b) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,
   (c) $|\int_{\mathbb{R}^d} a(x)dx| \leq \Big(\frac{r}{\rho(x_0)}\Big)^{\varepsilon}$.

Remark 2.1. *Let $1 < q \le \infty$ and $\varepsilon > 0$. Then, a (H^1_L, q)-atom will be a $\mathcal{C}_L$ times a generalized (H^1_L, q, ε)-atom related to the same ball.*

Remark 2.2. *Let $1 < q \le \infty$ and $\varepsilon > 0$. Then, a classical (H^1, q)-atom will be a generalized (H^1_L, q, ε)-atom related to the same ball, but not a (H^1_L, q)-atom in general.*

By Remark 2.1, Remark 2.2 and in what follows, it seems that the notation of generalized (H^1_L, q, ε)-atoms will be useful to study the theory of Hardy spaces associated with Schrödinger operators.

Definition 2.2. *Let $1 < q \le \infty$ and $\varepsilon > 0$.*

(1) The space $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$ is defined to be set of all functions f in $L^1(\mathbb{R}^d)$ which can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the a_j's are generalized (H^1_L, q, ε)-atoms and the λ_j's are complex numbers such that $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. As usual, the norm on $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$ is defined by

$$\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,at}} = \inf\Big\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \Big\}.$$

(2) The space $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is defined to be set of all $f = \sum_{j=1}^{k} \lambda_j a_j$, where the a_j's are generalized (H^1_L, q, ε)-atoms. Then, the norm of f in $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is defined by

$$\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}} = \inf\Big\{ \sum_{j=1}^{k} |\lambda_j| : f = \sum_{j=1}^{k} \lambda_j a_j \Big\}.$$

(3) The space $H^{1,q}_{L,at}(\mathbb{R}^d)$ is defined as in (1) with generalized (H^1_L, q, ε)-atoms replaced by (H^1_L, q)-atoms.

(4) The space $H^{1,q}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is defined as in (2) with generalized (H^1_L, q, ε)-atoms replaced by (H^1_L, q)-atoms.

(5) The space $H^{1,q}_{\mathrm{fin}}(\mathbb{R}^d)$ is defined as in (2) with generalized (H^1_L, q, ε)-atoms replaced by classical (H^1, q)-atoms.

Theorem 2.1. *Let $1 < q \le \infty$ and $\varepsilon > 0$. Then, $H^{1,q}_{L,at}(\mathbb{R}^d) = \mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d) = H^1_L(\mathbb{R}^d)$ and the norms are equivalent.*

In order to prove Theorem 2.1, we need the following two lemmas.

Lemma 2.2 (see [32], Lemma 2)**.** *Let $\sigma = \min\{1, 2-d/q_0\}/2 > 0$ for some $q_0 > d/2$ with $V \in RH_{q_0}$. Then, for all $|y-z| < |x-y|/2$ and $t > 0$,*

$$|T_t(x,y) - T_t(x,z)| \le C\Big(\frac{|y-z|}{\sqrt{t}}\Big)^{\sigma} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{t}} \le C\frac{|y-z|^{\sigma}}{|x-y|^{d+\sigma}}.$$

Lemma 2.3 (see (3.5) in [17]). *Given $\varepsilon > 0$. There exists a positive constant $C = C(\varepsilon, L)$ such that for every $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$|T_t(x,y)| \leq C \frac{1}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^{\varepsilon}} \frac{1}{|x-y|^d}.$$

Proof of Theorem 2.1. The proof is divided in three steps. Step 1: $H^{1,q}_{L,at}(\mathbb{R}^d) \subset \mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$ and the inclusion is continuous. Step 2: $\mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$ and the inclusion is continuous. Step 3: $H^1_L(\mathbb{R}^d) \subset H^{1,q}_{L,at}(\mathbb{R}^d)$ and the inclusion is continuous.

Step 1. It is an immediate consequence of Remark 2.1.

Step 2. Let a be a generalized (H^1_L, q, ε)-atom related to the ball $B = B(x_0, r)$, we would like to prove that

$$\|a\|_{H^1_L} = \|\mathcal{M}_L(a)\|_{L^1} \leq C. \tag{2.2}$$

Indeed, from the L^q-boundedness of the classical Hardy-Littlewood maximal operator $\mathcal{M}$, the estimate $\mathcal{M}_L(a) \leq C\mathcal{M}(a)$ and Hölder inequality,

$$\|\mathcal{M}_L(a)\|_{L^1(2B)} \leq C\|\mathcal{M}(a)\|_{L^1(2B)} \leq C|2B|^{1/q'}\|\mathcal{M}(a)\|_{L^q} \leq C, \tag{2.3}$$

where $1/q' + 1/q = 1$.

Let $x \notin 2B$ and $t > 0$, Lemma 2.2 and Lemma 2.3 give

$$\begin{aligned}
|T_t(a)(x)| &= \Big| \int_{\mathbb{R}^d} T_t(x,y)a(y)dy \Big| \\
&\leq \Big| \int_B (T_t(x,y) - T_t(x,x_0))a(y)dy \Big| + |T_t(x,x_0)| \Big| \int_B a(y)dy \Big| \\
&\leq C\frac{r^\sigma}{|x-x_0|^{d+\sigma}} + C\frac{r^\varepsilon}{|x-x_0|^{d+\varepsilon}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{M}_L(a)\|_{L^1((2B)^c)} &= \|\sup_{t>0}|T_t(a)|\|_{L^1((2B)^c)} \\
&\leq C \int_{(2B)^c} \frac{r^\sigma}{|x-x_0|^{d+\sigma}}dx + C\int_{(2B)^c} \frac{r^\varepsilon}{|x-x_0|^{d+\varepsilon}}dx \\
&\leq C. \qquad (2.4)
\end{aligned}$$

Then, (2.2) follows from (2.3) and (2.4).

Now, for every $f = \sum_j \lambda_j a_j \in \mathbb{H}^{1,q,\varepsilon}_{L,at}(\mathbb{R}^d)$. As $\mathcal{M}_L(f) \leq \sum_j |\lambda_j|\mathcal{M}_L(a_j)$, (2.2) implies that

$$\|\mathcal{M}_L(f)\|_{L^1} \leq \sum_j |\lambda_j| \|\mathcal{M}_L(a_j)\|_{L^1} \leq C\sum_j |\lambda_j|.$$

This prove that $f \in H^1_L(\mathbb{R}^d)$, moreover, $\|f\|_{H^1_L} \leq C\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,at}}$.

Step 3. It is an immediate consequence of Corollary 2.1 (see below) and the proof of Theorem 1.5 in [17]. We omit the details.

□

Theorem 2.2. *Let $1 < q < \infty$ and $\varepsilon > 0$. Then, the norms $\|\cdot\|_{H^1_L}$ and $\|\cdot\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}}$ are equivalent on $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d)$.*

Using Theorem 2.2, we immediately obtain the following result.

Proposition 2.1. *Let $1 < q < \infty$, $\varepsilon > 0$ and $\mathcal{X}$ be a Banach space. Suppose that $T : \mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d) \to \mathcal{X}$ is a sublinear operator with*

$$\sup\{\|Ta\|_{\mathcal{X}} : a \text{ is a generalized } (H^1_L, q, \varepsilon) - atom\} < \infty.$$

Then, T can be extended to a bounded sublinear operator $\widetilde{T}$ from $H^1_L(\mathbb{R}^d)$ into $\mathcal{X}$, moreover,

$$\|\widetilde{T}\|_{H^1_L \to \mathcal{X}} \leq C \sup\{\|Ta\|_{\mathcal{X}} : a \text{ is a generalized } (H^1_L, q, \varepsilon) - atom\}.$$

Remark 2.3. *It is not difficult to see that $\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}(\mathbb{R}^d) \equiv H^{1,q}_{L,\mathrm{fin}}(\mathbb{R}^d)$. Thus, Theorem 2.2 can be followed from Theorem 2.1 and Theorem 3.2 of* [45]. *However, we would also like to give a proof for two reasons:*

1. One has a direct proof in the setting of Euclidean space $\mathbb{R}^d$.

2. To prove Theorem 2.2, we give some lemmas and corollaries which are useful and will be used often in next sections.

Before giving the proof of Theorem 2.2. We would like to recall some notations and results of the paper from Dziubański and Zienkiewicz [17].

Let $P(x) = (4\pi)^{-d/2} e^{-|x|^2/4}$ be the Gauss function. For $n \in \mathbb{Z}$, the space $h^1_n(\mathbb{R}^d)$ denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x) = \sup_{0<t<2^{-n}} |P_{\sqrt{t}} * f(x)| = \sup_{0<t<2^{-n}} \Big| \int_{\mathbb{R}^d} p_t(x,y) f(y) dy \Big| \in L^1(\mathbb{R}^d),$$

where the kernel P_t is given by $p_t(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$. We equipped this space with the norm

$$\|f\|_{h^1_n} := \|\mathcal{M}_n f\|_{L^1}.$$

Definition 2.3. *For $1 < q \leq \infty$ and $n \in \mathbb{Z}$. A function a is said to be a (h^1_n, q)-atom related to the ball $B(x_0, r)$ if $r \leq 2^{1-n/2}$ and*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,

iii) if $r \leq 2^{-1-n/2}$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

The atomic space $h^{1,q}_{n,at}(\mathbb{R}^d)$ is defined as in (1) of Definition 2.2 with generalized (H^1_L, q, ε)-atoms replaced by (h^1_n, q)-atoms.

Lemma 2.4 (see [17], Theorem 4.5). *Let $1 < q \leq \infty$. Then, for all $n \in \mathbb{Z}$ we have $h^1_n(\mathbb{R}^d) = h^{1,q}_{n,at}(\mathbb{R}^d)$ with equivalent norms and constants are independent of n. Moreover, if $f \in h^1_n(\mathbb{R}^d)$, supp $f \subset B(x, 2^{1-n/2})$, then there are (h^1_n, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Corollary 2.1. *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are (H^1_L, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Proof. By Lemma 2.4, there are (h^1_n, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}.$$

As $x_j \in B(x, 2^{2-n/2})$ and $x \in \mathcal{B}_n$, Lemma 2.1 implies that $r_j \leq 2^{2-n/2} \leq \mathcal{C}_L \rho(x_j)$. In addition, if $r_j < \frac{1}{\mathcal{C}_L}\rho(x_j)$, then Lemma 2.1 implies that $r_j \leq 2^{-1-n/2}$, and thus $\int_{\mathbb{R}^d} a_j(x)dx = 0$ since a_j are (h^1_n, q)-atoms related to the balls $B(x_j, r_j)$. These prove that a_j are (H^1_L, q)-atoms related to the balls $B(x_j, r_j)$.

□

We next give three lemmas which are due to Dziubański and Zienkiewicz [17].

Lemma 2.5 (see [17], Lemma 2.3). *There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}, k = 1, 2, ...$, such that $x_{n,k} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B_{n,k}$, and*

$$card\,\{(n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n/2}) \neq \emptyset\} \leq R^C$$

for all n, k and $R \geq 2$.

Lemma 2.6 (see [17], Lemma 2.5). *There are nonnegative C^∞-functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, ...$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad \text{and} \quad \|\nabla\psi_{n,k}\|_{L^\infty} \leq C2^{n/2}.$$

Lemma 2.7 (see (4.7) in [17]). *For every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C\|f\|_{H^1_L}.$$

To prove Theorem 2.2, we also need some lemmas below.

Lemma 2.8. *Let $1 < q < \infty$. Then, the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^{1,q}_{\rm fin}}$ are equivalent on $H^{1,q}_{\rm fin}(\mathbb{R}^d)$.*

The proof of Lemma 2.8 can be found in [36].

Lemma 2.9. *Given $0 < R < \infty$. Then, there are two positive integer numbers N_R and K_R such that if $|n| > N_R$ or $k > K_R$,*

$$B(x_{n,k}, 2^{1-n/2}) \cap B(0,R) = \emptyset.$$

Deduce that for any f a function satisfying supp $f \subset B(0,R)$, we have

$$f = \sum_{n,k} \psi_{n,k} f = \sum_{n=-N_R}^{N_R} \sum_{k=1}^{K_R} \psi_{n,k} f.$$

Proof. As $\overline{B(0,R)}$ is a compact set, Lemma 2.5 follows that there is a finite set $\Gamma_R \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$\overline{B(0,R)} \subset \bigcup_{(n,k)\in\Gamma_R} B(x_{n,k}, 2^{-n/2}) \subset \bigcup_{(n,k)\in\Gamma_R} B(x_{n,k}, 2^{1-n/2}).$$

Again, using Lemma 2.5, the above inclusion implies that there is a finite set $\Gamma'_R \subset \mathbb{Z} \times \mathbb{Z}^+$ such that for every $(n,k) \notin \Gamma'_R$,

$$B(x_{n,k}, 2^{1-n/2}) \cap \overline{B(0,R)} = \emptyset,$$

which allows us to end the proof. □

Throughout the whole paper, we fix a non-negative function φ which belongs to $\mathcal{S}(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. We also assume that φ is a even function on $\mathbb{R}^d$, that is, $\varphi(x) = \varphi(-x)$ for all $x \in \mathbb{R}^d$. Then, we define the linear operator $\mathfrak{H}$ by

$$\mathfrak{H}(f) = \sum_{n,k} \Big(\psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f) \Big).$$

Lemma 2.10. *The linear operator $\mathfrak{H}$ is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.*

To prove Lemma 2.10, we need following lemma which proof can be found in [22].

Lemma 2.11. *There exists a constant $C = C(\varphi, d) > 0$ such that*

$$\|f - \varphi_{2^{-n/2}} * f\|_{H^1} \le C \|f\|_{h^1_n}, \quad \text{for all } n \in \mathbb{Z}, f \in h^1_n(\mathbb{R}^d).$$

Proof of Lemma 2.10. For every $f \in H^1_L(\mathbb{R}^d)$, it follows from Lemma 2.11 and Lemma 2.7 that

$$\begin{aligned} \|\mathfrak{H}(f)\|_{H^1} &\le \sum_{n,k} \Big\| \psi_{n,k} f - \varphi_{2^{-n/2}} * (\psi_{n,k} f) \Big\|_{H^1} \\ &\le C \sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \le C \|f\|_{H^1_L}, \end{aligned}$$

which finishes the proof. □

Lemma 2.12. *Let $1 < q < \infty$ and $\varepsilon > 0$. Suppose that a is a generalized (H^1_L, q, ε)-atom related to the ball $B(x_0, r)$. Then, $\mathfrak{H}(a)$ is a multiple of a classical (H^1, q)-atom, and thus generalized (H^1_L, q, ε)-atom.*

Proof. By Lemma 2.9, there are $N, K \in \mathbb{Z}^+$ such that

$$\mathfrak{H}(a) = \sum_{n=-N}^{N} \sum_{k=1}^{K} \Big(\psi_{n,k} a - \varphi_{2^{-n/2}} * (\psi_{n,k} a) \Big).$$

Therefore, the support of $\mathfrak{H}(a)$ is a compact set, moreover, $\|\mathfrak{H}(a)\|_{L^q} \leq K(2N + 1)(\|a\|_{L^q} + \|\varphi\|_{L^1}\|a\|_{L^q}) < \infty$. This together with Lemma 2.10 allow us to conclude that $\mathfrak{H}(a)$ is a multiple of a classical (H^1, q)-atom.

□

Definition 2.4. *For $1 < q \leq \infty$. A function a is said to be a (L^1, q)-atom related to the ball $B(x_0, r)$ if*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$.

Clearly, if a is a (H^1_L, q)-atom relate to the ball B then a is also a (L^1, q)-atom relate to the ball B.

Lemma 2.13. *Let $1 < q \leq \infty$. Then, for every $f \in L^1(\mathbb{R}^d)$ with supp $f \subset B(x, r)$, there are (L^1, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2r)$ and*

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{L^1},$$

where the constant C is independent of f.

The proof of Lemma 2.13 is classical and will be omitted.

Lemma 2.14. *Let $1 < q < \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Then, $\varphi_{2^{-n/2}} * a$ is C times a (H^1_L, q)-atom related to the ball $B(x, 5.2^{-n/2})$ for all (L^1, q)-atom a related to the ball $B(x_0, r) \subset B(x, 2^{2-n/2})$, where $C > 0$ independent of n, x, a.*

Proof. Obviously, $\frac{1}{\mathcal{C}_L}\rho(x) < 5.2^{-n/2} < \mathcal{C}_L\rho(x)$ since $x \in \mathcal{B}_n$. As supp $\varphi \subset B(0, 1)$, one get supp $\varphi_{2^{-n/2}} * a \subset B(x, 5.2^{-n/2})$. In addition,

$$\|\varphi_{2^{-n/2}} * a\|_{L^q} \leq \|\varphi_{2^{-n/2}}\|_{L^q}\|a\|_{L^1} \leq (2^{-n/2})^{d(1/q-1)}\|\varphi\|_{L^q} \leq C|B(x, 5.2^{-n/2})|^{1/q-1}.$$

These show that $\varphi_{2^{-n/2}} * a$ is C times a (H^1_L, q)-atom related to the ball $B(x, 5.2^{-n/2})$.

□

Lemma 2.13 and Lemma 2.14 give:

Corollary 2.2. *There exists a constant $C > 0$ such that*

$$\|\varphi_{2^{-n/2}} * (\psi_{n,k} f)\|_{H^1_L} \leq C\|\psi_{n,k} f\|_{L^1}$$

for all n, k and $f \in L^1(\mathbb{R}^d)$.

Proof of Theorem 2.2. It follows from Theorem 2.1 that $\mathbb{H}^{1,q,\varepsilon}_{L,\rm fin}(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$ and $\|f\|_{H^1_L} \le C\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\rm fin}}$ for all $f \in \mathbb{H}^{1,q,\varepsilon}_{L,\rm fin}(\mathbb{R}^d)$. Thus, we have to show that there exists a constant $C > 0$ such that

$$\|f\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\rm fin}} \le C\|f\|_{H^1_L} \tag{2.5}$$

for all $f \in \mathbb{H}^{1,q,\varepsilon}_{L,\rm fin}(\mathbb{R}^d)$. By homogeneity, we may assume that $\|f\|_{H^1_L} = 1$.

