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# Canonical path integral measures for Holst and Plebanski gravity.

## I. Reduced Phase Space Derivation

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### Abstract

An important aspect in defining a path integral quantum theory is the determination of the correct measure. For interacting theories and theories with constraints, this is non-trivial, and is normally not the heuristic “Lebesgue measure” usually used. There have been many determinations of a measure for gravity in the literature, but none for the Palatini or Holst formulations of gravity. Furthermore, the relations between different resulting measures for different formulations of gravity are usually not discussed.

In this paper we use the reduced phase technique in order to derive the path-integral measure for the Palatini and Holst formulation of gravity, which is different from the Lebesgue measure up to local measure factors which depend on the spacetime volume element and spatial volume element.

From this path integral for the Holst formulation of GR we can also give a new derivation of the Plebanski path integral and discover a discrepancy with the result due to Buffenoir, Henneaux, Noui and Roche (BHNR) whose origin we resolve. This paper is the first in a series that aims at better understanding the relation between canonical LQG and the spin foam approach.

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# 1 Introduction

Richard Feynman, in the course of his doctoral work, developed the path integral formulation of quantum mechanics as an alternative, space-time covariant description of quantum mechanics, which is nevertheless equivalent to the canonical approach [1]. It is thus not surprising that the path integral formulation has been of interest in the quantization of general relativity, a theory where space-time covariance plays a key role. However, once one departs from the regime of free, unconstrained systems, the equivalence of the path integral approach and canonical approach becomes more subtle than originally described by Feynman in [1]. In particular, in Feynman's original argument, the integration measure for the configuration path integral is a formal Lebesgue measure; in the interacting case, however, in order to have equivalence with the canonical theory, one cannot use the naive Lebesgue measure in the path integral, but must use a measure derived from the Liouville measure on the phase space [2].

Such a measure has yet to be incorporated into spin-foam models, which can be thought of as a path-integral version of loop quantum gravity (LQG) [3, 4]. Loop quantum gravity is an attempt to make a mathematically rigorous quantization of general relativity that preserves background independence — for reviews, see [8, 6, 7] and for books see [9, 10]. Spin-foams intend to be a path integral formulation for loop quantum gravity, directly motivated from the ideas of Feynman appropriately adapted to reparametrization-invariant theories [4, 5]. Only the kinematical structure of LQG is used in motivating the spin-foam framework. The dynamics one tries to encode in the amplitude factors appearing in the path integral which is being replaced by a sum in a regularisation step which depends on a triangulation of the spacetime manifold. Eventually one has to take a weighted average over these (generalised) triangulations for which the proposal at present is to use methods from group field theory [3]. The current spin foam approach is independent from the dynamical theory of canonical LQG [11] because the dynamics of canonical LQG is rather complicated, it uses an apparently much simpler starting point: Namely, in the Plebanski formulation [14], GR can be considered as a constrained BF theory and treating the so called simplicity constraints as a perturbation of BF theory, one can make use of the powerful toolbox that come with topological QFT's [12]. It is an unanswered question, however, and one of the most active research topics momentarily<sup>1</sup>, how canonical LQG and spin foams fit together. It is one the aims of this paper to make a contribution towards answering this question.

In LQG one is compelled to introduce a 1-parameter quantization ambiguity — the so-called Immirzi parameter [15, 16]. This enters the action through a necessary extra 'topological' term added to the Palatini action; the full action is termed the *Holst* action [17]. To properly incorporate the Immirzi parameter into spin-foams, one should in fact not start from the usual Plebanski formulation but rather an analogous generalization, in which an analogous topological term is added to the action, leading to what we call the Plebanski-Holst formulation of gravity [19, 20, 21].

In [22] we have shown (and partly reviewed) for a rather general theory that different canonical quantisation techniques for gauge theories, specifically Dirac's operator constraint method, the Master Constraint method and the reduced phase space method all lead to the same path integral. A prominent role in establishing this equivalence is played by what is called "the choice of gauge fixing" (from the reduced phase space point of view) or, equivalently, the choice of clocks (from the gauge invariant i.e. relational point of view [24]). After a long analysis, it transpires that the common basis for the path integral measure no matter from which starting point it is derived, is the Liouville measure on the reduced phase, which can be defined via gauge fixing of the first class constraints. This measure can be extended to the full phase space and one shows that the dependence on the gauge fixing disappears when one integrates gauge invariant functions<sup>2</sup>. From this point of view, that is, the equivalence between path-integral formulation and the canonical theory, it is obvious that formal path-integrals derived from the various formulations of gravity should all be equivalent, because all of them have the same reduced phase space — that of general relativity.

We thus apply the general reduced phase space framework to the Holst action as the starting point for deriving a formal path integral for *both* the Holst action and the Plebanski-Holst action. It turns out that the resulting path-integral for either the Holst action or the Plebanski-Holst action is not naively the Lebesgue measure integral times the exponentiated action. There are extra measure factors of spacetime volume element  $\mathcal{V}$  and spatial volume element  $V_s$ . The presence of a spatial volume element is especially surprising because it breaks the manifest spacetime covariance of the path-integral when we are off shell. The origin of this lack of covariance is in the mixture of dynamics and gauge invariance inherent to generally covariant systems

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<sup>1</sup>Here we are referring the spin-foam model for 4-dimensional gravity, while for 3-dimensional gravity the consistency is discussed in e.g. [13].

<sup>2</sup>However, the dependence on the gauge fixing is secretly there, *in a gauge invariant form*, since choices of algebras of Dirac observables (i.e. gauge invariant functions) are in one to one correspondence with choices of gauge fixing. The choice of such an algebra is the zeroth step in a canonical quantisation scheme and determines everything else such as the representation theory, see [22] for a comprehensive discussion.

with propagating degrees of freedom and it is well known that the gauge symmetries generated by the constraints only coincide on shell with spacetime diffeomorphism invariance. The quantum theory chooses to preserve the gauge symmetries generated by the constraints rather than spacetime diffeomorphism invariance when we take quantum corrections into account (go off shell).

This kind of extra measure factor (so called local measure) has appeared and been discussed in the literature since 1960s (see for instance [25, 26]) in the formalism of geometrodynamics and its *background-dependent* quantizations (stationary phase approximation). The outcome from the earlier investigations appears to be that in background-dependent, perturbative quantizations, these measure factors of  $\mathcal{V}$  and  $V_s$  only contribute to the divergent part of the higher loop-order amplitudes. Thus their meanings essentially depend on the regularization scheme used. One can of course try to choose certain regularization schemes such that, either the local measure factors never contribute to the transition amplitude, or that their effect is canceled by the divergence from the action [25, 26]. However, the power of renormalisation and the very reason we trust it is that its predictions are independent of the regularisation technique chosen. Therefore the status of these measure factors is very much unsettled, especially for non perturbative quantisation techniques. We here take the point of view that the measure factors should be taken seriously because they take the off shell symmetry generated by the constraints properly into account. In which sense this so called Bergmann – Komar “group” [33] is preserved in the path integral is the subject of the research conducted in [28]. In this article we confine ourselves to a brief discussion.

In the formalism of connection-dynamics, which is a preparation of background-independent quantization, a similar local measure factor also appears. It was first pointed out in [27], whose path-integral will be shown to be equivalent to our present formulation up to a discrepancy whose origin we resolve. When we perform *background-independent* quantization as in spin-foam models, therefore the local measure factor should not be simply ignored, because the regularization arguments in background-dependent quantization have no obvious bearing in the background-independent context anymore. For example, spin-foam models are defined on a triangulation of the spacetime manifold with finite number of vertices, where at each vertex the value of local measure is finite, and the action also does not show any divergence.

However, so far none of the existing spin-foam models implements this non-trivial local measure factor in the quantization<sup>3</sup>. The quantum effect implied by this measure factor has not been analyzed in the context of spin-foam models. But without it there is no chance to link spin foams with canonical LQG which at present is the only method we have in order to derive a path integral formulation of LQG from first principles. In ongoing work [32] we analyse the non-trivial effects caused by this measure factor in the context of spin-foam models, and try to give spin-foam amplitudes an unambiguous canonical interpretation by establishing a link between path-integral formulation and canonical quantization. In this article we also make a few comments on this.

The paper is organized as follows:

In section 2, after defining the reduced phase space path integral for a general theory, we begin with the Hamiltonian framework arising from the  $SO(\eta)$  Holst action<sup>4</sup> [36] (see also [37]). We then derive the path-integral formula for the Holst action in terms of spacetime field variables, i.e. the  $so(\eta)$  connection  $\omega_\mu^{IJ}$  and the co-tetrad  $e_\mu^I$ .

In section 3, starting from the Holst phase space path integral, we construct a path-integral formula for the Plebanski-Holst action by adding some extra fields and extra constraints.

In section 4, we discuss the consistency with the calculations in [27].

Finally, we summarize and conclude with an outlook to future research.

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<sup>3</sup>The ambiguities of the path integral measure in spin-foam models have been discussed in the literatures. In the context of spin-foam models, this issue of path integral measure can be translated into an ambiguity of the gluing amplitudes between 4-cells [29], while the quantum effects are discussed in [30]. And the relevance of the measure factor on the diffeomorphism symmetries of the spinfoam amplitudes is discussed in [31]. However in the present work we are concerning the measure factor which helps to make a connection with the canonical quantization, while the standard spin-foam approach doesn't rely on the canonical framework and 3+1 splitting of the spacetime manifold.

<sup>4</sup>Our discussions apply to both Euclidean and Lorentzian signatures.

## 2 The path-integral measure for the Holst action

### 2.1 Reduced Phase space path integral

To cut a long story short (see e.g. [2, 22]) the central ingredient for most applications of the path integral is the generating functional

$$\mathcal{Z}[j] := \int \mathcal{D}q \mathcal{D}p |\det(\{F, \xi\})| \sqrt{\det(\{S, S\})} \delta[S] \delta[F] \delta[\xi] \exp(i \int dt [p_a \dot{q}^a + j_a q^a]) \quad (2.1)$$

Here  $(q^a, p_a)$  denotes any instantaneous Darboux coordinates on phase space,  $S$  denotes the collection of all second class constraints  $S_\Sigma$ ,  $F$  the collection of all first class constraints  $F_\mu$ ,  $\xi$  any choice of gauge fixing conditions  $\xi^\mu$ , and  $j$  is a current which allows us to perform functional derivations at  $j = 0$  in order to define any object of physical interest. For instance the rigging kernel between initial and final kinematical states  $\psi_i(q)$ ,  $\psi_f(q)$  results by generating these two functions<sup>5</sup> through functional derivation at  $t = \pm\infty$ . In addition, as usual  $\mathcal{D}q = \prod_{t \in \mathbb{R}, a} dq^a(t)$  and  $\delta[F] = \prod_{t \in \mathbb{R}, \mu} \delta(F_\mu(t))$ , and likewise for  $\mathcal{D}p$  and  $\delta[S]$ . We will often write  $|D_1| = [\det(F, \xi)]^2$ ,  $|D_2| = \det(\{S, S\})$ . We will also drop the exponential of the current in what follows since it does not affect any of our manipulations, hence we will mostly deal with the partition function  $\mathcal{Z} = \mathcal{Z}[0]$ . Since what one is really interested in is  $\mathcal{Z}[j]/\mathcal{Z}$  we can drop overall constant factors from all subsequent formulas.

Applied to our situation, we restrict ourself to the case of pure gravity defined by the Holst action. We follow the notation employed in [17, 36]. Note that for the simplicity of the formulae, we skip “ $\prod_{x \in M}$ ” in almost all following path-integrals, where  $M$  is the spacetime manifold. Moreover, we will assume that all the gauge fixing conditions  $\xi_\alpha$  are functions independent of the connections  $\omega_a^{IJ}$  i.e. they are the functions of tetrad only. This assumption will simplify the following discussion. Then

$$\mathcal{Z} = \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^a \delta(C^{ab}) \delta(D^{ab}) \sqrt{|D_2|} \delta(G^{IJ}) \delta(H_a) \delta(H) \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int dt d^3x \pi_{IJ}^{(y)a} \dot{\omega}_a^{IJ} \quad (2.2)$$

where  $\pi_{IJ}^{(y)a} := (\pi - \frac{1}{\gamma} \star \pi)_{IJ}^a$ , and the expressions of the constraints  $G^{IJ}$ ,  $H_a$ ,  $H$ ,  $C^{ab}$ , and  $D^{ab}$  are given by [36]

$$\begin{aligned} G_{IJ} &= D_a^{(y)a} \pi_{IJ}^a := \partial_a \pi_{IJ}^{(y)a} + \omega_{aI}^{K(y)a} \pi_{JK}^a - \omega_{aJ}^{K(y)a} \pi_{IK}^a \\ H_a &= \frac{1}{2} F_{ab}^{IJ}[\omega] \pi_{IJ}^{(y)b} \\ H &= \frac{1}{4 \sqrt{\det q}} (F - \frac{1}{\gamma} \star F)_{ab}^{IJ}[\omega] \pi_{IK}^a \pi_{JL}^b \eta^{KL} \\ C^{ab} &= \epsilon^{IJKL} \pi_{IJ}^a \pi_{KL}^b \\ D^{ab} &= \frac{1}{2 \sqrt{\det q}} \star \pi_{IJ}^c (\pi^{aIK} D_c \pi^{bJL} + \pi^{bIK} D_c \pi^{aJL}) \eta_{KL} \end{aligned} \quad (2.3)$$

where  $D^{ab}$  is the secondary constraint with  $\{H(x), C^{ab}(x')\} = D^{ab}(x) \delta(x, x')$ . Note that the definition of  $H$  and  $D^{ab}$  is slightly different from [36], up to a factor of  $1/(2 \sqrt{\det q})$ . In rewriting the kinematical Liouville measure  $\mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^{(y)a}$  as  $\mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^a$ , an overall constant Jacobian factor has also been dropped.

