Solvability conditions for some non-Fredholm operators

Vitali Vougalter, Vitaly Volpert

Solvability Conditions for Some non Fredholm Operators

Vitali Vougalter[1], Vitaly Volpert[2]

[1]University of Toronto, Department of Mathematics, Toronto, ON, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

[2]Institute of Mathematics, UMR 5208 CNRS, University Lyon 1, 69622 Villeurbanne, France
e-mail: volpert@math.univ-lyon1.fr

Abstract. We obtain solvability conditions for some elliptic equations involving non Fredholm operators with the methods of spectral theory and scattering theory for Schrödinger type operators. One of the main results of the work concerns solvability conditions for the equation $-\Delta u+V(x)u-au=f$ where $a\geq 0$. They are formulated in terms of orthogonality of the function f to the solutions of the homogeneous adjoint equation.

Keywords: solvability conditions, non Fredholm operators, elliptic problems
AMS subject classification: 35J10, 35P10, 35P25

1. Introduction

Linear elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if and only if the ellipticity condition, proper ellipticity and the Lopatinskii conditions are satisfied. Fredholm property implies the solvability conditions: the nonhomogeneous operator equation $Lu = f$ is solvable if and only if the right-hand side f is orthogonal to all solutions of the homogeneous adjoint problem $L^*v = 0$. The orthogonality is understood in the sense of duality in the corresponding spaces.

In the case of unbounded domains, one more condition should be imposed in order to preserve the Fredholm property. This condition can be formulated in terms of limiting operators and requires that all limiting operators should be invertible or that the only bounded solution of limiting problems is trivial [VV06]. Limiting operators are the operators with limiting values of the coefficients at infinity, if such limiting values exist. Otherwise, limiting coefficients are determined by means of sequences of shifted coefficients and locally convergent subsequences.

If we consider for example the operator $Lu = -\Delta u - au$ in $\mathbb{R}^n$, where a is a positive constant, then its only limiting operator is the same operator L. Since the limiting equation $Lu = 0$ has a nonzero bounded solution, then the operator L, considered in Sobolev or in

Hölder spaces does not satisfy the Fredholm property. Therefore the solvability conditions are not applicable. However, the particular form of the equation $-\Delta u - au = f$ in $\mathbb{R}^n$ allows us to apply the Fourier transform and to find its solution. It can be easily verified that it has a solution $u \in L^2(\mathbb{R}^n)$ if and only if $\widehat{f}(\xi)/(\xi^2 - a) \in L^2(\mathbb{R}^n)$, where "hat" denotes the Fourier transform. In the other words, the solvability conditions are given by the equality

$$\int_{\mathbb{R}^n} e^{i\xi x} f(x)dx = 0$$

for any $\xi \in \mathbb{R}^n$ such that $|\xi|^2 = a$. This means that formally we obtain solvability conditions similar to those for Fredholm operators: the right-hand side is orthogonal to all solutions of the homogeneous formally adjoint problem.

In this example, we are able to obtain solvability conditions due to the fact that the operator has constant coefficients and we can apply the Fourier transform. In general, the question about solvability conditions for non Fredholm operators is open and represents one of the major challenges in the theory of elliptic problems. Some classes of reaction-diffusion operators without Fredholm property can be studied by the introduction of weighted spaces [VV06] or reducing them to integro-differential operators [DMV05], [DMV08]. Other types of solvability conditions, different from the usual orthogonality conditions, are obtained for some second order operators on the real axis or in cylinders [KV06]. Some elliptic problems in $\mathbb{R}^2$ are studied in [VKMP02] where the solvability conditions are obtained with the help of space decomposition of the operators.

A special class of elliptic operators in $\mathbb{R}^n$, $A = A_\infty + A_0$, where A_∞ is a homogeneous operator with constant coefficients and A_0 is an operator with rapidly decaying coefficients is studied in specially chosen spaces with a polynomial weight. The finiteness of the kernel is proved in [W1], [NW], their Fredholm property in [W2], [L], [LM] in the case of weighted Sobolev spaces and in [B], [BP] for weighted Hölder spaces. The Fredholm property and the index of such operators are determined by their principal part A_∞. The operator A_0 does not change them due to the rapid decay of the coefficients. Laplace operator in exterior domains is studied in [AB].

In this work we consider two classes of non Fredholm operators and establish the solvability conditions for the equations involving them. The methods cited above are not applicable here and we develop some new approaches. In the first case we study the operator H_a on $L^2(\mathbb{R}^3)$, such that

$$H_a u = -\Delta u + V(x)u - au$$

where $a \geq 0$ is a parameter, the potential $V(x)$ decays to zero as $x \to \infty$. We investigate the conditions on the function $f \in L^2(\mathbb{R}^3)$ under which the equations

$$H_a u = f \tag{1.1}$$

and

$$H_0 u = f, \tag{1.2}$$

the second one is the limiting case of the first one as $a \to 0$, have the unique solution in $L^2(\mathbb{R}^3)$. Since the potential equals zero at infinity, the operator H_a has a unique limiting operator $Lu = -\Delta u - au$, which is the same as discussed above. The limiting problem $Lu = 0$ has nonzero bounded solutions. Therefore, the operator H_a, $a \geq 0$ does not satisfy the Fredholm property, and the solvability of equations (1.1) and (1.2) is not known. The coefficients of the operators are not constant any more and we cannot simply apply the Fourier transform as in the example above. We will use the spectral decomposition of self-adjoint operators.