Suppose that $f = \sum_{j=1}^m \lambda_j a_j$, where a_j are generalized (H^1_L, q, ε)-atoms related to the balls $B_j = B(x_j, r_j)$. Then, by Lemma 2.9, there are $N, K \in \mathbb{Z}^+$ such that

$$\begin{aligned} f &= \sum_{n=-N}^{N}\sum_{k=1}^{K}\Big(\psi_{n,k}f - \varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big) + \sum_{n=-N}^{N}\sum_{k=1}^{K}\varphi_{2^{-n/2}} * (\psi_{n,k}f) \\ &= \sum_{j=1}^{m}\lambda_j\mathfrak{H}(a_j) + \sum_{n=-N}^{N}\sum_{k=1}^{K}\varphi_{2^{-n/2}} * (\psi_{n,k}f). \end{aligned} \tag{2.6}$$

On the other hand, by Lemma 2.8, Lemma 2.12, Remark 2.2 and Lemma 2.10, there are generalized (H^1_L, q, ε)-atoms $b_1, ..., b_\ell$ and $\beta_1, ..., \beta_\ell$ are complex numbers such that $\sum_{j=1}^m \lambda_j\mathfrak{H}(a_j) = \sum_{j=1}^\ell \beta_j b_j$, and

$$\sum_{j=1}^{\ell}|\beta_j| \le C\|\sum_{j=1}^{m}\lambda_j\mathfrak{H}(a_j)\|_{H^1} = C\|\mathfrak{H}(f)\|_{H^1} \le C\|f\|_{H^1_L} = C. \tag{2.7}$$

Let $n \in \{-N, ..., N\}$ and $k \in \{1, ..., K\}$. As supp $\psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$ and $x_{n,k} \in \mathcal{B}_n$, Corollary 2.1 and Remark 2.1 yield that there are generalized (H^1_L, q, ε)-atoms $\widetilde{a}^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ and complex numbers $\lambda^{n,k}_j$ such that

$$\psi_{n,k}f = \sum_{j=1}^{\infty}\lambda^{n,k}_j\widetilde{a}^{n,k}_j \quad \text{and} \quad \sum_{j=1}^{\infty}|\lambda^{n,k}_j| \le C\|\psi_{n,k}f\|_{h^1_n} \tag{2.8}$$

with a constant C independent of n, k, f. We deduce that

$$\|\varphi_{2^{-n/2}} * (\sum_{j=i}^{\infty}\lambda^{n,k}_j\widetilde{a}^{n,k}_j)\|_{L^q} \le 2^{Nd(1-1/q)}\|\varphi\|_{L^q}\sum_{j=i}^{\infty}|\lambda^{n,k}_j|$$

tends to 0 as $i \to \infty$. This implies there exists $N_{n,k} \in \mathbb{Z}^+$ such that

$$\|\varphi_{2^{-n/2}} * (\sum_{j=N_{n,k}}^{\infty}\lambda^{n,k}_j\widetilde{a}^{n,k}_j)\|_{L^q} \le \epsilon|B(x_{n,k}, 5.2^{-n/2})|^{1/q-1}$$

with $\epsilon = \frac{1}{K(2N+1)}$. By supp $\varphi_{2^{-n/2}} * (\sum_{j=N_{n,k}}^{\infty}\lambda^{n,k}_j\widetilde{a}^{n,k}_j) \subset B(x_{n,k}, 5.2^{-n/2})$ and $\frac{1}{\mathcal{C}_L}\rho(x_{n,k}) < 5.2^{-n/2} < \mathcal{C}_L\rho(x_{n,k})$, the above inequality and Remark 2.1 prove that $\varphi_{2^{-n/2}} * (\sum_{j=N_{n,k}}^{\infty}\lambda^{n,k}_j\widetilde{a}^{n,k}_j)$ is $C\epsilon$ times a generalized (H^1_L, q, ε)-atom related to the

ball $B(x_{n,k}, 5.2^{-n/2})$. This together with (2.8), Lemma 2.14, Remark 2.1 and Lemma 2.7 yield

$$\begin{aligned}
&\|\sum_{n=-N}^{N}\sum_{k=1}^{K}\varphi_{2^{-n/2}}*(\psi_{n,k}f)\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}}\\
&=\|\sum_{n=-N}^{N}\sum_{k=1}^{K}\sum_{j=1}^{N_{n,k}-1}\lambda_j^{n,k}\varphi_{2^{-n/2}}*\widetilde{a}_j^{n,k}+\sum_{n=-N}^{N}\sum_{k=1}^{K}C\epsilon\Big(\frac{1}{C\epsilon}\varphi_{2^{-n/2}}*(\sum_{j=N_{n,k}}^{\infty}\lambda_j^{n,k}\widetilde{a}_j^{n,k})\Big)\|_{\mathbb{H}^{1,q,\varepsilon}_{L,\mathrm{fin}}}\\
&\leq C\sum_{n,k}\|\psi_{n,k}f\|_{h^1_n}+CK(2N+1)\epsilon\leq C\|f\|_{H^1_L}+C\leq C. \qquad (2.9)
\end{aligned}$$

Finally, (2.5) follows from (2.6), (2.7) and (2.9). This finishes the proof. □

3. $VMO_L(\mathbb{R}^d)$ spaces and weak*-convergence in $H^1_L(\mathbb{R}^d)$

As usual, we denote by $C_c^\infty(\mathbb{R}^d)$ the space of all C^∞-functions with compact support, by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$ and by $C_0(\mathbb{R}^d)$ the space of all continuous functions vanishing at infinity.

3.1. **Discrete Riesz transforms.**

Recall that the linear operator $\mathfrak{H}$ is defined by

$$\mathfrak{H}(f)=\sum_{n,k}\Big(\psi_{n,k}f-\varphi_{2^{-n/2}}*(\psi_{n,k}f)\Big).$$

Lemma 3.1. *Let $1\leq p<\infty$. Then, $\mathfrak{H}$ is bounded on $L^p(\mathbb{R}^d)$.*

Proof. It is sufficient to show that

$$\Big\|\sum_{n,k}\varphi_{2^{-n/2}}*(\psi_{n,k}f)\Big\|_{L^p}\leq C\|f\|_{L^p}.$$

Let $z\in B(0,1)$, Lemma 2.5 yields

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big|\sum_{n,k}|f(x-2^{-n/2}z)|\chi_{B(x_{n,k},3.2^{-n/2})}(x)\Big|^p dx\\
\leq\ & C\int_{\mathbb{R}^d}\sum_{n,k}|f(x-2^{-n/2}z)|^p\chi_{B(x_{n,k},3.2^{-n/2})}(x)dx\\
\leq\ & C\sum_{n,k}\int_{B(x_{n,k},2^{2-n/2})}|f(y)|^p dy=C\int_{\mathbb{R}^d}|f(y)|^p\sum_{n,k}\chi_{B(x_{n,k},2^{2-n/2})}(y)dy\\
\leq\ & C\int_{\mathbb{R}^d}|f(y)|^p,
\end{aligned}$$

where the constants C are independent of f, z. This implies that

$$\left\|\sum_{n,k}|f(\cdot-2^{-n/2}z)|\chi_{B(x_{n,k},3.2^{-n/2})}\right\|_{L^p}\leq C\|f\|_{L^p}. \tag{3.1}$$

For every $x\in\mathbb{R}^d$, by $0\leq\psi_{n,k}\leq 1$ and supp $\psi_{n,k}\subset B(x_{n,k},2^{1-n/2})$,

$$\begin{aligned}\sum_{n,k}\Big|\Big(\varphi_{2^{-n/2}}*(\psi_{n,k}f)\Big)(x)\Big| &\leq \sum_{n,k}\int\limits_{B(0,1)}\varphi(z)|f(x-2^{-n/2}z)|\chi_{B(x_{n,k},2^{1-n/2})}(x-2^{-n/2}z)dz\\ &\leq \int\limits_{B(0,1)}\varphi(z)\sum_{n,k}|f(x-2^{-n/2}z)|\chi_{B(x_{n,k},3.2^{-n/2})}(x)dz.\end{aligned}$$

This together with (3.1) and Hölder inequality give

$$\left\|\sum_{n,k}\varphi_{2^{-n/2}}*(\psi_{n,k}f)\right\|_{L^p}\leq C\|\varphi\|_{L^1}\|f\|_{L^p}\leq C\|f\|_{L^p}.$$

□

By Lemma 3.1, we are now already to give the definition for discrete Riesz transforms $\widetilde{R}_j$ as follows. Here and in what follows we denote by $\mathcal{R}_j$ the classical Riesz transforms.

Definition 3.1. *Let $j=1,...,d$. For every $f\in\cup_{p\geq1}L^p(\mathbb{R}^d)$, we define*

$$\widetilde{R}_j(f)=\mathcal{R}_j(\mathfrak{H}(f)).$$

Then, by φ is a even function on $\mathbb{R}^d$, the adjoint operators of $\widetilde{R}_j$, $j=1,...,d$, have the forms

$$\widetilde{R}_j^*(f)=-\sum_{n,k}\psi_{n,k}\Big(\mathcal{R}_j(f)-\varphi_{2^{-n/2}}*(\mathcal{R}_j(f))\Big). \tag{3.2}$$

From Lemma 3.1, the boundedness on $L^p(\mathbb{R}^d)$ with $1<p<\infty$ and the boundedness from $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$ of the classical Riesz transforms, we deduce that:

Proposition 3.1. *Let $j=1,...,d$. Then, $\widetilde{R}_j$ is bounded on $L^p(\mathbb{R}^d)$, for $1<p<\infty$, and bounded from $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$.*

From Lemma 2.10 and the boundedness on $H^1(\mathbb{R}^d)$ of the classical Riesz transforms, one have:

Proposition 3.2. *Let $j=1,...,d$. Then, $\widetilde{R}_j$ is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.*

Corollary 3.1. *The operators $\widetilde{R}_j^*$ are bounded from $BMO(\mathbb{R}^d)$ into $BMO_L(\mathbb{R}^d)$.*

It is well-known (see [7]) that the Riesz transforms R_j are bounded on $BMO_L(\mathbb{R}^d)$, see also [34] for the setting of the Heisenberg group $\mathbb{H}^d$. Similarly, we also have:

Proposition 3.3. *The discrete Riesz transforms $\widetilde{R}_j$ are bounded on $BMO_L(\mathbb{R}^d)$.*

Proof. Let $f \in BMO_L(\mathbb{R}^d)$, let us first verify that

$$\|\varphi_{2^{-n/2}} * (\psi_{n,k} f)\|_{L^\infty} \leq C\|f\|_{BMO_L} \tag{3.3}$$

for all n, k. In fact, one only need to consider $x \in B(x_{n,k}, 3.2^{-n/2})$ since otherwise $\varphi_{2^{-n/2}} * (\psi_{n,k} f)(x) = 0$. For every $x \in B(x_{n,k}, 3.2^{-n/2})$,

$$\begin{aligned} |\varphi_{2^{-n/2}} * (\psi_{n,k} f)(x)| &\leq \|\varphi\|_{L^\infty} \frac{1}{(2^{-n/2})^d} \int\limits_{B(x_{n,k}, 2^{2-n/2})} |\psi_{n,k}(z) f(z)| dz \\ &\leq C \frac{1}{|B(x_{n,k}, 2^{2-n/2})|} \int\limits_{B(x_{n,k}, 2^{2-n/2})} |f(z)| dz \\ &\leq C\|f\|_{BMO_L}, \end{aligned}$$

since $x_{n,k} \in \mathcal{B}_n$, which verified (3.3). Then, Lemma 2.5 and (3.3) yield

$$\|\mathfrak{H}(f)\|_{BMO_L} \leq \|f\|_{BMO_L} + \|\sum_{n,k} \varphi_{2^{-n/2}} * (\psi_{n,k} f)\|_{L^\infty} \leq C\|f\|_{BMO_L},$$

and hence

$$\|\widetilde{R}_j(f)\|_{BMO} = \|\mathcal{R}_j(\mathfrak{H}(f))\|_{BMO} \leq C\|\mathfrak{H}(f)\|_{BMO} \leq C\|f\|_{BMO_L}.$$

Now, we only need to prove that for every ball $B_0 = B(x_0, r)$ with $r \geq \rho(x_0)$,

$$\frac{1}{|B_0|} \int\limits_{B_0} |\widetilde{R}_j(f)(x)| dx \leq C\|f\|_{BMO_L}. \tag{3.4}$$

By Lemma 2.1, there exists a constant $C = C(L) > 1$, depends only on L, such that for every $y \in B(x_{n,k}, 2^{1-n/2}) \cap (B_0)^c$,

$$2^{-n/2} \leq C\rho(x_0)^{\frac{1}{k_0+1}} |y - x_0|^{\frac{k_0}{k_0+1}} \leq C r^{\frac{1}{k_0+1}} |y - x_0|^{\frac{k_0}{k_0+1}}. \tag{3.5}$$

Deduce that

$$|y - x - z| \geq |y - x_0| - |x - x_0| - |z| \geq \frac{1}{4}|y - x_0| \tag{3.6}$$

for all $x \in B_0$, $z \in B(0, 2^{-n/2})$ and $y \in B(x_{n,k}, 2^{1-n/2}) \cap (\widetilde{C} B_0)^c$, where $\widetilde{C} := (2C)^{k_0+1}$ with C is the constant as in (3.5). Therefore, setting $f_2 := f\chi_{(\widetilde{C}B_0)^c}$, (3.6) allows

that for every $x \in B_0$,

$$\begin{aligned}
&|\mathcal{R}_j(\psi_{n,k} f_2 - \varphi_{2^{-n/2}} * (\psi_{n,k} f_2))(x)| \\
&= \left| \int_{\mathbb{R}^d} \left\{ \int_{B(0,2^{-n/2})} \varphi_{2^{-n/2}}(z) \Big(\frac{x_j - y_j}{|x-y|^{d+1}} - \frac{x_j - y_j - z_j}{|x-y-z|^{d+1}} \Big) dz \right\} \psi_{n,k}(y) f_2(y) dy \right| \\
&\leq C\|\varphi\|_{L^1} \int_{(\widetilde{C}B_0)^c} \frac{2^{-n/2}}{|y-x_0|^{d+1}} \psi_{n,k}(y)|f(y)|dy \\
&\leq C \int_{(\widetilde{C}B_0)^c} \frac{r^{\frac{1}{k_0+1}}}{|y-x_0|^{d+\frac{1}{k_0+1}}} \psi_{n,k}(y)|f(y)|dy
\end{aligned}$$

since (3.5). This implies that every $x \in B_0$,

$$\begin{aligned}
|\widetilde{R}_j(f_2)(x)| &\leq C \int_{(\widetilde{C}B_0)^c} \frac{r^{\frac{1}{k_0+1}}}{|y-x_0|^{d+\frac{1}{k_0+1}}} |f(y)|dy \\
&\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}B_0 \setminus 2^k B_0} \frac{r^{\frac{1}{k_0+1}}}{|y-x_0|^{d+\frac{1}{k_0+1}}} |f(y)|dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k\frac{1}{k_0+1}} \frac{1}{|2^{k+1}B_0|} \int_{2^{k+1}B_0} |f(y)|dy \\
&\leq C\|f\|_{BMO_L}
\end{aligned}$$

since $r \geq \rho(x_0)$. Deduce that

$$\frac{1}{|B_0|} \int_{B_0} |\widetilde{R}_j(f_2)(x)|dx \leq C\|f\|_{BMO_L}. \tag{3.7}$$

We next write $f = f_1 + f_2$ where $f_1 = f\chi_{\widetilde{C}B_0}$. Then, the L^2-boundedness of $\widetilde{R}_j$ and John-Nirenberg inequality (see Corollary 3.2) give

$$\begin{aligned}
\frac{1}{|B_0|} \int_{B_0} |\widetilde{R}_j(f_1)(x)|dx &\leq \Big(\frac{1}{|B_0|} \int_{B_0} |\widetilde{R}_j(f_1)(x)|^2 dx \Big)^{1/2} \\
&\leq C \Big(\frac{1}{|\widetilde{C}B_0|} \int_{\widetilde{C}B_0} |f(x)|^2 dx \Big)^{1/2} \\
&\leq C\|f\|_{BMO_L}. \qquad (3.8)
\end{aligned}$$

Finally, (3.4) follows from (3.7) and (3.8). This ends the proof.

□

Next, we give a characterization of $H^1_L(\mathbb{R}^d)$ via the discrete Riesz transforms.

Theorem 3.1. *A L^1-function f belongs to $H^1_L(\mathbb{R}^d)$ if and only if $\widetilde{R}_j(f) \in L^1(\mathbb{R}^d)$ for all $j = 1, ..., d$. Moreover, for every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\|f\|_{H^1_L} \approx \|f\|_{L^1} + \sum_{j=1}^{d} \|\widetilde{R}_j(f)\|_{L^1}.$$

Proof. Suppose that $f \in H^1_L(\mathbb{R}^d)$. Then, Proposition 3.2 implies that $\widetilde{R}_j(f) \in L^1(\mathbb{R}^d)$ for all $j = 1, ..., d$, moreover,

$$\|f\|_{L^1} + \sum_{j=1}^{d} \|\widetilde{R}_j(f)\|_{L^1} \leq C\|f\|_{H^1_L}.$$

Conversely, assume that $f \in L^1(\mathbb{R}^d)$ and $\mathcal{R}_j(\mathfrak{H}(f)) = \widetilde{R}_j(f) \in L^1(\mathbb{R}^d)$ for all $j = 1, ..., d$. Then, Lemma 3.1 implies that $\mathfrak{H}(f) \in L^1(\mathbb{R}^d)$, and thus $\mathfrak{H}(f) \in H^1(\mathbb{R}^d)$ since the characterization of $H^1(\mathbb{R}^d)$ via the classical Riesz transforms. In addition, Corollary 2.2 gives

$$\left\| \sum_{n,k} \varphi_{2^{-n/2}} * (\psi_{n,k} f) \right\|_{H^1_L} \leq C \sum_{n,k} \|\psi_{n,k} f\|_{L^1} \leq C\|f\|_{L^1}.$$

These prove that $f = \mathfrak{H}(f) + \sum_{n,k} \varphi_{2^{-n/2}} * (\psi_{n,k} f) \in H^1_L(\mathbb{R}^d)$, moreover,

$$\begin{aligned} \|f\|_{H^1_L} &\leq C\|\mathfrak{H}(f)\|_{H^1} + \left\| \sum_{n,k} \varphi_{2^{-n/2}} * (\psi_{n,k} f) \right\|_{H^1_L} \\ &\leq C\Big(\|f\|_{L^1} + \sum_{j=1}^{d} \|\widetilde{R}_j(f)\|_{L^1} \Big), \end{aligned}$$

which ends the proof. □

3.2. $VMO_L(\mathbb{R}^d)$ **spaces.** In [16] it was shown that the dual of $H^1_L(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x) \leq r} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)| dy < \infty.$$

As an immediate consequence of Theorem 2.1 and the duality, we have:

Corollary 3.2. *Let $1 \leq q < \infty$. For every $f \in BMO_L(\mathbb{R}^d)$, we have*

$$\sup_{B(x,r)} \Big(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|^q dy \Big)^{1/q} + \sup_{\rho(x) \leq r} \Big(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|^q dy \Big)^{1/q} \leq C\|f\|_{BMO_L}$$

and

$$\|f\|_{BMO_L} \approx \|f\|_{BMO} + \sup_{\frac{1}{\mathcal{C}_L}<\frac{r}{\rho(x)}\leq \mathcal{C}_L} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy.$$

Definition 3.2. *The space $VMO_L(\mathbb{R}^d)$ is defined to be the closure of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$.*

Remark 3.1. *The space $VMO_L(\mathbb{R}^d)$ coincides with the closure of $C_0(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$ since $C_0(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$ and BMO_L-norm is bounded by the L^∞-norm.*

Definition 3.3. *Let $n \in \mathbb{Z}$. The space $bmo_n(\mathbb{R}^d)$ is defined to be the set of all locally integrable functions f such that*

$$\|f\|_{bmo_n} = \|f\|_{BMO} + \sup_{x\in\mathbb{R}^d, 2^{-1-n/2}\leq r\leq 2^{1-n/2}} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy < \infty.$$

The space $vmo_n(\mathbb{R}^d)$ is the subspace of $bmo_n(\mathbb{R}^d)$ consisting of those f such that

$$\lim_{\sigma\to 0}\left(\sup_{x\in\mathbb{R}^d, r<\sigma} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)-f_{B(x,r)}|dy\right) = 0$$

and

$$\lim_{R\to\infty}\left(\sup_{2^{-1-n/2}\leq r\leq 2^{1-n/2}, B(x,r)\cap B(0,R)=\emptyset} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy\right) = 0.$$

Lemma 3.2 (see [14])**.** *Let $n \in \mathbb{Z}$. Then,*
i) The space $vmo_n(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ in $bmo_n(\mathbb{R}^d)$.
ii) The dual of $vmo_n(\mathbb{R}^d)$ is the space $h_n^1(\mathbb{R}^d)$.

Theorem 3.2. *The dual of $VMO_L(\mathbb{R}^d)$ is the space $H_L^1(\mathbb{R}^d)$.*

To prove Theorem 3.2, we need the following two lemmas.

Lemma 3.3. *The operators $\widetilde{R}_j^*$ map $\mathcal{S}(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$.*

Lemma 3.4. *Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. Then, $\psi_{n,k}$ is a multiplier from $vmo_n(\mathbb{R}^d)$ into $VMO_L(\mathbb{R}^d)$, that is, there exists a constant $C(n,k) > 0$ such that for all $f \in vmo_n(\mathbb{R}^d)$, we have*

$$\|\psi_{n,k}f\|_{VMO_L} \leq C(n,k)\|f\|_{vmo_n}.$$

Proof of Lemma 3.3. We first observe that the classical Riesz transforms $\mathcal{R}_j$ map $\mathcal{S}(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$. Then, by the fomular (3.2), we use the facts supp $\varphi_{2^{-n/2}} \subset B(0,2^{-n/2})$, $\|\varphi_{2^{-n/2}}\|_{L^1} = \|\varphi\|_{L^1}$ and $\|\varphi_{2^{-n/2}} * (\mathcal{R}_j(f))\|_{L^\infty} \leq \frac{1}{(2^{-n/2})^{d/2}}\|\varphi\|_{L^2}\|\mathcal{R}_j(f)\|_{L^2}$ to conclude that the operators $\widetilde{R}_j^*$ map $\mathcal{S}(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$. □

Proof of Lemma 3.4. Since $f \in vmo_n(\mathbb{R}^d)$ the closure of $C_c^\infty(\mathbb{R}^d)$ in $bmo_n(\mathbb{R}^d)$ (see Lemma 3.2), there exists a sequence $\{f_j\}_{j\geq 1}$ in $C_c^\infty(\mathbb{R}^d)$ such that $\|f_j - f\|_{bmo_n} \to 0$ as $j \to \infty$. Clearly, $\{\psi_{n,k} f_j\}_{j\geq 1}$ in $C_c^\infty(\mathbb{R}^d)$ since $\psi_{n,k} \in C_c^\infty(\mathbb{R}^d)$. Let us now show that $\|\psi_{n,k} f_j - \psi_{n,k} f\|_{BMO_L} \to 0$ as $j \to \infty$.