In Eq.(2.2)  $D_2$  is the determinant of the Dirac matrix

$$\begin{pmatrix} \{C^{ab}(x), C^{cd}(x')\} & \{C^{ab}(x), D^{cd}(x')\} \\ \{D^{ab}(x), C^{cd}(x')\} & \{D^{ab}(x), D^{cd}(x')\} \end{pmatrix} = \begin{pmatrix} 0 & \{C^{ab}(x), D^{cd}(x')\} \\ \{D^{ab}(x), C^{cd}(x')\} & \{D^{ab}(x), D^{cd}(x')\} \end{pmatrix} \quad (2.4)$$

Therefore  $|D_2| = [\det G]^2$  where  $G$  is the matrix

$$G^{ab, cd}(x, x') = \{C^{ab}(x), D^{cd}(x')\} \approx (\det q)^{3/2} \left[ q^{ab} q^{cd} - \frac{1}{2} q^{ac} q^{bd} - \frac{1}{2} q^{cb} q^{ad} \right] \delta^3(x, x'). \quad (2.5)$$

<sup>5</sup>Provided they are analytic. In case they are not, they are analytic functions times a reference vector  $\Omega_0$  in which case the reference vector must be included in (2.1). See [22] for details.

By the symmetry of  $q^{ab}$ , there exists an orthogonal matrix<sup>6</sup>  $M^a{}_b$  such that  $M^a{}_c M^b{}_d q^{cd} = \lambda^a \delta^{ab}$  for some  $\{\lambda^a\}$ , so that

$$M^a{}_e M^b{}_f M^c{}_g M^d{}_h G^{ef,gh} = (\det q)^{3/2} \left[ \lambda^a \lambda^c \delta^{ab} \delta^{cd} - \frac{1}{2} \lambda^a \lambda^b \delta^{ac} \delta^{bd} - \frac{1}{2} \lambda^c \lambda^a \delta^{cb} \delta^{ad} \right]. \quad (2.6)$$

Let  $\hat{G}^{ab,cd} := \lambda^a \lambda^c \delta^{ab} \delta^{cd} - \frac{1}{2} \lambda^a \lambda^b \delta^{ac} \delta^{bd} - \frac{1}{2} \lambda^c \lambda^a \delta^{cb} \delta^{ad}$  denote the portion in square brackets. Each of the rows (12), (13), (23) in  $\hat{G}$  has exactly one non-zero matrix element. Reducing  $\det \hat{G}$  by minors along these 3 rows,

$$\det \hat{G} = \hat{G}^{12,12} \hat{G}^{23,23} \hat{G}^{13,13} \det R = -2^{-9} (\lambda^1 \lambda^2 \lambda^3)^2 \det R$$

where  $R^{ab} = \hat{G}^{aa,bb}$  is the reduced 3 by 3 matrix.  $\det R$  has only 2 non-zero terms; evaluating it and substituting in the result gives

$$\det \hat{G} = \frac{-1}{4} (\lambda^1 \lambda^2 \lambda^3)^4 = \frac{-1}{4} (\det q^{ab})^4 = \frac{-1}{4} (\det q)^{-4}, \quad (2.7)$$

so that

$$\det G = \frac{-1}{4} (\det q)^{\frac{3}{2} \times 6} (\det q)^{-4} = \frac{-1}{4} (\det q)^5. \quad (2.8)$$

Thus, up to an overall factor,

$$\sqrt{|D_2|} = (\det q)^5 =: V_s^{10}. \quad (2.9)$$

Next we express the delta functions  $\delta(H)$  and  $\delta(D^{ab})$  in Eq.(2.2) as integrals of exponentials,

$$\mathcal{Z} = \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N \mathcal{D}d_{ab} V_s^{10} \delta(G^{IJ}) \delta(H_a) \delta(C^{ab}) \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int dt d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - NH + d_{ab} D^{ab} \right]. \quad (2.10)$$

Then we follow the strategy used in [27] to eliminate the secondary second class constraint  $D^{ab}$  in the path-integral. We consider a change of variables which is also a canonical transformation generated by the functional

$$F := - \int d^3x d_{ab} C^{ab} / N. \quad (2.11)$$

The integral measure is the Liouville measure on the phase space and thus is invariant under canonical transformation.  $\sqrt{|D_2|}$ ,  $G_{IJ}$ ,  $C^{ab}$ ,  $\sqrt{|D_1|}$  and  $\xi_\alpha$  are invariant because they strongly Poisson commute with  $C^{ab}$  (here we use the assumption that the gauge fixing conditions  $\xi_\alpha$  only depend on  $\pi_{IJ}^a$ ), and  $H_a$  is invariant because it weakly Poisson commutes with  $C^{ab}$ . The change of kinetic term  $\delta \int dt d^3x \pi_{IJ}^a \partial_t \omega_a^{IJ}(x, t)$  is proportional to  $\int dt d^3x C^{ab} \partial_t (d_{ab}/N)$  which also vanishes by the delta functions  $\delta(C^{ab})$  in front of the exponential. So  $H$  and  $D^{ab}$  are the only terms that change in the canonical transformation generated by  $F$ . Moreover because  $\{H(x), C^{ab}(x')\} = D^{ab}(x) \delta(x, x')$  and  $\{C^{ab}(x), D^{cd}(x')\} = G^{ab,cd}(x, x')$  we can obtain explicitly the transformation behavior

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<sup>6</sup>Here and later on in the paper, when we say a matrix is ‘orthogonal’, even if it has spatial-manifold indicies, we mean orthogonal in the standard matrix sense – i.e., ‘orthogonal’ with respect to  $\delta_{ab}$ , and not with respect to some covariantly determined metric.

of  $H(N)$  and  $D^{cd}(d_{cd})$

$$\begin{aligned}
\tilde{H}(N) &\equiv \sum_{n=0}^{\infty} \frac{1}{n!} \{F, H(N)\}_{(n)} \\
&= \int d^3x N(x) H(x) - \int d^3x \int d^3y \frac{N(x)}{N(y)} d_{ab}(y) \{C^{ab}(y), H(x)\} \\
&+ \frac{1}{2} \int d^3x \int d^3y \int d^3z \frac{N(x)}{N(y)N(z)} d_{ab}(y) d_{cd}(z) \{C^{ab}(y), \{C^{cd}(z), H(x)\}\} \\
&= \int d^3x N(x) H(x) + \int d^3x \int d^3y \frac{N(x)}{N(y)} d_{ab}(y) D^{ab}(x) \delta(x, y) \\
&- \frac{1}{2} \int d^3x \int d^3y \int d^3z \frac{N(x)}{N(y)N(z)} d_{ab}(y) d_{cd}(z) G^{ab,cd}(y, z) \delta(x, z) \\
&= \int d^3x N(x) H(x) + \int d^3x d_{ab}(x) D^{ab}(x) - \frac{1}{2} \int d^3y \int d^3z \frac{1}{N(y)} d_{ab}(y) d_{cd}(z) G^{ab,cd}(y, z) \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
\tilde{D}^{cd}(d_{cd}) &\equiv \sum_{n=0}^{\infty} \frac{1}{n!} \{F, D^{cd}(d_{cd})\}_{(n)} \\
&= \int d^3x d_{cd}(x) D^{cd}(x) - \int d^3x \int d^3y \frac{1}{N(y)} d_{cd}(x) d_{ab}(y) \{C^{ab}(y), D^{cd}(x)\} \\
&= \int d^3x d_{cd}(x) D^{cd}(x) - \int d^3x \int d^3y \frac{1}{N(y)} d_{cd}(x) d_{ab}(y) G^{ab,cd}(x, y) \quad (2.13)
\end{aligned}$$

here the series terminated because of  $\{C^{ab}(x), G^{cd,ef}(x', x'')\} = 0$ . Since  $G^{ab,cd}(x, y)$  is proportional to  $\delta(x, y)$  we have

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N \mathcal{D}d_{ab} V_s^{10} \delta(G^{IJ}) \delta(H_a) \delta(C^{ab}) \sqrt{|D_1|} \prod_{\alpha} \delta(\xi_{\alpha}) \exp i \int dt d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - NH - \frac{1}{2} d_{ab} d_{cd} G^{ab,cd} / N \right] \\
&= \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\omega_t^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N^a \mathcal{D}N \delta(C^{ab}) \frac{V_s^{10}}{\sqrt{|\det(G/N)|}} \sqrt{|D_1|} \prod_{\alpha} \delta(\xi_{\alpha}) \exp i \int dt d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - \omega_t^{IJ} G_{IJ} - N^a H_a - NH \right] \\
&= \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\omega_t^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N^a \mathcal{D}N \delta(C^{ab}) N^3 V_s^5 \sqrt{|D_1|} \prod_{\alpha} \delta(\xi_{\alpha}) \exp i \int dt d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - \omega_t^{IJ} G_{IJ} - N^a H_a - NH \right]. \quad (2.14)
\end{aligned}$$

This is the canonical phase space path integral for the Holst action, with secondary constraints removed as in [38]. The Palatini case is recovered by setting  $\gamma = \infty$  while holding  $G$  constant.

## 2.2 Configuration path integral in terms of spacetime $so(\eta)$ -connection and tetrad

It is too difficult in concretely performing the integrations in Eq.(2.14) to compute transition amplitudes. However if we transform the Eq.(2.14) to be an integral of the Lagrangian Holst action in terms of original configuration variables, i.e. the spacetime connection field  $\omega_{\mu}^{IJ}$  and tetrad field  $e_{\mu}^I$ , the integral will become easier to handle. To rewrite the the canonical path integral as a configuration path-integral for the Holst action, one proceeds in two steps: (1.) Replace the canonical variables and Lagrange multipliers with space-time variables and the simplicity constraint (2.) Integrate out the simplicity constraint.

### 2.2.1 Basic relations between variables

In this section we give the definitions of the new coordinates in terms of the old coordinates. These definitions will be motivated and explained, and the bijectivity of the coordinate transformation demonstrated, in the subsequent section.

When the simplicity constraint is imposed,

$$C^{ab} = \epsilon^{IJKL} \pi_{IJ}^a \pi_{KL}^b \approx 0 \quad (2.15)$$



$\pi_{IJ}^a$  takes one of the five forms<sup>7</sup>

$$\begin{aligned} (I\pm) \quad \pi_{IJ}^a &= \pm \epsilon^{abc} e_b^I e_c^J \\ (II\pm) \quad \pi_{IJ}^a &= \pm \frac{1}{2} \epsilon^{abc} e_b^K e_c^L \epsilon_{IJKL} \\ (Deg) \quad \pi_{IJ}^a &= 0. \end{aligned} \tag{2.16}$$

$$\tag{2.17}$$

Note that the appearance of the degenerated sector shows that the Hamiltonian constrained system derived from the Holst action is not regular, i.e. the rank of the Dirac matrix

$$\left( \begin{array}{cc} \{C^{ab}(x), C^{cd}(x')\} & , \{C^{ab}(x), D^{cd}(x')\} \\ \{D^{ab}(x), C^{cd}(x')\} & , \{D^{ab}(x), D^{cd}(x')\} \end{array} \right) \tag{2.18}$$

is not a constant on the whole phase space. We have to remove the degenerated sector in order to carry out the derivations in the reduced phase space. Therefore all the derivations in the last subsection hold only if the degenerate sector is removed. Now we restrict ourself in sector  $(II+)$ , and in addition stipulate  $\det e_a^I > 0$ , removing the sign ambiguity in the definition of  $e_a^I$ . The derivations for other sectors can be carried out in the same way. With the restriction to  $(II+)$ , the above relation can be inverted as

$$e_a^I = \frac{1}{4\sqrt{2}} |\det \pi_{0j}^b|^{-\frac{1}{2}} \epsilon^{IJKL} \epsilon_{abc} \pi_{0J}^b \pi_{KL}^c. \tag{2.19}$$

This equation can then be used to define  $e_a^I$  off-shell with respect to the simplicity constraint. One might ask whether  $e_a^I$  so defined, along with  $C^{ab}$ , form good coordinates on  $\pi_{IJ}^a$ . In fact, with the restrictions just stipulated, we will show  $\pi_{IJ}^a \mapsto (e_a^I, C^{ab})$  is bijective in the next subsection.

Lastly, we equip the internal space with a time orientation, and define  $n^I$  as the unique internal future-pointing unit vector satisfying  $n^I \pi_{IJ}^a = 0$ . Then one defines

$$e_t^I := N n^I + N^a e_a^I. \tag{2.20}$$

When the simplicity constraint is satisfied,  $e_t^I$  is equal to the  $t$  component of the physical space-time tetrad, so that the above definition is indeed an extension of the usual  $e_t^I$ .

