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# ON THE BOUNDARY OF THE ATTAINABLE SET OF THE DIRICHLET SPECTRUM 

LORENZO BRASCO, CARLO NITSCH, AND ALDO PRATELLI

Abstract. Denoting by $\mathcal{E} \subseteq \mathbb{R}^{2}$ the set of the pairs $\left(\lambda_{1}(\Omega), \lambda_{2}(\Omega)\right)$ for all the open sets $\Omega \subseteq \mathbb{R}^{N}$ with unit measure, and by $\Theta \subseteq \mathbb{R}^{N}$ the union of two disjoint balls of half measure, we give an elementary proof of the fact that $\partial \mathcal{E}$ has horizontal tangent at its lowest point $\left(\lambda_{1}(\Theta), \lambda_{2}(\Theta)\right)$.

## 1. Introduction

Given an open set $\Omega \subseteq \mathbb{R}^{N}$ with finite measure, its Dirichlet-Laplacian spectrum is given by the numbers $\lambda>0$ such that the boundary value problem

$$
-\Delta u=\lambda u \text { in } \Omega, \quad u=0 \text { on } \partial \Omega,
$$

has non trivial solutions. Such numbers $\lambda$ are called eigenvalues of the Dirichlet-Laplacian in $\Omega$, and form a discrete increasing sequence $0<\lambda_{1}(\Omega) \leq \lambda_{2}(\Omega) \leq \lambda_{3}(\Omega) \ldots$, diverging to $+\infty$ (see [4], for example). In this paper, we will work with the first two eigenvalues $\lambda_{1}$ and $\lambda_{2}$, for which we briefly recall the variational characterization: introducing the Rayleigh quotient as

$$
\mathcal{R}_{\Omega}(u)=\frac{\|\nabla u\|_{L^{2}(\Omega)}^{2}}{\|u\|_{L^{2}(\Omega)}^{2}}, \quad u \in H^{1}(\Omega)
$$

the first two eigenvalues of the Dirichlet-Laplacian satisfy

$$
\begin{gathered}
\lambda_{1}(\Omega)=\min \left\{\mathcal{R}_{\Omega}(u): u \in H_{0}^{1}(\Omega) \backslash\{0\}\right\}, \\
\lambda_{2}(\Omega)=\min \left\{\mathcal{R}_{\Omega}(u): u \in H_{0}^{1}(\Omega) \backslash\{0\}, \int_{\Omega} u(x) u_{1}(x) d x=0\right\},
\end{gathered}
$$

where $u_{1}$ is a first eigenfunction.
We are concerned about the attainable set of the first two eigenvalues $\lambda_{1}$ and $\lambda_{2}$, that is,

$$
\mathcal{E}:=\left\{\left(\lambda_{1}(\Omega), \lambda_{2}(\Omega)\right) \in \mathbb{R}^{2}:|\Omega|=\omega_{N}\right\},
$$

where $\omega_{N}$ is the volume of the ball of unit radius in $\mathbb{R}^{N}$. Of course, the set $\mathcal{E}$ depends on the dimension $N$ of the ambient space. The set $\mathcal{E}$ has been deeply studied (see for instance [1, 3, 6]); an approximate plot is shown in Figure 1. Let us recall now some of the most important known facts. In what follows, we will always denote by $B$ a ball of unit radius (then, of volume $\omega_{N}$ ), and by $\Theta$ a disjoint union of two balls of volume $\omega_{N} / 2$.

Basic properties of $\mathcal{E}$. The attainable set $\mathcal{E}$ has the following properties:
(i) for every $\left(\lambda_{1}, \lambda_{2}\right) \in \mathcal{E}$ and every $t \geq 1$, one has $\left(t \lambda_{1}, t \lambda_{2}\right) \in \mathcal{E}$;

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Figure 1. The attainable set $\mathcal{E}$
(ii)

$$
\mathcal{E} \subseteq\left\{x \geq \lambda_{1}(B), y \geq \lambda_{2}(\Theta), 1 \leq \frac{y}{x} \leq \frac{\lambda_{2}(B)}{\lambda_{1}(B)}\right\} ;
$$

