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► **To cite this version:**

Yasmina Bestaoui, Elie Kahale. Analysis of Time Optimal 3D Paths for an Autonomous Aircraft with a Piecewise Constant Acceleration. 49th AIAA Aerospace Sciences Meeting, American Institute of Aeronautics and Astronautics, Jan 2011, Orlando, FL, United States. 10.2514/6.2011-77 . hal-00654117

HAL Id: hal-00654117

<https://hal.science/hal-00654117v1>

Submitted on 20 Jan 2025

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Analysis of Time Optimal 3D Paths for an Autonomous Aircraft with a Piecewise Constant Acceleration

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Trajectory generation is a fundamental part of planning for an autonomous aerial vehicle. For the purpose of flight path generation, it is usually sufficient to treat only the translational motion. One component of the weather that greatly affects an aircraft trajectory is the wind. Study of the accessibility of this nonlinear affine system with drift makes use of the Lie algebra rank condition. The second part of this paper presents 3D time optimal translational trajectories characterization for an aircraft in steady wind. If unaccounted for, winds can substantially degrade the performance of an autonomous aircraft guidance system. We consider finding a time optimal trajectory for an airplane from some starting point and orientation to some final point and orientation, assuming that the system has independent bounded control over the acceleration as well as the turning rates for the flight path and heading angles. Through the use of the Pontryagin maximum principle, we characterize the time optimal trajectories for the system.

I. Introduction

FROM the earliest days of aeronautical experimentations, the natural wind proved itself to be a major parameter to successful flights. The wind mostly affects a trajectory through its speed. In general, the wind speed can be modeled as a sum of two components: a nominal deterministic component (available through meteorological forecasts or measured with a Doppler radar) and a stochastic component, representing deviations from the nominal one. The closed loop controller takes care of the stochastic part considered as perturbations, while the deterministic component is introduced into the motion planner. The path of the airplane with respect to the moving air frame will be referred to as the air path and with respect to the ground is referred as the ground path. In general, the optimality of a trajectory can be defined according to several objectives, like minimizing the transfer time or the energy. Traditionally, trajectories are optimized by the application of numerical optimal control methods that are based on the calculus of variations. Dubins¹ considered a particle moving at a constant velocity in the plane with a constraint of trajectory curvature. He proved the existence of shortest paths for his problem and showed that the optimal trajectories are a combination of arc of circles and segments of lines. Boukraa et al² presented a 3D trim trajectories planner algorithm for an autonomous plane. The proposed algorithm used a sequence of five elementary trim trajectories to generate a 3D global trajectory in space. A family of trim trajectories in level flight is used in all these references to construct paths. In the papers cited above, the atmosphere was considered to be an isotropic and homogeneous medium, i.e. when there is no wind and the air density is constant with altitude. However, wind cannot be ignored. McGee and Hedrick³ describe a method for finding the minimum time path from an initial position and orientation to a final position and orientation in the 2D plane for an airplane with a bounded turning rate in the presence of a known constant wind with a magnitude less than the airplane velocity. The problem statement is equivalent to finding the minimum time path from an initial configuration to a final one, over a moving virtual target, where the velocity of the virtual target is equal and opposite to the velocity of the wind. Nelson et al⁴ have introduced a method for a mini aerial vehicle path following based on the concept of vector field in the presence of constant wind disturbances. Rysdyk⁵ presents a path formulation for manoeuvring of a fixed wing aircraft in wind. Wind refers to an un-accelerated horizontally moving air mass. The inertial path of a fixed wing aircraft circling in wind can be formulated as a trochoid curve. In these papers, only 2D horizontal motion was considered. Seube et al⁶ formulated the take-off problem in a 2D vertical plane in the presence of wind shear as a

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differential game against nature. The first player is the relative angle of attack of the aircraft (considered as the control variable) and the second player is the disturbance caused by a wind shear.

This paper consists of 6 sections. Section 2 formulates the time optimal problem. Efforts are put in this paragraph on a variable velocity wind. Section 3 presents the analysis of the Lagrange multipliers while Section 4 proposes analysis of a set of solutions. Section 5 gives some information about singular control. Finally, some conclusions and perspectives are the subject of Section 6.

II. Aircraft Translational Dynamics

The translational equations of an aerospace vehicle through the atmosphere are directly derived from Newton's law. If the vehicle flies in the atmosphere with speeds less than Mach 5 (below hypersonic velocity), the Earth can be presumed an inertial reference frame. The aircraft equations of motion are expressed in a velocity coordinate frame

attached to the aircraft, considering the velocity of the wind $W = (W_x \ W_y \ W_z)^T$ (components of the wind velocity in the inertial frame). The kinematic equations of the aircraft are given by:

$$\dot{x} = V \cos \chi \cos \gamma + W_x \quad \dot{y} = V \sin \chi \cos \gamma + W_y \quad \dot{z} = V \sin \gamma + W_z \quad (1)$$

Where x (downrange), y (cross range) and z (altitude) are the vehicle's position, V the velocity magnitude, χ the heading angle. The aircraft flight path angle is the angle γ measured from the horizontal plane to the aircraft's velocity vector in inertial coordinates. The powered dynamic model used for flight over a flat Earth is the following

$$\dot{\gamma} = \frac{C_L(M, \alpha) A_{ref} \rho V \cos \sigma}{2m} - \frac{g \cos \gamma}{V} + \frac{T \sin \alpha \cos \sigma}{mV} + \frac{W_x \sin \gamma \cos \chi}{V} - \frac{W_y \sin \gamma \sin \chi}{V} - \frac{W_z \cos \gamma}{V}$$

$$\dot{\chi} = \frac{C_L(M, \alpha) A_{ref} \rho V \sin \sigma}{2m} + \frac{T \sin \alpha \sin \sigma}{mV \cos \gamma} + \frac{W_x \sin \chi}{V \cos \gamma} - \frac{W_y \cos \chi}{V \cos \gamma} \quad (2)$$

$$\dot{V} = -\frac{C_D(M, \alpha) A_{ref} \rho V^2}{2m} - g \sin \gamma + \frac{T \cos \alpha}{m} - W_x \cos \gamma \cos \chi - W_y \cos \gamma \sin \chi - W_z \sin \gamma$$

Where σ is the bank angle, ρ is the free stream mass density, m is the aircraft mass, A_{ref} is a characteristic area for the body, C_L, C_D are respectively the lift and drag coefficient functions that depend upon the Mach number M and the aerodynamic angle of attack α is measured from the aircraft x-y plane to the relative wind velocity vector.