### 2.2.2 Proof of bijectivity

For the purpose of making apparent the bijectivity of the coordinate transformation, and to aid in later calculations, define

$$\pi_i^a := \frac{1}{2} \pi_{0i}^a \tag{2.21}$$

$$\tilde{\pi}_i^a := \frac{1}{4} \epsilon_i^{jk} \pi_{jk}^a. \tag{2.22}$$

In terms of these, the ‘triad’  $e_a^I$  defined in the last subsection can be alternatively introduced via

$$\begin{aligned} 1. \quad f_i^a &:= |\det \pi_j^b|^{-\frac{1}{2}} \pi_i^a, & e_a^i &= (f_i^a)^{-1} \\ 2. \quad e_b^0 &:= \frac{1}{2} \epsilon_{abc} f_i^b \tilde{\pi}_i^c. \end{aligned}$$

<sup>7</sup>To see that the four sectors  $(I\pm)$  and  $(II\pm)$  are disjoint, define  $\pi_i^a := \frac{1}{2} \pi_{0i}^a$  and  $\tilde{\pi}_i^a := \frac{1}{4} \epsilon_i^{jk} \pi_{jk}^a$ . Then one has

$$\begin{aligned} (I+) &\Rightarrow \det \pi_i^a = 0 \text{ and } (\det \tilde{\pi}_i^a)(\det e_a^I) > 0, \\ (I-) &\Rightarrow \det \pi_i^a = 0 \text{ and } (\det \tilde{\pi}_i^a)(\det e_a^I) < 0, \\ (II+) &\Rightarrow \det \tilde{\pi}_i^a = 0 \text{ and } (\det \pi_i^a)(\det e_a^I) > 0, \\ (II-) &\Rightarrow \det \tilde{\pi}_i^a = 0 \text{ and } (\det \pi_i^a)(\det e_a^I) < 0. \end{aligned}$$

Note that here  $e_a^i$  denotes simply the  $I = 1, 2, 3$  components of  $e_a^I$ , *not* the co-triad.<sup>8</sup>

The map  $f_i^a \mapsto \pi_i^a$  is manifestly bijective. The definition of  $e_b^0$  uses precisely the information contained in the anti-symmetric part of  $f_i^a \tilde{\pi}^{bi}$ , whence the remaining information in  $\pi_{IJ}^a$  is exactly the symmetric part of  $f_i^a \tilde{\pi}^{bi}$ :

$$S^{ab} := f_i^{(a} \tilde{\pi}^{b)i} \quad (2.23)$$

In terms of this, the simplicity constraint is given by

$$C^{ab} = -2\pi_j^{(a} \tilde{\pi}^{b)j} = -2(\det e_c^k) S^{ab}. \quad (2.24)$$

From this one sees that  $\pi_{IJ}^a \mapsto (e_a^I, C^{ab})$  is bijective.

### 2.2.3 Rewriting the measure

We have

$$d\pi_{IJ}^a = d\pi_i^a d\tilde{\pi}_i^a \quad (2.25)$$

The inverse of the relation between  $\pi_i^a$  and  $e_a^i$ ,  $\pi_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k$ , gives

$$\frac{\partial \pi_i^a}{\partial e_b^j} = \epsilon^{abc}\epsilon_{ijk}e_c^k. \quad (2.26)$$

Note  $(ai)$  labels rows and  $(bj)$  labels columns. Let  $J^{ai}_{bj}$  denote this matrix. From the *singular value decomposition theorem*, there exist orthogonal matrices  $O^a_b$  and  $O^i_j$  such that  $O^b_a O^i_j e_b^j$  is diagonal, that is

$$O^b_a O^i_j e_b^j = \lambda_a \delta_a^i. \quad (2.27)$$

Let  $O^{ai}_{bj} := O^a_b O^i_j$ . Then  $O^{ai}_{bj}$  is also an orthogonal matrix, and we use it to define

$$\tilde{J}^{ai}_{bj} := O^{ai}_{bj} J^{bj}_{ck} O^{ck}_{dl} = \sum_{c,k} \epsilon^{abc}\epsilon_{ijk}(\lambda_c \delta_c^k) = \sum_c \epsilon^{abc}\epsilon_{ijc}\lambda_c \quad (2.28)$$

where the symmetry of  $\epsilon^{abc}$  and  $\epsilon_{ijk}$  under orthogonal transformations has been used. From the above equation,  $\tilde{J}^{ai}_{bj} = 0$  when  $(i = j)$  or  $(a = b)$  or  $\{i, j\} \neq \{a, b\}$ . From this one can deduce that, for  $i \neq a$ , the row  $(a, i)$  in  $\tilde{J}^{ai}_{bj}$  has only one non-zero element: the one in column  $(b = i, j = a)$ . Reducing by minors along these 6 rows then gives

$$\det \tilde{J} = \tilde{J}^{12}_{21} \tilde{J}^{21}_{12} \tilde{J}^{13}_{31} \tilde{J}^{31}_{13} \tilde{J}^{23}_{32} \tilde{J}^{32}_{23} \det R \quad (2.29)$$

where  $R^i_j = \tilde{J}^{ii}_{jj}$  is the reduced  $3 \times 3$  matrix. The diagonal elements of  $R$  are zero, so that  $\det R$  has only 2 non-zero terms,

$$\det R = \tilde{J}^{11}_{22} \tilde{J}^{22}_{33} \tilde{J}^{33}_{11} + \tilde{J}^{11}_{33} \tilde{J}^{22}_{11} \tilde{J}^{33}_{22} \quad (2.30)$$

As one can check,  $\tilde{J}^{11}_{22} = \tilde{J}^{22}_{11} = -\tilde{J}^{12}_{12} = -\tilde{J}^{21}_{21} = \lambda_3$ , and similarly for cyclic permutations of 1, 2, 3. Plugging this into (2.30) and then (2.29) gives

$$\det \tilde{J} = 2(\lambda_1 \lambda_2 \lambda_3)^3 = 2(\det e_a^i)^3. \quad (2.31)$$

$\det J = \det \tilde{J}$ , so that dropping the irrelevant 2 factor,

$$\mathcal{D}\pi^{ai} = (\det e_c^k)^3 \mathcal{D}e_b^j. \quad (2.32)$$

<sup>8</sup>One can see that  $e_a^i$  may not be taken as the co-triad from the following. For  $v^a, w^b$  tangent to the spatial slice  $\tilde{M}$ ,

$$e_a^i e_{bi} v^a w^b = e_a^I e_{bI} v^a w^b + e_a^0 e_{b0} v^a w^b = g_{ab} v^a w^b + \sigma^2 (e_a^0 v^a)(e_b^0 w^b) \equiv q_{ab} v^a w^b + s(e_a^0 v^a)(e_b^0 w^b)$$

however,  $e_a^0$  is in general arbitrary, so that the second term on the right hand side is in general non-zero, whence in general

$$e_a^i e_{ai} \neq q_{ab}.$$

Next, define  $\mathcal{G}^{ab} := f_a^i \tilde{\pi}^{bi}$ , so that

$$\frac{\partial \mathcal{G}^{ab}}{\partial \tilde{\pi}^{ci}} = f_i^a \delta_c^b. \quad (2.33)$$

This is again block diagonal, whence

$$\det \left( \frac{\partial \mathcal{G}^{ab}}{\partial \tilde{\pi}^{ci}} \right) = (\det f_i^a)^3 = (\det e_a^i)^{-3} \quad (2.34)$$

so that

$$\mathcal{D}\tilde{\pi}^{ai} = (\det e_a^i)^3 \mathcal{D}\mathcal{G}^{ab} = (\det e_a^i)^3 \mathcal{D}\mathcal{G}^{(ab)} \mathcal{D}\mathcal{G}^{[ab]} = (\det e_a^i)^3 \mathcal{D}S^{ab} \mathcal{D}e_b^0. \quad (2.35)$$

Lastly, from (2.24),  $\mathcal{D}C^{ab} = (\det e_a^i)^6 \mathcal{D}S^{ab}$ , so that

$$\mathcal{D}\tilde{\pi}^{ai} = (\det e_a^i)^{-3} \mathcal{D}C^{ab} \mathcal{D}e_b^0. \quad (2.36)$$

Coming to the lapse and shift, the Jacobian of the transformation  $(N, N^a) \mapsto e_t^I$  is

$$J = \frac{\partial e_t^I}{\partial (N, N^a)} = (n^I, e_a^I) \quad (2.37)$$

On the other hand, the 4-volume element

$$\det e_a^I = \det (e_t^I, e_a^I) = \det (N n_I + N^a e_{aI}, e_a^I) = \det (N n_I, e_a^I) = N \det J \quad (2.38)$$

thus  $|\det J| = V_s$  and  $dN dN^a = de_t^I / V_s$ .

Putting all the above relations together, we have

$$\mathcal{D}\pi_{IJ}^a \mathcal{D}N \mathcal{D}N^a = \frac{1}{V_s} \mathcal{D}e_\mu^I \mathcal{D}C^{ab}. \quad (2.39)$$

Note the above measure is  $SO(\eta)$  covariant, consistent with the  $SO(\eta)$  covariance of the starting point.

#### 2.2.4 Final path integral

Inserting (2.39) into (2.14), and integrating out  $C^{ab}$  finally gives

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\omega_\mu^{IJ} \mathcal{D}e_\mu^I N^3 V_s^4 \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int e^I \wedge e^J \wedge \left( {}^*F_{IJ} - \frac{1}{\gamma} F_{IJ} \right) [\omega] \\ &= \int \mathcal{D}\omega_\mu^{IJ} \mathcal{D}e_\mu^I \mathcal{V}^3 V_s \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int e^I \wedge e^J \wedge \left( {}^*F_{IJ} - \frac{1}{\gamma} F_{IJ} \right) [\omega]. \end{aligned} \quad (2.40)$$

Note that the integral in Eq.(2.40) is restricted in the sector  $(II+)$ . But if we want the integral to be over both the sectors  $(II+)$  and  $(II-)$ , we will obtain

$$\mathcal{Z}_\pm = \int_{II\pm} \mathcal{D}\omega_\mu^{IJ} \mathcal{D}e_\mu^I \mathcal{V}^3 V_s \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \cos \int e^I \wedge e^J \wedge \left( {}^*F_{IJ} - \frac{1}{\gamma} F_{IJ} \right) [\omega]. \quad (2.41)$$

In the existing spin-foam models in the literature [3, 18, 21], sectors  $(II+)$  and  $(II-)$  are not distinguished. One can see from the above equation, therefore, why it is generally the Cosine of the action and not the exponential of the action that is expected to appear (and does appear) in the asymptotic analysis of vertex amplitudes [39], see also the discussions of the issue in some other different perspectives [40]

In the follows, we always use  $\mathcal{Z}_\pm$  and  $\int_{II\pm}$  to denote the integral over both sectors, and  $\mathcal{Z}, \int$  only to denote the integral over a single sector  $(II+)$ .

### 3 The construction of path-integral measure for Plebanski-Holst, by way of Holst

In this section we would like to relate the previous Holst action partition function with the partition function for the Plebanski-Holst action. Our starting point for the reconstruction is Eq.(2.14) (we first only consider a single sector  $II+$  for simplicity)

$$\mathcal{Z} = \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\omega_i^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N^a \mathcal{D}N \delta(C_{\pi\pi}^{ab}) N^3 V_s^5 \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int dt d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} + \omega_i^{IJ} G_{IJ} - N^a H_a - NH \right] \quad (3.1)$$

where we use new notation for the simplicity constraint  $C_{\pi\pi}^{ab} := C^{ab}$ , anticipating the introduction of further simplicity constraints. To remind the reader,

$$\begin{aligned} G_{IJ} &= D_a^{(\gamma)a} \pi_{IJ}^{(\gamma)b} \\ H_a &= \frac{1}{2} F_{ab}^{IJ}[\omega] \pi_{IJ}^{(\gamma)b} \\ H &= \frac{1}{4 \sqrt{\det q}} (F - \frac{1}{\gamma} * F)_{ab}^{IJ}[\omega] \pi_{IK}^a \pi_{JL}^b \eta^{KL} \\ C_{\pi\pi}^{ab} &= \epsilon^{IJKL} \pi_{IJ}^a \pi_{KL}^b. \end{aligned} \quad (3.2)$$

#### 3.1 Basic strategy and some definitions

In order to rewrite this path integral as a (generalized) Plebanski path integral, one needs to change the variables  $\pi_{IJ}^a, N, N^a$  in favor of a constrained Plebanski two-form  $X_{\mu\nu}^{IJ}$ . If we define  $\langle Y, Z \rangle := \frac{1}{4} \epsilon_{IJKL} Y^{IJ} Z^{KL}$ , the constraint on  $X_{\mu\nu}^{IJ}$  is

$$\langle X_{\mu\nu}, X_{\rho\sigma} \rangle = \frac{\mathcal{V}}{4!} \epsilon_{\mu\nu\rho\sigma} \quad (3.3)$$

where  $\mathcal{V} := \epsilon^{\mu\nu\rho\sigma} \langle X_{\mu\nu}, X_{\rho\sigma} \rangle$ . This constraint implies  $X_{\mu\nu}^{IJ}$  takes one of the four forms

$$X_{\mu\nu}^{IJ} = \begin{cases} \pm 2 e_\mu^I e_\nu^J & (I\pm) \\ \pm \epsilon^{IJ}_{KL} e_\mu^K e_\nu^L & (II\pm) \end{cases} \quad (3.4)$$

for some tetrad  $e_\mu^I$ . On-shell,  $\mathcal{V} = \det e_\mu^I$ , the 4-volume element. Following [27], we decompose  $X_{\mu\nu}^{IJ}$  into

$$\pi_{IJ}^a := \frac{1}{2} \epsilon^{abc} (X_{bc})_{IJ} \quad (3.5)$$

$$\beta_a^{IJ} := X_{Ia}^{IJ} \quad (3.6)$$

and (3.3) becomes

$$\begin{aligned} C_{\pi\pi}^{ab} &:= \langle \pi^a, \pi^b \rangle \approx 0 \\ (C_{\beta\beta})_{ab} &:= \langle \beta_a, \beta_b \rangle \approx 0 \\ (C_{\beta\pi})_a^b &:= \text{traceless part of } \langle \beta_a, \pi^b \rangle \approx 0. \end{aligned} \quad (3.7)$$

The first of these constraints was imposed in section 2.2.1; the four sectors appearing there are the same four sectors here in (3.4). As in section 2.2, we restrict to sector  $(II+)$ . The last two of the constraints (3.7) are new.

As  $\pi_{IJ}^a$  was coordinatized by  $e_a^I$  and the simplicity constraint  $C_{\pi\pi}^{ab}$  in section 2.2, similarly in this section we introduce coordinates for  $\beta_a^{IJ}$ . Specifically, we will define a change of variables

$$\beta_a^{IJ} \leftrightarrow (N, N^a, \tilde{C}_{\beta\beta}, \tilde{C}_{\beta\pi}) \quad (3.8)$$

where  $\tilde{C}_{\beta\beta}$  and  $\tilde{C}_{\beta\pi}$  have the properties

1. If  $C_{\pi\pi} = 0$ , then  $\tilde{C}_{\beta\pi} = C_{\beta\pi}$
2. If  $\tilde{C}_{\beta\pi} = 0$ , then  $\tilde{C}_{\beta\beta} = C_{\beta\beta}$ .