We first note that, by Theorem 3 of [37], there exists a constant $C(n,k) > 0$ such that for all $g \in bmo_n(\mathbb{R}^d)$,

$$\|\psi_{n,k} g\|_{bmo_n} \leq C(n,k)\|g\|_{bmo_n}.$$

This implies that

$$\|\psi_{n,k} f_j - \psi_{n,k} f\|_{BMO} \leq \|\psi_{n,k}(f_j - f)\|_{bmo_n} \leq C(n,k)\|f_j - f\|_{bmo_n} \to 0$$

as $j \to \infty$. Therefore, it is now sufficient to prove that

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\psi_{n,k}(y)(f_j(y) - f(y))|dy \leq C.C(n,k)\|f_j - f\|_{bmo_n} \tag{3.9}$$

for all $x \in \mathbb{R}^d$ and $r \geq \rho(x)$. Inequality (3.9) is obvious if $B(x,r) \cap B(x_{n,k}, 2^{1-n/2}) = \emptyset$ since supp $\psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. Otherwise, that is $B(x,r) \cap B(x_{n,k}, 2^{1-n/2}) \neq \emptyset$. Taking $z \in B(x,r) \cap B(x_{n,k}, 2^{1-n/2})$, Lemma 2.1 gives

$$2^{-1-n/2} < \rho(x_{n,k}) \leq C\rho(z)$$

since $x_{n,k} \in \mathcal{B}_n$ and $z \in B(x_{n,k}, 2^{1-n/2})$, and

$$\rho(z) \leq C\rho(x)\Big(1 + \frac{|z-x|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}} \leq Cr$$

since $z \in B(x,r)$ and $\rho(x) \leq r$. This implies that $2^{-n/2} \leq Cr$, and thus

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |\psi_{n,k}(y)(f_j(y) - f(y))|dy &\leq C\|\psi_{n,k}(f_j - f)\|_{bmo_n} \\ &\leq C.C(n,k)\|f_j - f\|_{bmo_n} \to 0,\end{aligned}$$

as $j \to \infty$, which ends the proof.

□

Proof of Theorem 3.2. Since $VMO_L(\mathbb{R}^d)$ is a subspace of $BMO_L(\mathbb{R}^d)$, which is the dual of $H_L^1(\mathbb{R}^d)$, every function f in $H_L^1(\mathbb{R}^d)$ determines a bounded linear functional on $VMO_L(\mathbb{R}^d)$ of norm bounded by $\|f\|_{H_L^1}$.

Conversely, given a bounded linear functional $\mathcal{T}$ on $VMO_L(\mathbb{R}^d)$. Then, for every $n \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$, Lemma 3.4 follows that the linear functional $\mathcal{T}_{n,k}(g) \mapsto \mathcal{T}(\psi_{n,k} g)$ is continuous on $vmo_n(\mathbb{R}^d)$. Consequently, by Lemma 3.2, there exists $f_{n,k} \in h_n^1(\mathbb{R}^d)$ such that for all $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{T}(\psi_{n,k}\phi) = \mathcal{T}_{n,k}(\phi) = \int\limits_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy.$$

Note that supp $f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$ since supp $\psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. Corollary 2.1 allows that $f_{n,k} \in H^1_L(\mathbb{R}^d)$. Let $R > 0$, by Lemma 2.9, there exist $N_R, K_R \in \mathbb{Z}^+$ such that

$$\phi = \sum_{n=-N_R}^{N_R} \sum_{k=1}^{K_R} \psi_{n,k}\phi$$

for all $\phi \in C^\infty(B(0,R))$. Therefore, for all $\phi \in C^\infty(B(0,R))$,

$$\mathcal{T}(\phi) = \int_{\mathbb{R}^d} f_R(y)\phi(y)dy,$$

where $f_R = \sum_{n=-N_R}^{N_R} \sum_{k=1}^{K_R} f_{n,k} \in H^1_L(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. This follows that there exists $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that for all $\phi \in C^\infty_c(\mathbb{R}^d)$,

$$\mathcal{T}(\phi) = \int_{\mathbb{R}^d} f(y)\phi(y)dy.$$

From the VMO_L-norm (i.e., BMO_L-norm) is bounded by the L^∞-norm, the above prove that f_R tends f, as $R \to \infty$, in $L^1(\mathbb{R}^d)$, and $\|f\|_{L^1} \leq \|\mathcal{T}\|$.

Next, as $f_R \in H^1_L(\mathbb{R}^d)$ for all $R > 0$, Lemma 3.3 and Remark 3.1 allow us to define the "discrete Riesz transforms" of $\mathcal{T}$ by

$$\widetilde{R}_j(\mathcal{T})(\phi) := \mathcal{T}(\widetilde{R}^*_j(\phi)) = \lim_{R\to\infty} \int_{\mathbb{R}^d} R_j(f_R)(y)\phi(y)dy$$

whenever $\phi \in C^\infty_c(\mathbb{R}^d)$, $j = 1, ..., d$. Then, by Corollary 3.1, we get

$$|\widetilde{R}_j(\mathcal{T})(\phi)| \leq \|\mathcal{T}\|\|\widetilde{R}^*_j(\phi)\|_{BMO_L} \leq C\|\mathcal{T}\|\|\phi\|_{VMO_L}$$

for all $\phi \in C^\infty_c(\mathbb{R}^d)$. This show that $\widetilde{R}_j(\mathcal{T})$ can be extended to a bounded linear functional on $VMO_L(\mathbb{R}^d)$, moreover,

$$\|\widetilde{R}_j(\mathcal{T})\| \leq C\|\mathcal{T}\|.$$

Therefore, similarly to the previous argument, there are $f_j \in L^1(\mathbb{R}^d)$, $j = 1, ..., d$, such that $R_j(f_R)$ tends f_j in $L^1(\mathbb{R}^d)$ and $\|f_j\|_{L^1} \leq \|\widetilde{R}_j(\mathcal{T})\| \leq C\|\mathcal{T}\|$. Deduce that $R_j(f) = f_j$ since $R_j(f_R)$ tends $R_j(f)$ in $L^{1,\infty}(\mathbb{R}^d)$ by f_R tends f in $L^1(\mathbb{R}^d)$. Then, we use Theorem 3.1 to conclude that $f \in H^1_L(\mathbb{R}^d)$, moreover,

$$\|f\|_{H^1_L} \leq C\Big(\|f\|_{L^1} + \sum_{j=1}^{d} \|f_j\|_{L^1}\Big) \leq C\|\mathcal{T}\|,$$

which ends the proof.

□

3.3. **Comparison with the space $\widetilde{VMO}_L(\mathbb{R}^d)$.** Following Deng *et al.* [15], a function $f \in BMO_L(\mathbb{R}^d)$ is in $\widetilde{VMO}_L(\mathbb{R}^d)$ if it satisfies the limiting conditions $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^d, r\le t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{t\to \infty}\left(\sup_{x\in\mathbb{R}^d, r\ge t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_3(f) = \lim_{t\to \infty}\left(\sup_{B(x,r)\subset(B(0,t))^c}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right),$$

we endow $\widetilde{VMO}_L(\mathbb{R}^d)$ with the norm of $BMO_L(\mathbb{R}^d)$. It was shown in [15] that $H^1_L(\mathbb{R}^d)$ is just the dual of $\widetilde{VMO}_L(\mathbb{R}^d)$.

Theorem 3.3. *The space $VMO_L(\mathbb{R}^d)$ coincides with the space $\widetilde{VMO}_L(\mathbb{R}^d)$.*

Proof. As $VMO_L(\mathbb{R}^d)$ and $\widetilde{VMO}_L(\mathbb{R}^d)$ (see [15]) are two closed subspaces of $BMO_L(\mathbb{R}^d)$ and they define the same dual space $H^1_L(\mathbb{R}^d)$, by Hahn-Banach theorem, it suffices to show that $C^\infty_c(\mathbb{R}^d) \subset \widetilde{VMO}_L(\mathbb{R}^d)$. Indeed, for every $f \in C^\infty_c(\mathbb{R}^d)$ with supp $f \subset B(0, R_0)$ for some $R_0 > 0$, one need to establish the following three steps:

Step 1: For every $x \in \mathbb{R}^d$ and $r > 0$,

$$\begin{aligned}&\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\\ \le\ & 2\Big(\sup_{|y-z|<r^{1/4}}|f(y)-f(z)|\Big)^2 + 2\Big(2\|f\|_{L^\infty}|B(0,R_0)|\Big)^2\frac{1}{(4\pi r)^d}e^{-\frac{1}{2\sqrt{r}}}.\end{aligned}$$

By the uniformly continuity of f, the above proves that

$$\gamma_1(f) = \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^d, r\le t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right) = 0.$$

Step 2: For every $x \in \mathbb{R}^d$ and $r > (2R_0)^2$,

$$\begin{aligned}
&\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)-T_r(f)(y)|^2dy\\
\leq\ & \frac{1}{|B(x,r)|}\int\limits_{B(x,r)\cap B(0,\sqrt{r})}\Big(\frac{1}{(4\pi r)^{d/2}}\int\limits_{\mathbb{R}^d} e^{-\frac{|y-z|^2}{4r}}|f(y)-f(z)|dz\Big)^2dy+\\
&+\frac{1}{|B(x,r)|}\int\limits_{B(x,r)\cap (B(0,\sqrt{r}))^c}\Big(\frac{1}{(4\pi r)^{d/2}}\int\limits_{B(0,R_0)} e^{-\frac{|y-z|^2}{4r}}|f(y)-f(z)|dz\Big)^2dy\\
\leq\ & \frac{|B(0,\sqrt{r})|}{|B(x,r)|}(2\|f\|_{L^\infty})^2+C\|f\|_{L^\infty}^2\frac{1}{r^{d/2}}.
\end{aligned}$$

This follows that

$$\gamma_2(f)=\lim_{t\to\infty}\left(\sup_{x\in\mathbb{R}^d,r\geq t}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right)=0.$$

Step 3: For every $t > 2R_0$ and $B(x,r)\subset (B(0,t))^c$,

$$\begin{aligned}
&\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\\
\leq\ & \frac{1}{|B(x,r)|}\int\limits_{B(x,r)}\Big(\frac{1}{(4\pi r)^{d/2}}\int\limits_{B(0,R_0)} e^{-\frac{(t-R_0)^2}{4r}}|f(z)|dz\Big)^2dy\\
\leq\ & C(\|f\|_{L^\infty}|B(0,R_0)|)^2\frac{1}{r^{d/2}}e^{-\frac{t^2}{16r}}\\
\leq\ & C(\|f\|_{L^\infty}|B(0,R_0)|)^2\frac{1}{(\frac{t^2}{8d})^{d/2}}e^{-\frac{t^2}{16\frac{t^2}{8d}}}.
\end{aligned}$$

Therefore,

$$\gamma_3(f)=\lim_{t\to\infty}\left(\sup_{B(x,r)\subset(B(0,t))^c}\Big(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)-T_r(f)(y)|^2dy\Big)^{1/2}\right)=0,$$

which finishes the proof.

□

3.4. **Weak*-convergence in $H^1_L(\mathbb{R}^d)$.** Now, we are ready to state the main theorem of Section 3.

Theorem 3.4. *Suppose that $\{f_m\}_{m\geq 1}$ is a bounded sequence by $\mathfrak{A} > 0$ in $H^1_L(\mathbb{R}^d)$ and that $f_m(x) \to f(x)$ almost everywhere. Then, $f \in H^1_L(\mathbb{R}^d)$ and $\{f_m\}_{m\geq 1}$ weak*-converges to f, that is,*

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} f_m(x)\phi(x)dx = \int_{\mathbb{R}^d} f(x)\phi(x)dx$$

for all $\phi \in C^\infty_c(\mathbb{R}^d)$ since the space $C^\infty_c(\mathbb{R}^d)$ is dense in $VMO_L(\mathbb{R}^d)$ the predual of $H^1_L(\mathbb{R}^d)$ and f_m are uniformly bounded in $H^1_L(\mathbb{R}^d)$.

Corollary 3.3. *Suppose that $\{f_m\}_{m\geq 1}$ is a bounded sequence by $\mathfrak{A} > 0$ in $H^1_L(\mathbb{R}^d)$ and that f_m converges to f in $L^1(\mathbb{R}^d)$. Then, f belongs to $H^1_L(\mathbb{R}^d)$, moreover,*

$$\|f\|_{H^1_L} \leq \varliminf_{m\to\infty} \|f_m\|_{H^1_L} \leq \mathfrak{A}.$$

Proof of Theorem 3.4. For every $n \in \mathbb{Z}, k \in \mathbb{Z}^+$, one has $\psi_{n,k}f_m(x) \to \psi_{n,k}f(x)$ a.e since $f_m(x) \to f(x)$ a.e. Consequently, it follows from Theorem 11 of [14] that $\{\psi_{n,k}f_m\}_m$ weak*-converges to $\psi_{n,k}f$ in $h^1_n(\mathbb{R}^d)$, and thus

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} \psi_{n,k}(x)f_m(x)\phi(x)dx = \int_{\mathbb{R}^d} \psi_{n,k}(x)f(x)\phi(x)dx, \tag{3.10}$$

for all $\phi \in C^\infty_c(\mathbb{R}^d) \subset vmo_n(\mathbb{R}^d)$, moreover,

$$\|\psi_{n,k}f\|_{h^1_n} \leq \varliminf_{m\to\infty} \|\psi_{n,k}f_m\|_{h^1_n}. \tag{3.11}$$

Then, Corollary 2.1 implies that there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$\psi_{n,k}f = \sum_j \lambda^{n,k}_j a^{n,k}_j, \quad \sum_j |\lambda^{n,k}_j| \leq C\|\psi_{n,k}f\|_{h^1_n}.$$

This together with (3.11) and Lemma 2.7 follow that for every $N, K \in \mathbb{Z}^+$, there exist $m_{N,K} \in \mathbb{Z}^+$ such that

$$\begin{aligned}
\sum_{n=-N}^{N}\sum_{k=1}^{K}\sum_j |\lambda^{n,k}_j| &\leq \sum_{n=-N}^{N}\sum_{k=1}^{K} C\Big(\frac{\mathfrak{A}}{(1+n^2)(1+k^2)} + \|\psi_{n,k}f_{m_{N,K}}\|_{h^1_n}\Big) \\
&\leq C\sum_{n,k}\frac{\mathfrak{A}}{(1+n^2)(1+k^2)} + C\|f_{m_{N,K}}\|_{H^1_L} \\
&\leq C\mathfrak{A},
\end{aligned}$$

where the constants C are independent of N, K, which allows to conclude that

$$f = \sum_{n,k}\psi_{n,k}f \in H^1_L(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H^1_L} \leq \sum_{n,k}\sum_j |\lambda^{n,k}_j| \leq C\mathfrak{A}.$$

Finally, for every $\phi \in C_c^\infty(\mathbb{R}^d)$, Lemma 2.9 implies that there are two positive integer numbers N, K depend only on support of ϕ such that

$$f\phi = \sum_{n=-N}^{N}\sum_{k=1}^{K}\psi_{n,k}f\phi \quad \text{and} \quad f_m\phi = \sum_{n=-N}^{N}\sum_{k=1}^{K}\psi_{n,k}f_m\phi$$

for all $m \geq 1$. Consequently, by (3.10), we obtain that

$$\begin{aligned}
\lim_{m\to\infty}\int_{\mathbb{R}^d} f_m(x)\phi(x)dx &= \lim_{m\to\infty}\int_{\mathbb{R}^d}\sum_{n=-N}^{N}\sum_{k=1}^{K}\psi_{n,k}(x)f_m(x)\phi(x)dx \\
&= \sum_{n=-N}^{N}\sum_{k=1}^{K}\lim_{m\to\infty}\int_{\mathbb{R}^d}\psi_{n,k}(x)f_m(x)\phi(x)dx \\
&= \sum_{n=-N}^{N}\sum_{k=1}^{K}\int_{\mathbb{R}^d}\psi_{n,k}(x)f(x)\phi(x)dx \\
&= \int_{\mathbb{R}^d} f(x)\phi(x)dx,
\end{aligned}$$

which ends the proof.

□

4. The bilinear decomposition for $H_L^1(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$

The main result of this section, namely the bilinear decomposition theorem for the product space $H_L^1(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$, is as follows:

Theorem 4.1. *There exist two continuous bilinear operators on the product space $H_L^1(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d)$, respectively $S_L : H_L^1(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and $T_L : H_L^1(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \to H^{\log}(\mathbb{R}^d)$ such that*

$$f \times g = S_L(f, g) + T_L(f, g).$$

In applications to nonlinear PDEs, the distribution $f \times g \in \mathcal{S}'(\mathbb{R}^d)$ is used to justify weak continuity properties of the pointwise product fg. It is therefore important to recover fg from the action of the distribution $f \times g$ on the test functions. An idea that naturally comes to mind is to look at the mollified distributions

$$(f \times g)_\epsilon = (f \times g) * \phi_\epsilon, \tag{4.1}$$

and let $\epsilon \to 0$. Here $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\phi(x)dx = 1$.

As a consequence of Theorem 4.1, we will see that the limit (4.1) exists and equals fg almost everywhere.

Theorem 4.2. *Let $f \in H^1_L(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$. For almost every $x \in \mathbb{R}^d$ it holds*

$$\lim_{\epsilon\to 0}(f \times g)_\epsilon(x) = f(x)g(x).$$

Noting that $H^{\log}(\mathbb{R}^d) \subset H^\Phi(\mathbb{R}^d, d\mu)$ with continuous embedding, the proof of Theorem 4.2 follows directly from the one of Theorem 1.8 of [6]. We leave the details to interested reader. It should also be pointed out that Theorem 2 of [32] can be seen as an immediate consequence of Theorem 4.2 since $V \in RH_q$, for any $q \geq 1$, if V is a nonnegative nonzero polynomial on $\mathbb{R}^d$. Let us now start to prove Theorem 4.1. First, we need the following key lemma.

Lemma 4.1. *There exists a constant $C = C(\varphi, d) > 0$ such that for all $(n,k) \in \mathbb{Z}\times\mathbb{Z}^+$, $g \in BMO_L(\mathbb{R}^d)$ and $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x_{n,k}, 2^{1-n/2})$, we have*

$$\Big\|(\varphi_{2^{-n/2}} * f)g\Big\|_{H^1_L} \leq C\|f\|_{h^1_n}\|g\|_{BMO_L}.$$

Proof. As $x_{n,k} \in \mathcal{B}_n$, it follows from Corollary 2.1 that there are $(H^1_L, 2)$-atoms a_j related to the balls $B(x_j, r_j) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h^1_n}. \tag{4.2}$$

Then, Lemma 2.14 yields that $\varphi_{2^{-n/2}} * a_j$ are C times $(H^1_L, 2)$-atoms related to the ball $B(x_{n,k}, 5.2^{-n/2})$, and hence

$$\|\varphi_{2^{-n/2}} * a_j\|_{H^1_L} \leq C. \tag{4.3}$$

Furthermore, supp $(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k},5.2^{-n/2})}) \subset B(x_{n,k}, 5.2^{-n/2})$ and

$$\begin{aligned} &\Big\|(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k},5.2^{-n/2})})\Big\|_{L^{3/2}} \\ \leq\ & \|\varphi_{2^{-n/2}} * a_j\|_{L^2}\|(g - g_{B(x_{n,k},5.2^{-n/2})})\chi_{B(x_{n,k},5.2^{-n/2})}\|_{L^6} \\ \leq\ & C\|g\|_{BMO}|B(x_{n,k}, 5.2^{-n/2})|^{-1/3} \end{aligned}$$

where $C > 0$ is independent of g, f, n. This proves that $(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k},5.2^{-n/2})})$ are $C\|g\|_{BMO}$ times $(H^1_L, 3/2)$-atoms, and thus

$$\|(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k},5.2^{-n/2})})\|_{H^1_L} \leq C\|g\|_{BMO}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) allows to conclude that

$$\begin{aligned} \Big\|(\varphi_{2^{-n/2}} * f)g\Big\|_{H^1_L} \leq\ & C\sum_j |\lambda_j|\|(\varphi_{2^{-n/2}} * a_j)(g - g_{B(x_{n,k},5.2^{-n/2})})\|_{H^1_L} \\ & + C\sum_j |\lambda_j|\|\varphi_{2^{-n/2}} * a_j\|_{H^1_L}|g_{B(x_{n,k},5.2^{-n/2})}| \\ \leq\ & C\|f\|_{h^1_n}\|g\|_{BMO_L}, \end{aligned}$$

since $|g_{B(x_{n,k},5.2^{-n/2})}| \leq \|g\|_{BMO_L}$ as $\rho(x_{n,k}) \leq 5.2^{-n/2}$.