(iii) $\mathcal{E}$ is horizontally and vertically convex, i.e., for every $0 \leq t \leq 1$

$$
\begin{aligned}
& \left(x_{0}, y\right),\left(x_{1}, y\right) \in \mathcal{E} \Longrightarrow\left((1-t) x_{0}+t x_{1}, y\right) \in \mathcal{E}, \\
& \left(x, y_{0}\right),\left(x, y_{1}\right) \in \mathcal{E} \Longrightarrow\left(x,(1-t) y_{0}+t y_{1}\right) \in \mathcal{E} .
\end{aligned}
$$

The first property is a simple consequence of the scaling property $\lambda_{i}(t \Omega)=t^{-2} \lambda_{i}(\Omega)$, valid for any open set $\Omega \subseteq \mathbb{R}^{N}$ and any $t>0$. The second property is true because, for every open set $\Omega$ of unit measure, the Faber-Krahn inequality ensures $\lambda_{1}(\Omega) \geq \lambda_{1}(B)$, the Krahn-Szego inequality (see [5, 7, 8]) ensures $\lambda_{2}(\Omega) \geq \lambda_{2}(\Theta)=\lambda_{1}(\Theta)$, and a celebrated result by Ashbaugh and Benguria (see [2]) ensures

$$
1 \leq \frac{\lambda_{2}(\Omega)}{\lambda_{1}(\Omega)} \leq \frac{\lambda_{2}(B)}{\lambda_{1}(B)} .
$$

Finally, the third property is proven in [3]. It has been conjectured also that the set $\mathcal{E}$ is convex, as it seems reasonable by a numerical plot, but a proof for this fact is still not known.

Thanks to the above listed properties, the set $\mathcal{E}$ is completely known once one knows its "lower boundary"

$$
\mathcal{C}:=\left\{\left(\lambda_{1}, \lambda_{2}\right) \in \overline{\mathcal{E}}: \forall t<1,\left(t \lambda_{1}, t \lambda_{2}\right) \notin \mathcal{E}\right\}
$$

therefore studying $\mathcal{E}$ is equivalent to study $\mathcal{C}$. Notice in particular that $\partial \mathcal{E}$ consists of the union of $\mathcal{C}$ with the two half-lines

$$
\left\{(t, t): t \geq \lambda_{1}(\Theta)\right\} \quad \text { and } \quad\left\{\left(t, \frac{\lambda_{2}(B)}{\lambda_{1}(B)} t\right): t \geq \lambda_{1}(B)\right\} .
$$

Let us call for brevity $P$ and $Q$ the endpoints of $\mathcal{C}$, that is, $P \equiv\left(\lambda_{1}(\Theta), \lambda_{2}(\Theta)\right)$ and $Q \equiv$ $\left(\lambda_{1}(B), \lambda_{2}(B)\right)$.

The plot of the set $\mathcal{E}$ seems to suggest that the curve $\mathcal{C}$ reaches the point $Q$ with vertical tangent, and the point $P$ with horizontal tangent. In fact, Wolf and Keller in [6, Section 5]
proved the first fact, and they also suggested that the second fact should be true, providing a numerical evidence. The aim of the present paper is to give a short proof of this fact.

Theorem. For every dimension $N \geq 2$, the curve $\mathcal{C}$ reaches the point $P$ with horizontal tangent.
The rest of the paper is devoted to prove this result: the proof will be achieved by exhibiting a suitable family $\left\{\widetilde{\Omega}_{\varepsilon}\right\}_{\varepsilon>0}$ of deformations of $\Theta$ having measure $\omega_{N}$ and such that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{\lambda_{2}\left(\widetilde{\Omega}_{\varepsilon}\right)-\lambda_{2}(\Theta)}{\lambda_{1}(\Theta)-\lambda_{1}\left(\widetilde{\Omega}_{\varepsilon}\right)}=0 \tag{1.1}
\end{equation*}
$$

## 2. Proof of the Theorem

Throughout this section, for any given $x=\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{R}^{N}$, we will write $x=\left(x_{1}, x^{\prime}\right)$ where $x_{1} \in \mathbb{R}$ and $x^{\prime} \in \mathbb{R}^{N-1}$.