These properties will be used to replace  $\tilde{C}_{\beta\pi}, \tilde{C}_{\beta\beta}$  in favor of  $C_{\beta\pi}, C_{\beta\beta}$  in the final path integral in section 3.4.  $N$  and  $N^a$  are defined as follows. First define, in order,

1.  $e_t^i := \frac{1}{2}\epsilon^{ij}_k f_j^a \beta_a^{0k}$
2.  $e_t^0 := \frac{1}{3}f_i^a \left( \frac{1}{2}\epsilon^i_{jk} \beta_a^{jk} + e_a^0 e_t^i \right)$

where  $f_i^a := (e_a^i)^{-1} = \left| \det \pi_{0j}^b \right|^{-\frac{1}{2}} \pi_{0i}^a$  and  $e_a^0$  are as defined in §2.2. One can verify that when  $X_{\mu\nu}^{IJ}$  is of the form (II+) above, this definition of  $e_t^I$  coincides with the  $t$  component of the tetrad.  $N, N^a$  are then defined by the relation

$$e_t^I =: N n^I + N^a e_a^I \quad (3.9)$$

where recall from §2.2.1 that  $n^I$  is determined by  $\pi_{IJ}^a$  essentially via  $n^I \pi_{IJ}^a = 0$ .

### 3.2 The coordinate transformation and its bijectivity

We begin by decomposing  $\beta_a^{IJ}$  as

$$\beta_a^i := \beta_a^{0i} \quad (3.10)$$

$$\tilde{\beta}_a^i := \frac{1}{2}\epsilon^i_{jk} \beta_a^{jk}. \quad (3.11)$$

Then one sees that  $e_t^i$  and  $e_t^0$

$$e_t^i = \frac{1}{2}\epsilon^{ij}_k f_j^a \beta_a^k \quad (3.12)$$

$$e_t^0 = \frac{1}{3}f_i^a (\tilde{\beta}_a^i + e_a^0 e_t^i) \quad (3.13)$$

contain precisely the information about the skew-symmetric part of  $f^{ai}\beta_a^k$  and the trace of  $f_i^a \tilde{\beta}_b^i$ . This leaves only the symmetric part of  $f^{ai}\beta_a^k$  and the trace-free part of  $f_i^a \tilde{\beta}_b^i$ :

$$C^{ij} := f^{a(i}\beta_a^{j)} \quad (3.14)$$

$$K_b^a := f_i^a \tilde{\beta}_b^i - \frac{1}{3}\delta_b^a f_i^c \tilde{\beta}_c^i \quad (3.15)$$

so that there is a manifest isomorphism

$$\beta_a^{IJ} \equiv (\beta_a^i, \tilde{\beta}_a^i) \leftrightarrow (e_t^i, e_t^0, C^{ij}, K_b^a). \quad (3.16)$$

It is convenient to replace  $K_b^a$  with a translation depending only on  $\pi_{IJ}^a, e_t^i$

$$L_b^a = K_b^a + f_i^a e_b^0 e_t^i - \frac{1}{3}\delta_b^a f_i^c e_c^0 e_t^i = \text{trace-free part of } f_i^a (\tilde{\beta}_b^i + e_b^0 e_t^i). \quad (3.17)$$

$$\beta_a^{IJ} \leftrightarrow (e_t^i, e_t^0, C^{ij}, L_b^a). \quad (3.18)$$

We next invert the relations (3.12)-(3.17),

$$\beta_a^i = \epsilon^i_{jk} e_t^k e_a^j + C^{ij} e_{aj} \quad (3.19)$$

$$\tilde{\beta}_a^i = e_b^i L_a^b + 2e_a^{[i} e_t^{0]} \quad (3.20)$$

and substitute them into the expressions for  $C_{\beta\beta}, C_{\beta\pi}$ . This gives

$$\frac{s}{2}(C_{\beta\beta})_{ab} = \epsilon_{ijk} e_t^k e_c^j e_{(a}^i L_{b)}^c + C_{ij} e_c^i e_{(a}^j L_{b)}^c + 2C_{ij} e_{(a}^j e_{b)}^{[i} e_t^{0]} \quad (3.21)$$

$$s(C_{\beta\pi})_b^a = \epsilon_{ijk} e_t^k e_b^j S^{ac} e_c^i + C_{ij} \epsilon^{acd} e_b^j e_c^i e_d^0 + (\det e_d^k) L_b^a + C_{ij} e_b^j e_c^i S^{ac} - \frac{1}{3}\delta_b^a C_{ij} e_c^j e_d^i S^{cd} \quad (3.22)$$

where  $S^{ab}$  is determined by  $\pi_{IJ}^a$  as in section 2.2.2. Dropping the  $S^{ab}$  terms from the second of these equations leads to the definition of  $\tilde{C}_{\beta\pi}$ :

$$s(\tilde{C}_{\beta\pi})_b^a = C_{ij}\epsilon^{acd}e_b^j e_c^i e_d^0 + (\det e_d^k)L_b^a. \quad (3.23)$$

So that  $\tilde{C}_{\beta\pi} = C_{\beta\pi}$  when  $C_{\pi\pi} = 0$ . With  $e_t^i, e_t^0, C^{ij}$  fixed,  $L_b^a \mapsto (\tilde{C}_{\beta\pi})_b^a$  is manifestly bijective, so that, composing with (3.16),

$$\beta_a^{IJ} \leftrightarrow (e_t^i, e_t^0, C^{ij}, \tilde{C}_{\beta\pi}) \quad (3.24)$$

is bijective.

Solving (3.22) for  $L_b^a$  and substituting it into (3.21) gives, after some manipulation,

$$\frac{S}{2}(C_{\beta\beta})_{ab} = (\det e_d^k)^{-1}(\epsilon_{ijk}e_t^k + C_{ij})e_c^j e_a^i (\tilde{C}_{\beta\pi})_b^c + (e_t^0 - e_t^k f_k^e e_e^0)e_a^j e_b^i C_{ij}. \quad (3.25)$$

Dropping the  $\tilde{C}_{\beta\pi}$  term leads to the definition

$$\frac{S}{2}\tilde{C}_{\beta\beta} := (e_t^0 - e_t^k f_k^e e_e^0)e_a^j e_b^i C_{ij} \quad (3.26)$$

so that  $\tilde{C}_{\beta\beta} = C_{\beta\beta}$  when  $\tilde{C}_{\beta\pi} = 0$ . In order to understand the significance of the prefactor in (3.26), we prove the following

**Lemma 3.1.**  $(\det e_a^i)(e_t^0 - e_h^0 f_k^h e_t^k)$  is equal to the 4-volume element  $\mathcal{V} = \det e_\alpha^I$ .

**Proof.**

$$\begin{aligned} \det e_\alpha^I &= \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{IJKL}e_\alpha^I e_\beta^J e_\gamma^K e_\delta^L \\ &= \frac{1}{3!}\epsilon^{tabc}\epsilon_{IJKL}e_t^I e_a^J e_b^K e_c^L = \frac{1}{3!}\epsilon^{abc}\epsilon_{IJKL}e_t^I e_a^J e_b^K e_c^L \\ &= \frac{1}{3!}\epsilon^{abc}\epsilon_{jkl}e_t^0 e_a^j e_b^k e_c^l + \frac{1}{2}\epsilon^{abc}\epsilon_{i0kl}e_t^i e_a^0 e_b^k e_c^l \\ &= (\det e_a^i)e_t^0 - e_t^i e_a^0 (\det e_b^j)f_i^a \\ &= (\det e_a^i)(e_t^0 - e_t^i f_i^a e_a^0). \end{aligned}$$

■

By assumption  $\mathcal{V}$  and  $\det e_a^i$  are non-zero, so the prefactor in (3.26) is non-zero, whence, from (3.26),  $C^{ij} \mapsto (\tilde{C}_{\beta\beta})_{ab}$  is manifestly bijective. Thus

$$\beta_a^{IJ} \leftrightarrow (e_t^i, e_t^0, \tilde{C}_{\beta\beta}, \tilde{C}_{\beta\pi}) \quad (3.27)$$

is a bijection.

Finally,

$$e_t^I \leftrightarrow (N, N^a) \quad (3.28)$$

as defined in (3.9) is clearly an isomorphism. Putting these together, we see that

$$\beta_a^{IJ} \leftrightarrow (N, N^a, \tilde{C}_{\beta\beta}, \tilde{C}_{\beta\pi})$$

is a bijection as claimed.

### 3.3 The change of measure

In this subsection, we calculate the measure  $\mathcal{D}\beta_a^{IJ}$  in terms of  $\mathcal{D}N\mathcal{D}N^a\mathcal{D}\tilde{C}_{\beta\beta}\mathcal{D}\tilde{C}_{\beta\pi}$ . First,

$$\mathcal{D}\beta_a^{IJ} = \mathcal{D}\beta_a^i \mathcal{D}\tilde{\beta}_a^i. \quad (3.29)$$

We next wish to relate  $\mathcal{D}\beta_a^k$  and  $\mathcal{D}e_t^i \mathcal{D}C_{jk}$ . Our strategy is to first define

$$\mathcal{F}_{ij} := \beta_{ai} f_j^a, \quad (3.30)$$

find the relation between  $\mathcal{D}\beta_{ai}$  and  $\mathcal{D}\mathcal{F}_{ij}$ , and then use  $\mathcal{D}\mathcal{F}_{ij} = \mathcal{D}\mathcal{F}_{(ij)}\mathcal{D}\mathcal{F}_{[ij]}$ . As  $C_{ij} = \mathcal{F}_{(ij)}$  and  $e_t^j = \frac{1}{2}\epsilon^{jkl}\mathcal{F}_{[kl]}$ , in fact this becomes  $\mathcal{D}\mathcal{F}_{ij} = \mathcal{D}C_{ij}\mathcal{D}e_t^j$ . First,

$$\frac{\partial \mathcal{F}_{ij}}{\partial \beta_{ak}} = \delta_i^k f_j^a. \quad (3.31)$$

This is a block diagonal matrix, with three blocks each equal to  $f_i^a$ :

$$\det \frac{\partial \mathcal{F}_{ij}}{\partial \beta_{ak}} = (\det f_j^a)^3 = (\det e_a^j)^{-3} \quad (3.32)$$

so that

$$\mathcal{D}\beta_{ai} = (\det e_a^j)^3 \mathcal{D}\mathcal{F}_{ij} = (\det e_a^j)^3 \mathcal{D}C_{ij} \mathcal{D}e_t^k. \quad (3.33)$$

Next, to rewrite  $\mathcal{D}\tilde{\beta}_a^i$ , first define  $\mathcal{H}_b^a := f_i^a \tilde{\beta}_b^i$ . Then

$$\frac{\partial \mathcal{H}_b^a}{\partial \tilde{\beta}_c^i} = f_j^b \delta_c^i \quad (3.34)$$

so that the matrix is block diagonal with three blocks, each equal to  $f_j^a$ :

$$\det \left( \frac{\partial \mathcal{H}_b^a}{\partial \tilde{\beta}_c^i} \right) = (\det f_j^a)^3 = (\det e_a^j)^{-3}, \quad (3.35)$$

so that

$$\mathcal{D}\tilde{\beta}_a^k = (\det e_c^j)^3 \mathcal{D}\mathcal{H}_b^a = (\det e_a^j)^3 \mathcal{D}K_b^a \mathcal{D}\text{tr}\mathcal{H} \quad (3.36)$$

where we have used that  $K_b^a$  is the traceless part of  $\mathcal{H}_b^a$ . As  $L_b^a$  is just a translation of  $K_b^a$  (3.17) by a term involving only  $e_t^i$ , which is being held constant, one can replace  $\mathcal{D}K_b^a$  with  $\mathcal{D}L_b^a$ . Likewise  $e_t^0$  is just a translation of  $\text{tr}\mathcal{H}$  by a term involving only  $e_t^i$ , so that  $\mathcal{D}\text{tr}\mathcal{H} = \mathcal{D}e_t^0$ , whence

$$\mathcal{D}\tilde{\beta}_a^i = (\det e_a^j)^3 \mathcal{D}L_b^a \mathcal{D}e_t^0. \quad (3.37)$$

Putting together (3.29, 3.33, 3.37),

$$\mathcal{D}\beta_a^{IJ} = (\det e_a^j)^6 \mathcal{D}e_t^I \mathcal{D}C_{ij} \mathcal{D}L_b^a. \quad (3.38)$$

To perform the change of variables  $(C_{ij}, L_b^a) \rightarrow (\tilde{C}_{\beta\beta}, \tilde{C}_{\beta\pi})$ , we proceed in two steps,  $(C_{ij}, L_b^a) \rightarrow (C_{ij}, \tilde{C}_{\beta\pi}) \rightarrow (\tilde{C}_{\beta\beta}, \tilde{C}_{\beta\pi})$ . Holding  $C_{ij}$  constant, the Jacobian of the first transformation is

$$\frac{\partial(\tilde{C}_{\beta\pi})_a^b}{\partial L_d^c} \approx (\det e_d^k)(\delta_a^c \delta_d^b - \frac{1}{3} \delta_a^b \delta_d^c) \quad (3.39)$$

Here  $\delta_a^c \delta_d^b - \frac{1}{3} \delta_a^b \delta_d^c$  is the identity matrix on the space of trace-less matrices. The space of trace-less matrices is 8-dimensional, so that

$$\det \left( \frac{\partial(\tilde{C}_{\beta\pi})_a^b}{\partial L_j^i} \right) \approx (\det e_d^k)^8 \quad (3.40)$$

whence

$$\mathcal{D}L_b^a = (\det e_d^k)^{-8} \mathcal{D}\tilde{C}_{\beta\pi}. \quad (3.41)$$

Next, to replace  $C_{ij}$  in favor of  $C_{\beta\beta}$ . We want the Jacobian of the matrix

$$\frac{\partial(\tilde{C}_{\beta\beta})_{ab}}{\partial C_{ij}} = 2s(e_t^0 - e_h^0 f_k^h e_t^k) e_a^{(i} e_b^{j)}, \quad (3.42)$$

that is,

$$\det \left( \frac{\partial(\tilde{C}_{\beta\beta})_{ab}}{\partial C_{ij}} \right) = 2^6 (e_t^0 - e_h^0 f_k^h e_t^k)^6 \det_{(ab,ij)} e_a^{(i} e_b^{j)}. \quad (3.43)$$

Let  $H^{ab}_{ij} := e^{(i}_a e^{j)}_b$ . Recall from §2.2.3 the orthogonal matrices  $O^b_a$  and  $O^i_j$  satisfying  $O^b_a O^i_j e^j_b = \lambda_a \delta^i_a$ ; let  $S^{ab}_{cd} := O^{(a}_c O^{b)}_d$  and  $S^{ij}_{kl} := O^k_{(i} O^l_{j)}$ , orthogonal matrices. Then

$$\tilde{H}^{ab}_{ij} := S^{ab}_{cd} H^{cd}_{kl} S^{kl}_{ij} = \lambda_a \lambda_b \delta^i_a \delta^j_b \quad (3.44)$$

which is a diagonal 6 by 6 matrix.

$$\det \tilde{H} = (\lambda_1^2)(\lambda_2^2)(\lambda_3^2)(\lambda_1 \lambda_2)(\lambda_1 \lambda_3)(\lambda_2 \lambda_3) = (\det e^i_a)^4 \quad (3.45)$$

so

$$\mathcal{D}C_{ij} = (e^0_t - e^0_{hJ_k} e^k_t)^{-6} (\det e^i_a)^{-4} \mathcal{D}\tilde{C}_{\beta\beta} \quad (3.46)$$

where an irrelevant overall numerical coefficient was dropped.