□

The proof of Lemma 4.1 gives a corollary which is useful for studying the theory of commutators in the next sections.

Corollary 4.1. *Assume that $g \in BMO(\mathbb{R}^d)$ and $f \in h^1_n(\mathbb{R}^d)$ as in Lemma 4.1. Then, we have*

$$\|\varphi_{2^{-n/2}} * f\|_{H^1_L} \leq C\|f\|_{h^1_n}$$

and

$$\Big\|(\varphi_{2^{-n/2}} * f)(g - g_{B(x_{n,k},5.2^{-n/2})})\Big\|_{H^1_L} \leq C\|f\|_{h^1_n}\|g\|_{BMO}.$$

Recall that the set $\mathcal{E} = \{0,1\}^d \setminus \{(0,\cdots,0)\}$ and $\{\psi^\sigma\}_{\sigma\in\mathcal{E}}$ is the wavelet with compact support as in Section 3 of [4]. Suppose that ψ^σ is supported in the cube $(\frac{1}{2} - \frac{c}{2}, \frac{1}{2} - \frac{c}{2})^d$ for all $\sigma \in \mathcal{E}$. As it is classical, for $\sigma \in \mathcal{E}$ and I a dyadic cube of $\mathbb{R}^d$ which may be written as the set of x such that $2^j x - k \in (0,1)^d$, we note

$$\psi^\sigma_I(x) = 2^{dj/2}\psi^\sigma(2^j x - k).$$

In the sequel, the letter I always refers to dyadic cubes. Moreover, we note kI the cube of same center dilated by the coefficient k.

Remark 4.1. *For every $\sigma \in \mathcal{E}$ and I a dyadic cube. Because of the assumption on the support of ψ^σ, the function ψ^σ_I is supported in the cube cI.*

In [4] (see also [31]), Bonami et al. established the following.

Proposition 4.1. *The bounded bilinear operator Π, defined by*

$$\Pi(f,g) = \sum_I \sum_{\sigma\in\mathcal{E}} \langle f, \psi^\sigma_I\rangle\langle g, \psi^\sigma_I\rangle(\psi^\sigma_I)^2,$$

is bounded from $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

To prove Theorem 4.1, we need to recall a recent result of Bonami, Grellier and Ky [4] which can be stated as follows:

Theorem 4.3 (see [4], Theorem 1.1)**.** *There is a bilinear operator $\mathfrak{T}$ maps continuously $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $H^{\log}(\mathbb{R}^d)$ such that for every $(f,g) \in H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$,*

$$f \times g = \Pi(f,g) + \mathfrak{T}(f,g).$$

Before giving the proof of Theorem 4.1, we should point out that the bilinear operator $\mathfrak{T}$ in Theorem 4.3 satisfies

$$\|\mathfrak{T}(f,g)\|_{H^{\log}} \leq C\|f\|_{H^1}(\|g\|_{BMO} + |g_{\mathbb{Q}}|) \tag{4.5}$$

where $\mathbb{Q} := [0,1)^d$ is the unit cube. To prove this, the authors in [4] used the generalized Hölder inequality (see also [6])

$$\|fg\|_{L^{\log}} \leq C\|f\|_{L^1}\|g\|_{L^\Xi}$$

and the fact that $\|g-g_{\mathbb{Q}}\|_{L^\Xi} \le C\|g\|_{BMO}$, where $L^\Xi(\mathbb{R}^d)$ denotes the space of all measurable functions g such that $\int_{\mathbb{R}^d}(e^{|g(x)|/\lambda}-1)\frac{dx}{(1+|x|)^{2d}} < \infty$ for some $\lambda > 0$ with the norm

$$\|g\|_{L^\Xi} = \inf\Big\{\lambda > 0 : \int\limits_{\mathbb{R}^d}\Big(e^{|g(x)|/\lambda}-1\Big)\frac{dx}{(1+|x|)^{2d}} \le 1\Big\}.$$

In fact, Inequality (4.5) also holds when we replace the unit cube $\mathbb{Q}$ by $B(0,r)$ for every $r > 0$ since $\|g-g_{B(0,r)}\|_{L^\Xi} \le C\|g\|_{BMO}$. More precisely, there exists a constant $C > 0$ such that

$$\|fg\|_{L^{\log}} \le C\|f\|_{L^1}(\|g\|_{BMO} + |g_{B(0,\rho(0))}|) \le C\|f\|_{L^1}\|g\|_{BMO_L} \tag{4.6}$$

for all $f \in L^1(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$. As a consequence, we have

$$\|\mathfrak{T}(f,g)\|_{H^{\log}} \le C\|f\|_{H^1}\|g\|_{BMO_L} \tag{4.7}$$

for all $f \in H^1(\mathbb{R}^d)$ and $g \in BMO_L(\mathbb{R}^d)$.

Proof of Theorem 4.1. We define two bilinear operators S_L and T_L by

$$S_L(f,g) = \Pi(\mathfrak{H}(f),g) + \sum_{n,k}\Big(\varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big)g$$

and

$$T_L(f,g) = \mathfrak{T}(\mathfrak{H}(f),g),$$

for all $(f,g) \in H^1_L(\mathbb{R}^d)\times BMO_L(\mathbb{R}^d)$. Then, it follows from Proposition 4.1, Lemma 4.1, Lemma 2.10 and Lemma 2.7 that

$$\begin{aligned}
\|S_L(f,g)\|_{L^1} &\le \|\Pi(\mathfrak{H}(f),g)\|_{L^1} + C\sum_{n,k}\Big\|\Big(\varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big)g\Big\|_{H^1_L} \\
&\le C\|g\|_{BMO}\|\mathfrak{H}(f)\|_{H^1} + C\|g\|_{BMO_L}\sum_{n,k}\|\psi_{n,k}f\|_{h^1_n} \\
&\le C\|f\|_{H^1_L}\|g\|_{BMO_L},
\end{aligned}$$

and as (4.7), Theorem 4.3 yields

$$\begin{aligned}
\|T_L(f,g)\|_{H^{\log}} &\le C\|\mathfrak{H}(f)\|_{H^1}\|g\|_{BMO_L} \\
&\le C\|f\|_{H^1_L}\|g\|_{BMO_L}.
\end{aligned}$$

Furthermore, in the sense of distributions, we have

$$\begin{aligned}
& S_L(f,g) + T_L(f,g) \\
= \ & \left(\sum_{n,k}\Big(\psi_{n,k}f - \varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big)\right) \times g + \sum_{n,k}\Big(\varphi_{2^{-n/2}} * (\psi_{n,k}f)\Big)g \\
= \ & \left(\sum_{n,k}\psi_{n,k}f\right) \times g = f \times g,
\end{aligned}$$

which ends the proof.

□

5. Bilinear, subbilinear decompositions and commutators

Recall that $\mathcal{K}_L$ is the set of all sublinear operators T bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ and that there are $q \in (1,\infty]$ and $\varepsilon > 0$ such that

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO}$$

for all $b \in BMO(\mathbb{R}^d)$, any generalized (H^1_L, q, ε)-atom a related to the ball B, where $C > 0$ a constant independent of b, a.

5.1. **Two decomposition theorems.** Let b be a locally integrable function and $T \in \mathcal{K}_L$. As usual, the (sublinear) commutator $[b,T]$ of the operator T is defined by $[b,T](f)(x) := T\Big((b(x) - b(\cdot))f(\cdot)\Big)(x)$. Here and in what follows, we denote the bilinear operator

$$\mathfrak{S}(f,g) := -\Pi(\mathfrak{H}(f), g).$$

Then, by Proposition 4.1 and Lemma 2.10, we obtain that:

Proposition 5.1. *The bilinear operator $\mathfrak{S}$ is bounded from $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Theorem 5.1 (Subbilinear decomposition)**.** *Let $T \in \mathcal{K}_L$. There exists a bounded subbilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have*

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq \mathfrak{R}(f,b) + |T(\mathfrak{S}(f,b))|.$$

Corollary 5.1. *Suppose that $T \in \mathcal{K}_L$ and T is of weak type $(1,1)$. Then, the operator $\mathfrak{P}(f,g) = [g,T](f)$ maps continuously $H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$. In particular, the commutator $[b,T]$ is of weak type (H^1_L, L^1) if $b \in BMO(\mathbb{R}^d)$.*

When T is linear and belongs to $\mathcal{K}_L$, we obtain the bilinear decomposition for the linear commutator $[b,T]$ of f, $[b,T](f) = bT(f) - T(bf)$, instead of the subbilinear decomposition as stated in Theorem 5.1.

Theorem 5.2 (Bilinear decomposition)**.** *Let T be a linear operator in $\mathcal{K}_L$. Then, there exists a bounded bilinear operator $\mathfrak{R} = \mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have*

$$[b,T](f) = \mathfrak{R}(f,b) + T(\mathfrak{S}(f,b)).$$

5.2. **The space $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$.** Using Theorem 5.1 and Theorem 5.2, we find the largest subspace $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ of $H^1_L(\mathbb{R}^d)$ such that all commutators of Schrödinger-Calderón-Zygmund operators and the Riesz transforms are bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. We also find all functions b in $BMO(\mathbb{R}^d)$ such that $\mathcal{H}^1_{L,b}(\mathbb{R}^d) \equiv H^1_L(\mathbb{R}^d)$.

Definition 5.1. *Let b be a non-constant BMO-function. The space $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ consists of all f in $H^1_L(\mathbb{R}^d)$ such that $[b,\mathcal{M}_L](f)(x)=\mathcal{M}_L(b(x)f(\cdot)-b(\cdot)f(\cdot))(x)$ belongs to $L^1(\mathbb{R}^d)$. We equipped $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ with the norm*

$$\|f\|_{\mathcal{H}^1_{L,b}}=\|f\|_{H^1_L}\|b\|_{BMO}+\|[b,\mathcal{M}_L](f)\|_{L^1}.$$

Here, we just define for b is a non-constant BMO-function since $[b,T]=0$ if b is a constant function.

Theorem 5.3. *Let b be a non-constant BMO-function, we have:*

i) For every $T\in\mathcal{K}_L$, the commutator $[b,T]$ is bounded from $\mathcal{H}^1_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

ii) Assume that $\mathcal{X}$ is a subspace of $H^1_L(\mathbb{R}^d)$ such that all commutators of the Riesz transforms are bounded from $\mathcal{X}$ into $L^1(\mathbb{R}^d)$. Then, $\mathcal{X}\subset\mathcal{H}^1_{L,b}(\mathbb{R}^d)$.

Theorem 5.3 gives an anwser for *Question 1* in Introduction.

Theorem 5.4. *i) Let $b\in BMO^{\log}_L(\mathbb{R}^d)$. Then, for every $T\in\mathcal{K}_L$, the commutator $[b,T]$ is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Deduce that $\mathcal{H}^1_{L,b}(\mathbb{R}^d)\equiv H^1_L(\mathbb{R}^d)$ since $\mathcal{M}_L\in\mathcal{K}_L$.*

ii) Suppose that b is a non-constant BMO-function such that $\mathcal{H}^1_{L,b}(\mathbb{R}^d)\equiv H^1_L(\mathbb{R}^d)$. Then, b belongs to $BMO^{\log}_L(\mathbb{R}^d)$.

Theorem 5.4 gives an anwser for *Question 2* in Introduction.

5.3. **Hardy estimates for linear commutators.** Let us first recall (see [8]) that $BMO^{\log}_L(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO^{\log}_L}=\sup_{B(x,r)}\left(\log\Big(e+\frac{\rho(x)}{r}\Big)MO(f,B(x,r))\right)<\infty,$$

where and in what follows

$$MO(f,B):=\frac{1}{|B|}\int\limits_B|f(y)-f_B|dy.$$

Observe that Ma *et al.* [35] have very recently showed that g is a pointwise multiplier for $BMO_L(\mathbb{R}^d)$ if and only if g belongs to $L^\infty(\mathbb{R}^d)\cap BMO^{\log}_L(\mathbb{R}^d)$.

Our first main result of this subsection is the following theorem.

Theorem 5.5. *i) Let $b\in BMO^{\log}_L(\mathbb{R}^d)$ and T be a L-Calderón-Zygmund operator satisfying $T^*(1)=0$. Then, the linear commutator $[b,T]$ is bounded on $H^1_L(\mathbb{R}^d)$.*

ii) When $V\in RH_d$, we have the conversion. More precisely, assume that $b\in BMO(\mathbb{R}^d)$ and that $[b,T]$ is bounded on $H^1_L(\mathbb{R}^d)$ for every L-Calderón-Zygmund operator T satisfying $T^*(1)=0$. Then, $b\in BMO^{\log}_L(\mathbb{R}^d)$, moreover,

$$\|b\|_{BMO^{\log}_L}\approx\|b\|_{BMO}+\sum_{j=1}^d\|[b,R_j]\|_{H^1_L\to H^1_L}.$$

In order to study commutators of general Schrödinger-Calderón-Zygmund operators, we always assume that $b \in BMO(\mathbb{R}^d)$. However, when $T = R_j = \partial_{x_j} L^{-1/2}$, $j = 1, ..., d$, one can find a class of functions which is larger $BMO_L^{\log}(\mathbb{R}^d)$ such that the commutators of the Riesz transforms $[b, R_j]$ are bounded on $H_L^1(\mathbb{R}^d)$ whenever b is in this class. To be more precise, we need to recall some notations from [8].

Let $\theta \geq 0$. Following Bongioanni *et al.* [8], we denote by $BMO_{L,\theta}(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO_{L,\theta}} = \sup_{B(x,r)} \left(\frac{1}{\left(1 + \frac{r}{\rho(x)}\right)^{\theta}} MO(f, B(x,r)) \right) < \infty,$$

and $BMO_{L,\theta}^{\log}(\mathbb{R}^d)$ the set of all locally integrable functions f such that

$$\|f\|_{BMO_{L,\theta}^{\log}} = \sup_{B(x,r)} \left(\frac{\log\left(e + \frac{\rho(x)}{r}\right)}{\left(1 + \frac{r}{\rho(x)}\right)^{\theta}} MO(g, B(x,r)) \right) < \infty.$$

Then, we define

$$BMO_{L,\infty}(\mathbb{R}^d) = \bigcup_{\theta \geq 0} BMO_{L,\theta}(\mathbb{R}^d)$$

and

$$BMO_{L,\infty}^{\log}(\mathbb{R}^d) = \bigcup_{\theta \geq 0} BMO_{L,\theta}^{\log}(\mathbb{R}^d).$$

Clearly, $BMO_{L,0}(\mathbb{R}^d)$ is just the classical $BMO(\mathbb{R}^d)$, and $BMO_{L,0}^{\log}(\mathbb{R}^d)$ is just $BMO_L^{\log}(\mathbb{R}^d)$. Moreover, for any $0 \leq \theta \leq \theta' < \infty$,

$$BMO_{L,\theta}^{\log}(\mathbb{R}^d) = BMO_{L,\theta}(\mathbb{R}^d) \cap BMO_{L,\infty}^{\log}(\mathbb{R}^d) \tag{5.1}$$

and

$$BMO_{L,\theta}(\mathbb{R}^d) \subset BMO_{L,\theta'}(\mathbb{R}^d), \quad BMO_{L,\theta}^{\log}(\mathbb{R}^d) \subset BMO_{L,\theta'}^{\log}(\mathbb{R}^d). \tag{5.2}$$

Remark that the inclusions in (5.2) are strict in general. In particular, $BMO_{L,\infty}^{\log}(\mathbb{R}^d)$ is in general larger than $BMO_L^{\log}(\mathbb{R}^d)$. As an example, when $L = -\Delta + 1$, it is easy to check that the functions $b_j = |x_j|$, $j = 1, ..., d$, belong to $BMO_{L,\infty}^{\log}(\mathbb{R}^d)$ but not to $BMO_L^{\log}(\mathbb{R}^d)$.

Now, we are ready to give the necessary and sufficient condition for the H_L^1-boundedness of the (linear) commutators of the Riesz transforms.

Theorem 5.6. *i) Let $b \in BMO_{L,\infty}^{\log}(\mathbb{R}^d)$. Then, the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H_L^1(\mathbb{R}^d)$.*

ii) Conversely, assume that b belongs to $BMO_{L,\infty}(\mathbb{R}^d)$ and that the commutators $[b, R_j]$, $j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$. Then, b is in $BMO^{\log}_{L,\infty}(\mathbb{R}^d)$. Furthermore, when $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$, we have

$$\|b\|_{BMO^{\log}_{L,\theta}} \approx \|b\|_{BMO_{L,\theta}} + \sum_{j=1}^{d} \|[b, R_j]\|_{H^1_L \to H^1_L}.$$

Remark that the constants in the above equivalence depend on θ.

As an immediate consequence of Theorem 5.6, the following gives a positive anwser for *Question 3* in Introduction.

Theorem 5.7. *Let $b \in BMO^{\log}_{L,\infty}(\mathbb{R}^d)$. Then, the commutators $[b, R^*_j]$, $j = 1, ..., d$, are bounded from $BMO_L(\mathbb{R}^d)$ into itself.*

Recall that $LMO(\mathbb{R}^d)$ is the set of all locally integrable functions f such that

$$\|f\|_{LMO} := \sup_{B(x,r)} \left(\log\Big(e + \frac{1}{r}\Big) MO(f, B(x, r)) \right) < \infty.$$

It should be pointed out that LMO type spaces appear naturally when studying the boundedness of Hankel operators on the Hardy spaces $H^1(\mathbb{T}^d)$ and $H^1(\mathbb{B}^d)$ (where $\mathbb{B}^d$ is the unit ball in $\mathbb{C}^d$), the characterization of pointwise multipliers for BMO type spaces, the endpoint estimates for commutators of singular integrals operators and their applications to PDEs (see for example [10, 5, 26, 27, 31, 38, 41, 43]). Remark that when $L = -\Delta + 1$, then $H^1_L(\mathbb{R}^d)$ is just the space $h^1(\mathbb{R}^d)$ of Goldberg (see [22]). The following gives a characterization of the space $LMO(\mathbb{R}^d)$.

Corollary 5.2. *Let $b \in BMO(\mathbb{R}^d)$. Then, b belongs to $LMO(\mathbb{R}^d)$ if and only if the vector-valued commutator $[b, \nabla(-\Delta + 1)^{-1/2}]$ maps continuously $h^1(\mathbb{R}^d)$ into $h^1(\mathbb{R}^d, \mathbb{R}^d) = (h^1(\mathbb{R}^d), ..., h^1(\mathbb{R}^d))$. Furthermore,*

$$\|b\|_{LMO} \approx \|b\|_{BMO} + \|[b, \nabla(-\Delta + 1)^{-1/2}]\|_{h^1(\mathbb{R}^d) \to h^1(\mathbb{R}^d, \mathbb{R}^d)}.$$

In [37], Nakai and Yabuta characterized the pointwise multipliers for $BMO(\mathbb{R}^d)$: they proved that g is a pointwise multiplier for $BMO(\mathbb{R}^d)$ if and only if g belong to $L^\infty(\mathbb{R}^d) \cap BMO^{\log}(\mathbb{R}^d)$, with

$$\|g\|_{BMO^{\log}} = \sup_B \Big((\log(e + |B|^{-1}) + \sup_{x \in B} \log(e + |x|)) MO(g, B) \Big) < \infty, \tag{5.3}$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. It should be pointed out that the $BMO^{\log}$-norm studied in [37] is different from (5.3), however they are equivalent (see [30]). Moreover, in [30] the author established also that $BMO^{\log}(\mathbb{R}^d)$ is just the dual of $H^{\log}(\mathbb{R}^d)$. Thus, from Theorem 4.1 and Proposition 3.2 of [35], it is easy to see that

$$L^\infty(\mathbb{R}^d) \cap BMO^{\log}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \cap BMO^{\log}_L(\mathbb{R}^d).$$

A natural question arises: is $BMO^{\log}(\mathbb{R}^d)$ a subspace of $BMO^{\log}_L(\mathbb{R}^d)$? The following theorem gives an answer.