The dynamic pressure is $\bar{q} = 0.5 \rho V^2$ where the air density ρ at altitude h is approximated using an exponential model $\rho = \rho_0 e^{-\beta h}$ where ρ_0 is the air density at sea level and β is the atmospheric density scale. Generally the lift coefficient is a linear function of the angle of attack and the drag coefficient is a quadratic function of the lift coefficient. These equations have an important place in aerospace vehicle study because they can be assembled from trimmed aerodynamic data and simple autopilot designs. Nevertheless, they give a realistic picture of the translational and rotational dynamics unless large angles and cross coupling effects dominate the simulations. Trajectory studies, navigation and guidance evaluations can be successfully executed with simulations of these equations. The limitations on thrust and velocity will be used in the path planning via a transformation on limitations on $\dot{\gamma}, \dot{\chi}, \dot{V}$. The following physical bounds must be taken into account in the following analysis:

$$|\dot{\gamma}| \leq u_{1\max} \quad |\dot{\chi}| \leq u_{2\max} \quad |\dot{V}| \leq u_{3\max} \quad V_{\min} \leq V \leq V_{\max} \quad (3)$$

The shape of a space curve can be completely captured by its curvature and torsion. Using the Frenet-Serret formulation, curvature κ can be deduced ('represents the derivation versus s):

$$\kappa(s) = \frac{\|C \times C''\|}{\|C'\|^3} = \sqrt{\dot{\gamma}^2 + \dot{\chi}^2 \cos^2 \gamma(s)} \quad (4)$$

as well as torsion τ

$$\tau(s) = \frac{(C \times C'') \cdot C'''}{\|C \times C'\|^2} = \frac{\dot{\chi} \dot{\gamma} \cos \gamma + 2 \dot{\chi} \dot{\gamma}^2 \sin \gamma - \dot{\chi} \dot{\chi} \cos \gamma}{\dot{\gamma}^2 + \dot{\chi}^2 \cos^2 \gamma} + \frac{-\dot{\chi} \dot{\chi}^2 \cos \chi \cos \gamma \sin \chi \sin^2 \gamma + \dot{\chi}^3 \cos^2 \gamma \sin \gamma}{\dot{\gamma}^2 + \dot{\chi}^2 \cos^2 \gamma} \quad (5)$$

If a non vanishing curvature and a torsion are given as smooth functions of s , theoretically both equations can be integrated to find the numerical values of the corresponding space curve (up to a rigid motion).

III. Controllability

Controllability is an important notion for systems. Jurdjevic⁸ introduced the theory of Lie groups and their associated Lie algebras into the context of nonlinear control to express notions such as controllability, observability and realization theory. Some of the early works on nonlinear controllability was based on linearization of nonlinear systems. It was observed that if the linearization of a nonlinear system at an equilibrium point is controllable, the system itself is locally controllable. Later, a differential geometric approach to the problem was adopted in which a control system was viewed as a family of vector fields. It was observed that a lot of the interesting control theoretic information was contained in the Lie brackets of these vector fields.

Driftless nonholonomic control systems have been extensively studied in recent years [9, 10]. Chow's theorem leads to the characterization of controllability for systems without drift. It provides a Lie algebra rank test, for controllability of nonlinear systems without drift, similar in spirit to that of Kalman rank condition for linear systems. In the setting of controlled mechanical systems, the Lagrangian dynamics, being second order, necessarily include drift. In this setting, Chow's theorem cannot be used to conclude controllability. Studying controllability of general systems with drift is usually a hard problem. The discussion of nonholonomic system with drift in the literature has been concentrated on the so-called dynamic extension of drift-free systems with the addition of integrators. Sufficient conditions for the controllability of a conservative dynamical nonlinear affine control system on a compact Riemannian manifold are presented, if the drift vector field is assumed to be weakly positively Poisson stable. Let's begin with a brief review of some concepts in controllability of nonlinear systems applied to the affine nonlinear control systems of the form:

$$\dot{X} = f(X) + \sum_{i=1}^m g_i u_i \quad (6)$$

Several important results have been derived based on the structure of the Lie algebra generated by the control vector fields. Assume $X \in M \subset R^n$ where M is a smooth manifold. Let $x(t)$ denote the solution of (6) for $t \geq 0$, a particular input function u and initial condition $x(0) = x_0$.

Generally, we can say that the nonlinear system (6) is called controllable if for two points x_1 & x_2 in M there exists a finite time T and an admissible control function $u : [0, T] \rightarrow U$ such that $x(T) = x_2$.

Let ψ be a neighborhood of the point $X \in M$ and $R^\psi(x_0, t)$ indicate the set of reachable points at the time t by trajectories remaining inside ψ and satisfying the equations (6). So we can define the reachable set from x_0 at time T as: $R^M(x_0, T) = \bigcup_{0 \leq t \leq T} R^M(x_0, t)$.

The **accessibility algebra** A of the system (6) is the smallest Lie algebra of vector fields on M that contains the vector fields f and g_1, g_2, \dots, g_m .

We define the **accessibility distribution** Δ_A of (6) to be the distribution generated by the vector fields in A ; i.e. $\Delta_A(x)$ is the span of vector fields v in A at x . So, we can determine Δ_A as:

$$\Delta_A = \text{span}\{v \mid v \in A\} \quad (7)$$

In other words, Δ_A is the involutive closure of $\Delta = \text{span}\{f, g_1, \dots, g_m\}$.

The computation of Δ_A may be organized as an iterative procedure: $\Delta_A = \text{span}\{v \mid v \in \Delta_i, \forall i \geq 1\}$, With:

$$\Delta_1 = \Delta = \text{span}\{f, g_1, \dots, g_m\}$$

$$\Delta_i = \Delta_{i-1} + \text{span}\{[g, v] \mid g \in \Delta_1, v \in \Delta_{i-1}\}, \quad i \geq 2$$

This procedure stops after K steps, where K is the smallest integer such that $\Delta_{K+1} = \Delta_K > \Delta_A$. This number is called the non-holonomy degree of the system and is related to the ‘level’ of Lie brackets that must be included in Δ_A .

Let us now define the different accessibility notions:

- We say that the system (6) is **accessible** from $x_0 \in M$ if for every $T > 0$, $R^M(x_0, T)$ contains a nonempty open set.
- We can say that the system (6) is **locally accessible** from $x \in M$ if for every $T > 0$, $R^\psi(x_0, T)$ contains a nonempty open set.

To prove the local accessibility property we can use the following theorem:

Consider the system (6) and assume that the vector fields are C^∞ . If $\dim \Delta_A(x_0) = n$ (i.e. the accessibility algebra spans the tangent space to M at x_0), then for any $T > 0$, the set $R^\psi(x_0, T)$ has a nonempty interior; (i.e., the system has the accessibility property from x_0 for all neighborhoods ψ of x_0 and all $T > 0$).