Finally one performs the change of variables  $e^i_t \rightarrow (N, N^a)$ . Using (2.37, 2.38) from the last section, and (3.41, 3.46, 3.38), one has finally

$$\mathcal{D}\beta^{IJ}_a = (e^0_t - e^0_{cJ_k} e^k_t)^{-6} (\det e^i_a)^{-6} V_s \mathcal{D}N \mathcal{D}N^a \mathcal{D}\tilde{C}_{\beta\beta} \mathcal{D}\tilde{C}_{\beta\pi}. \quad (3.47)$$

With lemma 3.1 this becomes

$$\mathcal{D}\beta^{IJ}_a = \mathcal{V}^{-6} V_s \mathcal{D}N \mathcal{D}N^a \mathcal{D}\tilde{C}_{\beta\beta} \mathcal{D}\tilde{C}_{\beta\pi}. \quad (3.48)$$

### 3.4 Final path integral

Starting with the canonical path integral (2.14), and inserting  $\int \mathcal{D}\tilde{C}_{\beta\pi} \mathcal{D}\tilde{C}_{\beta\beta} \delta(\tilde{C}_{\beta\pi}) \delta(\tilde{C}_{\beta\beta}) = 1$ , we have

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\omega^{IJ}_\mu \mathcal{D}\pi^a_{IJ} \mathcal{D}N^a \mathcal{D}N \mathcal{D}\tilde{C}_{\beta\pi} \mathcal{D}\tilde{C}_{\beta\beta} \delta(C_{\pi\pi}) \delta(\tilde{C}_{\beta\pi}) \delta(\tilde{C}_{\beta\beta}) N^3 V_s^5 \exp i \int dt d^3x \left[ \pi^a_{IJ} \dot{\omega}^{IJ}_a - \omega^{IJ}_t G_{IJ} - N^a H_a - NH \right] \\ &\quad \times \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha). \end{aligned} \quad (3.49)$$

Using (3.48) then gives

$$\mathcal{Z} = \int \mathcal{D}\omega^{IJ}_\mu \mathcal{D}\pi^a_{IJ} \mathcal{D}\beta^{IJ}_a \delta(C_{\pi\pi}) \delta(\tilde{C}_{\beta\pi}) \delta(\tilde{C}_{\beta\beta}) N^9 V_s^{10} \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int dt d^3x \left[ \pi^a_{IJ} \dot{\omega}^{IJ}_a - \omega^{IJ}_t G_{IJ} - N^a H_a - NH \right]. \quad (3.50)$$

We next use the presence of  $\delta(\tilde{C}_{\beta\pi})$  and the fact that this enforces  $\tilde{C}_{\beta\beta} = C_{\beta\beta}$  to replace  $\tilde{C}_{\beta\beta}$  in favor of  $C_{\beta\beta}$ ; then we use the presence of  $\delta(C_{\pi\pi})$  and the fact that it enforces  $\tilde{C}_{\beta\pi} = C_{\beta\pi}$  to replace  $\tilde{C}_{\beta\pi}$  in favor of  $C_{\beta\pi}$ , yielding

$$\mathcal{Z} = \int \mathcal{D}\omega^{IJ}_\mu \mathcal{D}\pi^a_{IJ} \mathcal{D}\beta^{IJ}_a \delta(C_{\pi\pi}) \delta(C_{\beta\pi}) \delta(C_{\beta\beta}) N^9 V_s^{10} \exp i \int dt d^3x \left[ \pi^a_{IJ} \dot{\omega}^{IJ}_a - \omega^{IJ}_t G_{IJ} - N^a H_a - NH \right]. \quad (3.51)$$

**Lemma 3.2.** *When the simplicity constraints are satisfied,*

$$X^{IJ}_{ta} = \frac{1}{2} N^c \epsilon_{abc} \pi^{bIJ} - \frac{N}{\sqrt{\det q}} \epsilon_{abc} \pi^{bI}{}_K \pi^{cKJ} \quad (3.52)$$

**Proof.** With the simplicity constraints satisfied,

$$X^{IJ}_{\mu\nu} = \epsilon^{IJ}{}_{KL} e^K_\mu e^L_\nu$$

so that

$$\begin{aligned} X^{IJ}_{ta} &= \epsilon^{IJ}{}_{KL} e^K_t e^L_a \\ &= \epsilon^{IJ}{}_{KL} (N^b e^K_b + N n^K) e^L_a \\ &= N^b X^{IJ}_{bc} + N n^K \epsilon_K{}^{IJ}{}_L e^L_a \\ &= \frac{1}{2} N^b \epsilon_{acb} \pi^{bIJ} + N n^K \epsilon_K{}^{IJ}{}_L e^L_a. \end{aligned} \quad (3.53)$$



The first term here matches the first term on the right hand side of (3.52). The second term on the right hand side of (3.52) is

$$\begin{aligned} \frac{-N}{\sqrt{\det q}} \epsilon_{abc} \pi^{bl}{}_K \pi^{cKJ} &= \frac{-N}{4\sqrt{\det q}} \epsilon_{abc} \epsilon^{bde} \epsilon^{cfs} \epsilon^I{}_{KMN} \epsilon^{KJ}{}_{PQ} e_d^M e_e^N e_f^P e_g^Q \\ &= \frac{-N}{2\sqrt{\det q}} \epsilon^{bde} \epsilon^I{}_{KMN} \epsilon^{KJ}{}_{PQ} e_d^M e_e^N e_a^P e_b^Q. \end{aligned} \quad (3.54)$$

Now,

$$\epsilon^{bde} e_b^Q e_d^M e_e^N = \epsilon^{tbde} e_b^Q e_d^M e_e^N = \mathcal{V} \eta^{tbde} e_b^Q e_d^M e_e^N = \mathcal{V} e_R^t \epsilon^{RQMN}$$

where  $\eta^{\mu\nu\rho\sigma}$  is the inverse volume form, and  $e_R^t$  is the indicated component of  $e_I^\mu := (e_I^\mu)^{-1}$ . But  $e_R^t = (\partial_\mu t) e_R^\mu = \frac{1}{N} n_\mu e_R^\mu = \frac{1}{N} n_R$ , so that

$$\epsilon^{bde} e_b^Q e_d^M e_e^N = -\sqrt{\det q} n_R \epsilon^{RQMN}. \quad (3.55)$$

Thus (3.54) becomes

$$\begin{aligned} \frac{-N}{\sqrt{\det q}} \epsilon_{abc} \pi^{bl}{}_K \pi^{cKJ} &= \frac{N}{2} \epsilon^{bde} \epsilon^I{}_{KMN} \epsilon^{KJ}{}_{PQ} n_R \epsilon^{RQMN} e_a^P \\ &= N n^K \epsilon_K{}^{IJ}{}_P e_a^P \end{aligned}$$

matching the second term in (3.53) and completing the proof.  $\blacksquare$

**Corollary 3.3.** *On-shell with respect to the simplicity constraints,*

$$\int dt d^3x \left[ \pi_{IJ}^{(a} \dot{\omega}_a^{IJ} + \omega_t^{IJ} G_{IJ} - N^a H_a - NH \right] = \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ} =: \int B_{IJ} \wedge F^{IJ}$$

**Proof.**

$$\begin{aligned} \int B_{IJ} \wedge F^{IJ} &= \frac{1}{4} \int dt d^3x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu IJ} F_{\rho\sigma}^{IJ} = \frac{1}{2} \int dt d^3x \left[ \epsilon^{abc} B_{abIJ} \partial_t \omega_c^{IJ} + \omega_t^{IJ} (D_c B_{ab})_{IJ} \epsilon^{abc} + \epsilon^{abc} B_{tcIJ} F_{ab}^{IJ} \right] \\ &= \int dt d^3x \left[ \pi_{IJ}^{(\gamma c} \dot{\omega}_c^{IJ} + \omega_t^{IJ} \pi_{IJ}^{(\gamma c} + \frac{1}{2} \epsilon^{abc} (X - \frac{1}{\gamma} X)_{tcIJ} F_{ab}^{IJ} \right] \\ &= \int dt d^3x \left[ \pi_{IJ}^{(\gamma c} \dot{\omega}_c^{IJ} + \omega_t^{IJ} \pi_{IJ}^{(\gamma c} + \frac{1}{4} \epsilon^{abc} N^e \epsilon_{cde} \pi_{IJ}^{(\gamma d} F_{ab}^{IJ} - \frac{N}{4\sqrt{\det q}} \epsilon^{abc} \epsilon_{cde} \left[ \pi^d \pi^e - \frac{1}{\gamma} \star(\pi^d \pi^e) \right]_{IJ} F_{ab}^{IJ} \right] \\ &= \int dt d^3x \left[ \pi_{IJ}^{(\gamma c} \dot{\omega}_c^{IJ} + \omega_t^{IJ} G_{IJ} - N^a H_a - NH \right] \end{aligned}$$

Substituting this into the path integral (3.51), one has finally

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\omega_\mu^{IJ} \mathcal{D}X_{\mu\nu}^{IJ} \delta(C_{\pi\pi}) \delta(C_{\beta\pi}) \delta(C_{\beta\beta}) N^9 V_s^{10} \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ} \\ &= \int \mathcal{D}\omega_\mu^{IJ} \mathcal{D}X_{\mu\nu}^{IJ} \delta(C_{\pi\pi}) \delta(C_{\beta\pi}) \delta(C_{\beta\beta}) \mathcal{V}^9 V_s \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ}. \end{aligned} \quad (3.56)$$

Note that this integral is restricted to the solution sector  $(II+)$  of the simplicity constraint. We can extend the integral to include both sectors  $(II+)$  and  $(II-)$  without changing the form of the integrand, i.e. we obtain

$$\mathcal{Z}_\pm = \int_{II\pm} \mathcal{D}\omega_\mu^{IJ} \mathcal{D}X_{\mu\nu}^{IJ} \delta(C_{\pi\pi}) \delta(C_{\beta\pi}) \delta(C_{\beta\beta}) \mathcal{V}^9 V_s \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \exp i \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ}. \quad (3.57)$$

In the following subsection, we show another way to construct the Plebanski-Holst path-integral from the Holst path-integral, where we implement both sectors  $(II+)$  and  $(II-)$ .