Theorem 5.8. *Let $b \in BMO^{\log}(\mathbb{R}^d)$. Then, the commutators $[b, R_j], j = 1, ..., d$, are bounded on $H^1_L(\mathbb{R}^d)$.*

Then, Theorem 5.6 and Theorem 5.8 allow us to conclude that

$$BMO^{\log}(\mathbb{R}^d) \subset BMO^{\log}_L(\mathbb{R}^d)$$

and the inclusion is continuous. Remark that the above inclusion is strict in general. For example, when $L = -\Delta + 1$, it is not difficult to see that the function $b(x) = \log(1 + |x|^2)$ belongs to $BMO^{\log}_L(\mathbb{R}^d)$ but not to $BMO^{\log}(\mathbb{R}^d)$.

6. Some fundamental operators and the class $\mathcal{K}_L$

The purpose of this section is to give some examples of (sublinear) operators which are in the class $\mathcal{K}_L$.

6.1. **Schrödinger-Calderón-Zygmund operators.** Let $\delta \in (0, 1]$. Following [35], a continuous function $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{C}$ is said to be a (δ, L)-Calderón-Zygmund singular integral kernel if for each $N > 0$,

$$|K(x, y)| \leq \frac{C(N)}{|x - y|^d}\Big(1 + \frac{|x - y|}{\rho(x)}\Big)^{-N} \tag{6.1}$$

for all $x \neq y$, and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C\frac{|x - x'|^\delta}{|x - y|^{d+\delta}} \tag{6.2}$$

for all $2|x - x'| \leq |x - y|$.

A linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is said to be a (δ, L)-Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbb{R}^d)$ and if there exists a (δ, L)-Calderón-Zygmund singular integral kernel K such that for all $f \in C^\infty_c(\mathbb{R}^d)$ and all $x \notin \text{supp } f$, we have

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy.$$

We say that T is a L-Calderón-Zygmund operator if it is a (δ, L)-Calderón-Zygmund operator for some $\delta \in (0, 1]$, and that T satisfies the condition $T^*(1) = 0$ (in the sense of $BMO_L(\mathbb{R}^d)$, see the paper of Bernicot [2] for the details) if there are $q \in (1, \infty]$ and $\varepsilon > 0$ such that $\int_{\mathbb{R}^n} Ta(x)dx = 0$ for all generalized (H^1_L, q, ε)-atom a.

Remark 6.1. *i) Using Lemma 2.1, Inequality (6.1) is equivalent to*

$$|K(x, y)| \leq \frac{C(N)}{|x - y|^d}\Big(1 + \frac{|x - y|}{\rho(y)}\Big)^{-N}$$

for all $x \neq y$.

ii) If T is a L-Calderón-Zygmund operator then it is also a classical Calderón-Zygmund operator, and thus T is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and bounded from $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$.

Proposition 6.1. *Let T be any L-Calderón-Zygmund operator. Then, T belongs to the class $\mathcal{K}_L$.*

Proposition 6.2. *The Riesz transforms R_j are in the class $\mathcal{K}_L$.*

The proof of Proposition 6.2 follows directly from Lemma 9.6 and the fact that the Riesz transforms R_j are bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

To prove Proposition 6.1, we need the following two lemmas.

Lemma 6.1. *Let $1 \leq q < \infty$. Then, there exists a constant $C > 0$ such that for every ball B, $f \in BMO(\mathbb{R}^d)$ and $k \in \mathbb{Z}^+$,*

$$\Big(\frac{1}{|2^k B|}\int\limits_{2^k B} |f(y) - f_B|^q dy\Big)^{1/q} \leq Ck\|f\|_{BMO}.$$

Lemma 6.2. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. Assume that T is a (δ, L)-Calderón-Zygmund operator and a is a generalized (H^1_L, q, ε)-atom related to the ball $B = B(x_0, r)$. Then,*

$$\|Ta\|_{L^q(2^{k+1}B\setminus 2^k B)} \leq C2^{-k\delta_0}|2^k B|^{1/q-1}$$

for all $k = 1, 2, ...$, where $\delta_0 = \min\{\varepsilon, \delta\}$.

Using the classical John-Nirenberg inequality, the proof of Lemma 6.1 is elementary and left to reader.

Proof of Lemma 6.2. Let $x \in 2^{k+1}B \setminus 2^k B$, so that $|x - x_0| \geq 2r$. Since T is a (δ, L)-Calderón-Zygmund operator, we get

$$\begin{aligned}
|Ta(x)| &\leq \Big|\int\limits_B (K(x,y) - K(x,x_0))a(y)dy\Big| + |K(x,x_0)|\Big|\int\limits_{\mathbb{R}^d} a(y)dy\Big| \\
&\leq C\int\limits_B \frac{|y-x_0|^\delta}{|x-x_0|^{d+\delta}}|a(y)|dy + C\frac{1}{|x-x_0|^d}\Big(1 + \frac{|x-x_0|}{\rho(x_0)}\Big)^{-\varepsilon}\Big(\frac{r}{\rho(x_0)}\Big)^{\varepsilon} \\
&\leq C\frac{r^\delta}{|x-x_0|^{d+\delta}} + C\frac{r^\varepsilon}{|x-x_0|^{d+\varepsilon}} \leq C\frac{r^{\delta_0}}{|x-x_0|^{d+\delta_0}}.
\end{aligned}$$

Consequently,

$$\|Ta\|_{L^q(2^{k+1}B\setminus 2^k B)} \leq C\frac{r^{\delta_0}}{(2^k r)^{d+\delta_0}}|2^{k+1}B|^{1/q} \leq C2^{-k\delta_0}|2^k B|^{1/q-1}.$$

□

Proof of Proposition 6.1. Assume that T is a (δ, L)-Calderón-Zygmund for some $\delta \in (0, 1]$. Let us first verify that T is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. By Proposition 2.1, it is sufficient to show that

$$\|Ta\|_{L^1} \leq C$$

for all generalized $(H^1_L, 2, \delta)$-atom a related to the ball B. Indeed, from the L^2-boundedness of T and Lemma 6.2, we obtain that

$$\begin{aligned}\|Ta\|_{L^1} &= \|Ta\|_{L^1(2B)} + \sum_{k=1}^{\infty} \|Ta\|_{L^1(2^{k+1}B\setminus 2^k B)} \\ &\leq C|2B|^{1/2}\|T\|_{L^2\to L^2}\|a\|_{L^2} + C\sum_{k=1}^{\infty} |2^{k+1}B|^{1/2}2^{-k\delta}|2^k B|^{-1/2} \\ &\leq C.\end{aligned}$$

Let us next establish that

$$\|(f-f_B)Ta\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \delta)$-atom a related to the ball $B = B(x_0, r)$. Indeed, by Hölder inequality, Lemma 6.1 and Lemma 6.2, we get

$$\begin{aligned}&\|(f-f_B)Ta\|_{L^1} \\ =\ &\|(f-f_B)Ta\|_{L^1(2B)} + \sum_{k\geq 1} \|(f-f_B)Ta\|_{L^1(2^{k+1}B\setminus 2^k B)} \\ \leq\ &\|(f-f_B)\chi_{2B}\|_{L^2}\|T\|_{L^2\to L^2}\|a\|_{L^2} + \sum_{k\geq 1} \|f-f_B\|_{L^2(2^{k+1}B)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^k B)} \\ \leq\ &C\|f\|_{BMO} + \sum_{k\geq 1} C(k+1)\|f\|_{BMO}|2^{k+1}B|^{1/2}2^{-k\delta}|2^k B|^{-1/2} \\ \leq\ &C\|f\|_{BMO},\end{aligned}$$

which ends the proof.

□

6.2. **The maximal operators.** Recall that $\{T_t\}_{t>0}$ be heat semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL}f(x) = \int_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t>0.$$

Then the "heat" maximal operator is defined by

$$\mathcal{M}_L f(x) = \sup_{t>0} |T_t f(x)|,$$

and the "Poisson" maximal operator is defined by

$$\mathcal{M}^P_L f(x) = \sup_{t>0} |P_t f(x)|,$$

where

$$P_t f(x) = e^{-t\sqrt{L}}f(x) = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} T_u f(x)du.$$

Proposition 6.3. *The "heat" maximal operator $\mathcal{M}_L$ is in the class $\mathcal{K}_L$.*

Proposition 6.4. *The "Poisson" maximal operator $\mathcal{M}_L^P$ is in the class $\mathcal{K}_L$.*

Here we just give the proof for Proposition 6.3. For the one of Proposition 6.4, we leave the details to the interested reader.

Proof of Proposition 6.3. Obviously, $\mathcal{M}_L$ is bounded from $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Now, let us prove that

$$\|(f-f_B)\mathcal{M}_L(a)\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \sigma)$-atom a related to the ball $B = B(x_0, r)$, where the constant σ is as in Lemma 2.2. Indeed, by Step 2 in the proof of Theorem 2.1, for every $x \notin 2B$,

$$\mathcal{M}_L(a)(x) \leq C\frac{r^\sigma}{|x-x_0|^{d+\sigma}}.$$

Therefore, using Lemma 6.1, the L^2-boundedness of the classical Hardy-Littlewood maximal operator $\mathcal{M}$ and the estimate $\mathcal{M}_L(a) \leq C\mathcal{M}(a)$, we obtain that

$$\begin{aligned}
& \|(f-f_B)\mathcal{M}_L(a)\|_{L^1} \\
= \;& \|(f-f_B)\mathcal{M}_L(a)\|_{L^1(2B)} + \|(f-f_B)\mathcal{M}_L(a)\|_{L^1((2B)^c)} \\
\leq \;& C\|f-f_B\|_{L^2(2B)}\|\mathcal{M}(a)\|_{L^2} + C\int\limits_{|x-x_0|\geq 2r} |f(x)-f_{B(x_0,r)}|\frac{r^\sigma}{|x-x_0|^{d+\sigma}}dx \\
\leq \;& C\|f\|_{BMO},
\end{aligned}$$

where we have used the following classical inequality, which proof can be found in [19],

$$\int\limits_{|x-x_0|\geq 2r} |f(x)-f_{B(x_0,r)}|\frac{r^\sigma}{|x-x_0|^{d+\sigma}}dx \leq C\|f\|_{BMO}.$$

This allows us to end the proof of Proposition 6.3.

□

6.3. **The L-square funcfions.** Recall (see [16]) that the L-square funcfions $\mathfrak{g}$ and $\mathcal{G}$ are defined by

$$\mathfrak{g}(f)(x) = \left(\int_0^\infty |t\partial_t T_t(f)(x)|^2\frac{dt}{t}\right)^{1/2}$$

and

$$\mathcal{G}(f)(x) = \left(\int_0^\infty \int\limits_{|x-y|<t} |t\partial_t T_t(f)(y)|^2\frac{dydt}{t^{d+1}}\right)^{1/2}.$$

Proposition 6.5. *The L-square function $\mathfrak{g}$ is in the class $\mathcal{K}_L$.*

Proposition 6.6. *The L-square function $\mathcal{G}$ is in the class $\mathcal{K}_L$.*

Here we just give the proof for Proposition 6.5. For the one of Proposition 6.6, we leave the details to the interested reader.

In order to prove Proposition 6.5, we need the following lemma.

Lemma 6.3. *There exists a constant $C > 0$ such that*

$$|t\partial_t T_t(x, y+h) - t\partial_t T_t(x,y)| \leq C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-y|^2}{t}}, \tag{6.3}$$

for all $|h| < \frac{|x-y|}{2}$, $0 < t$. Here and in the proof of Proposition 6.5, the constants $\delta, c \in (0,1)$ are as in Proposition 4 of [16].

Proof. One only need to consider the case $\sqrt{t} < |h| < \frac{|x-y|}{2}$. Otherwise, (6.3) follows directly from (b) in Proposition 4 of [16].

For $\sqrt{t} < |h| < \frac{|x-y|}{2}$. By (a) in Proposition 4 of [16], we get

$$\begin{aligned} |t\partial_t T_t(x, y+h) - t\partial_t T_t(x,y)| &\leq Ct^{-d/2} e^{-c\frac{|x-y-h|^2}{t}} + Ct^{-d/2} e^{-c\frac{|x-y|^2}{t}} \\ &\leq C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-y|^2}{t}}. \end{aligned}$$

□

Proof of Proposition 6.5. The $(H^1_L - L^1)$ type boundedness of $\mathfrak{g}$ is well-known, see for example [16, 24]. Let us now show that

$$\|(f - f_B)\mathfrak{g}(a)\|_{L^1} \leq C\|f\|_{BMO}$$

for all $f \in BMO(\mathbb{R}^d)$, any generalized $(H^1_L, 2, \delta)$-atom a related to the ball $B = B(x_0, r)$. Indeed, it follows from Lemma 6.3 and (a) in Proposition 4 of [16] that for every $t > 0$, $x \notin 2B$,

$$\begin{aligned} & |t\partial_t T_t(a)(x)| \\ = & \Big| \int_B (t\partial_t T_t(x,y) - t\partial_t T_t(x,x_0))a(y)dy + t\partial_t T_t(x,x_0) \int_B a(y)dy \Big| \\ \leq & C\Big(\frac{r}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-x_0|^2}{t}} \|a\|_{L^1} + Ct^{-d/2} e^{-c\frac{|x-x_0|^2}{t}} \Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(x_0)}\Big)^{-\delta} \Big(\frac{r}{\rho(x_0)}\Big)^{\delta} \\ \leq & C\Big(\frac{r}{\sqrt{t}}\Big)^{\delta} t^{-d/2} e^{-\frac{c}{4}\frac{|x-x_0|^2}{t}}. \end{aligned}$$

Therefore, as $0<\delta<1$, using the estimate $e^{-\frac{c}{2}\frac{|x-x_0|^2}{t}}\le C(c,d)(\frac{t}{|x-x_0|^2})^{d+2}$,

$$\begin{aligned}\mathfrak{g}(a)(x) &\le C\left\{\int_0^\infty \Big(\frac{r^2}{t}\Big)^\delta t^{-d}e^{-\frac{c}{2}\frac{|x-x_0|^2}{t}}\frac{dt}{t}\right\}^{1/2}\\ &\le C\left\{\int_0^{|x-x_0|^2}\Big(\frac{r^2}{t}\Big)^\delta t^{-d}\Big(\frac{t}{|x-x_0|^2}\Big)^{d+2}\frac{dt}{t}+\int_{|x-x_0|^2}^\infty\Big(\frac{r^2}{t}\Big)^\delta t^{-d}\frac{dt}{t}\right\}^{1/2}\\ &\le C\frac{r^\delta}{|x-x_0|^{d+\delta}}.\end{aligned}$$

Therefore, the L^2-boundedness of $\mathfrak{g}$ and Lemma 6.1 yield

$$\begin{aligned}&\|(f-f_B)\mathfrak{g}(a)\|_{L^1}\\ =\;&\|(f-f_B)\mathfrak{g}(a)\|_{L^1(2B)}+\|(f-f_B)\mathfrak{g}(a)\|_{L^1((2B)^c)}\\ \le\;&\|f-f_B\|_{L^2(2B)}\|\mathfrak{g}(a)\|_{L^2}+C\int_{|x-x_0|\ge 2r}|f(x)-f_{B(x_0,r)}|\frac{r^\delta}{|x-x_0|^{d+\delta}}dx\\ \le\;&C\|f\|_{BMO},\end{aligned}$$

which allows us to ends the proof.

□

7. Some applications of two decomposition theorems

7.1. Atomic Hardy spaces related to $b\in BMO(\mathbb{R}^d)$.

Definition 7.1. *Let $1<q\le\infty$, $\varepsilon>0$ and $b\in BMO(\mathbb{R}^d)$. A function a is called a $(H^1_{L,b},q,\varepsilon)$-atom related to the ball $B=B(x_0,r)$ if a is a generalized (H^1_L,q,ε)-atom related to the same ball B and*

$$\Big|\int_{\mathbb{R}^d}a(x)(b(x)-b_B)dx\Big|\le\Big(\frac{r}{\rho(x_0)}\Big)^\varepsilon. \tag{7.1}$$

Then, the space $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ is defined as in (1) of Definition 2.2 with generalized (H^1_L,q,ε)-atoms replaced by $(H^1_{L,b},q,\varepsilon)$-atoms.

Obviously, $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)\subset H^1_L(\mathbb{R}^d)$ and the inclusion is continuous.

Theorem 7.1. *Let $1<q\le\infty$, $\varepsilon>0$, $b\in BMO(\mathbb{R}^d)$ and $T\in\mathcal{K}_L$. Then, the commutator $[b,T]$ is bounded from $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Remark 7.1. *The space $H^1_b(\mathbb{R}^d)$ which has been considered by Wang et al. [44] is a strictly subspace of $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ in general. As an example, let us take $1<q\le\infty$, $\varepsilon>0$, $L=-\Delta+1$, and b be a non-constant bounded function, then it is easy to*

check that the function $f = \chi_{B(0,1)}$ belongs to $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ but not to $H^1_b(\mathbb{R}^d)$. Thus, Theorem 7.1 is an improvement of Theorem 5 of [44].

We should also point out that the authors in [44] proved Theorem 5 in their paper by establishing that

$$\|[b, R_j](a)\|_{L^1} \leq C\|b\|_{BMO}$$

for all H^1_b-atom a. However, as pointed in [9] and [31] that such arguments are not enough to conclude that $[b, R_j]$ is bounded from $H^1_b(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ in general.

Proof of Theorem 7.1. Let a be a $(H^1_{L,b}, q, \varepsilon)$-atom related to the ball $B = B(x_0, r)$. We first prove that $(b-b_B)a$ is $C\|b\|_{BMO}$ times a generalized $(H^1_L, (\widetilde{q}+1)/2, \varepsilon)$-atom, where $\widetilde{q} \in (1, \infty)$ will be defined later and the positive constant C is independent of b, a. Indeed, one has supp $(b - b_B)a \subset$ supp $a \subset B$. In addition, from Hölder inequality and John-Nirenberg (classical) inequality,

$$\|(b - b_B)a\|_{L^{(\widetilde{q}+1)/2}} \leq \|(b - b_B)\chi_B\|_{L^{\widetilde{q}(\widetilde{q}+1)/(\widetilde{q}-1)}}\|a\|_{L^{\widetilde{q}}} \leq C\|b\|_{BMO}|B|^{(-\widetilde{q}+1)/(\widetilde{q}+1)},$$

where $\widetilde{q} = q$ if $1 < q < \infty$ and $\widetilde{q} = 2$ if $q = \infty$. These together with (7.1) yield that $(b - b_B)a$ is $C\|b\|_{BMO}$ times a generalized $(H^1_L, (\widetilde{q} + 1)/2, \varepsilon)$-atom, and thus $\|(b - b_B)a\|_{H^1_L} \leq C\|b\|_{BMO}$.

We now prove that $\mathfrak{S}(a, b)$ belongs to $H^1_L(\mathbb{R}^d)$.