Notes:

- (1) We call the previous condition ($\dim \Delta_A(x_0) = n$) the **Lie Algebra Rank Condition** (LARC).
- (2) If the system (6) is driftless (i.e. $f(X) = 0$), the accessibility property characterizes the controllability.
- (3) As we motioned above, the accessibility property implies the controllability of the linear systems with the Kalman's rank condition. Assuming that we have the following system:

$$\dot{X} = AX + \sum_{i=1}^m b_i u_i = AX + BU$$

Where: $X \in \mathbb{R}^n$, and $A \in \mathbb{R}^n \times \mathbb{R}^n$ and $B \in \mathbb{R}^n \times \mathbb{R}^m$ are constant matrices, b_i being the columns of B .

The Lie bracket of the drift vector field AX with b_i is readily checked to be the constant vector field $-Ab_i$. Bracketing the latter field with AX and so on tells us that the accessibility algebra is spanned by $A, b_i, Ab_i, \dots, A^{n-1}b_i$; $i = 1, \dots, m$. Thus the accessibility rank condition at the origin is equivalent to Kalman's rank condition: $rank [B, AB, \dots, A^{n-1}B]$.

In the preceding discussion, we note that the term $\text{span}\{AX\}$ is not present in the controllability rank condition. This motivates a slightly stronger definition of accessibility in the nonlinear setting, where g_i play a more prominent role in the rank condition.

- The system (6) is said to be **strongly accessible** from $x_0 \in M$ if the set $R^M(x_0, T)$ contains a nonempty open set for any $T > 0$ sufficiently small.
- The system (6) is said to be **locally strongly accessible** from $x_0 \in M$ if for any neighborhood ψ of x_0 the set $R^\psi(x_0, T)$ contains a nonempty open set for any $T > 0$ sufficiently small.

Let A to be the accessibility algebra of (6). We define the **strong accessibility algebra** A_0 to be the smallest sub-algebra containing g_1, g_2, \dots, g_m and satisfies $[f, v_0] \in A_0$ for all $v_0 \in A_0$.

We define the **strong accessibility distribution** Δ_{A_0} of (6) to be the involutive distribution generated by the vector fields in A_0 .

If $\dim \Delta_{A_0}(x_0) = n$, then the system (6) is locally strongly accessible from x_0 .

Let us now try to apply the previous notions to our system.

We can write the system (1)-(2) as an affine nonlinear control system:

$$\dot{X} = f(X) + g_1 u_1 + g_2 u_2 + g_3 u_3 = f(X) + \sum_{i=1}^3 g_i u_i \quad (8)$$

With:

$$X = \begin{bmatrix} x \\ y \\ z \\ \gamma \\ \chi \\ V \end{bmatrix}, u = \begin{bmatrix} \dot{\gamma} \\ \dot{\chi} \\ \dot{V} \end{bmatrix}, f = \begin{bmatrix} V \cos \chi \cos \gamma + W_x \\ V \sin \chi \cos \gamma + W_y \\ V \sin \gamma + W_z \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We are going now to find Δ_A of this system:

$$\Delta_1 = \text{span}\{f, g_1, g_2, g_3\} \quad (9)$$

At the second level, the following relationship can be written:

$$\Delta_2 = \Delta_1 + \text{span}\{[f, g_1], [f, g_2], [f, g_3], [g_1, g_2], [g_1, g_3], [g_2, g_3]\} \quad (10)$$

Straightforward calculations allow us to write: $[g_1, g_2] = [g_1, g_3] = [g_2, g_3] = 0_{6 \times 1}$

$$[f, g_1] = \begin{bmatrix} V \cos \chi \sin \gamma \\ V \sin \chi \sin \gamma \\ -V \cos \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}, [f, g_2] = \begin{bmatrix} V \sin \chi \cos \gamma \\ -V \cos \chi \cos \gamma \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [f, g_3] = \begin{bmatrix} -\cos \chi \cos \gamma \\ -\sin \chi \cos \gamma \\ -\sin \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By continuing the calculations we obtain:

$$\Delta_A = \Delta_3 = \text{span} \left\{ \begin{array}{l} f, g_1, g_2, g_3, \\ g_4 = [f, g_1], g_5 = [f, g_2], g_6 = [f, g_3] \\ g_7 = [g_2, [f, g_2]] \end{array} \right\} \quad (11)$$

We must check if $\dim(\Delta_A) = n = 6$

$$\Leftrightarrow \text{rank} \begin{bmatrix} f & g_1 & g_2 & g_3 & [f, g_1] & [f, g_2] \end{bmatrix} = 6 \text{ ???}$$

$$\Leftrightarrow \det \begin{bmatrix} V \cos \chi \cos \gamma + w_x & 0 & 0 & 0 & V \cos \chi \sin \gamma & V \sin \chi \cos \gamma \\ V \sin \chi \cos \gamma + w_y & 0 & 0 & 0 & V \sin \chi \sin \gamma & -V \cos \chi \cos \gamma \\ V \sin \gamma + w_z & 0 & 0 & 0 & -V \cos \gamma & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \neq 0$$

$$\Leftrightarrow -V^3 \cos \gamma - W_z V^2 \sin \gamma \cos \gamma - W_y V^2 \sin \chi \cos^2 \gamma - W_x V^2 \cos \chi \cos^2 \gamma \neq 0$$

So we must study the previous relation and verify that the determinate is not equal to zero. With this condition, system (8) verifies the Lie Algebra rank condition and is locally accessible. Therefore the non-holonomy degree of the system is: $K = 3$.

Let us now verifies the strong accessibility property of our system (8).

Firstly, we have to find the strong accessibility distribution Δ_{A_0} :

$$\Delta_{A_0} = \text{span} \left\{ \begin{array}{l} g_1, g_2, g_3, \\ g_4 = [f, g_1], g_5 = [f, g_2], g_6 = [f, g_3] \\ g_7 = [g_2, [f, g_2]] \end{array} \right\} \quad (12)$$

The following condition must be checked : $\dim(\Delta_{A_0}) = n = 6$

$$\Leftrightarrow \text{rank} \begin{bmatrix} g_1 & g_2 & g_3 & [f, g_1] & [f, g_2] & [f, g_3] \end{bmatrix} = 6 \text{ ????$$

$$\Leftrightarrow \det \begin{bmatrix} 0 & 0 & 0 & V \cos \chi \sin \gamma & V \sin \chi \cos \gamma & -\cos \chi \cos \gamma \\ 0 & 0 & 0 & V \sin \chi \sin \gamma & -V \cos \chi \cos \gamma & -\sin \chi \cos \gamma \\ 0 & 0 & 0 & -V \cos \gamma & 0 & -\sin \gamma \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \neq 0$$

$$\Leftrightarrow -V^2 \cos \gamma \neq 0$$

So, either: $V^2 \neq 0$ and it's true because $V_{\min} < V < V_{\max}$; where $V_{\min}, V_{\max} \in \mathbb{R}^+$.