### 3.5 An alternative way to construct Plebanski-Holst path-integral from Holst

In this subsection we would like to give another derivation from the the Holst action partition function to the partition of Plebanski-Holst action. Such a derivation is made by transforming delta functions in the integral. Our starting point for this alternative derivation is also Eq.(2.14), but with a integral over both sectors ( $II+$ ) and ( $II-$ )

$$\mathcal{Z}_{\pm} = \int_{II\pm} \mathcal{D}\omega_a^{IJ} \mathcal{D}\omega_t^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}N^a \mathcal{D}N \delta(C^{ab}) \mathcal{V}^3 V_s^2 \sqrt{|D_1|} \prod_{\alpha} \delta(\xi_{\alpha}) \exp i \int dt \int d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - \omega_t^{IJ} G_{IJ} - N^a H_a + NH \right] \quad (3.58)$$

We define a new tensor field  $X_{tc}^{IJ}$  by

$$X_{tc}^{IJ} := \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{\sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \quad (3.59)$$

then the action on the exponential is again expressed as a BF action as it is shown above

$$S := \int dt \int d^3x \left[ \pi_{IJ}^a \dot{\omega}_a^{IJ} - \omega_t^{IJ} G_{IJ} - N^a H_a + NH \right] = \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ} =: \int B_{IJ} \wedge F^{IJ} \quad (3.60)$$

Therefore in terms of the new field  $X_{tc}^{IJ}$  we can re-express the path-integral as a constrained BF theory:

$$\begin{aligned} \mathcal{Z}_{\pm} &= \int_{II\pm} \mathcal{D}\omega_a^{IJ} \mathcal{D}\omega_t^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}X_{tc}^{IJ} \mathcal{D}N^a \mathcal{D}N \mathcal{V}^3 V_s^2 \sqrt{|D_1|} \prod_{\alpha} \delta(\xi_{\alpha}) \exp i \int (X - \frac{1}{\gamma} \star X)_{IJ} \wedge F^{IJ} \\ &\quad \times \delta^6 \left( \epsilon^{IJKL} \pi_{IJ}^a \pi_{KL}^b \right) \delta^{18} \left( X_{tc}^{IJ} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \right) \end{aligned} \quad (3.61)$$

As a first step for recovering the full Plebanski simplicity constraints, we divide the 18  $\delta$ -functions into two collections, each of which has 9  $\delta$ -functions. Then we transform the two collections of  $\delta$ -functions in two different way, i.e.

$$\begin{aligned} &\delta^{18} \left( X_{tc}^{IJ} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \right) \\ &= \delta^9 \left( X_{tc}^{0i} + \frac{1}{2} \epsilon_{abc} N^a \pi^{b0i} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{a0l} \pi^{bil} \right) \delta^9 \left( X_{tc}^{ij} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bij} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \right) \\ &= (\text{Jacobian}) \\ &\quad \times \delta^9 \left( X_{tc}^{0i} X_{td}^{jk} \epsilon_{ijk} + X_{tc}^{ij} X_{td}^{0k} \epsilon_{ijk} + \frac{1}{2} N^a \epsilon_{abc} \left[ \pi^{b0i} X_{td}^{jk} + \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \left[ \pi^{a0l} \pi^{bil} X_{td}^{jk} + \pi^{aIK} \pi^{bJL} \eta_{KL} X_{td}^{0k} \right] \epsilon_{ijk} \right) \\ &\quad \times \delta^9 \left( X_{tc}^{ij} \pi^{d0k} \epsilon_{ijk} + X_{tc}^{0i} \pi^{djk} \epsilon_{ijk} + \frac{1}{2} \epsilon_{abc} N^a \left[ \pi^{bij} \pi^{d0k} + \pi^{b0i} \pi^{djk} \right] \epsilon_{ijk} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \left[ \pi^{aIK} \pi^{bj} \pi^{d0k} + \pi^{a0l} \pi^{bil} \pi^{djk} \right] \epsilon_{ijk} \right) \end{aligned}$$

We define some notations:

$$\begin{aligned} \chi_c^i &:= X_{tc}^{0i} + \frac{1}{2} \epsilon_{abc} N^a \pi^{b0i} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{a0l} \pi^{bil} \\ \tilde{\chi}_c^{ij} &:= X_{tc}^{ij} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bij} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \\ \mathcal{F}_{cd} &:= X_{tc}^{0i} X_{td}^{jk} \epsilon_{ijk} + X_{tc}^{ij} X_{td}^{0k} \epsilon_{ijk} + \frac{1}{2} N^a \epsilon_{abc} \left[ \pi^{b0i} X_{td}^{jk} + \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \left[ \pi^{a0l} \pi^{bil} X_{td}^{jk} + \pi^{aIK} \pi^{bJL} \eta_{KL} X_{td}^{0k} \right] \epsilon_{ijk} \\ &= \chi_c^i X_{td}^{jk} \epsilon_{ijk} + \tilde{\chi}_c^{ij} X_{td}^{0k} \epsilon_{ijk} \\ \tilde{\mathcal{F}}_c^d &:= X_{tc}^{ij} \pi^{d0k} \epsilon_{ijk} + X_{tc}^{0i} \pi^{djk} \epsilon_{ijk} + \frac{1}{2} \epsilon_{abc} N^a \left[ \pi^{bij} \pi^{d0k} + \pi^{b0i} \pi^{djk} \right] \epsilon_{ijk} - \frac{N}{4\sqrt{\det q}} \epsilon_{abc} \left[ \pi^{aIK} \pi^{bj} \pi^{d0k} + \pi^{a0l} \pi^{bil} \pi^{djk} \right] \epsilon_{ijk} \\ &= \tilde{\chi}_c^{ij} \pi^{d0k} \epsilon_{ijk} + \chi_c^i \pi^{djk} \epsilon_{ijk} \end{aligned} \quad (3.62)$$

then the Jacobian from above  $\delta$ -function transformation is the determinant of the transformation matrix on the constraint surface

$$\frac{\partial(\mathcal{F}_{cd}, \tilde{\mathcal{F}}_c^d)}{\partial(\chi_a^i, \tilde{\chi}_a^{jk})} = \begin{pmatrix} \delta_c^a (X_{td}^{jk} + \tilde{\chi}_d^{jk}) \epsilon_{ijk} & \delta_d^a (X_{tc}^{0i} + \chi_c^i) \epsilon_{ijk} \\ \delta_c^a \pi^{djk} \epsilon_{ijk} & \delta_d^a \pi^{d0i} \epsilon_{ijk} \end{pmatrix} \approx \begin{pmatrix} \delta_c^a X_{td}^{jk} \epsilon_{ijk} & \delta_d^a X_{tc}^{0i} \epsilon_{ijk} \\ \delta_c^a \pi^{djk} \epsilon_{ijk} & \delta_d^a \pi^{d0i} \epsilon_{ijk} \end{pmatrix}. \quad (3.63)$$

We can see that

$$\det \left[ \begin{pmatrix} \delta_c^a X_{td}^{jk} \epsilon_{ijk} & \delta_d^a X_{tc}^{0i} \epsilon_{ijk} \\ \delta_c^a \pi^{djk} \epsilon_{ijk} & \delta_c^a \pi^{d0i} \epsilon_{ijk} \end{pmatrix} \right] = \left( \det \left[ \begin{pmatrix} X_{td}^{jk} \epsilon_{ijk} & X_{tc}^{0i} \epsilon_{ijk} \\ \pi^{djk} \epsilon_{ijk} & \pi^{d0i} \epsilon_{ijk} \end{pmatrix} \right] \right)^3 = (\det[X_{\alpha\beta}^{IJ}])^3 = \mathcal{V}^9 \quad (3.64)$$

Therefore we insert back this result, and further divide the first collection into its symmetric and anti-symmetric parts. After some manipulation, we have

$$\begin{aligned} & \delta^{18} \left( X_{tc}^{IJ} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{4 \sqrt{\det q}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \right) \\ &= \mathcal{V}^9 \delta^3 \left( \frac{1}{2} N^a \left[ \epsilon_{ab[c} \pi^{b0i} X_{td}^{jk} + \epsilon_{ab[c} \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} - \frac{N}{4 \sqrt{\det q}} \left[ \epsilon_{ab[c} \pi^{a0l} \pi^{bil} X_{td}^{jk} + \epsilon_{ab[c} \pi^{aik} \pi^{bjl} \eta_{KL} X_{td}^{0k} \right] \epsilon_{ijk} \right) \\ & \times \delta^6 \left( X_{tc}^{IJ} X_{td}^{KL} \epsilon_{IJKL} + \frac{1}{2} N^a \left[ \epsilon_{ab(c} \pi^{b0i} X_{td}^{jk} + \epsilon_{ab(d} \pi^{bjk} X_{tc}^{0i} \right] \epsilon_{ijk} - \frac{N}{4 \sqrt{\det q}} \left[ \epsilon_{ab(c} \pi^{a0l} \pi^{bil} X_{td}^{jk} + \epsilon_{ab(d} \pi^{ajk} \pi^{bkl} \eta_{KL} X_{tc}^{0i} \right] \epsilon_{ijk} \right) \\ & \times \delta^9 \left( \frac{1}{2} X_{tc}^{IJ} \pi^{dKL} \epsilon_{IJKL} - \frac{N}{4 \sqrt{\det q}} \epsilon_{abc} \left[ \pi^{aiK} \pi^{bj} \pi^{d0k} + \pi^{a0l} \pi^{bil} \pi^{djk} \right] \epsilon_{ijk} \right) \end{aligned} \quad (3.65)$$

Since the simplicity constraint  $C^{ab} = \epsilon^{IJKL} \pi_{IJ}^a \pi_{KL}^b = 0$  implies that there exists a non-degenerated  $so(3)$ -valued one form  $e_a^i$  and an another independent 1-form  $e_a^0$  such that (we are working in both two sectors  $(II\pm)$ )  $\pi_{IJ}^a = \pm \epsilon^{abc} e_b^K e_c^L \epsilon_{IJKL}$ . Then we obtain the following lemma:

**Lemma 3.4.** *On the constraint surface defined by the delta function  $\delta(C^{ab})$ , the field  $X_{tc}^{IJ}$  can be written as*

$$X_{tc}^{IJ} = \pm e_t^K e_c^L \epsilon_{KL}^{IJ} = \pm (N n^K + N^a e_a^K) e_c^L \epsilon_{KL}^{IJ} \quad (3.66)$$

where  $e_a^l$   $\alpha = t, 1, 2, 3$  form a non-degenerate tetrad field for non-vanished  $N$ . Thus the 18 delta functions

$$\delta^{18} \left( X_{tc}^{IJ} + \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{4 \sqrt{|\det q|}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \right) \quad (3.67)$$

can essentially be written as  $\delta^{18} (X_{tc}^{IJ} - e_t^K e_c^L \epsilon_{KL}^{IJ}) + \delta^{18} (X_{tc}^{IJ} + e_t^K e_c^L \epsilon_{KL}^{IJ})$ .

**Proof.** The lemma follows straight-forwardly from the definition of  $X_{tc}^{IJ}$ :

$$X_{tc}^{IJ} = \frac{1}{2} \epsilon_{abc} N^a \pi^{bIJ} - \frac{N}{\sqrt{|\det q|}} \epsilon_{abc} \pi^{aIK} \pi^{bJL} \eta_{KL} \quad (3.68)$$

with the solution  $\pi_{IJ}^a = \epsilon^{abc} e_b^K e_c^L \epsilon_{IJKL}$ . First of all, we check the first term

$$-\frac{1}{2} \epsilon_{abc} N^a \pi_{IJ}^b = \mp \frac{1}{2} \epsilon_{abc} N^a \epsilon^{bde} e_d^K e_e^L \epsilon_{IJKL} = \pm \delta_a^d \delta_c^e N^a e_d^K e_e^L \epsilon_{IJKL} = \pm N^a e_a^K e_c^L \epsilon_{IJKL} \quad (3.69)$$

And then the second term:

$$\begin{aligned} & \frac{N}{4 \sqrt{|\det q|}} \epsilon_{abc} \pi_{IK}^a \pi_{JL}^b \eta^{KL} \\ &= \frac{N}{4 \sqrt{|\det q|}} \epsilon_{abc} \epsilon^{ade} e_d^M e_e^N \epsilon_{IKMN} \epsilon^{bfh} e_f^P e_h^Q \epsilon_{JLPQ} \eta^{KL} = -\frac{N}{2 \sqrt{|\det q|}} \delta_a^f \delta_c^h \epsilon^{ade} e_d^M e_e^N e_f^P e_h^Q \epsilon_{IKMN} \epsilon_{JLPQ} \eta^{KL} \\ &= -\frac{N}{2 \sqrt{|\det q|}} \epsilon^{dea} e_d^M e_e^N e_a^P e_c^Q \epsilon_{IKMN} \epsilon_{JLPQ} \eta^{KL} \end{aligned}$$

here now we define a new field  $e_H^t$  or  $n_H$  such that  $\epsilon^{dea} e_d^M e_e^N e_a^P = \pm N \sqrt{|\det q|} \epsilon^{HMPN} e_H^t = \mp \sqrt{|\det q|} \epsilon^{HMPN} n_H$  and  $n^H n_H = -1$ . Thus

$$\begin{aligned} & \frac{N}{4 \sqrt{|\det q|}} \epsilon_{abc} \pi_{IK}^a \pi_{JL}^b \eta^{KL} \\ &= \pm \frac{N}{2 \sqrt{|\det q|}} \sqrt{|\det q|} \epsilon^{HMPN} n_H e_c^Q \epsilon_{IKMN} \epsilon_{JLPQ} \eta^{KL} = \mp N [\delta_I^H \delta_K^P - \delta_K^H \delta_I^P] n_H e_c^Q \epsilon_{JLPQ} \eta^{KL} \\ &= \pm N n_K e_c^Q \epsilon_{JLIQ} \eta^{KL} = \pm N n^L e_c^Q \epsilon_{IJLQ} \end{aligned} \quad (3.70)$$

As a result:

$$X_{ic}^{IJ} = \pm (N n^K + N^a e_a^K) e_c^L \epsilon_{KL}^{IJ} \equiv \pm e_i^K e_c^L \epsilon_{KL}^{IJ} \quad (3.71)$$

To check the non-degeneracy of  $e_\alpha^I$ , we calculate its determinate

$$\begin{aligned} \det e_\alpha^I &= \frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta} e_\alpha^I e_\beta^J e_\gamma^K e_\delta^L \epsilon_{IJKL} = \frac{1}{3!} \epsilon^{tabc} e_i^I e_a^J e_b^K e_c^L \epsilon_{IJKL} = \frac{1}{3!} \epsilon^{tabc} (N n^I + N^d e_d^I) e_a^J e_b^K e_c^L \epsilon_{IJKL} \\ &= N \frac{1}{3!} \epsilon^{tabc} n^I e_a^J e_b^K e_c^L \epsilon_{IJKL} = \pm N \sqrt{|\det q|} \end{aligned} \quad (3.72)$$

which is nonzero for non-vanished  $N$ . ■

By this Lemma, we can immediately simplify the expression of Eq.(3.65) to be

$$\begin{aligned} &\mathcal{V}^9 \delta^3 \left( \frac{1}{2} N^a \left[ \epsilon_{ab[c} \pi^{b0i} X_{td}^{jk} + \epsilon_{ab[c} \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} - \frac{N}{4 \sqrt{|\det q|}} \left[ \epsilon_{ab[c} \pi^{a0l} \pi^{bil} X_{td}^{jk} + \epsilon_{ab[c} \pi^{aiK} \pi^{bjL} \eta_{KL} X_{td}^{0k} \right] \epsilon_{ijk} \right) \\ &\times \delta^6 (X_{ic}^{IJ} X_{td}^{KL} \epsilon_{IJKL}) \delta^9 \left( X_{ic}^{IJ} \pi^{dKL} \epsilon_{IJKL} - \frac{1}{3} \delta_c^d N \frac{1}{3!} \epsilon^{tabc} n^I e_a^J e_b^K e_c^L \epsilon_{IJKL} \right) \end{aligned} \quad (3.73)$$