We will make use of the sets $\left\{\Omega_{\varepsilon}\right\} \subseteq \mathbb{R}^{N}$, shown in Figure 2, defined by

$$
\begin{aligned}
\Omega_{\varepsilon}:= & \left\{\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{+} \times \mathbb{R}^{N-1}:\left(x_{1}-1+\varepsilon\right)^{2}+\left|x^{\prime}\right|^{2}<1\right\} \\
& \cup\left\{\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{-} \times \mathbb{R}^{N-1}:\left(x_{1}+1-\varepsilon\right)^{2}+\left|x^{\prime}\right|^{2}<1\right\} \\
= & : \Omega_{\varepsilon}^{+} \cup \Omega_{\varepsilon}^{-} .
\end{aligned}
$$

for every $\varepsilon>0$ sufficiently small. The sets $\widetilde{\Omega}_{\varepsilon}$ for which we will eventually prove (1.1) will be rescaled copies of $\Omega_{\varepsilon}$, in order to have measure $\omega_{N}$.

To get our thesis, we need to provide an upper bound to $\lambda_{1}\left(\Omega_{\varepsilon}\right)$ and an upper bound to $\lambda_{2}\left(\Omega_{\varepsilon}\right)$; this will be the content of Lemmas 2.1 and 2.2 respectively.


Figure 2. The sets $\Omega_{\varepsilon}=\Omega_{\varepsilon}^{+} \cup \Omega_{\varepsilon}^{-}$

Lemma 2.1. There exists a constant $\gamma_{1}>0$ such that for every $\varepsilon \ll 1$ it is

$$
\begin{equation*}
\lambda_{1}\left(\Omega_{\varepsilon}\right) \leq \lambda_{1}(B)-\gamma_{1} \varepsilon^{N / 2} . \tag{2.1}
\end{equation*}
$$

Proof. Let $B_{\varepsilon}$ be the ball of unit radius centered at ( $1-\varepsilon, 0$ ), so that $B_{\varepsilon} \subseteq \Omega_{\varepsilon}$ and in particular $\Omega_{\varepsilon}^{+}=B_{\varepsilon} \cap\left\{x_{1}>0\right\}$. Let also $u$ be a first Dirichlet eigenfunction of $B_{\varepsilon}$ with unit $L^{2}$ norm, and denote by $\mathbb{T}$ the region (shaded in Figure 3) bounded by the right circular conical surface $\left\{\sqrt{2 \varepsilon-\varepsilon^{2}}-x_{1}-\left|x^{\prime}\right|=0\right\}$ and by the plane $\left\{x_{1}=0\right\}$.

Since the normal derivative of $u$ is constantly $\kappa$ on $\partial B_{\varepsilon}^{+}$, we know that

$$
\begin{equation*}
D u\left(x_{1}, x^{\prime}\right)=D u\left(0, x^{\prime}\right)+O(\sqrt{\varepsilon})=(\kappa, 0)+O(\sqrt{\varepsilon}) \quad \text { on } \mathbb{T} . \tag{2.2}
\end{equation*}
$$



Figure 3. The ball $B_{\varepsilon}$ and the cone $\mathbb{T}$ (shaded) in the proof of Lemma 2.1
Let us now define the function $\tilde{u}: \Omega_{\varepsilon}^{+} \rightarrow \mathbb{R}$ as

$$
\tilde{u}\left(x_{1}, x^{\prime}\right):= \begin{cases}u\left(x_{1}, x^{\prime}\right) & \text { if }\left(x_{1}, x^{\prime}\right) \notin \mathbb{T} \\ u\left(x_{1}, x^{\prime}\right)+\frac{\kappa}{2}\left(\sqrt{2 \varepsilon-\varepsilon^{2}}-x_{1}-\left|x^{\prime}\right|\right) & \text { if }\left(x_{1}, x^{\prime}\right) \in \mathbb{T}\end{cases}
$$

It is immediate to observe that $\tilde{u}=u$ on the surface $\left\{\sqrt{2 \varepsilon-\varepsilon^{2}}-x_{1}-\left|x^{\prime}\right|=0\right\} \cap\left\{x_{1}>0\right\}$, so that $\tilde{u} \in H^{1}\left(\Omega_{\varepsilon}^{+}\right)$. Notice that $\tilde{u} \notin H_{0}^{1}\left(\Omega_{\varepsilon}^{+}\right)$since $\tilde{u}$ does not vanish on $\left\{x_{1}=0\right\} \cap \partial \Omega_{\varepsilon}^{+}$. By construction and recalling (2.2),

$$
\begin{equation*}
D \tilde{u}\left(x_{1}, x^{\prime}\right)=D u\left(x_{1}, x^{\prime}\right)+\left(-\frac{\kappa}{2},-\frac{\kappa}{2} \frac{x^{\prime}}{\left|x^{\prime}\right|}\right)=\left(\frac{\kappa}{2},-\frac{\kappa}{2} \frac{x^{\prime}}{\left|x^{\prime}\right|}\right)+O(\sqrt{\varepsilon}) \quad \text { on } \mathbb{T} . \tag{2.3}
\end{equation*}
$$