Or: $-\cos \gamma \neq 0 \Rightarrow \gamma \neq \frac{\pi}{2}$.

With this condition, system (8) verifies the Lie Algebra rank condition and is locally strongly accessible.

Poisson stability and controllability:

As we noted above, it is well known that for a driftless affine nonlinear control system (*i.e.* $f = 0$) the accessibility implies the controllability. For the general case (*i.e.* $f \neq 0$), the situation is more complicated and the accessibility property is not sufficient to guarantee the controllability. In fact, the affine nonlinear control system (6) is controllable if the drift f is a weakly Poisson stable (WPPS) vector field, and if the accessibility Lie Algebra Rank Condition LARC is satisfied.

Let X be a smooth vector field on a smooth connected M and. Let $\phi^X(\cdot)$ denote its flow:

$$\phi^X : \mathbb{R} \times M \rightarrow M ; \{t, p\} \rightarrow \phi_t^X(p)$$

Assume that X is complete or $\phi_t^X(p)$ is defined for all $t \geq 0$. A point $p \in M$ is called **positively Poisson stable** for X if for all $T > 0$ and for any neighborhood ψ_p of p , there exists a time $t > T$, such that $\phi_t^X(p) \in \psi_p$.

The vector field X is called **positively Poisson stable (PPS)** if the set of Poisson stable points for X is dense in M .

A point $p \in M$ is called a **non-wandering point** of X if for all $T > 0$, and for any neighborhood ψ_p of p , there exists a time $t > T$ such that:

$$\phi_t^X(\psi_p) \cap \psi_p \neq \emptyset,$$

Where $\phi_t^X(\psi_p) = \{\phi_t^X(q) : q \in \psi_p\}$. A positively Poisson stable point is necessarily a non-wandering point.

Let Γ_X denote the non-wandering set, which is defined to be the set of all the non-wandering points of X .

One should observe here that though positive Poisson stability of a vector field is a sufficient condition that the non-wandering set (Γ_X) is the entire manifold M , there could exist weaker condition under which the non-wandering set (Γ_X) is M . This give rise to the definition: a vector field X is called **weakly positively Poisson stable (WPPS)** if its non-wandering set is M . So, if the drift f is positively Poisson stable (or weakly positively Poisson stable) vector field, and as the accessibility Lie algebra rank condition (LARC) is satisfied, the system (6) is controllable.

IV. Optimal Path Planning

In this section, an algorithm for open-loop path planning is derived for the system presented in the previous section. The idea is to use the structure and to apply simple bang-bang controls in the planning [11-13]. The amount of control available is a concern in the planning for this system due to the drift term. The class of bang-bang controls is often a sufficiently rich class of controls for analysis of nonlinear systems. This simple class of controls makes it possible to integrate the equations forward in a simple manner.

1. Time Optimal Problem formulation

The subject of this paragraph is to formulate the trajectory generation problem in minimum time as this system has bounds on the magnitudes of the inputs. The velocity is assumed to be linearly variable. As the set of allowable inputs is convex, the time optimal paths result from saturating the inputs at all times (or zero for singular control). For a linear time- invariant controllable system with bounded control inputs, the time-optimal control solution to a typical two point boundary value problem is a bang-bang function with a finite number of switches.

Time optimal trajectory generation can be formulated as follows

$$\min \int_0^T dt \quad (13)$$

Subject to System

$$\begin{cases} \dot{x} = V \cos \chi \cos \gamma + W_x \\ \dot{y} = V \sin \chi \cos \gamma + W_y \\ \dot{z} = V \sin \gamma + W_z \end{cases} \quad \begin{cases} \dot{\gamma} = u_1 \\ \dot{\chi} = u_2 \\ \dot{V} = u_3 \end{cases} \quad (14)$$

Initial conditions

$$x(0) = x_0, y(0) = y_0, z(0) = z_0, \chi(0) = \chi_0, \gamma(0) = \gamma_0 \text{ and } V(0) = V_0 \quad (15)$$

Final condition

$$x(T) = x_f, y(T) = y_f, z(T) = z_f, \chi(T) = \chi_f, \gamma(T) = \gamma_f, V(T) = V_f \quad (16)$$

Limitations on the control input and state

$$|u_1| \leq u_{1\max}, |u_2| \leq u_{2\max}, |u_3| \leq u_{3\max}, V_{\min} \leq V \leq V_{\max} \quad (17)$$

This formulation is a generalization of Zermelo's navigation problem, where the problem consists of finding the quickest nautical path for a ship at sea in the presence of currents.

For points that are reachable, the resolution is based on the Pontryagin Minimum Principle which constitutes a generalization of Lagrange problem of the calculus of variations. It is a local reasoning based on the comparison of trajectories corresponding to infinitesimally close control laws. It provides necessary conditions for paths to be optimal. Of course, the kinematic model used below implies a perfect response to the turn commands. A major reason for using the kinematic model is the fact that only necessary conditions for optimality exist for the second order model (given by Pontryagin minimum principle).

The Hamiltonian is classically defined as follows

$$H = 1 + \lambda_1 (V \cos \chi \cos \gamma + W_x) + \lambda_2 (V \sin \chi \cos \gamma + W_y) + \lambda_3 (V \sin \gamma + W_z) + \lambda_4 u_1 + \lambda_5 u_2 + \lambda_6 u_3 \quad (18)$$

where λ represents the Lagrange multiplier. The optimal control input must satisfy the following set of necessary conditions:

$$\dot{X} = \left(\frac{\partial H}{\partial \lambda} \right)^T, \quad \dot{\lambda} = - \left(\frac{\partial H}{\partial X} \right) \quad \& \quad X(0), X(T) \text{ specified}$$

With the transversality condition $H(T) = 0$

A first interesting result is the determination of a sufficient family of trajectories, i.e. a family of trajectories containing an optimal solution for linking any two configurations.