By Theorem 5.2, there exist d bounded bilinear operators $\mathfrak{R}_j : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, $j = 1, ..., d$, such that

$$[b, R_j](a) = \mathfrak{R}_j(a, b) + R_j(\mathfrak{S}(a, b)),$$

since R_j is linear and belongs to $\mathcal{K}_L$ (see Proposition 6.2). Consequently, for every $j = 1, ..., d$, as $R_j \in \mathcal{K}_L$,

$$\begin{aligned}\|R_j(\mathfrak{S}(a, b))\|_{L^1} &= \|(b - b_B)R_j(a) - R_j((b - b_B)a) - \mathfrak{R}_j(a, b)\|_{L^1}\\ &\leq \|(b - b_B)R_j(a)\|_{L^1} + \|R_j\|_{H^1_L \to L^1}\|(b - b_B)a\|_{H^1_L} + \|\mathfrak{R}_j(a, b)\|_{L^1}\\ &\leq C\|b\|_{BMO}.\end{aligned}$$

This together with Proposition 5.1 prove that $\mathfrak{S}(a, b) \in H^1_L(\mathbb{R}^d)$, and moreover that

$$\|\mathfrak{S}(a, b)\|_{H^1_L} \leq C\|b\|_{BMO}. \tag{7.2}$$

Now, for any $f \in H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$, there exists an expansion $f = \sum_{k=1}^{\infty} \lambda_k a_k$ where the a_k are $(H^1_{L,b}, q, \varepsilon)$-atoms and $\sum_{k=1}^{\infty} |\lambda_k| \leq 2\|f\|_{H^{1,q,\varepsilon}_{L,b}}$. Then, the sequence $\{\sum_{k=1}^{n} \lambda_k a_k\}_{n\geq 1}$ converges to f in $H^{1,q,\varepsilon}_{L,b}(\mathbb{R}^d)$ and thus in $H^1_L(\mathbb{R}^d)$. This together with Proposition 5.1 imply that the sequence $\Big\{\mathfrak{S}\Big(\sum_{k=1}^{n} \lambda_k a_k, b\Big)\Big\}_{n\geq 1}$ converges to $\mathfrak{S}(f, b)$ in $L^1(\mathbb{R}^d)$. In addition, by (7.2),

$$\left\|\mathfrak{S}\Big(\sum_{k=1}^{n} \lambda_k a_k, b\Big)\right\|_{H^1_L} \leq \sum_{k=1}^{n} |\lambda_k|\|\mathfrak{S}(a_k, b)\|_{H^1_L} \leq C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO}.$$

We then use Corollary 3.3 and Theorem 5.1 to conclude that

$$\begin{aligned}\|[b,T](f)\|_{L^1} &\leq \|\mathfrak{R}_T(f,b)\|_{L^1} + \|T\|_{H^1_L\to L^1}\|\mathfrak{S}(f,b)\|_{H^1_L}\\ &\leq C\|f\|_{H^1_L}\|b\|_{BMO} + C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO}\\ &\leq C\|f\|_{H^{1,q,\varepsilon}_{L,b}}\|b\|_{BMO},\end{aligned}$$

which allows us to end the proof.

□

7.2. The spaces $H^{\log}_{L,\alpha}(\mathbb{R}^d)$.

Definition 7.2. *Let $\alpha \in \mathbb{R}$. We say that a is a $H^{\log}_{L,\alpha}$-atom related to the ball $B = B(x_0,r)$ if $r \leq \mathcal{C}_L\rho(x_0)$ and*

i) supp $a \subset B$,

ii) $\|a\|_{L^2} \leq \Big(\log(e+\frac{\rho(x_0)}{r})\Big)^{\alpha}|B|^{-1/2}$,

iii) if $r \leq \frac{1}{\mathcal{C}_L}\rho(x_0)$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

Then, the space $H^{\log}_{L,\alpha}(\mathbb{R}^d)$ is defined as in (1) of Definition 2.2 with generalized (H^1_L,q,ε)-atoms replaced by $H^{\log}_{L,\alpha}$-atoms.

Clearly, $H^{\log}_{L,0}(\mathbb{R}^d)$ is just $H^1_L(\mathbb{R}^d)$. Moreover, $H^{\log}_{L,\alpha}(\mathbb{R}^d) \subset H^{\log}_{L,\alpha'}(\mathbb{R}^d)$ for all $\alpha \leq \alpha'$. It should be pointed out that when $L = -\Delta + 1$ and $\alpha > 0$, then $H^{\log}_{L,\alpha}(\mathbb{R}^d)$ is just the space of all distributions f such that

$$\int\limits_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\Big(\log(e+\frac{\mathfrak{M}f(x)}{\lambda})\Big)^{\alpha}}dx < \infty$$

for some $\lambda > 0$, moreover (see [30] for the details),

$$\|f\|_{H^{\log}_{L,\alpha}} \approx \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^d} \frac{\frac{\mathfrak{M}f(x)}{\lambda}}{\Big(\log(e+\frac{\mathfrak{M}f(x)}{\lambda})\Big)^{\alpha}}dx \leq 1\right\}.$$

Theorem 7.2. *Let $T \in \mathcal{K}_L$ and $b \in BMO(\mathbb{R}^d)$. Then, the commutator $[b,T]$ is bounded from $H^{\log}_{L,-1}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Proof. Let a be a $H^{\log}_{L,-1}$-atom related to the ball $B = B(x_0,r)$. Let us first prove that $(b-b_B)a \in H^1_L(\mathbb{R}^d)$. By $H^1_L(\mathbb{R}^d)$ is the dual of $VMO_L(\mathbb{R}^d)$, it is sufficient to show that for every $g \in C^{\infty}_c(\mathbb{R}^d)$,

$$\|(b-b_B)ag\|_{L^1} \leq C\|b\|_{BMO}\|g\|_{BMO_L}.$$

Indeed, thanks to John-Nirenberg inequality and Lemma 2 of [16], we get

$$\begin{aligned}\|(b-b_B)ag\|_{L^1} &\leq \|(g-g_B)(b-b_B)a\|_{L^1}+|g_B|\|(b-b_B)a\|_{L^1}\\ &\leq \|(g-g_B)\chi_B\|_{L^4}\|(b-b_B)\chi_B\|_{L^4}\|a\|_{L^2}+\\ &\quad +C\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|g\|_{BMO_L}\|(b-b_B)\chi_B\|_{L^2}\|a\|_{L^2}\\ &\leq C\|b\|_{BMO}\|g\|_{BMO_L},\end{aligned}$$

which prove that $(b-b_B)a\in H^1_L(\mathbb{R}^d)$, moreover, $\|(b-b_B)a\|_{H^1_L}\leq C\|b\|_{BMO}$.

Similarly to the proof of Theorem 7.1, we also obtain that

$$\|\mathfrak{S}(f,b)\|_{H^1_L}\leq C\|f\|_{H^{\log}_{L,-1}}\|b\|_{BMO}$$

for all $f\in H^{\log}_{L,-1}(\mathbb{R}^d)$. Therefore, Theorem 5.1 allows to conclude that

$$\|[b,T](f)\|_{L^1}\leq C\|f\|_{H^{\log}_{L,-1}}\|b\|_{BMO},$$

which ends the proof. □

7.3. **The Hardy-Sobolev space $H^{1,1}_L(\mathbb{R}^d)$.** Following Hofmann et al. [25], we say that f belongs to the (inhomogeneous) Hardy-Sobolev $H^{1,1}_L(\mathbb{R}^d)$ if $f,\partial_{x_1}f,...,\partial_{x_d}f\in H^1_L(\mathbb{R}^d)$. Then, the norm on $H^{1,1}_L(\mathbb{R}^d)$ is defined by

$$\|f\|_{H^{1,1}_L}=\|f\|_{H^1_L}+\sum_{j=1}^d\|\partial_{x_j}f\|_{H^1_L}.$$

It should be pointed out that the authors in [25] proved that the space $H^{1,1}_{-\Delta}(\mathbb{R}^d)$ is just the classical (inhomogeneous) Hardy-Sobolev $H^{1,1}(\mathbb{R}^d)$ (see for example [1]), and can be identified with the (inhomogeneous) Triebel-Lizorkin space $F^{1,2}_1(\mathbb{R}^d)$ (see [29]). More precisely, f belongs to $H^{1,1}(\mathbb{R}^d)$ if and only if

$$\mathcal{W}_\psi(f)=\left\{\sum_I\sum_{\sigma\in\mathcal{E}}|\langle f,\psi^\sigma_I\rangle|^2(1+|I|^{-1/d})^2|I|^{-1}\chi_I\right\}^{1/2}\in L^1(\mathbb{R}^d),$$

moreover, $\|f\|_{H^{1,1}}\approx\|\mathcal{W}_\psi(f)\|_{L^1}$. Here $\{\psi^\sigma\}_{\sigma\in\mathcal{E}}$ is the wavelet as in Section 4.

Theorem 7.3. *Let $L=-\Delta+1$, $T\in\mathcal{K}_L$ and $b\in BMO(\mathbb{R}^d)$. Then, the commutator $[b,T]$ is bounded from $H^{1,1}_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Remark 7.2. *When $L=-\Delta+1$, we can define $\mathfrak{H}(f)=f-\varphi*f$ instead of $\mathfrak{H}(f)=\sum_{n,k}(\psi_{n,k}f-\varphi_{2^{-n/2}}*(\psi_{n,k}f))$ as in Section 3. Then, all results in this paper are still holding. Moreover, it is easy to see that*

$$\partial_{x_j}(\mathfrak{H}(f))=\mathfrak{H}(\partial_{x_j}f).$$

Here and in what follows, for any cube $Q[y,r] := \{x \in \mathbb{R}^d : \max_{1\le j\le d} |x_j - y_j| \le r\}$, we denote

$$B_Q := \Big\{x \in \mathbb{R}^d : |x-y| < 2\sqrt{d}r\Big\}.$$

To prove Theorem 7.3, we need the following lemma.

Lemma 7.1. *Let $L = -\Delta + 1$. Then, the bilinear Π maps continuously $H^{1,1}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$ into $H^1_L(\mathbb{R}^d)$.*

Proof. Note that $\rho(x) = 1$ for all $x \in \mathbb{R}^d$ since $V \equiv 1$. We first prove that there exists a constant $C > 0$ such that

$$\|(1+|I|^{-1/d})^{-1}(\psi_I^\lambda)^2\|_{H^1_L} \le C \tag{7.3}$$

for all dyadic $I = Q[x_0, r]$ and $\lambda \in E$. Indeed, it follows from Remark 4.1 that supp $(1+|I|^{-1/d})^{-1}(\psi_I^\lambda)^2 \subset cI \subset cB_I$, and it is clear that $\|(1+|I|^{-1/d})^{-1}(\psi_I^\lambda)^2\|_{L^\infty} \le |I|^{-1}\|\psi\|_{L^\infty} \le C|cB_I|^{-1}$. In addition,

$$\Big|\int_{\mathbb{R}^d} (1+|I|^{-1/d})^{-1}(\psi_I^\lambda(x))^2 dx\Big| = (1+|I|^{-1/d})^{-1} \le C\frac{r}{\rho(x_0)}.$$

These prove that $(1+|I|^{-1/d})^{-1}(\psi_I^\lambda)^2$ is C times a generalized $(H^1_L, \infty, 1)$-atom related to the ball cB_I, and thus (7.3) holds.

Now, for every $(f,g) \in H^{1,1}(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, (7.3) implies that

$$\begin{aligned}
\|\Pi(f,g)\|_{H^1_L} &= \|\sum_I \sum_{\lambda\in E} \langle f, \psi_I^\lambda\rangle \langle g, \psi_I^\lambda\rangle (\psi_I^\lambda)^2\|_{H^1_L} \\
&\le C\sum_I \sum_{\lambda\in E} \Big(|\langle f, \psi_I^\lambda\rangle|(1+|I|^{-1/d})\Big)|\langle g, \psi_I^\lambda\rangle| \\
&\le C\|\mathcal{W}_\psi(f)\|_{L^1}\|g\|_{\dot F^{0,2}_\infty} \\
&\le C\|f\|_{H^{1,1}}\|g\|_{BMO},
\end{aligned}$$

where we have used the fact that $BMO(\mathbb{R}^d) \equiv \dot F^{0,2}_\infty(\mathbb{R}^d)$ is the dual of $H^1(\mathbb{R}^d) \equiv \dot F^{0,2}_1(\mathbb{R}^d)$, we refer the reader to [20] for more details.

□

Proof of Theorem 7.3. Let $(f,b) \in H^{1,1}_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$. Thanks to Lemma 7.1, Remark 7.2 and Lemma 2.10, one get

$$\begin{aligned}
\|\mathfrak{S}(f,b)\|_{H^1_L} &\le C\|\mathfrak{H}(f)\|_{H^{1,1}}\|b\|_{BMO} \\
&\le C\|f\|_{H^{1,1}_L}\|b\|_{BMO}.
\end{aligned}$$

Then we use Theorem 5.1 to conclude that

$$\begin{aligned}
\|[b,T](f)\|_{L^1} &\le \|\mathfrak{R}_T(f,b)\|_{L^1} + \|T\|_{H^1_L\to L^1}\|\mathfrak{S}(f,b)\|_{H^1_L} \\
&\le C\|f\|_{H^{1,1}_L}\|b\|_{BMO},
\end{aligned}$$

which ends the proof.

□

8. Proof of Theorem 5.1, Theorem 5.2, Theorem 5.3 and Theorem 5.4

First, we recall (see [31]) that $\mathcal{K}$ is the set of all sublinear operators T bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and there is $q \in (1,\infty]$ such that

$$\|(b-b_B)Ta\|_{L^1} \le C\|b\|_{BMO},$$

for all $b \in BMO(\mathbb{R}^n)$, any classical (H^1,q)-atom a related to the ball B, where $C>0$ a constant independent of b, a.

Remark 8.1. *By Remark 2.2 and $H^1(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$, we obtain that $\mathcal{K}_L \subset \mathcal{K}$.*

Lemma 8.1. *Let $T \in \mathcal{K}_L$. Then,*

$$\left\|(g-g_{B(x_{n,k},5.2^{-n/2})})T(\varphi_{2^{-n/2}}*f)\right\|_{L^1} \le C\|f\|_{h^1_n}\|g\|_{BMO}$$

for all $g \in BMO(\mathbb{R}^d)$ and $f \in h^1_n(\mathbb{R}^d)$ as in Lemma 4.1.

Proof. Since $T \in \mathcal{K}_L$, there are $q \in (1,\infty]$ and $\varepsilon > 0$ such that

$$\|(g-g_B)a\|_{L^1} \le C\|g\|_{BMO} \tag{8.1}$$

for all generalized (H^1_L,q,ε)-atom a related to the ball B. Then, it follows from Remark 2.1 and the proof of Lemma 4.1 that there are generalized (H^1_L,q,ε)-atoms a_j related to the ball $B(x_{n,k},5.2^{-n/2})$ such that

$$\varphi_{2^{-n/2}}*f = \sum_j \lambda_j a_j \quad \text{and} \quad \sum_j |\lambda_j| \le C\|f\|_{h^1_n}.$$

Therefore, (8.1) implies that

$$\begin{aligned}\|(g-g_{B(x_{n,k},5.2^{-n/2})})T(\varphi_{2^{-n/2}}*f)\|_{L^1} &\le C\|g\|_{BMO}\sum_j|\lambda_j| \\ &\le C\|f\|_{h^1_n}\|g\|_{BMO}.\end{aligned}$$

□

Lemma 8.2. *Let $T \in \mathcal{K}_L$. Then, the subbilinear operator*

$$\mathcal{U}(f,b) := [b,T](f-\mathfrak{H}(f))$$

is bounded from $H^1_L(\mathbb{R}^d)\times BMO(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Proof. As $\mathbb{H}^{1,2,1}_{L,\mathrm{fin}}(\mathbb{R}^d)$ is dense in $H^1_L(\mathbb{R}^d)$, it is sufficient to show that

$$\|\mathcal{U}(f,b)\|_{L^1} \le C\|f\|_{H^1_L}\|b\|_{BMO}$$

for all $(f,b) \in \mathbb{H}^{1,2,1}_{L,\mathrm{fin}}(\mathbb{R}^d)\times BMO(\mathbb{R}^d)$. Indeed, it follows from Lemma 8.1 and Corollary 4.1 that for every n,k,

$$\begin{aligned}\|[b,T](\varphi_{2^{-n/2}}*(\psi_{n,k}f))\|_{L^1} &\le \|(b-b_{B(x_{n,k},5.2^{-n/2})})T(\varphi_{2^{-n/2}}*(\psi_{n,k}f))\|_{L^1} \\ &\quad + \|T((b-b_{B(x_{n,k},5.2^{-n/2})})(\varphi_{2^{-n/2}}*(\psi_{n,k}f)))\|_{L^1} \\ &\le C\|\psi_{n,k}f\|_{h^1_n}\|b\|_{BMO}. \qquad (8.2)\end{aligned}$$

As $f \in \mathbb{H}^{1,2,1}_{L,\mathrm{fin}}(\mathbb{R}^d)$, thanks to Lemma 2.9, there exist $N, K \in \mathbb{Z}^+$ such that

$$\psi_{n,k} f = 0$$

if $|n| > N$ or $k > K$. Hence, (8.2) and Lemma 2.7 allow us to conclude that

$$\begin{aligned} \|\mathcal{U}(f,b)\|_{L^1} &\leq \left\| \sum_{n=-N}^{N} \sum_{k=1}^{K} \left| [b,T](\varphi_{2^{-n/2}} * (\psi_{n,k} f)) \right| \right\|_{L^1} \\ &\leq C\|b\|_{BMO} \sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C\|f\|_{H^1_L} \|b\|_{BMO}, \end{aligned}$$

which ends the proof.

□

The proof of Theorem 5.1. As $T \in \mathcal{K}_L \subset \mathcal{K}$ (see Remark 8.1), it follows from Theorem 3.1 of [31] that there exists a bounded subbilinear operator $\mathcal{V} : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f,b) \in H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have

$$|T(-\Pi(f,b))| - \mathcal{V}(f,b) \leq |[b,T](f)| \leq \mathcal{V}(f,b) + |T(-\Pi(f,b))|. \tag{8.3}$$

Let us now define the bilinear operator $\mathfrak{R}$ by

$$\mathfrak{R}(f,b) := |\mathcal{U}(f,b)| + \mathcal{V}(\mathfrak{H}(f), b)$$

for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, where $\mathcal{U}$ is the subbilinear operator as in Lemma 8.2. Then, the subbilinear decomposition (8.3) gives

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}(f,b) \leq |[b,T](f)| \leq |T(\mathfrak{S}(f,b))| + \mathfrak{R}(f,b).$$

Moreover, by Lemma 8.2, the boundedness of $\mathcal{V}$ and Lemma 2.10, we get

$$\begin{aligned} \|\mathfrak{R}(f,b)\|_{L^1} &\leq \|\mathcal{U}(f,b)\|_{L^1} + \|\mathcal{V}(\mathfrak{H}(f),b)\|_{L^1} \\ &\leq C\|f\|_{H^1_L} \|b\|_{BMO} + C\|\mathfrak{H}(f)\|_{H^1} \|b\|_{BMO} \\ &\leq C\|f\|_{H^1_L} \|b\|_{BMO}, \end{aligned}$$

which ends the proof.

□

The proof of Theorem 5.2. As T is a linear operator in $\mathcal{K}_L \subset \mathcal{K}$, it follows from Theorem 3.2 of [31] that there exists a bounded bilinear operator $\mathcal{W} : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that for all $(f,b) \in H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, we have

$$[b,T](f) = \mathcal{W}(f,b) + T(-\Pi(f,b)). \tag{8.4}$$

Let us now define the bilinear operator $\mathfrak{R}$ by

$$\mathfrak{R}(f,b) := \mathcal{U}(f,b) + \mathcal{W}(\mathfrak{H}(f), b)$$

for all $(f,b) \in H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$. Then, the bilinear decomposition (8.4) gives

$$[b,T](f) = \mathfrak{R}(f,b) + T(\mathfrak{S}(f,b)).$$

Moreover, by Lemma 8.2, the boundedness of $\mathcal{W}$ and Lemma 2.10, we get

$$\begin{aligned}\|\mathfrak{R}(f,b)\|_{L^1} &\leq \|\mathcal{U}(f,b)\|_{L^1} + \|\mathcal{W}(\mathfrak{H}(f),b)\|_{L^1}\\ &\leq C\|f\|_{H^1_L}\|b\|_{BMO} + C\|\mathfrak{H}(f)\|_{H^1}\|b\|_{BMO}\\ &\leq C\|f\|_{H^1_L}\|b\|_{BMO},\end{aligned}$$

which ends the proof.

□

To prove Theorem 5.3 and Theorem 5.4, we need the following lemma.

Lemma 8.3. *Let b be a non-constant BMO-function and $f \in H^1_L(\mathbb{R}^d)$. Then, the following conditions are equivalent:*

i) $f \in \mathcal{H}^1_{L,b}(\mathbb{R}^d)$.

ii) $\mathfrak{S}(f,b) \in H^1_L(\mathbb{R}^d)$.

iii) $[b,R_j](f) \in L^1(\mathbb{R}^d)$ for all $j = 1,...,d$.

Furthermore, if one of these conditions is satisfied, then

$$\begin{aligned}\|f\|_{\mathcal{H}^1_{L,b}} &= \|f\|_{H^1_L}\|b\|_{BMO} + \|[b,\mathcal{M}_L](f)\|_{L^1}\\ &\approx \|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1_L}\\ &\approx \|f\|_{H^1_L}\|b\|_{BMO} + \sum_{j=1}^{d}\|[b,R_j](f)\|_{L^1},\end{aligned}$$

where the constants are independent of b and f.