Since $\tilde{u} \geq u$ on $\Omega_{\varepsilon}^{+}$, and recalling that $u \in H_{0}^{1}\left(B_{\varepsilon}^{+}\right)$, one clearly has

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}^{+}} \tilde{u}^{2} d x \geq \int_{\Omega_{\varepsilon}^{+}} u^{2} d x=\int_{B_{\varepsilon}^{+}} u^{2} d x+O\left(\varepsilon^{(N+5) / 2}\right)=1+O\left(\varepsilon^{(N+5) / 2}\right), \tag{2.4}
\end{equation*}
$$

since the small region $B_{\varepsilon} \backslash \Omega_{\varepsilon}^{+}$has volume $O\left(\varepsilon^{(N+1) / 2}\right)$, and on this region $u=O(\varepsilon)$.
On the other hand, comparing (2.2) and (2.3), one has

$$
|D \tilde{u}|^{2}=|D u|^{2}-\frac{\kappa^{2}}{2}+O(\sqrt{\varepsilon}) \quad \text { on } \mathbb{T},
$$

and since the volume of $\mathbb{T}$ is $\frac{\omega_{N-1}}{N}\left(2 \varepsilon-\varepsilon^{2}\right)^{N / 2}$ we deduce

$$
\begin{align*}
\int_{\Omega_{\varepsilon}^{+}}|D \tilde{u}|^{2} d x & =\int_{\Omega_{\varepsilon}^{+}}|D u|^{2} d x-\frac{\omega_{N-1}}{N}\left(2 \varepsilon-\varepsilon^{2}\right)^{N / 2}\left(\frac{\kappa^{2}}{2}+O(\sqrt{\varepsilon})\right) \\
& =\int_{\Omega_{\varepsilon}^{+}}|D u|^{2} d x-\frac{\omega_{N-1}}{N} \kappa^{2} 2^{(N / 2-1)} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right)  \tag{2.5}\\
& =\int_{B_{\varepsilon}^{+}}|D u|^{2} d x-C_{N} \kappa^{2} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right)
\end{align*}
$$

where $C_{N}=\frac{\omega_{N-1}}{N} 2^{(N / 2-1)}$.
Therefore, by (2.4) and (2.5) we obtain

$$
\begin{aligned}
\mathcal{R}_{\Omega_{\varepsilon}^{+}}(\tilde{u})=\frac{\int_{\Omega_{\varepsilon}^{+}}|D \tilde{u}|^{2} d x}{\int_{\Omega_{\varepsilon}^{+}} \tilde{u}^{2} d x} & \leq \mathcal{R}_{B_{\varepsilon}^{+}}(u)-C_{N} \kappa^{2} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right) \\
& =\lambda_{1}(B)-C_{N} \kappa^{2} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right)
\end{aligned}
$$

We can finally extend $\tilde{u}$ to the whole $\Omega_{\varepsilon}$, simply defining $\tilde{u}\left(x_{1}, x^{\prime}\right)=\tilde{u}\left(\left|x_{1}\right|, x^{\prime}\right)$ on $\Omega_{\varepsilon}^{-}$. By construction, $\tilde{u} \in H_{0}^{1}\left(\Omega_{\varepsilon}\right)$, and

$$
\lambda_{1}\left(\Omega_{\varepsilon}\right) \leq \mathcal{R}_{\Omega_{\varepsilon}}(\tilde{u})=\mathcal{R}_{\Omega_{\varepsilon}^{+}}(\tilde{u}) \leq \lambda_{1}(B)-C_{N} \kappa^{2} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right),
$$

so that (2.1) follows and the proof is concluded.
Lemma 2.2. There exists a constant $\gamma_{2}>0$ such that for every $\varepsilon \ll 1$, it is

$$
\begin{equation*}
\lambda_{2}\left(\Omega_{\varepsilon}\right) \leq \lambda_{1}(B)+\gamma_{2} \varepsilon^{(N+1) / 2} \tag{2.6}
\end{equation*}
$$