The adjoint equations are the first part of the necessary conditions: $\dot{\lambda} = -\frac{\partial H}{\partial X}$ where λ is the Lagrange multiplier

vector, or more explicitly

$$\begin{aligned} \dot{\lambda}_1 &= 0 \quad \dot{\lambda}_2 = 0 \quad \dot{\lambda}_3 = 0 \\ \dot{\lambda}_4 &= \frac{\lambda_1 V}{2} [\sin(\chi + \gamma) - \sin(\chi - \gamma)] - \lambda_3 V \cos \gamma - \frac{\lambda_2 V}{2} [\cos(\chi + \gamma) - \cos(\chi - \gamma)] \\ \dot{\lambda}_5 &= \lambda_1 V \sin \chi \cos \gamma - \lambda_2 V \cos \chi \cos \gamma = \lambda_1 (y - W_y) - \lambda_2 (x - W_x) \\ \dot{\lambda}_6 &= -\frac{\lambda_1}{2} [\cos(\chi + \gamma) + \cos(\chi - \gamma)] - \lambda_3 \sin \gamma - \frac{\lambda_2}{2} [\sin(\chi + \gamma) + \sin(\chi - \gamma)] \end{aligned} \quad (19)$$

Defining the Hamiltonian and multiplier dynamics in this way, the minimum principle of Pontryagin states that the control variable must be chosen to minimize the Hamiltonian at every instant.

$$H(X, \lambda, u^*) \leq H(X, \lambda, u) \quad (20)$$

On the optimal trajectory (X^*, u^*) , the optimal control u^* must satisfy:

$$\lambda_4 u_1^* + \lambda_5 u_2^* + \lambda_6 u_3^* \leq \lambda_4 u_1 + \lambda_5 u_2 + \lambda_6 u_3 \quad (21)$$

Leading to the following solution

$$u_1^* = -\text{sign}(\lambda_4) u_{1\max} = \delta_1 u_{1\max} \quad u_2^* = -\text{sign}(\lambda_5) u_{2\max} = \delta_2 u_{2\max} \quad u_3^* = -\text{sign}(\lambda_6) u_{3\max} = \delta_3 u_{3\max} \quad (22)$$

With: $\delta_i \in \{+1, 0, -1\}$; $i = 1, 2, 3$

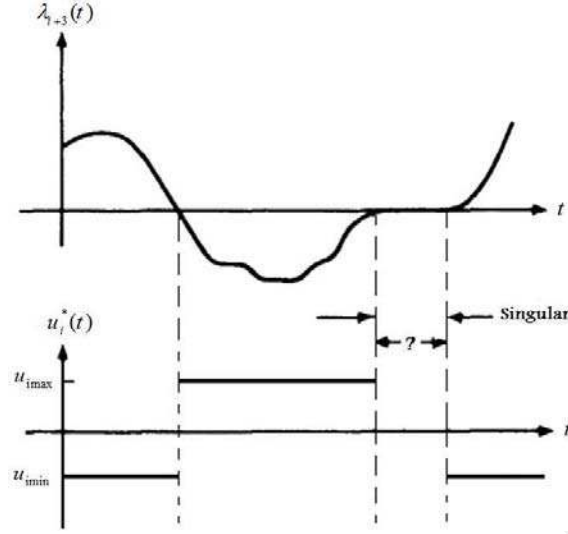
In this paper, we will follow the approach of Sussmann, stating that minimum length paths are LRL and RLR (L for Left and R for Right). These solutions can be written in the following form:

$$u_1^* \in \left\{ \begin{array}{l} \delta_1 u_{1\max} \\ -\delta_1 u_{1\max} \\ \delta_1 u_{1\max} \end{array} \right\}, u_2^* \in \left\{ \begin{array}{l} \delta_2 u_{2\max} \\ -\delta_2 u_{2\max} \\ \delta_2 u_{2\max} \end{array} \right\}, u_3^* \in \left\{ \begin{array}{l} \delta_3 u_{3\max} \\ -\delta_3 u_{3\max} \end{array} \right\}; \text{ With a determined number of switching times (from } \lambda_6) \quad (23)$$

All sub paths are allowed to have zero length. The following section presents the analysis of the Lagrange multipliers as the number of switches of the optimal controls depend on them.

2. Lagrange Multipliers Analysis

In the previous section we have seen that the minimum condition implies that $u_i^*(t) = -\text{sign}(\lambda_{i+3}(t)) u_{i\max}$; $i = 1, 2, 3$. In other words, the commands $u_i^*(t)$; $i = 1, 2, 3$ depend on the Lagrange Multipliers $\lambda_{i+3}(t)$; $i = 1, 2, 3$, which determine the number of switching. So we called these multipliers the **switching functions**. The following figure shows the relation between the switching function and the commands.



We note that when $\lambda_{i+3}(t)$ passes through zero, a switching time of the control $u_i^*(t)$ is indicated. If $\lambda_{i+3}(t)$ is zero for some finite time interval, then the minimal condition provides no information about how to select $u_i^*(t)$, and we call the control in this section *singular control*.

Let us now study the case when we don't have a singular control (i.e. $\delta_i \neq 0$; $i = 1, 2, 3$). By the integration of the adjoint equations (19) we obtain the following results:

$$\lambda_1 = \text{const}, \lambda_2 = \text{const}, \lambda_3 = \text{const}$$

$$\lambda_4(t) = \frac{\lambda_1}{2} \left(\begin{aligned} & -\frac{V_0 \cos(\chi + \gamma)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\delta_3 \mu_{3\max}}{(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} [-t(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}) \cos(\chi + \gamma) + \sin(\chi + \gamma)] + \\ & + \frac{V_0 \cos(\chi - \gamma)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} - \frac{\delta_3 \mu_{3\max}}{(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} [-t(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}) \cos(\chi - \gamma) + \sin(\chi - \gamma)] \end{aligned} \right) -$$

$$- \frac{\lambda_2}{2} \left(\begin{aligned} & \frac{V_0 \sin(\chi + \gamma)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\delta_3 \mu_{3\max}}{(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} [t(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}) \sin(\chi + \gamma) + \cos(\chi + \gamma)] - \\ & \frac{V_0 \sin(\chi - \gamma)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} - \frac{\delta_3 \mu_{3\max}}{(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} [t(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}) \sin(\chi - \gamma) + \cos(\chi - \gamma)] \end{aligned} \right) -$$

$$- \lambda_3 \left(\frac{V_0 \sin(\gamma)}{\delta_1 \mu_{1\max}} + \frac{\delta_3 \mu_{3\max}}{u_{1\max}^2} [t \delta_1 \mu_{1\max} \sin(\gamma) + \cos(\gamma)] \right) + \lambda_{40} \quad (\text{a.24})$$

$$\lambda_5(t) = \lambda_1 y - \lambda_2 x - \lambda_1 W_y t + \lambda_2 W_x t + \lambda_{50} \quad (\text{b.24})$$

$$\lambda_6(t) = -\frac{\lambda_1}{2} \left(\frac{\sin(\chi - \gamma)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\sin(\chi + \gamma)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) + \frac{\lambda_2}{2} \left(\frac{\cos(\chi - \gamma)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\cos(\chi + \gamma)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) + \frac{\lambda_3}{\delta_1 \mu_{1\max}} \cos \gamma + \lambda_{60} \quad (\text{c.24})$$

Because the values of $\lambda_1, \lambda_2, \lambda_3, \lambda_{50}$ are constant, so the equation $\lambda_5(t) = 0$ defines the line on which some switching and straight line travel must occur.

$$y = \frac{\lambda_2}{\lambda_1} x + W_y t - \frac{\lambda_2}{\lambda_1} W_x t - \frac{\lambda_{50}}{\lambda_1} \quad (\text{25})$$

The remaining problem is to find the optimal values of $\lambda_1, \lambda_2, \lambda_3, \lambda_{40}, \lambda_{50}$ and λ_{60} such that the two-point boundary value problem is solved (i.e. finding the values that insure that the correspond switching times allow to steer the system from the given initial point to desired final point). The solution of this problem will obtain by numerical methods.