Proof. $(i) \Leftrightarrow (ii)$. As $\mathcal{M}_L \in \mathcal{K}_L$ (see Proposition 6.3), by Theorem 5.1, there is a bounded subbilinear operator $\mathfrak{R} : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

$$\mathcal{M}_L(\mathfrak{S}(f,b)) - \mathfrak{R}(f,b) \leq |[b,\mathcal{M}_L](f)| \leq \mathcal{M}_L(\mathfrak{S}(f,b)) + \mathfrak{R}(f,b).$$

Consequently, $[b,\mathcal{M}_L](f) \in L^1(\mathbb{R}^d)$ iff $\mathfrak{S}(f,b) \in H^1_L(\mathbb{R}^d)$, moreover,

$$\|f\|_{\mathcal{H}^1_{L,b}} \approx \|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1_L}.$$

$(ii) \Leftrightarrow (iii)$. As the Riesz transforms R_j are in $\mathcal{K}_L$ (see Proposition 6.2), by Theorem 5.2, there are d bounded subbilinear operator $\mathfrak{R}_j : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, $j = 1,...,d$, such that

$$[b,R_j](f) = \mathfrak{R}_j(f,b) + R_j(\mathfrak{S}(f,b)).$$

Therefore, $\mathfrak{S}(f,b) \in H^1_L(\mathbb{R}^d)$ iff $[b,R_j](f) \in L^1(\mathbb{R}^d)$ for all $j = 1,...,d$, moreover,

$$\|f\|_{H^1_L}\|b\|_{BMO} + \|\mathfrak{S}(f,b)\|_{H^1_L} \approx \|f\|_{H^1_L}\|b\|_{BMO} + \sum_{j=1}^{d}\|[b,R_j](f)\|_{L^1}.$$

□

Proof of Theorem 5.3. (i). By Theorem 5.1, there is a bounded subbilinear operator $\mathfrak{R}_T : H^1_L(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ such that

$$|T(\mathfrak{S}(f,b))| - \mathfrak{R}_T(f,b) \leq |[b,T](f)| \leq |T(\mathfrak{S}(f,b))| + \mathfrak{R}_T(f,b).$$

Hence, Lemma 8.3 allows that for every $f \in \mathcal{H}^1_{L,b}(\mathbb{R}^d)$,

$$\begin{aligned} \|[b,T](f)\|_{L^1} &\leq \|T\|_{H^1_L \to L^1} \|\mathfrak{S}(f,b)\|_{H^1_L} + \|\mathfrak{R}_T(f,b)\|_{L^1} \\ &\leq C\|f\|_{\mathcal{H}^1_{L,b}} + C\|f\|_{H^1_L}\|b\|_{BMO} \leq C\|f\|_{\mathcal{H}^1_{L,b}}, \end{aligned}$$

which ends the proof of (i).

(ii). It follows directly from Lemma 8.3, and thus the proof of Theorem 5.3 is finished. □

Proof of Theorem 5.4. The proof of Theorem 5.4 can be seen a consequence of Theorem 5.6 and Lemma 8.3. We leave the details to the interested reader.

□

9. Proof of Theorem 5.5, Theorem 5.6 and Theorem 5.8

We start by recalling the notation of molecules which is a slightly modified version of the classical one.

Definition 9.1. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. A function a is called a (H^1, q, ε)-molecule related to the ball B if*

i) $\|a\|_{L^q(2^4 B)} \leq |B|^{1/q-1}$,

ii) $\|a\|_{L^q(2^{k+1}B \setminus 2^k B)} \leq 2^{-k\varepsilon}|2^k B|^{1/q-1}$, $k = 4, 5, ...,$

iii) $\int_{\mathbb{R}^d} a(x)dx = 0$.

Lemma 9.1. *Let $1 < q \leq \infty$ and $\varepsilon > 0$. Then, there is a constant $C = C(q,\varepsilon) > 1$ such that for every (H^1, q, ε)-molecule f related to the ball B,*

$$\|f\|_{H^1} \leq C.$$

Furthermore, there exists a sequence of classical (H^1, q)-atoms $a_1, a_2, ..., a_k, ...$, related to $2^4 B, 2^5 B, ..., 2^{3+k} B, ...$, respectively, satisfying

$$f = C \sum_{k=1}^{\infty} 2^{-k\varepsilon} a_k.$$

The proof of Lemma 9.1 is similar to the one of Theorem 4.7 of [40]. We omit the details.

9.1. **Proof of Theorem 5.5.** In order to prove Theorem 5.5, we need the following lemma.

Lemma 9.2. *Let $1 < q < \infty$, $\varepsilon > 0$ and T be a L-Calderón-Zygmund operator. Then,*

$$\|(f - f_B)(g - g_B)Ta\|_{L^1} \leq C\|f\|_{BMO}\|g\|_{BMO}$$

for all $f, g \in BMO(\mathbb{R}^d)$, generalized (H^1_L, q, ε)-atom a related to the ball B.

Proof. By Lemma 6.1, Lemma 6.2 and Hölder inequality, we get

$$\begin{aligned}
&\|(f-f_B)(g-g_B)Ta\|_{L^1}\\
=\;&\|(f-f_B)(g-g_B)Ta\|_{L^1(2B)}+\sum_{k\geq 1}\|(f-f_B)(g-g_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\leq\;&\|f-f_B\|_{L^{2q'}(2B)}\|g-g_B\|_{L^{2q'}(2B)}\|T(a)\|_{L^q}+\\
&+\sum_{k\geq 1}\|f-f_B\|_{L^{2q'}(2^{k+1}B)}\|g-g_B\|_{L^{2q'}(2^{k+1}B)}\|T(a)\|_{L^q(2^{k+1}B\setminus 2^kB)}\\
\leq\;&C\|f\|_{BMO}\|g\|_{BMO}+\sum_{k\geq 1}C(k+1)^2\|f\|_{BMO}\|g\|_{BMO}|2^{k+1}B|^{1/q'}2^{-k\delta_0}|2^kB|^{1/q-1}\\
\leq\;&C\|f\|_{BMO}\|g\|_{BMO},
\end{aligned}$$

where $1/q+1/q'=1$ and $\delta_0>0$ in Lemma 6.2.

□

Proof of Theorem 5.5. (i). Suppose that T is a (δ,L)-Schrödinger-Calderón-Zygmund operator. Let us first prove that T is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$. Indeed, for every generalized $(H^1_L,2,\delta)$-atom a related to the ball B. As $T^*(1)=0$, Lemma 6.2 implies that Ta is C times a $(H^1,2,\delta)$-molecule related to the ball B. Therefore, Lemma 9.1 and Proposition 2.1 yield that T is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$. By this, the proof of (i) will be reduced to showing that

$$\|(b-b_B)a\|_{H^1_L}\leq C\|b\|_{BMO^{\log}_L} \tag{9.1}$$

and

$$\|(b-b_B)Ta\|_{H^1_L}\leq C\|b\|_{BMO^{\log}_L} \tag{9.2}$$

for all generalized $(H^1_L,2,\delta)$-atom a related to the ball $B=B(x_0,r)$. Note that the constants C in (9.1) and (9.2) are independent of b,a. Indeed, if (9.1) and (9.2) are true, then

$$\begin{aligned}
\|[b,T](a)\|_{H^1_L}&\leq\|(b-b_B)Ta\|_{H^1_L}+\|T((b-b_B)a)\|_{H^1_L}\\
&\leq C\|b\|_{BMO^{\log}_L}+C\|T\|_{H^1_L\to H^1}\|(b-b_B)a\|_{H^1_L}\\
&\leq C\|b\|_{BMO^{\log}_L}.
\end{aligned}$$

Therefore, Proposition 2.1 yields that $[b,T]$ is bounded on $H^1_L(\mathbb{R}^d)$, moreover, $\|[b,T]\|_{H^1_L\to H^1_L}\leq C$ with the constant C is independent of b.

Verifying (9.1) is similar to (9.2) but its proof uses an easier argument, we leave the details to the interested reader. Let us now establish (9.2). As $H^1_L(\mathbb{R}^d)$ is the dual of $VMO_L(\mathbb{R}^d)$ (see Theorem 3.2), it is sufficient to prove that

$$\|\phi(b-b_B)Ta\|_{L^1}\leq C\|b\|_{BMO^{\log}_L}\|\phi\|_{BMO_L} \tag{9.3}$$

for all $\phi\in C^\infty_c(\mathbb{R}^d)$. Besides, from Lemma 9.2,

$$\|(\phi-\phi_B)(b-b_B)Ta\|_{L^1}\leq C\|b\|_{BMO}\|\phi\|_{BMO}\leq C\|b\|_{BMO^{\log}_L}\|\phi\|_{BMO_L}.$$

This together with Lemma 2 of [16] allow us to reduce (9.3) to showing that

$$\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)Ta\|_{L^1}\le C\|g\|_{BMO_L^{\log}}. \tag{9.4}$$

Setting $\varepsilon=\delta/2$, it is easy to check that there exists a constant $C=C(\varepsilon)>0$ such that

$$\log(e+kt)\le Ck^{\varepsilon}\log(e+t)$$

for all $k\ge 2, t>0$. Consequently, for all $k\ge 1$,

$$\log\Big(e+\frac{\rho(x_0)}{r}\Big)\le C2^{k\varepsilon}\log\left(e+\Big(\frac{\rho(x_0)}{2^{k+1}r}\Big)^{k_0+1}\right). \tag{9.5}$$

Then, by Lemma 6.2 and Lemma 9.7 (see below), we get

$$\begin{aligned}
&\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)Ta\|_{L^1}\\
=\;&\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)Ta\|_{L^1(2B)}+\\
&+\sum_{k\ge1}\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\le\;&C\log\left(e+\Big(\frac{\rho(x_0)}{2r}\Big)^{k_0+1}\right)\|b-b_B\|_{L^2(2B)}\|Ta\|_{L^2}+\\
&+C\sum_{k\ge1}2^{k\varepsilon}\log\left(e+\Big(\frac{\rho(x_0)}{2^{k+1}r}\Big)^{k_0+1}\right)\|b-b_B\|_{L^2(2^{k+1}B)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)}\\
\le\;&C|2B|^{1/2}\|b\|_{BMO_L^{\log}}\|a\|_{L^2}+C\sum_{k\ge1}2^{k\varepsilon}(k+1)|2^{k+1}B|^{1/2}\|b\|_{BMO_L^{\log}}2^{-k\delta}|2^kB|^{-1/2}\\
\le\;&C\|b\|_{BMO_L^{\log}},
\end{aligned}$$

where we used $\delta=2\varepsilon$. This allows us to end the proof of (i).

(ii). As $V\in RH_d$, it is well-known (see [35, 39]) that the Riesz transforms R_j are Schrödinger-Calderón-Zygmund operators satisfying $R_j^*(1)=0$, and thus (ii) can be seen as an immediate consequence of Theorem 5.6.

□

9.2. **Proof of Theorem 5.6.** Here and in what follows, $N_0=\log_2 C_0+1$ with C_0 the constant in (2.1) and the constant $\delta\in(0,1)$ is as in Lemma 2.6 of [33] (see below). Let us now recall the following two lemmas.

Lemma 9.3 (see [23], Lemma 1)**.** *Let $V\in RH_{d/2}$. Suppose that $N\ge N_0$. Then, there exists $C(N)>0$ such that for all ball $B(x,R)$,*

$$\frac{1}{\Big(1+\frac{R}{\rho(x)}\Big)^N}\int\limits_{B(x,R)}V(y)dy\le C(N)R^{d-2}.$$

Lemma 9.4 (see [33], Lemma 2.6)**.** *Let $V \in RH_{d/2}$. Then, there exists $\delta \in (0,1)$ such that for any positive number N and $0 < h < |x-y|/16$, we have*

$$|K_j(x,y)| \le \frac{C(N)}{\Big(1+\frac{|x-y|}{\rho(y)}\Big)^N}\frac{1}{|x-y|^{d-1}}\Big(\int\limits_{B(x,|x-y|)}\frac{V(z)}{|x-z|^{d-1}}dz+\frac{1}{|x-y|}\Big)$$

and

$$|K_j(x,y+h)-K_j(x,y)| \le \frac{C(N)}{\Big(1+\frac{|x-y|}{\rho(y)}\Big)^N}\frac{h^\delta}{|x-y|^{\delta+d-1}}\Big(\int\limits_{B(x,|x-y|)}\frac{V(z)}{|x-z|^{d-1}}dz+\frac{1}{|x-y|}\Big),$$

where $K_j(x,y)$, $j=1,...,d$, are the kernels of the Riesz transforms R_j.

Next lemma will be useful for proving Theorem 5.6.

Lemma 9.5. *Let $1 < q \le d/2$. Then, $R_j(a)$ is C times a (H^1,q,δ)-molecule for all generalized (H^1_L,q,δ)-atom a related to the ball $B=B(x_0,r)$. Furthermore, for any $N>0$, there exists $C(N)>0$ such that for every $k \ge 4$,*

$$\|R_j(a)\|_{L^q(2^{k+1}B\setminus 2^kB)} \le \frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^N}2^{-k\delta}|2^kB|^{1/q-1}. \tag{9.6}$$

Proof. It is well-known that R_j is bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$, inparticular, one has $\int_{\mathbb{R}^d} R_j(a)(x)dx = 0$. Moreover, by the L^q-boundedness of R_j (see [39], Theorem 0.5) one has $\|R_j(a)\|_{L^q} \le C|B|^{1/q-1}$. Therefore, it is sufficient to verify (9.6). Thanks to Lemma 9.4, as a is a generalized (H^1_L,q,δ)-atom related to the ball B, for every $x \in 2^{k+1}B\setminus 2^kB$,

$$\begin{aligned}
|R_j(a)(x)| &\le \Big|\int\limits_B (K_j(x,y)-K_j(x,x_0))a(y)dy\Big| + |K_j(x,x_0)|\Big|\int\limits_B a(y)dy\Big| \\
&\le \int\limits_B \frac{C(N)}{\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{N+4N_0}}\frac{|y-x_0|^\delta}{|x-x_0|^{d+\delta-1}}\Big\{\int\limits_{B(x,|x-x_0|)}\frac{V(z)}{|x-z|^{d-1}}dz+\frac{1}{|x-x_0|}\Big\}|a(y)|dy \\
&\quad + \frac{C(N)}{\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{N+4N_0+\delta}}\frac{1}{|x-x_0|^{d-1}}\Big(\int\limits_{B(x,|x-x_0|)}\frac{V(z)}{|x-z|^{d-1}}dz+\frac{1}{|x-x_0|}\Big)\Big(\frac{r}{\rho(x_0)}\Big)^\delta \\
&\le \frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^N}\left(\frac{1}{\Big(1+\frac{2^{k+2}r}{\rho(x_0)}\Big)^{N_0}}\frac{r^\delta}{(2^kr)^{d+\delta-1}}\int\limits_{B(x,|x-x_0|)}\frac{V(z)}{|x-z|^{d-1}}dz+\frac{2^{-k\delta}}{|2^kB|}\right).
\end{aligned} \tag{9.7}$$

Here and in what follows, the constants $C(N)$ depend only on N, but may change from line to line. Note that for every $x \in 2^{k+1}B\setminus 2^kB$, one has $B(x,|x-x_0|) \subset$

$B(x,2^{k+1}r)\subset B(x_0,2^{k+2}r)$. The fact $V\in RH_{d/2}$, $d/2\geq q>1$, and Hölder inequality yield

$$\begin{aligned}
&\left\|\int\limits_{B(x,|x-x_0|)}\frac{V(z)}{|x-z|^{d-1}}dz\right\|_{L^q(2^{k+1}B\setminus 2^kB,dx)}\\
&\leq C(2^{k+1}r)^{1-\frac{2}{d}}\left\{\int\limits_{2^{k+1}B\setminus 2^kB}\Big(\int\limits_{B(x,2^{k+1}r)}\frac{|V(z)|^{d/2}}{|x-z|^{d-1}}dz\Big)^{\frac{2q}{d}}dx\right\}^{1/q}\\
&\leq C(2^kr)^{1-\frac{2}{d}}|2^{k+1}B|^{\frac{1}{q}-\frac{2}{d}}\left\{\int\limits_{B(z,2^{k+1}r)}dx\int\limits_{B(x_0,2^{k+2}r)}\frac{|V(z)|^{d/2}}{|x-z|^{d-1}}dz\right\}^{2/d}\\
&\leq C2^kr|2^kB|^{1/q-1}\int\limits_{B(x_0,2^{k+2}r)}V(z)dz. \qquad (9.8)
\end{aligned}$$

Combining (9.7), (9.8) and Lemma 9.3, we obtain that

$$\begin{aligned}
&\|R_j(a)\|_{L^q(2^{k+1}B\setminus 2^kB)}\\
\leq\ &\frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^N}\left(\frac{r^\delta 2^kr|2^kB|^{1/q-1}}{(2^kr)^{d+\delta-1}}\frac{1}{\Big(1+\frac{2^{k+2}r}{\rho(x_0)}\Big)^{N_0}}\int\limits_{B(x_0,2^{k+2}r)}V(z)dz+\frac{2^{-k\delta}}{|2^kB|}|2^{k+1}B|^{1/q}\right)\\
\leq\ &\frac{C(N)}{\Big(1+\frac{2^kr}{\rho(x_0)}\Big)^N}2^{-k\delta}|2^kB|^{1/q-1},
\end{aligned}$$

which completes the proof. □

In order to prove Theorem 5.6, we need the following two technical lemmas.

Lemma 9.6. *Let $1<q\leq d/2$ and $\theta\geq 0$. Then,*

$$\|(g-g_B)R_j(a)\|_{L^1}\leq C\|g\|_{BMO_{L,\theta}}$$

and

$$\|(f-f_B)(g-g_B)R_j(a)\|_{L^1}\leq C\|f\|_{BMO}\|g\|_{BMO_{L,\theta}}$$

for all $f\in BMO(\mathbb{R}^d)$, $g\in BMO_{L,\theta}(\mathbb{R}^d)$ and generalized (H^1_L,q,δ)-atom a related to the ball $B=B(x_0,r)$.

Lemma 9.7. *Let $1 \le q < \infty$ and $\theta \ge 0$. Then, for every $f \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$, $B = B(x,r)$ and $k \in \mathbb{Z}^+$, we have*

$$\Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_B|^q dy\Big)^{1/q} \le Ck\frac{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x)}{2^kr})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}},$$

where the constant k_0 is as in Lemma 2.1.

To prove Lemma 9.6, we need to recall the following lemma.

Lemma 9.8 (see [8], Lemma 1)**.** *Let $1 \le q < \infty$ and $\theta \ge 0$. Then, for every $f \in BMO_{L,\theta}(\mathbb{R}^d)$, $B = B(x,r)$ and $k \in \mathbb{Z}^+$, we have*

$$\Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_B|^q dy\Big)^{1/q} \le Ck\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}\|f\|_{BMO_{L,\theta}}.$$

Proof of Lemma 9.6. Noting that $r \le \mathcal{C}_L\rho(x_0)$ since a is a generalized (H^1_L,q,δ)-atom related to the ball $B = B(x_0,r)$, choosing $N = (k_0+1)\theta$ in (9.6) together with Hölder inequality and Lemma 9.8 allow to conclude that

$$\begin{aligned}
&\|(g-g_B)R_j(a)\|_{L^1}\\
=\;&\|(g-g_B)R_j(a)\|_{L^1(2^4B)} + \sum_{k=4}^{\infty}\|(g-g_B)R_j(a)\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\le\;&\|g-g_B\|_{L^{q'}(2^4B)}\|R_j\|_{L^q\to L^q}\|a\|_{L^q} + \sum_{k=4}^{\infty}\|g-g_B\|_{L^{q'}(2^{k+1}B\setminus 2^kB)}\|R_j(a)\|_{L^q(2^{k+1}B\setminus 2^kB)}\\
\le\;&C\|g\|_{BMO_{L,\theta}} +\\
&+C\sum_{k=4}^{\infty}(k+1)|2^{k+1}B|^{1/q'}\Big(1+\frac{2^{k+1}r}{\rho(x)}\Big)^{(k_0+1)\theta}\|g\|_{BMO_{L,\theta}}\frac{1}{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}2^{-k\delta}|2^kB|^{1/q-1}\\
\le\;&C\|g\|_{BMO_{L,\theta}},
\end{aligned}$$

where $1/q + 1/q' = 1$. Similarly, we also obtain that

$$\begin{aligned}
&\|(f-f_B)(g-g_B)R_j(a)\|_{L^1}\\
=\;&\|(f-f_B)(g-g_B)R_j(a)\|_{L^1(2^4B)} + \sum_{k=4}^{\infty}\|(f-f_B)(g-g_B)R_j(a)\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\le\;&\|f-f_B\|_{L^{2q'}(2^4B)}\|g-g_B\|_{L^{2q'}(2^4B)}\|R_j(a)\|_{L^q} +\\
&+\sum_{k=4}^{\infty}\|f-f_B\|_{L^{2q'}(2^{k+1}B)}\|g-g_B\|_{L^{2q'}(2^{k+1}B)}\|R_j(a)\|_{L^q(2^{k+1}B\setminus 2^kB)}\\
\le\;&C\|f\|_{BMO}\|g\|_{BMO_{L,\theta}},
\end{aligned}$$

which ends the proof.