Now we are ready to integral over  $N$  and  $N^a$  and obtain

$$\begin{aligned} \mathcal{Z}_\pm &= \int_{II\pm} \mathcal{D}\omega_\alpha^{IJ} \mathcal{D}X_{\alpha\beta}^{IJ} \prod_{x \in M} \mathcal{V}^{12} V_s \frac{1}{|\det(M)|} \delta^{20} \left( \epsilon_{IJKL} X_{\alpha\beta}^{IJ} X_{\gamma\delta}^{KL} - \frac{1}{4!} \mathcal{V} \epsilon_{\alpha\beta\gamma\delta} \right) \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \\ &\times \exp i \int \left( X - \frac{1}{\gamma} \star X \right)_{IJ} \wedge F^{IJ} \end{aligned} \quad (3.74)$$

where

$$\begin{aligned} M_a^e &:= \left[ \epsilon_{abc} \pi^{b0i} X_{td}^{jk} + \epsilon_{abc} \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} \epsilon^{cde} \\ &= \left[ (\delta_a^d \delta_b^e - \delta_b^d \delta_a^e) \pi^{b0i} X_{td}^{jk} + (\delta_a^d \delta_b^e - \delta_b^d \delta_a^e) \pi^{bij} X_{td}^{0k} \right] \epsilon_{ijk} \\ &= \left[ \pi^{e0i} X_{ta}^{jk} - \delta_a^e \pi^{b0i} X_{tb}^{jk} + \pi^{eij} X_{ta}^{0k} - \delta_a^e \pi^{bij} X_{tb}^{0k} \right] \epsilon_{ijk} \\ &= \frac{1}{6} \mathcal{V} \delta_a^e - \frac{1}{2} \mathcal{V} \delta_a^e = -\frac{1}{3} \mathcal{V} \delta_a^e \end{aligned} \quad (3.75)$$

Thus  $|\det(M)| = \mathcal{V}^3$  and the final result is

$$\begin{aligned} \mathcal{Z}_\pm &= \int_{II\pm} \mathcal{D}\omega_\alpha^{IJ} \mathcal{D}X_{\alpha\beta}^{IJ} \prod_{x \in M} \mathcal{V}^9 V_s \delta^{20} \left( \epsilon_{IJKL} X_{\alpha\beta}^{IJ} X_{\gamma\delta}^{KL} - \frac{1}{4!} \mathcal{V} \epsilon_{\alpha\beta\gamma\delta} \right) \sqrt{|D_1|} \prod_\alpha \delta(\xi_\alpha) \\ &\times \exp i \int \left( X - \frac{1}{\gamma} \star X \right)_{IJ} \wedge F^{IJ} \end{aligned} \quad (3.76)$$

which is resulting path-integral of Plebanski-Holst action on both sector ( $II\pm$ ). Considering both sectors is the preparation for the spin-foam construction.

## 4 Consistency with Buffenoir, Henneaux, Noui and Roche

On setting  $\gamma = \infty$ , the path integral of the last section becomes a Plebanski path integral. However, at first glance, this path integral is different from the one derived in the paper of Buffenoir, Henneaux, Noui and Roche (BHNr) [27], having a different measure factor. In this section we will show that this discrepancy is only apparent, and show how the two path integrals are in fact equivalent. Because in this section we set  $\gamma = \infty$ ,  $B = X$ .

The key difference in the analysis of [27] is that  $B_{ia}$  is made into a *dynamical variable* by introducing a conjugate variable  $P_{IJ}^\mu$  constrained, however, to be zero. This leads, in a precise sense, to the presence of “two lapses and two shifts.” First, lapse

and shift appear as certain components of  $B_{Ia}$ ; we shall call these ‘physical lapse and shift’ and shall denote them  $N_p, N_p^a$ . More precisely, we define  $N_p, N_p^a$  to be functions of  $B_{Ia}$  in the same way  $N, N^a$  depended on  $X_{Ia}$  in the last section. A second lapse and shift appear on writing the Hamiltonian and vector constraint delta functions in exponential form; we shall call these ‘langrange multiplier lapse and shift’ and shall denote them  $N, N^a$ . ( $N_p, N_p^a, N, N^a$  will always be undensitized.)

In the last section, by contrast, only one lapse and shift appeared. For this reason, the path integral of [27] is not directly comparable with the path integral of the last section, but rather first one of the extra lapse and shift needs to be removed before comparison. We will carry this out in the present section, and see how the path integral of [27] in fact reduces to that of the last section. Of course, we knew that these two path integrals must be equivalent since they are constructed from a single reduced phase space — that of GR. It is nevertheless instructive to see explicitly how the equivalence comes about. This also provides a valuable check against errors, by deriving the final path integral from two independent starting points.

#### Choice of gauge-fixings and manipulation of constraints

We start from equation (75) in [27]:

$$\mathcal{Z} = \int \mathcal{D}\omega_a^{IJ} \mathcal{D}\pi_{IJ}^a \mathcal{D}\beta_a^{IJ} \left( \sqrt{D_1^{(\psi, \xi)}}|_{P=0} \delta(C'_0) \delta(C'_a) \delta(G) \delta(\xi_\alpha) \right) \left( N_p^{11} V_s^{15} \delta(C_{\pi\pi}) \delta(C_{\beta\pi}) \delta(C_{\beta\beta}) \delta(\tilde{T}) \right) \exp i \int dt d^3x \pi_{IJ}^a \dot{\omega}_a^{IJ}, \quad (4.1)$$

where we have used  $\mathcal{V} = N_p V_s$  and  $(\sqrt{D_2})_{BHN R} = N_p^{20} V_s^{24}$ , and where we have also used the presence of constraint delta functions to remove a weakly vanishing term  $\tilde{H}$  that is present in the exponent in equation (75) of [27]. (Indeed, in [27],  $\tilde{H}$  is *introduced* into the exponent in this way, using that  $\tilde{H}$  vanishes weakly.) Here  $C'_0$ ,  $C'_a$ , and  $\tilde{T}^{ab}$  are as defined in [27], and are essentially the scalar constraint, vector constraint, and secondary constraint generated by  $C_{\pi\pi}$ , respectively.  $\xi_\alpha = (\xi_{\kappa_0}, \xi_{\kappa_a}, \xi_S, \xi_V, \xi_G)$  are the gauge-fixing functions corresponding to the full set of first class constraints  $\kappa_0, \kappa_a, C'_0, C'_a, G_{IJ}$  originally present in [27]. (By the time one reaches the above equation in [27],  $\delta(\kappa_0)\delta(\kappa_a)$  have already been integrated out, but their corresponding gauge-fixing functions have not.) To review,  $\kappa_0$  and  $\kappa_a$  are defined by

$$\kappa_0 := \frac{1}{2} P_{IJ}^a B_{Ia}^{IJ} \quad (4.2)$$

$$\kappa_a := \frac{1}{2} \epsilon_{abc} P_{IJ}^b \pi^{cIJ} \quad (4.3)$$

and the gauge they generate correspond precisely to the freedom in the choice of physical lapse and shift  $N_p, N_p^a$ .  $D_1^{(\psi, \xi)}$  in the above path integral denotes the determinant of the poisson bracket matrix

$$\begin{pmatrix} \{\psi_\alpha, \psi_\beta\} & \{\psi_\alpha, \xi_\beta\} \\ \{\xi_\alpha, \psi_\beta\} & \{\xi_\alpha, \xi_\beta\} \end{pmatrix}$$

where  $\psi_\alpha$  collectively denotes the first class constraints  $\kappa_0, \kappa_a, C'_0, C'_a, G$ , so that  $\{\psi_\alpha, \psi_\beta\} \approx 0$ , and we have  $D_1^{(\psi, \xi)} = (\det\{\psi_\alpha, \xi_\beta\})^2$ . Because of the argument in [22] without loss of generality we may assume for convenience a particular choice of gauge fixing:

$$\xi_{\kappa_0} := N_p - 1 \approx 0 \quad (4.4)$$

$$\xi_{\kappa_a} := N_p^a \approx 0. \quad (4.5)$$

With this choice, one can check

$$\begin{aligned} \{\kappa_0, \xi_{\kappa_0}\} &= -\frac{1}{2} N_p & \{\kappa_0, \xi_{\kappa_b}\} &= -\frac{1}{2} N_p^b \\ \{\kappa_a, \xi_{\kappa_0}\} &= 0 & \{\kappa_a, \xi_{\kappa_b}\} &= -\delta_a^b. \end{aligned}$$

We furthermore assume that the gauge-fixing functions  $\xi_S, \xi_V, \xi_G$  are chosen to depend only on  $\pi_{IJ}^a$ ; this is clearly possible due to the fact that the scalar, vector, and Gauss constraints are also present in the Hamiltonian framework of Barros e Sa [36] derived from the Holst action, and there it is possible to choose pure momentum gauge-fixing conditions, hence depending only on  $\pi_{IJ}^a$ .

Second, recall that

$$\sqrt{D_1^{(\psi, \xi)}} \prod_\alpha \delta(\psi_\alpha) \delta(\xi_\alpha) \quad (4.6)$$

is invariant under the choice of functions  $\psi_\alpha, \xi_\alpha$  enforcing the chosen gauge-fixed constraint surface. We use this to replace  $C'_0, C'_a$  in favor of the constraints  $H, H_a$  defined in the foregoing sections. That this replacement is valid can be seen in two steps:



which is precisely equation (2.2), which in the last section was in turn shown to be equal to (3.56).

*Remark on BHNR*

In the above we began with equation (75) in BHNR [27], and not the final answer (76) in BHNR. This is because in BHNR the Henneaux-Slavnov trick [38] was not applied correctly in passing from (75) to (76). Specifically, as already mentioned earlier in this section, BHNR introduces the Hamiltonian constraint into the path integral exponential ‘for free’ by using the presence of  $\delta(C_0)$  in the path integral, instead of by ‘exponentiating’  $\delta(C_0)$ . As a consequence,  $\delta(C_0)$  remains explicitly in the path integral; but  $\delta(C_0)$  is not invariant under the canonical transformation used in the Henneaux-Slavnov trick introduced in [27], even on-shell. ([27] explicitly calculates the change of  $C_0$  under the canonical transformation.) This was overlooked in [27] and presents an obstacle to using the Henneaux-Slavnov trick.

In section 2.1 of the present paper, however, the Hamiltonian constraint is brought into the exponential by casting the associated delta function in exponential form. As a consequence, no similar problem arises when performing the Henneaux-Slavnov trick, and the trick goes through.

## 5 Discussion

The goal of the present work has been to calculate the appropriate formal path integrals for Holst gravity and for Plebanski gravity with Immirzi parameter — which we call Holst-Plebanski gravity — as determined by canonical analysis. This has been done, starting from the  $SO(\eta)$  covariant framework of [36]. The final Holst-Plebanski path integral was shown to be consistent with the calculations of [27], modulo a slight oversight in [27] which we corrected. We used the well known reduced phase space method [2] in our derivation of which a compact account adapted to the notation employed here can be found in [22].

The main difference between the formal path integral expression for Holst gravity derived in this paper and the “new spin foam models” [21] that are also supposed to be quantisations of Holst gravity are<sup>10</sup> 1. the appearance of the local measure factor, 2. the continuum rather than discrete formulation (triangulation) and the lack of manifest spacetime covariance<sup>11</sup>. The next steps in our programme are therefore clear: In [32] we propose a discretisation of the path integral derived in this paper which does take the proper measure factor into account. We will do this using a new method designed to take care of the simplicity constraints of Plebanski gravity and which lies somewhere between the spirits of [21] and [23]. As we have said before, we interpret the lack of manifest spacetime covariance even in the continuum as an unavoidable consequence of the mixture of dynamics and gauge invariance in background independent (generally covariant) theories with propagating degrees of freedom. In the classical theory it requires some work to establish that spacetime covariance actually does hold on, albeit on shell only. However, the quantum corrections are about the off shell physics and thus lack of spacetime diffeomorphism invariance will presumably prevail off the classical limit. The question is then whether there is a different symmetry group of the quantum theory, which coincides with spacetime diffeomorphism invariance on shell in the classical theory. The obvious candidate for this “quantum diffeomorphism group” is the quantisation of the Bergmann – Komar group (BKG) [33] as was proposed in [10] and in [28] it is analysed if and in which sense the BKG is a symmetry of the quantum theory.

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<sup>10</sup>The origin of the differences is that the approach in the present paper relies on the canonical quantization, while the standard spin-foam approach doesn’t.

<sup>11</sup>In fact discrete models are also never spacetime covariant unless they are topological, however, the continuum limit of spin foam models, if one could take it, should be spacetime covariant.