3. One switching time case resolution

Let us take the following set of controls:

$$u_1^*(t) = \begin{cases} \delta_1 u_{1\max} & 0 \leq t \leq t_1 \\ -\delta_1 u_{1\max} & t_1 \leq t \leq T \end{cases} \quad u_2^*(t) = \begin{cases} \delta_2 u_{2\max} & 0 \leq t \leq t_1' \\ -\delta_2 u_{2\max} & t_1' \leq t \leq T \end{cases} \quad (26)$$

$$u_3^*(t) = \begin{cases} \delta_3 u_{3\max} & 0 \leq t \leq t_1'' \\ -\delta_3 u_{3\max} & t_1'' \leq t \leq T \end{cases}$$

By integration:

$$\gamma^*(t) = \begin{cases} \delta_1 u_{1\max} t + \gamma_0 & 0 \leq t \leq t_1 \\ -\delta_1 u_{1\max} t + \gamma_1 & t_1 \leq t \leq T \end{cases}, \quad \chi^*(t) = \begin{cases} \delta_2 u_{2\max} t + \chi_0 & 0 \leq t \leq t_1' \\ -\delta_2 u_{2\max} t + \chi_1 & t_1' \leq t \leq T \end{cases}, \quad V^*(t) = \begin{cases} \delta_3 u_{3\max} t + V_0 & 0 \leq t \leq t_1'' \\ -\delta_3 u_{3\max} t + V_1 & t_1'' \leq t \leq T \end{cases} \quad (27)$$

The integration constants $\gamma_0, \gamma_1, \chi_0, \chi_1, V_0, V_1$ will be determined using the initial and final conditions as well as the continuity condition.

$$\gamma_0 = \gamma_0, \quad \chi_0 = \chi_0, \quad \gamma_1 = 2\delta_1 u_{1\max} t_1, \quad \chi_1 = 2\delta_2 u_{2\max} t_1', \quad V_0 = V_0, \quad V_1 = \delta_3 u_{3\max} t_1'' + V_0$$

The final conditions give

$$\gamma_f = -\delta_1 u_{1\max} T + \gamma_1, \quad \chi_f = -\delta_2 u_{2\max} T + \chi_1, \quad V_f = -\delta_3 u_{3\max} T + V_1 \quad (28)$$

With this relation, one obtains the first equation relying the transition times with the final time.

$$t_1 = \frac{1}{2} \left(T + \frac{\gamma_f - \gamma_0}{\delta_1 u_{1\max}} \right), \quad t_1' = \frac{1}{2} \left(T + \frac{\chi_f - \chi_0}{\delta_2 u_{2\max}} \right), \quad t_1'' = \frac{1}{2} \left(T + \frac{V_f - V_0}{\delta_3 u_{3\max}} \right) \quad (29)$$

Let's present the different respective positions of t_1 versus t_1' and t_1'' giving way to 6 different intervals.

$$P1: 0 \leq t_1 \leq t_1' \leq t_1'' \leq T$$

$$P2: 0 \leq t_1 \leq t_1'' \leq t_1' \leq T$$

$$P3: 0 \leq t_1' \leq t_1 \leq t_1'' \leq T$$

$$P4: 0 \leq t_1' \leq t_1'' \leq t_1 \leq T$$

$$P5: 0 \leq t_1'' \leq t_1' \leq t_1 \leq T$$

$$P6: 0 \leq t_1'' \leq t_1 \leq t_1' \leq T$$

Calculations are similar for the six intervals. In the following detailed analysis, we are interested by the interval P1.

I. **The vertical motion:** This motion is obtained by the integration of $\dot{z} = V \sin \gamma + W_z$.

The first interval: For $0 \leq t \leq t_1$, we obtain:

$$z(t) = W_z t - \frac{V_0 \cos(\delta_1 u_{1\max} t + \gamma_0)}{\delta_1 u_{1\max}} + z_0 + \delta_3 u_{3\max} z_{11} \quad (30)$$

$$\text{With: } z_{11} = \frac{-t \cos(\delta_1 u_{1\max} t + \gamma_0)}{\delta_1 u_{1\max}} + \frac{\sin(\delta_1 u_{1\max} t + \gamma_0)}{u_{1\max}^2} \quad \text{and}$$

$$z_0 = z_0 + \frac{V_0 \cos \gamma_0}{\delta_1 u_{1\max}} - \frac{\delta_3 u_{3\max}}{u_{1\max}^2} \sin \gamma_0$$

The second interval: For $t_1 \leq t \leq t_1''$, we can write

$$z(t) = W_z t + z_1 + \frac{V_0 \cos(-\delta_1 u_{1\max} t + \gamma_1)}{\delta_1 u_{1\max}} + \delta_3 u_{3\max} z_{21} \quad (31)$$

With:

$$z_{21} = \frac{t \cos(-\delta_1 u_{1\max} t + \gamma_1)}{\delta_1 u_{1\max}} + \frac{\sin(-\delta_1 u_{1\max} t + \gamma_1)}{u_{1\max}^2} \quad \text{and} \quad z_1 = z_0' - \left(\frac{2V_0}{\delta_1 u_{1\max}} + \frac{2t_1 \delta_3 u_{3\max}}{\delta_1 u_{1\max}} \right) \cos(\delta_1 u_{1\max} t_1 + \gamma_0)$$

The third interval: For $t_1'' \leq t \leq T$, we have

$$z(t) = W_z t + z_2 + \frac{V_1 \cos(-\delta_1 u_{1\max} t + \gamma_1)}{\delta_1 u_{1\max}} - \delta_3 u_{3\max} z_{31} \quad (32)$$

$$\text{With: } z_{31} = z_{21} \quad \& \quad z_2 = z_1 + \frac{2\delta_3 u_{3\max}}{u_{1\max}^2} \sin(-\delta_1 u_{1\max} t_1'' + \gamma_1)$$