Proof. First of all, we start underlining that

$$
\begin{equation*}
\lambda_{2}\left(\Omega_{\varepsilon}\right) \leq \lambda_{1}\left(\Omega_{\varepsilon}^{+}\right) ; \tag{2.7}
\end{equation*}
$$

in fact if we define

$$
\tilde{u}\left(x_{1}, x^{\prime}\right):= \begin{cases}u_{\varepsilon}\left(x_{1}, x^{\prime}\right), & \text { if } x_{1} \in \Omega_{\varepsilon}^{+}, \\ -u_{\varepsilon}\left(-x_{1}, x^{\prime}\right), & \text { if } x_{1} \in \Omega_{\varepsilon}^{-}\end{cases}
$$

then by construction it readily follows that $-\Delta \tilde{u}=\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right) \tilde{u}$. As a consequance $\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right)$is an eigenvalue of $\Omega_{\varepsilon}$, say $\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right)=\lambda_{\ell}\left(\Omega_{\varepsilon}\right)$. Since $\Omega_{\varepsilon}$ is connected and $\tilde{u}$ changes sign, it is not possible $\ell=1$, hence

$$
\lambda_{2}\left(\Omega_{\varepsilon}\right) \leq \lambda_{\ell}\left(\Omega_{\varepsilon}\right)=\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right) .
$$

It is then enough for us to estimate $\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right)$. To this aim, define the set

$$
\mathcal{O}_{\varepsilon}:=\left\{\left(x_{1}, x^{\prime}\right) \in \Omega_{\varepsilon}^{+}: x_{1} \geq \varepsilon\right\}
$$

and take a Lipschitz cut-off function $\xi_{\varepsilon} \in W^{1, \infty}\left(\Omega_{\varepsilon}^{+}\right)$such that

$$
0 \leq \xi_{\varepsilon} \leq 1 \text { on } \Omega_{\varepsilon}^{+}, \quad \xi_{\varepsilon} \equiv 1 \text { on } \mathcal{O}_{\varepsilon}, \quad \xi_{\varepsilon} \equiv 0 \text { on } \partial \Omega_{\varepsilon}^{+} \cap\left\{x_{1}=0\right\}, \quad\left\|\nabla \xi_{\varepsilon}\right\|_{\infty} \leq L \varepsilon^{-1}
$$

As in Lemma 2.1, let again $u$ be a first eigenfunction of the ball $B_{\varepsilon}$ of radius 1 centered at $(1-\varepsilon, 0)$ having unit $L^{2}$ norm, and define on $\Omega_{\varepsilon}$ the function $\varphi=u \xi_{\varepsilon}$. Since by construction $\varphi$ belongs to $H_{0}^{1}\left(\Omega_{\varepsilon}\right)$, we obtain

$$
\begin{equation*}
\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right) \leq \mathcal{R}\left(\varphi, \Omega_{\varepsilon}^{+}\right)=\frac{\int_{\Omega_{\varepsilon}^{+}}\left[|\nabla u|^{2} \xi_{\varepsilon}^{2}+\left|\nabla \xi_{\varepsilon}\right|^{2} u^{2}+2 u \xi_{\varepsilon}\left\langle\nabla u, \nabla \xi_{\varepsilon}\right\rangle\right] d x}{\int_{\Omega_{\varepsilon}^{+}} u^{2} \xi_{\varepsilon}^{2} d x} \tag{2.8}
\end{equation*}
$$

We can start estimating the denominator very similarly to what already done in (2.4). Indeed, recalling that $\left|\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}\right|=O\left(\varepsilon^{(N+1) / 2}\right)$ and that in that small region $u=O(\varepsilon)$, we have

$$
\int_{\Omega_{\varepsilon}^{+}} u^{2} \xi_{\varepsilon}^{2} d x=\int_{B_{\varepsilon}} u^{2} d x-\int_{B_{\varepsilon} \backslash \Omega_{\varepsilon}^{+}} u^{2} d x-\int_{\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}} u^{2}\left(1-\xi_{\varepsilon}^{2}\right) d x=1+O\left(\varepsilon^{(N+5) / 2}\right) .
$$