□

Proof of Lemma 9.7. First, we claim that for every ball $B_0 = B(x_0, r_0)$,

$$\Big(\frac{1}{|B_0|}\int\limits_{B_0}|f(y)-f_{B_0}|^q dy\Big)^{1/q} \le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}. \tag{9.9}$$

Assume that (9.9) for a moment. Then,

$$\begin{aligned}
&\Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_B|^q dy\Big)^{1/q}\\
\le\ &\Big(\frac{1}{|2^kB|}\int\limits_{2^kB}|f(y)-f_{2^kB}|^q dy\Big)^{1/q}+\sum_{j=0}^{k-1}|f_{2^{j+1}B}-f_{2^jB}|\\
\le\ &\frac{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x)}{2^kr})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}+\sum_{j=0}^{k-1}2^d\frac{\Big(1+\frac{2^{j+1}r}{\rho(x)}\Big)^{\theta}}{\log\Big(e+\frac{\rho(x)}{2^{j+1}r}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}\\
\le\ &Ck\frac{\Big(1+\frac{2^kr}{\rho(x)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x)}{2^kr})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}.
\end{aligned}$$

Now, it remains to prove (9.9).

Let us define the function h on $\mathbb{R}^d$ as follows

$$h(x)=\begin{cases}1, & x\in B_0,\\ \frac{2r_0-|x-x_0|}{r_0}, & x\in 2B_0\setminus B_0,\\ 0, & x\notin 2B_0,\end{cases}$$

and remark that

$$|h(x)-h(y)|\le\frac{|x-y|}{r_0}. \tag{9.10}$$

Setting $\widetilde{f}:=f-f_{2B_0}$. By the classical John-Nirenberg inequality, there exists a constant $C=C(d,q)>0$ such that

$$\begin{aligned}
\Big(\frac{1}{|B_0|}\int\limits_{B_0}|f(y)-f_{B_0}|^q dy\Big)^{1/q} &= \Big(\frac{1}{|B_0|}\int\limits_{B_0}|h(y)\widetilde{f}(y)-(h\widetilde{f})_{B_0}|^q dy\Big)^{1/q}\\
&\le C\|h\widetilde{f}\|_{BMO}.
\end{aligned}$$

Therefore, the proof of the lemma will be reduced to showing that

$$\|h\widetilde{f}\|_{BMO} \le C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}},$$

namely, for every ball $B = B(x,r)$,

$$\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy \le C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}. \tag{9.11}$$

Now, let us focus on Inequality (9.11). Noting that supp $h \subset 2B_0$, Inequality (9.11) is obvious if $B \cap 2B_0 = \emptyset$. Hence, we only consider the case $B \cap 2B_0 \neq \emptyset$. Then, we have the following two cases:

The case $r > r_0$: the fact $B \cap 2B_0 \neq \emptyset$ implies that $2B_0 \subset 5B$, and thus

$$\begin{aligned}
\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy &\le 2\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y)|dy \\
&\le 2.5^d\frac{1}{|2B_0|}\int_{2B_0}|f(y)-f_{2B_0}|dy \\
&\le C\frac{\left(1+\frac{2r_0}{\rho(x_0)}\right)^{\theta}}{\log\left(e+\frac{\rho(x_0)}{2r_0}\right)}\|f\|_{BMO^{\log}_{L,\theta}} \\
&\le C\frac{\left(1+\frac{r_0}{\rho(x_0)}\right)^{(k_0+1)\theta}}{\log\left(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\right)}\|f\|_{BMO^{\log}_{L,\theta}}.
\end{aligned}$$

The case $r \le r_0$: Inequality (9.10) yields

$$\begin{aligned}
\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy &\le 2\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y) - h_B\widetilde{f}_B|dy \\
&\le 2\frac{1}{|B|}\int_B |h(y)(\widetilde{f}(y)-\widetilde{f}_B)|dy + \\
&\quad + 2|\widetilde{f}_B|\frac{1}{|B|}\int_B \frac{1}{|B|}\Big|\int_B (h(x)-h(y))dy\Big|dx \\
&\le 2\frac{1}{|B|}\int_B |f(y)-f_B|dy + 4\frac{r}{r_0}|f_B - f_{2B_0}|. \qquad (9.12)
\end{aligned}$$

By $r \le r_0$, $B = B(x,r) \cap B(x_0,r_0) \neq \emptyset$, Lemma 2.1 gives

$$\frac{r}{\rho(x)} \le \frac{r_0}{\rho(x)} \le C\frac{r_0}{\rho(x_0)}\Big(1+\frac{|x-x_0|}{\rho(x_0)}\Big)^{k_0} \le C\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{k_0+1}.$$

Consequently,

$$\begin{aligned}\frac{1}{|B|}\int_B |f(y)-f_B|dy &\le \frac{\Big(1+\frac{r}{\rho(x)}\Big)^{\theta}}{\log(e+\frac{\rho(x)}{r})}\|f\|_{BMO^{\log}_{L,\theta}}\\ &\le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}, \qquad (9.13)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{|B(x,2^3r_0)|}\int_{B(x,2^3r_0)} |f(y)-f_{B(x,2^3r_0)}|dy &\le \frac{\Big(1+\frac{2^3r_0}{\rho(x)}\Big)^{\theta}}{\log(e+\frac{\rho(x)}{2^3r_0})}\|f\|_{BMO^{\log}_{L,\theta}}\\ &\le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}. \qquad (9.14)\end{aligned}$$

Noting that for every $k \in \mathbb{N}$ with $2^{k+1}r \le 2^3r_0$,

$$\begin{aligned}|f_{2^{k+1}B}-f_{2^kB}| &\le 2^d\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y)-f_{2^{k+1}B}|dy\\ &\le C\frac{\Big(1+\frac{2^3r_0}{\rho(x)}\Big)^{\theta}}{\log(e+\frac{\rho(x)}{2^3r_0})}\|f\|_{BMO^{\log}_{L,\theta}}\\ &\le C\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}},\end{aligned}$$

allows us to conclude that

$$|f_{B(x,r)}-f_{B(x,2^3x_0)}| \le C\log\Big(e+\frac{r_0}{r}\Big)\frac{\Big(1+\frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e+(\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}}. \qquad (9.15)$$

Then, the inclusion $2B_0 \subset B(x, 2^3 r_0)$ together with the inequalities (9.12), (9.13), (9.14) and (9.15) yield

$$\begin{aligned}\frac{1}{|B|}\int_B |h(y)\widetilde{f}(y) - (h\widetilde{f})_B|dy &\leq 2\frac{1}{|B|}\int_B |f(y) - f_B|dy + \\ &\quad +4\frac{r}{r_0}\Big(|f_{B(x,r)} - f_{B(x,2^3x_0)}| + 4^d MO(f, B(x, 2^3x_0))\Big) \\ &\leq C\Big(1 + \frac{r}{r_0}\log(e + \frac{r_0}{r})\Big)\frac{\Big(1 + \frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e + (\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}} \\ &\leq C\frac{\Big(1 + \frac{r_0}{\rho(x_0)}\Big)^{(k_0+1)\theta}}{\log\Big(e + (\frac{\rho(x_0)}{r_0})^{k_0+1}\Big)}\|f\|_{BMO^{\log}_{L,\theta}},\end{aligned}$$

we have used $\frac{r}{r_0}\log(e + \frac{r_0}{r}) \leq \sup_{t\leq 1} t\log(e + 1/t) < \infty$. This ends the proof.

□

Proof of Theorem 5.6. (i). Suppose that $b \in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$ for some $\theta \geq 0$. By the Riesz transforms R_j are bounded from $H^1_L(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$, similarly to the proof of Theorem 5.5, it is sufficient to show that

$$\|(b - b_B)a\|_{H^1_L} \leq C\|b\|_{BMO^{\log}_{L,\theta}} \tag{9.16}$$

and

$$\|(b - b_B)R_j(a)\|_{H^1_L} \leq C\|b\|_{BMO^{\log}_{L,\theta}} \tag{9.17}$$

for all generalized $(H^1_L, d/2, \delta)$-atom a related to the ball $B = B(x_0, r)$. Note that the constants C in (9.16) and (9.17) are independent of b, a.

Verifying (9.16) is similar to (9.17) but its proof uses an easier argument, we leave the details to the interested reader. Let us now establish (9.17). Using the ideas from the proof of Theorem 5.5 together with Lemma 9.6 allow us to reduce (9.17) to showing that

$$\log\Big(e + \frac{\rho(x_0)}{r}\Big)\|(b - b_B)R_j(a)\|_{L^1} \leq C\|b\|_{BMO^{\log}_{L,\theta}}. \tag{9.18}$$

Setting $\varepsilon = \delta/2$, there is a constant $C = C(\varepsilon) > 0$ such that for all $k \geq 1$,

$$\log\Big(e + \frac{\rho(x_0)}{r}\Big) \leq C2^{k\varepsilon}\log\left(e + \Big(\frac{\rho(x_0)}{2^{k+1}r}\Big)^{k_0+1}\right). \tag{9.19}$$

Note that $r \leq \mathcal{C}_L\rho(x_0)$ since a is a $(H^1_L, d/2)$-atom related to the ball $B(x_0, r)$. Choosing $N = (k_0 + 1)\theta$ in (9.6) together with Hölder inequality, (9.19) and Lemma

9.7 allow us to conclude that

$$\begin{aligned}
&\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)R_j(a)\|_{L^1}\\
=\;&\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)R_j(a)\|_{L^1(2^4B)}+\\
&+\sum_{k\geq 4}\log\Big(e+\frac{\rho(x_0)}{r}\Big)\|(b-b_B)R_j(a)\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\leq\;&C\log\left(e+\Big(\frac{\rho(x_0)}{2^4r}\Big)^{k_0+1}\right)\|b-b_B\|_{L^{\frac{d}{d-2}}(2^4B)}\|R_j(a)\|_{L^{d/2}}+\\
&+C\sum_{k\geq 4}2^{k\varepsilon}\log\left(e+\Big(\frac{\rho(x_0)}{2^{k+1}r}\Big)^{k_0+1}\right)\|b-b_B\|_{L^{\frac{d}{d-2}}(2^{k+1}B)}\|R_j(a)\|_{L^{d/2}(2^{k+1}B\setminus 2^kB)}\\
\leq\;&C\|b\|_{BMO^{\log}_{L,\theta}}+C\|b\|_{BMO^{\log}_{L,\theta}}\sum_{k\geq 4}k2^{-k\varepsilon}\\
\leq\;&C\|b\|_{BMO^{\log}_{L,\theta}}
\end{aligned}$$

where we used $\delta=2\varepsilon$. This ends the proof of (i).

(ii). Although (ii) can be followed from the duality and Theorem 2 of [8], we would also like to give a direct proof for the completeness.

Suppose that $b\in BMO_{L,\theta}(\mathbb{R}^d)$ for some $\theta\geq 0$. For every $(H^1_L,d/2)$-atom a related to some ball $B=B(x_0,r)$ in $\mathbb{R}^d$, Remark 2.1 and Lemma 9.6 give

$$\begin{aligned}
\|R_j((b-b_B)a)\|_{L^1}&\leq\|(b-b_B)R_j(a)\|_{L^1}+C\|[b,R_j](a)\|_{H^1_L}\\
&\leq C\|b\|_{BMO_{L,\theta}}+C\|[b,R_j]\|_{H^1_L\to H^1_L}
\end{aligned}\tag{9.20}$$

for all $j=1,...,d$. On the other hand, noting that $r\leq\mathcal{C}_L\rho(x_0)$ since a is a $(H^1_L,d/2)$-atom related to some ball $B=B(x_0,r)$, Hölder inequality and Lemma 9.8 give

$$\|(b-b_B)a\|_{L^1}\leq\|b-b_B\|_{L^{\frac{d}{d-2}}(B)}\|a\|_{L^{d/2}(B)}\leq C\|b\|_{BMO_{L,\theta}}.$$

This together with (9.20) prove that $(b-b_B)a\in H^1_L(\mathbb{R}^d)$, moreover,

$$\|(b-b_B)a\|_{H^1_L}\leq C\left(\|b\|_{BMO_{L,\theta}}+\sum_{j=1}^d\|[b,R_j]\|_{H^1_L\to H^1_L}\right)\tag{9.21}$$

where the constant $C>0$ independent of b,a.

Now, we prove that $b\in BMO^{\log}_{L,\theta}(\mathbb{R}^d)$. More precisely, the following

$$\frac{\log\Big(e+\frac{\rho(x_0)}{r}\Big)}{\Big(1+\frac{r}{\rho(x_0)}\Big)^{\theta}}MO(b,B(x_0,r))\leq C\left(\|b\|_{BMO_{L,\theta}}+\sum_{j=1}^d\|[b,R_j]\|_{H^1_L\to H^1_L}\right)\tag{9.22}$$

holds for any ball $B(x_0,r)$ in $\mathbb{R}^d$. In fact, we only need to establish (9.22) for $0<r<\rho(x_0)/2$ since $b\in BMO_{L,\theta}(\mathbb{R}^d)$.

Indeed, in (9.21) we choose $B = B(x_0, r)$ and $a = (2|B|)^{-1}(f - f_B)\chi_B$, where $f = \text{sign}\,(b - b_B)$. Then, it is easy to see that a is a $(H^1_L, d/2)$-atom related to the ball B. We next consider

$$g_{x_0,r}(x) = \chi_{[0,r]}(|x - x_0|)\log\Big(\frac{\rho(x_0)}{r}\Big) + \chi_{(r,\rho(x_0)]}(|x - x_0|)\log\Big(\frac{\rho(x_0)}{|x - x_0|}\Big).$$

Then, thanks to Lemma 2.5 in [35], one has $\|g_{x_0,r}\|_{BMO_L} \le C$. Moreover, it is clear that $g_{x_0,r}(b - b_B)a \in L^1(\mathbb{R}^d)$. Consequently, (9.21) together with the fact that $BMO_L(\mathbb{R}^d)$ is the dual of $H^1_L(\mathbb{R}^d)$ allows us to conclude that

$$\begin{aligned}
\frac{\log\Big(e + \frac{\rho(x_0)}{r}\Big)}{\Big(1 + \frac{r}{\rho(x_0)}\Big)^{\theta}} MO(b, B(x_0, r)) &\le 3\log\Big(\frac{\rho(x_0)}{r}\Big) MO(b, B(x_0, r)) \\
&= 6\Big|\int_{\mathbb{R}^d} g_{x_0,r}(x)(b(x) - b_B)a(x)dx\Big| \\
&\le 6\|g_{x_0,r}\|_{BMO_L}\|(b - b_B)a\|_{H^1_L} \\
&\le C\left(\|b\|_{BMO_{L,\theta}} + \sum_{j=1}^{d}\|[b, R_j]\|_{H^1_L \to H^1_L}\right),
\end{aligned}$$

where we used $r < \rho(x_0)/2$ and

$$\int_{\mathbb{R}^d}(b(x) - b_B)a(x)dx = \frac{1}{2|B(x_0, r)|}\int_{B(x_0,r)}|b(x) - b_{B(x_0,r)}|dx.$$

This ends the proof.

□

9.3. **Proof of Theorem 5.8.** To prove Theorem 5.8, we need the following technical lemma.

Lemma 9.9. *There exists a constant $C > 0$ such that*

$$\|(g - g_B)\mathfrak{M}f\|_{L^1} \le C\|g\|_{BMO} \tag{9.23}$$

and

$$\|(g - g_B)\mathfrak{M}(R_j(a))\|_{L^1} \le C\|g\|_{BMO} \tag{9.24}$$

for all $g \in BMO(\mathbb{R}^d)$, classical $(H^1, d/2)$-atom f related to the ball B and generalized $(H^1_L, d/2, \delta)$-atom a related to the ball B.

Proof. The proof of (9.23) is classical. It can be found in [19] (see also [42], Chapter IV).

Let us now look at (9.24). By Lemma 9.1 and Lemma 9.5, there exists a sequence of classical $(H^1, d/2)$-atoms $a_1, a_2, ..., a_k, ...$, related respectively to the balls $2^4B, 2^5B, ..., 2^{3+k}B, ...$, such that

$$R_j(a) = C\sum_{k=1}^{\infty} 2^{-k\delta} a_k.$$

Therefore, (9.23) implies that

$$\begin{aligned}
\|(g-g_B)\mathfrak{M}(R_j(a))\|_{L^1} &\leq C\sum_{k=1}^{\infty} 2^{-k\delta}(\|(g-g_{2^{k+3}B})\mathfrak{M}(a_k)\|_{L^1} + |g_{2^{k+3}B}-g_B|\|\mathfrak{M}(a_k)\|_{L^1}) \\
&\leq C\|g\|_{BMO}\left(\sum_{k=1}^{\infty} 2^{-k\delta} + \sum_{k=1}^{\infty} 2^{-k\delta}(k+3)\right) \\
&\leq C\|g\|_{BMO}.
\end{aligned}$$

□

Proof of Theorem 5.8. Similarly to the proof of Theorem 5.6, it is sufficient to show that

$$\|(b-b_B)a\|_{H^1_L} \leq C\|b\|_{BMO^{\log}} \tag{9.25}$$

and

$$\|(b-b_B)R_j(a)\|_{H^1_L} \leq C\|b\|_{BMO^{\log}} \tag{9.26}$$

for all generalized $(H^1_L, d/2, \delta)$-atom a related to the ball B.

Verifying (9.25) is similar to (9.26) but its proof uses an easier argument, we leave the details to the interested reader. Let us now establish (9.26), namely, the following

$$\left|\int_{\mathbb{R}^d} (g(x)-g_B)R_j(a)(x)v(x)dx\right| \leq C\|g\|_{BMO^{\log}}\|v\|_{BMO_L} \tag{9.27}$$

holds for all $v \in C_c^\infty(\mathbb{R}^d)$ since $H^1_L(\mathbb{R}^d)$ is the dual of $VMO_L(\mathbb{R}^d)$ the completion of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$. By Lemma 9.6, $(g-g_B)R_j(a)v \in L^1(\mathbb{R}^d)$, moreover,

$$\begin{aligned}
\|(v-v_B)(g-g_B)R_j(a)\|_{L^1} &\leq C\|g\|_{BMO}\|v\|_{BMO} \\
&\leq C\|g\|_{BMO^{\log}}\|v\|_{BMO_L}.
\end{aligned} \tag{9.28}$$

Now, let us look at $f := v_B R_j(a)$. From Lemma 9.9 and (4.6),

$$\begin{aligned}
\|\mathfrak{M}f\|_{L^{\log}} &= \|v\mathfrak{M}(R_j(a)) - (v-v_B)\mathfrak{M}(R_j(a))\|_{L^{\log}} \\
&\leq C\|v\mathfrak{M}(R_j(a))\|_{L^{\log}} + C\|(v-v_B)\mathfrak{M}(R_j(a))\|_{L^1} \\
&\leq C\|\mathfrak{M}(R_j(a))\|_{L^1}\|v\|_{BMO_L} + C\|v\|_{BMO} \\
&\leq C\|v\|_{BMO_L}.
\end{aligned}$$

This together with the fact that $BMO^{\log}(\mathbb{R}^d)$ is the dual of $H^{\log}(\mathbb{R}^d)$ (see Theorem 3.3 of [30]) allows to conclude that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}(g(x)-g_B)v_BR_j(a)(x)dx\Big| &= \Big|\int_{\mathbb{R}^d}(g(x)-g_B)f(x)dx\Big| \\ &\leq C\|g-g_B\|_{BMO^{\log}}\|\mathfrak{M}f\|_{L^{\log}} \\ &\leq C\|g\|_{BMO^{\log}}\|v\|_{BMO_L}. \end{aligned} \tag{9.29}$$

Finally, (9.27) follows from (9.28) and (9.29). This ends the proof.

□

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MAPMO-UMR 6628, Département de Mathématiques, Université d'Orleans, 45067 Orléans Cedex 2, France
E-mail address: dangky@math.cnrs.fr