Thus the final vertical condition gives

$$z_f = W_z T + z_2 + \frac{V_1}{\delta_1 u_{1\max}} \cos(\gamma_f) - \delta_3 u_{3\max} z_{31}(T) \quad (33)$$

$$\text{Where: } z_{31}(T) = \frac{T \cos(\gamma_f)}{\delta_1 u_{1\max}} + \frac{\sin(\gamma_f)}{u_{1\max}^2}$$

II. The horizontal motion :

This motion is obtained by integrating $\dot{x} = V \cos \chi \cos \gamma + W_x$ & $\dot{y} = V \sin \chi \cos \gamma + W_y$ for each interval:

The first interval: For $0 \leq t \leq t_1$, we have:

$$x = x_0' + W_x t + \frac{\delta_3 u_{3\max}}{2} (x_{11} + x_{12}) + \frac{V_0}{2} \left[\frac{\sin((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\sin((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} \right] \quad (34)$$

$$y = y_0' + W_y t + \frac{\delta_3 u_{3\max}}{2} (y_{11} + y_{12}) - \frac{V_0}{2} \left[\frac{\cos((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\cos((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} \right]$$

With

$$x_{11} = \frac{t \sin((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\cos((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{(\delta_2 u_{2\max} - \delta_1 u_{1\max})^2}, \quad x_{12} = \frac{t \sin((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} + \frac{\cos((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{(\delta_2 u_{2\max} + \delta_1 u_{1\max})^2}$$

$$y_{11} = -\frac{t \cos((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\sin((\delta_2 u_{2\max} - \delta_1 u_{1\max})t + \chi_0 - \gamma_0)}{(\delta_2 u_{2\max} - \delta_1 u_{1\max})^2}, \quad y_{12} = -\frac{t \cos((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} + \frac{\sin((\delta_2 u_{2\max} + \delta_1 u_{1\max})t + \chi_0 + \gamma_0)}{(\delta_2 u_{2\max} + \delta_1 u_{1\max})^2}$$

Using the initial condition gives the integration constants:

$$x_0' = x_0 - \frac{V_0}{2} \left[\frac{\sin(\chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\sin(\chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} \right] - \frac{\delta_3 u_{3\max}}{2} \left[\frac{\cos(\chi_0 - \gamma_0)}{(\delta_2 u_{2\max} - \delta_1 u_{1\max})^2} + \frac{\cos(\chi_0 + \gamma_0)}{(\delta_2 u_{2\max} + \delta_1 u_{1\max})^2} \right]$$

$$y_0' = y_0 + \frac{V_0}{2} \left[\frac{\cos(\chi_0 - \gamma_0)}{\delta_2 u_{2\max} - \delta_1 u_{1\max}} + \frac{\cos(\chi_0 + \gamma_0)}{\delta_2 u_{2\max} + \delta_1 u_{1\max}} \right] - \frac{\delta_3 u_{3\max}}{2} \left[\frac{\sin(\chi_0 - \gamma_0)}{(\delta_2 u_{2\max} - \delta_1 u_{1\max})^2} + \frac{\sin(\chi_0 + \gamma_0)}{(\delta_2 u_{2\max} + \delta_1 u_{1\max})^2} \right]$$

The second interval: For $t_1 \leq t \leq t_1'$, we have:

$$\begin{aligned} x &= x_1 + W_x t + \frac{\delta_3 \mu_{3\max}}{2} (x_{21} + x_{22}) + \frac{V_0}{2} \left[\frac{\sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] \\ y &= y_1 + W_y t + \frac{\delta_3 \mu_{3\max}}{2} (y_{21} + y_{22}) - \frac{V_0}{2} \left[\frac{\cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] \end{aligned} \quad (35)$$

With:

$$\begin{aligned} x_{21} &= \frac{t \sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2}, \quad x_{22} = \frac{t \sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \\ y_{21} &= -\frac{t \cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_0 - \gamma_1)}{(\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2}, \quad y_{22} = -\frac{t \cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_0 + \gamma_1)}{(\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \end{aligned}$$

While the continuity condition gives the integration constants:

$$\begin{aligned} x_1 &= x_0 + \frac{\delta_3 \mu_{3\max}}{2} x_1' + \frac{V_0 \delta_1 \mu_{1\max}}{u_{2\max}^2 - u_{1\max}^2} \left[\sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right] \\ y_1 &= y_0 + \frac{\delta_3 \mu_{3\max}}{2} y_1' - \frac{V_0 \delta_1 \mu_{1\max}}{u_{2\max}^2 - u_{1\max}^2} \left[\cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right] \end{aligned}$$

Where:

$$\begin{aligned} x_1' &= \frac{2t_1 \delta_1 \mu_{1\max}}{u_{2\max}^2 - u_{1\max}^2} \left(\sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right) + \frac{4\delta_1 \delta_2 \mu_{1\max} \mu_{2\max}}{(u_{2\max}^2 - u_{1\max}^2)^2} \left(\cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right) \\ y_1' &= -\frac{2t_1 \delta_1 \mu_{1\max}}{u_{2\max}^2 - u_{1\max}^2} \left(\cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right) + \frac{4\delta_1 \delta_2 \mu_{1\max} \mu_{2\max}}{(u_{2\max}^2 - u_{1\max}^2)^2} \left(\sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1 + \chi_0 - \gamma_0) - \sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1 + \chi_0 + \gamma_0) \right) \end{aligned}$$

The third interval: For $t_1' \leq t \leq t_1''$, we have:

$$\begin{aligned} x &= x_2 + W_x t + \frac{\delta_3 \mu_{3\max}}{2} (x_{31} + x_{32}) + \frac{V_0}{2} \left[\frac{\sin((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] \\ y &= y_2 + W_y t + \frac{\delta_3 \mu_{3\max}}{2} (y_{31} + y_{32}) - \frac{V_0}{2} \left[\frac{\cos((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] \end{aligned} \quad (36)$$

Where:

$$\begin{aligned} x_{31} &= \frac{t \sin((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} \\ x_{32} &= \frac{t \sin((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\cos((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \\ y_{31} &= -\frac{t \cos((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} \\ y_{32} &= -\frac{t \cos((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\sin((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \end{aligned}$$

With the continuity condition:

$$\begin{aligned} x_2 &= x_1 + \frac{\delta_3 \mu_{3\max}}{2} x_2' - \frac{V_0 \delta_2 \mu_{2\max}}{u_{1\max}^2 - u_{2\max}^2} \left(\sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right) \\ y_2 &= y_1 + \frac{\delta_3 \mu_{3\max}}{2} y_2' + \frac{V_0 \delta_2 \mu_{2\max}}{u_{1\max}^2 - u_{2\max}^2} \left(\cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right) \end{aligned}$$