Let us pass to study the numerator: first of all, being $0 \leq \xi_{\varepsilon} \leq 1$ we have

$$
\int_{\Omega_{\varepsilon}^{+}}|\nabla u|^{2} \xi_{\varepsilon}^{2} d x \leq \int_{B_{\varepsilon}}|\nabla u|^{2} d x=\lambda_{1}(B)
$$

Moreover,

$$
\int_{\Omega_{\varepsilon}^{+}}\left|\nabla \xi_{\varepsilon}\right|^{2} u^{2} d x=\int_{\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}}\left|\nabla \xi_{\varepsilon}\right|^{2} u^{2} d x \leq \frac{L^{2}}{\varepsilon^{2}}\left|\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}\right|\|u\|_{L^{\infty}\left(\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}\right)}^{2}=O\left(\varepsilon^{(N+1) / 2}\right),
$$

and in the same way

$$
\int_{\Omega_{\varepsilon}^{+}} u \xi_{\varepsilon}\left\langle\nabla u, \nabla \xi_{\varepsilon}\right\rangle d x \leq \int_{\Omega_{\varepsilon}^{+} \backslash \mathcal{O}_{\varepsilon}}|u||\nabla u|\left|\nabla \xi_{\varepsilon}\right| d x=O\left(\varepsilon^{(N+1) / 2}\right) .
$$

Summarizing, by (2.8) we deduce

$$
\lambda_{1}\left(\Omega_{\varepsilon}^{+}\right) \leq \lambda_{1}(B)+O\left(\varepsilon^{(N+1) / 2}\right)
$$

thus by (2.7) we get the thesis.
We are now ready to conclude the paper by giving the proof of the Theorem.
Proof of the Theorem. For any small $\varepsilon>0$, we define $\widetilde{\Omega}_{\varepsilon}=t_{\varepsilon} \Omega_{\varepsilon}$, where $t_{\varepsilon}=\sqrt[N]{\omega_{N} /\left|\Omega_{\varepsilon}\right|}$ so that $\left|\widetilde{\Omega}_{\varepsilon}\right|=\omega_{N}$. Notice that

$$
\left|\Omega_{\varepsilon}\right|=2 \omega_{N}+O\left(\varepsilon^{(N+1) / 2}\right),
$$

thus $t_{\varepsilon}=2^{-1 / N}+O\left(\varepsilon^{(N+1) / 2}\right)$. Recalling the trivial rescaling formula $\lambda_{i}(t \Omega)=t^{-2} \lambda_{i}(\Omega)$, valid for any natural $i$, any positive $t$ and any open set $\Omega$, we can then estimate by Lemma 2.1 and Lemma 2.2

$$
\begin{gathered}
\lambda_{1}\left(\widetilde{\Omega}_{\varepsilon}\right)=\left(\frac{\left|\Omega_{\varepsilon}\right|}{\omega_{N}}\right)^{2 / N} \lambda_{1}\left(\Omega_{\varepsilon}\right) \leq 2^{2 / N} \lambda_{1}(B)-2^{2 / N} \gamma_{1} \varepsilon^{N / 2}+O\left(\varepsilon^{(N+1) / 2}\right) \\
\lambda_{2}\left(\widetilde{\Omega}_{\varepsilon}\right)=\left(\frac{\left|\Omega_{\varepsilon}\right|}{\omega_{N}}\right)^{2 / N} \lambda_{2}\left(\Omega_{\varepsilon}\right) \leq 2^{2 / N} \lambda_{1}(B)+O\left(\varepsilon^{(N+1) / 2}\right)
\end{gathered}
$$

Since $\lambda_{1}(\Theta)=\lambda_{2}(\Theta)=2^{2 / N} \lambda_{1}(B)$, the two above estimates give

$$
\lim _{\varepsilon \rightarrow 0} \frac{\lambda_{2}\left(\widetilde{\Omega}_{\varepsilon}\right)-\lambda_{2}(\Theta)}{\lambda_{1}(\Theta)-\lambda_{1}\left(\widetilde{\Omega}_{\varepsilon}\right)}=0
$$

which as already noticed in (1.1) implies the thesis.
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