## A An example for checking the equivalences between the path-integrals of the Holst Hamiltonian, Ashtekar-Barbero-Immirzi Hamiltonian and triad-ADM Hamiltonian formalisms: Imposing the time-gauge

In order to check these equivalences we need to fix the boost part of the internal gauge transformations by imposing the time-gauge, i.e. inserting the delta function  $\delta(e_a^0)$  and the corresponding Faddeev-Popov determinant into the path-integral formula Eq.(2.2). In order to do that, we need the time gauge condition written in terms of canonical variables of Holst action. But it is not hard to find:

$$-2(\det e_a^i)e_c^0 = \tilde{\pi}^{ai}\pi_i^b\epsilon_{bac} = -\frac{1}{8}\epsilon^{ijk}\pi_{0i}^a\pi_{jk}^b\epsilon_{abc} \quad (\text{A.1})$$

then we denote the time-gauge condition  $T_c := \epsilon^{ijk}\pi_{0i}^a\pi_{jk}^b\epsilon_{abc} = 0$  instead of  $e_c^0 = 0$ , in the sector that  $e_a^i$  is non degenerate. On the other hand, the boost part of the Gauss constraint reads:

$$G_{0j} = \partial_a(\pi - \frac{1}{\gamma} * \pi)_{0j}^a + \omega_{a0}^k\pi_{jk}^a - \omega_{aj}^k\pi_{0k}^a \quad (\text{A.2})$$

The Faddeev-Popov determinant  $\Delta_{FP}$  is defined by:

$$\Delta_{FP} = |\det(\{G_{0i}(x), T_c(x')\}_D)| \quad (\text{A.3})$$

where  $\{, \}_D$  is the Dirac bracket with respect to the second class constraints  $C^{ab}$  and  $D^{ab}$ . However since both  $C^{ab}$  and  $D^{ab}$  are  $SO(\eta)$  gauge invariant, they are Poisson-commutative with the Gauss constraint  $G_{IJ}$ . Therefore the Dirac bracket between  $G_{0i}$  and  $T_c$  is identical to their Poisson bracket, so that

$$\begin{aligned} \{G_{0j}(x), T_c(x')\}_D &= \{G_{0j}(x), T_b(x')\} = \left\{ \left( \omega_{a0}^k\pi_{jk}^a - \omega_{aj}^k\pi_{0k}^a \right) (x), \epsilon^{imn}\pi_{0i}^d\pi_{mn}^b\epsilon_{dbc}(x') \right\} \\ &= -\epsilon_{dbc}\epsilon^{imn}\pi_{jk}^a\pi_{mn}^b \left\{ \omega_a^{0k}(x), \pi_{0i}^d(x') \right\} - \epsilon_{dbc}\epsilon^{imn}\pi_{0k}^a\pi_{0i}^d \left\{ \omega_a^{jk}(x), \pi_{mn}^b(x') \right\} \\ &= -\epsilon_{dbc}\epsilon^{imn}\pi_{jk}^a\pi_{mn}^b\delta_a^d \left( \delta_0^0\delta_i^k - \delta_i^0\delta_0^k \right) \delta(x, x') - \epsilon_{dbc}\epsilon^{imn}\pi_{0k}^a\pi_{0i}^d\delta_a^b \left( \delta_m^j\delta_n^k - \delta_n^j\delta_m^k \right) \delta(x, x') \\ &= \epsilon_{abc}\epsilon^{imn}\pi_{ij}^a\pi_{mn}^b\delta(x, x') - 2\epsilon_{abc}\epsilon^{ijk}\pi_{0i}^a\pi_{0k}^b\delta(x, x') \end{aligned}$$

where the first term vanishes by the time-gauge  $e_a^0 = 0$  (our analysis is for the sector of solutions in which  $\pi_{IJ}^a = \epsilon^{abc}e_b^K e_c^L \epsilon_{IJKL}$ ). And the second term

$$-2\epsilon_{abc}\epsilon^{ijk}\pi_{0i}^a\pi_{0k}^b = -2 \left[ \det e_a^i \right]^2 \epsilon_{abc}\epsilon^{ijk} f_i^a f_k^b = -2 \left[ \det e_a^i \right] \epsilon_{ikl}\epsilon^{ijk} e_c^l = 4 \left[ \det e_a^i \right] e_c^j \quad (\text{A.4})$$

As a result, we obtain that  $\Delta_{FP} = \prod_{x \in M} \left[ \det e_a^i \right]^4$ . Thus we can insert the gauge fixing term  $\Delta_{FP}\delta^3(T_c)$  into the phase space path-integral Eq.(2.2).

$$\begin{aligned} Z_T &= \int [D\omega_a^{IJ}] [D\pi_{IJ}^a] \prod_{x \in M} \delta(G^{IJ}) \delta(H_a) \delta(H) \delta(C^{ab}) \delta(D^{ab}) \delta(T_c) \Delta_{FP} \sqrt{|D_2|} \exp i \int dt \int d^3x \pi_{IJ}^a \dot{\omega}_a^{IJ} \\ &= \int [D\omega_a^{IJ}] [D\pi_{IJ}^a] \prod_{x \in M} \delta(G^{IJ}) \delta(H_a) \delta(H) \delta(C^{ab}) \delta(D^{ab}) \delta(T_c) V_s^{14} \exp i \int dt \int d^3x \pi_{IJ}^a \dot{\omega}_a^{IJ} \end{aligned} \quad (\text{A.5})$$

Following the same strategy as in section 2.2, we can obtain the time-gauge-fixed path-integral for the Holst action which is expressed as

$$\begin{aligned} Z_T &= \int [D\omega_a^{IJ}] [De_a^I] \prod_{x \in M} \mathcal{V}^3 V_s^5 \delta^3((\det e_a^i)e_c^0) \exp i \int e^I \wedge e^J \wedge \left( *F_{IJ} - \frac{1}{\gamma} F_{IJ} \right) [\omega] \\ &= \int [D\omega_a^{IJ}] [De_a^I] \prod_{x \in M} \mathcal{V}^3 V_s^2 \delta^3(e_c^0) \exp i \int e^I \wedge e^J \wedge \left( *F_{IJ} - \frac{1}{\gamma} F_{IJ} \right) [\omega] \end{aligned} \quad (\text{A.6})$$

where  $\mathcal{V} = |\det e_a^I|$  is the 4-dimensional volume element, and  $V_s = |\det e_a^i|$  is the spatial volume element when we have imposed the time-gauge fixing condition.



In the following, starting from Eq.(A.5), we try to derive a canonical path-integral formula for the Hamiltonian framework used in canonical LQG, i.e. the Ashtekar-Barbero-Immirzi Hamiltonian. The product of two  $\delta$ -functions in (A.5) can be rewritten

$$\begin{aligned}\delta(T_c)\delta(C^{ab}) &= \delta^3(\epsilon^{ijk}\pi_{0i}^a\pi_{jk}^b\epsilon_{abc})\delta^6(\epsilon^{ijk}\pi_{0i}^a\pi_{jk}^b) = \delta^9(\epsilon^{ijk}\pi_{0i}^a\pi_{jk}^b) \\ &= [\det\pi_{0i}^a]^{-3}\delta^9(\pi_{jk}^b) = V_s^{-6}\delta^9(\pi_{jk}^b)\end{aligned}\quad (\text{A.7})$$

then we integrate over  $\pi_{jk}^b$  in Eq.(A.5) and denote  $\pi_{0i}^a = \sqrt{|\det q|}f_i^a \equiv E_i^a$

$$Z_T = \int [DA_a^i][D\Gamma_a^i][DE_i^a] \prod_{x \in M} \delta(G^{IJ}) \delta(H_a) \delta(H) \delta(D^{ab}) V_s^8 \exp i \int dt \int d^3x E_i^a \dot{A}_a^i \quad (\text{A.8})$$

where  $A_a^i \equiv \omega^{0i} = A_a^{0i} - \frac{1}{2\gamma}\epsilon^{ijk}A_a^{jk}$  and  $\Gamma_a^i \equiv \frac{1}{2}\epsilon^{ijk}A_a^{jk}$ . We then obtain the relation:

$$\omega_a^{ij} = \epsilon^{ijk}[(1 + \gamma^{-2})\Gamma_a^k + \gamma^{-1}A_a^k] \quad (\text{A.9})$$

Then the Gauss constraint  $G_{IJ}$  and the secondary constraint  $D^{ab}$  become

$$\begin{aligned}G_{0i} &= \partial_a E_i^a + \epsilon^{ijk}[(1 + \gamma^{-2})\Gamma_a^j + \gamma^{-1}A_a^j]E_k^a \\ G_i &= \frac{1}{2}\epsilon_{ijk}G_{ij} = -\gamma^{-1}\partial_a E_k^a + \epsilon_{kij}A_a^i E_j^a \\ D^{ab} &= \frac{1}{\sqrt{|\det q|}}[\epsilon_{ijk}E_i^c(\partial_c E_j^a)E_k^b + \epsilon_{ijk}E_i^c(\partial_c E_j^b)E_k^a + \epsilon_{ijk}E_i^c\epsilon_{jmn}\Gamma_c^m E_n^a E_k^b + \epsilon_{ijk}E_i^c\epsilon_{jmn}\Gamma_c^m E_n^b E_k^a]\end{aligned}\quad (\text{A.10})$$

And the result in [36] means that

$$\delta^6(G^{IJ})\delta^6(D^{ab}) = \left[ \frac{\partial(G_{0i}, D^{ab})}{\partial\Gamma_a^i} \right]^{-1} \delta^9\left(\Gamma_a^i - \frac{1}{2}\epsilon_{ijk}E_j^b[\partial_{[b}E_{a]}^k + E_{a[l}E_{k]}^c\partial_b E_{cl}]\right)\delta^3(G_i) \quad (\text{A.11})$$

which shows that  $\Gamma_a^i$  is the spin-connection compatible with the triad  $E_i^a$ . And we have defined  $E_a^i = e_a^i / \sqrt{|\det q|}$  as the inverse of  $E_i^a$ . In order to compute the Jacobian factor, we observe that:

$$\begin{aligned}\delta^6(D^{ab})\delta^3(G_{0i}) &= |\det E_{(a}^i E_{b)}^j| \delta^6(D^{ab}E_a^i E_b^j) \delta^3\left(\frac{\gamma}{1 + \gamma^2}[\gamma G_{0i} - G_i]\right) \\ &= V_s^{-8} \delta^6\left(\frac{4}{\sqrt{|\det q|}}[E_{(i}^a \Gamma_{|a|j)} - \delta_{ij}E_k^a \Gamma_a^k] + \dots\right) \delta^3(\epsilon_{ijk}\Gamma_{aj}E_k^a + \dots) \\ &= V_s^{-2} \delta^6(4[E_{(i}^a \Gamma_{|a|j)} - \delta_{ij}E_k^a \Gamma_a^k] + \dots) \delta^3(\epsilon_{ijk}\Gamma_{aj}E_k^a + \dots) \\ &= V_s^{-2} \delta(4[E_k^a \Gamma_a^k - 3E_k^a \Gamma_a^k] + \dots) \delta^5\left(4\left[E_{(i}^a \Gamma_{|a|j)} - \frac{1}{3}\delta_{ij}E_k^a \Gamma_a^k\right] + \dots\right) \delta^3(\epsilon_{ijk}\Gamma_{aj}E_k^a + \dots) = V_s^{-2} \delta^9(E_i^a \Gamma_{aj} + \dots) \\ &= \frac{V_s^{-2}}{|\det E_i^a|^3} \delta^9\left(\Gamma_a^i - \frac{1}{2}\epsilon_{ijk}E_j^b[\partial_{[b}E_{a]}^k + E_{a[l}E_{k]}^c\partial_b E_{cl}]\right) = V_s^{-8} \delta^9\left(\Gamma_a^i - \frac{1}{2}\epsilon_{ijk}E_j^b[\partial_{[b}E_{a]}^k + E_{a[l}E_{k]}^c\partial_b E_{cl}]\right)\end{aligned}\quad (\text{A.12})$$

where in the fourth step we split the six  $\delta$ -functions for a symmetric matrix into a  $\delta$ -function of its trace product five  $\delta$ -functions of its traceless part. With this result, we can integral over  $\Gamma_a^i$  and obtain the desired result:

$$Z_T = \int [DA_a^i][DE_i^a] \prod_{x \in M} \delta(G_i) \delta(H_a) \delta(H) \exp i \int dt \int d^3x E_i^a \dot{A}_a^i \quad (\text{A.13})$$

where the constraint  $G_i$ ,  $H_a$  and  $H$  take the form as we used in canonical LQG ( $s = \eta_{00}$ )

$$\begin{aligned}G_i &= \partial_a E_i^a + \epsilon_{ijk}A_a^j E_k^a & H_a &= F_{ab}^i E_i^b - A_a^i G_i \\ H &= \frac{1}{2} \frac{E_i^a E_j^b}{\sqrt{|\det q|}} [\epsilon^{ij}_k F_{ab}^k - 2(\gamma^2 - s) K_{[a}^i K_{b]}^j]\end{aligned}\quad (\text{A.14})$$

Therefore we obtain equivalence between the time-gauge fixed Holst action path-integral Eq.(A.6) and the canonical path-integral of the Ashtekar-Barbero-Immirzi Hamiltonian. Furthermore, the Ashtekar-Barbero-Immirzi Hamiltonian formalism is symplectically equivalent to the triad ADM Hamiltonian formalism, i.e. there is a canonical transformation  $A_a^i = \Gamma_a^i + \gamma K_a^i$  relates the conjugate pair  $(A_a^i, E_i^a)$  to  $(E_i^a, K_a^i)$ , where the  $su(2)$ -valued 1-form  $K_a^i$  relates the extrinsic curvature via  $-sK_{ab} = K_{(a}^i e_{b)}^i$ . Then it is trivial that the time-gauge fixed Holst action path-integral Eq.(A.6) also equivalent to the canonical path-integral of the triad ADM Hamiltonian, i.e.

$$Z_T = \int [DK_a^i][DE_i^a] \prod_{x \in M} \delta(G_i) \delta(H_a) \delta(H) \exp i \int dt \int d^3x K_a^i \dot{E}_i^a \quad (\text{A.15})$$

where the constraint  $G_i$ ,  $H_a$  and  $H$  takes the form:

$$\begin{aligned} G_i &= \epsilon_{ijk} K_a^j E_k^a & H_a &= 2sD_b [K_a^j E_j^b - \delta_a^b K_c^j E_j^c] \\ H &= -\frac{s}{\sqrt{|\det q|}} (K_a^i K_b^j - K_b^i K_a^j) E_i^a E_j^b - \sqrt{|\det q|} R \end{aligned} \quad (\text{A.16})$$

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