Where:

$$\begin{aligned} x_2' &= \frac{-2t_1' \delta_2 \mu_{2\max}}{u_{1\max}^2 - u_{2\max}^2} \left(\sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right) + \\ &+ \frac{4\delta_1 \delta_2 \mu_{1\max} \mu_{2\max}}{(u_{2\max}^2 - u_{1\max}^2)^2} \left(-\cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right) \end{aligned}$$

$$y_2' = \frac{2t_1 \delta_2 \mu_{2\max}}{u_{1\max}^2 - u_{2\max}^2} \left(\cos((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \cos((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right) + \frac{4\delta_1 \delta_2 \mu_{1\max} u_{2\max}}{(u_{2\max}^2 - u_{1\max}^2)^2} \left(-\sin((\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1' + \chi_0 - \gamma_1) + \sin((\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1' + \chi_0 + \gamma_1) \right)$$

The fourth interval: For $t_1'' \leq t \leq T$, we have:

$$x = x_3 + W_x t - \frac{\delta_3 \mu_{3\max}}{2} (x_{41} + x_{42}) + \frac{V_1}{2} \left[\frac{\sin((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] \quad (37)$$

$$y = y_3 + W_y t - \frac{\delta_3 \mu_{3\max}}{2} (y_{41} + y_{42}) - \frac{V_1}{2} \left[\frac{\cos((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t + \chi_1 - \gamma_1)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t + \chi_1 + \gamma_1)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right]$$

Where: $x_{41} = x_{31}$ & $x_{42} = x_{32}$ & $y_{41} = y_{31}$ & $y_{42} = y_{32}$

$$x_3 = x_2 + \delta_3 \mu_{3\max} \left[\frac{\cos((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1'' + \chi_1 - \gamma_1)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} + \frac{\cos((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1'' + \chi_1 + \gamma_1)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \right]$$

$$y_3 = y_2 + \delta_3 \mu_{3\max} \left[\frac{\sin((-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})t_1'' + \chi_1 - \gamma_1)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} + \frac{\sin((-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})t_1'' + \chi_1 + \gamma_1)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \right]$$

With the final condition, transcendental equations are obtained:

$$x_f = x_3 + W_x T + \frac{V_f}{2} \left[\frac{\sin(\chi_f - \gamma_f)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\sin(\chi_f + \gamma_f)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] - \frac{\delta_3 \mu_{3\max}}{2} \left[\frac{\cos(\chi_f - \gamma_f)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} + \frac{\cos(\chi_f + \gamma_f)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \right] \quad (38)$$

$$y_f = y_3 + W_y T - \frac{V_f}{2} \left[\frac{\cos(\chi_f - \gamma_f)}{-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} + \frac{\cos(\chi_f + \gamma_f)}{-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} \right] - \frac{\delta_3 \mu_{3\max}}{2} \left[\frac{\sin(\chi_f - \gamma_f)}{(-\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max})^2} + \frac{\sin(\chi_f + \gamma_f)}{(-\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max})^2} \right]$$

The transversality condition $H(T) = 0 \Rightarrow$

$$-1 = \lambda_1 (V_f \cos \chi_f \cos \gamma_f + W_x + \delta_2 \mu_{2\max} (y_f - W_y T)) + \lambda_2 (V_f \sin \chi_f \cos \gamma_f + W_y - \delta_2 \mu_{2\max} (x_f - W_x T)) + \lambda_3 (V_f \sin \gamma_f + W_z) + \delta_1 \mu_{1\max} \lambda_{4f} + \delta_2 \mu_{2\max} \lambda_{50} \pm \delta_3 \mu_{3\max} \lambda_{6f} \quad (39)$$

where

$$\lambda_{4f} = \frac{\lambda_1 V_f}{2} \left(\frac{\cos(\chi_f - \gamma_f)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} - \frac{\cos(\chi_f + \gamma_f)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) - \frac{\lambda_2 V_f}{\delta_1 \mu_{1\max}} \sin \gamma_f + \frac{\lambda_2 V_f}{2} \left(\frac{\sin(\chi_f - \gamma_f)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} - \frac{\sin(\chi_f + \gamma_f)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) + \lambda_{40}$$

$$\lambda_{6f} = -\frac{\lambda_1}{2} \left(\frac{\sin(\chi_f - \gamma_f)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\sin(\chi_f + \gamma_f)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) + \frac{\lambda_3}{\delta_1 \mu_{1\max}} \cos \gamma_f + \frac{\lambda_2}{2} \left(\frac{\cos(\chi_f - \gamma_f)}{\delta_2 \mu_{2\max} - \delta_1 \mu_{1\max}} + \frac{\cos(\chi_f + \gamma_f)}{\delta_2 \mu_{2\max} + \delta_1 \mu_{1\max}} \right) + \lambda_{60}$$

To improve the accuracy of the direct optimization solutions and to enlarge the convergence domain of the indirect methods, a hybrid approach is proposed to solve the optimal control problem. This cascaded computational scheme has become widely applied. The key idea is to extract the co-states and other control structure information from a nonlinear programming approach as a first step. The indirect shooting method is then used to refine the solutions. The three major steps to solve for the optimal maneuver solutions and to validate the results based on the first order optimality conditions.

1. The kinematic and dynamic differentiation equations are discretized using the trapezoidal method. Commercially available software is used to get the preliminary and approximate control structures, switching times and initial co-states.
2. Using the results from step 1 as the initial guess, this software is used as a shooting method to solve the two point boundary value problem. The constraints include the final time conditions and the invariance of the Hamiltonian.
3. The results from step 2, together with the originally known initial time state conditions, are used to solve for the dynamic system response by integrating the kinematic and dynamic equations forward in time. The Hamiltonian history and the final state errors are the validation criteria.

This approach can only guarantee that the found solutions are local extrema.

V. Conclusion

This paper presents an analysis of the time optimal trajectories of an aircraft considering a constant velocity wind. Geometric characterization of the candidate paths satisfying the necessary conditions for time optimality is presented. As long as the wind velocity is small versus the aircraft velocity, solutions of set S1 can be used.

One can apply this technique to discrete gusts assuming one value before the steep gradient and another one after it. If one assumes that it is a constant velocity wind, there will be no consequence onto the dynamic study. However, at the joining configuration, infinite accelerations would be encountered. An obvious generalization of this work is to include dynamics. It will help into energy savings as the wind variations are used as inputs in the trajectory generation for the vehicle motion.

Another motivation for determining these elementary pieces is for use as motion primitives for determining these elementary pieces is for use as motion primitives in modern planning and control algorithms that consider obstacles.

Acknowledgments

The authors gratefully acknowledge the financial support of the CNES (National Center of Spatial Studies), France.

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