We note that in the case where $a = 0$ and the potential is rapidly decaying at infinity, the operator H_0 belong to the class of operators $A_\infty + A_0$ discussed above. The results of this work differ from the results in the cited papers. We do not work in the weighted spaces and obtain solvability conditions without proving the Fredholm property which may not hold. However, more important difference is that we consider also the case $a > 0$. It is essentially different and the previous methods are not applicable. To the best of our knowledge, solvability conditions for equation (1.1) with $a > 0$ and $n \geq 2$ were not obtained before. The solvability conditions are formulated in terms of the orthogonality of the right-hand side f to all solutions of the homogeneous adjoint equation $H_a v = 0$ (the operator is self-adjoint).

For a function $\psi(x)$ belonging to a $L^p(\mathbb{R}^d)$ space with $1 \leq p \leq \infty,\ d \in \mathbb{N}$ its norm is being denoted as $\|\psi\|_{L^p(\mathbb{R}^d)}$. As technical tools for estimating the appropriate norms of functions we will be using, in particular the Young's inequality

$$\|f_1 * f_2\|_{L^\infty(\mathbb{R}^3)} \leq \|f_1\|_{L^4(\mathbb{R}^3)} \|f_2\|_{L^{\frac{4}{3}}(\mathbb{R}^3)},\ f_1 \in L^4(\mathbb{R}^3),\ f_2 \in L^{\frac{4}{3}}(\mathbb{R}^3)\ ,$$

where * stands for the convolution and the Hardy-Littlewood-Sobolev inequality

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \Big| \leq c_{HLS} \|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3)$$

with the constant c_{HLS} given on p.98 of [LL97]. In our notations $(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f_1(x) \bar{f}_2(x) dx$ and for a vector function $A(x) = (A_1(x), A_2(x), A_3(x))$ the inner product $(f_1(x), A(x))_{L^2(\mathbb{R}^3)}$ is the vector with the coordinates $\int_{\mathbb{R}^3} f_1(x) \bar{A}_i(x) dx,\ i = 1, 2, 3$. Note that with a slight abuse the same notation will be used even if the functions above are not square integrable, like the so called perturbed plane waves $\varphi_k(x)$ which are normalized to a delta function (see the equation (2.1) in the Section 2). We make the following technical assumption.

Assumption 1.1. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the bound* $|V(x)| \leq \dfrac{C}{1 + |x|^{3.5+\varepsilon}}$ *with some* $\varepsilon > 0$ *and* $x \in \mathbb{R}^3$ *a.e. such that*

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{\frac{-2}{3}} \|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)} \|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1 \quad \text{and} \quad \sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi$$

The function $f(x) \in L^2(\mathbb{R}^3)$ *and* $|x| f(x) \in L^1(\mathbb{R}^3)$.

Here and further down C stands for a finite positive constant. Since under our assumptions on the potential the essential spectrum $\sigma_{ess}(H_a)$ of the Schrödinger type operator $H_a = H_0 - a$ fills the interval $[-a, \infty)$ (see e.g. [JMST]), the Fredholm alternative theorem fails to work in this case. The problem can be easily handled by the method of the Fourier transform in the absence of the potential term $V(x)$. We show that this method can be generalized in the presence of a shallow, short-range $V(x)$ by means of replacing the Fourier harmonics by the functions $\varphi_k(x),\ k \in \mathbb{R}^3$ of the continuous spectrum of the operator H_0, which are the solutions of the Lippmann-Schwinger equation (see (2.1) in Section 2 and the explicit formula (2.2)).

While the wave vector k attains all the possible values in $\mathbb{R}^3$, the function $\varphi_0(x)$ corresponds to $k = 0$ in the formulas (2.1) and (2.2). The sphere of radius r in $\mathbb{R}^d,\ d \in \mathbb{N}$ centered at the origin is being designated as S^d_r, the unit one as S^d and $|S^d|$ stands for its Lebesgue measure. Our first main result is as follows.

Theorem 1. *Let the Assumption 1.1 hold. Then*
a) The problem (1.1) admits a unique solution $u \in L^2(\mathbb{R}^3)$ if and only if

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0 \quad for \quad k \in S^3_{\sqrt{a}} \quad a.e.$$

b) The problem (1.2) has a unique solution $u \in L^2(\mathbb{R}^3)$ if and only if

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0$$

In the second part of the article we consider the operator $\mathcal{L} = -\Delta_x - \Delta_y + \mathcal{V}(y)$ on $L^2(\mathbb{R}^{n+m})$ with the Laplacian operators Δ_x and Δ_y in $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$ and prove the necessary and sufficient conditions for the solvability in $L^2(\mathbb{R}^{n+m})$ of the inhomogeneous problem

$$\mathcal{L}u = g(x, y), \tag{1.3}$$

where $g(x, y) \in L^2(\mathbb{R}^{n+m})$. We assume the following.

Assumption 1.2. *The function $\mathcal{V}(y) : \mathbb{R}^m \to \mathbb{R}$ is continuous and $lim_{y\to\infty}\mathcal{V}(y) = \mathcal{V}_+ > 0$.*

Thus for the operator $h := -\Delta_y + \mathcal{V}(y)$ the essential spectrum $\sigma_{ess}(h) = [\mathcal{V}_+, \infty)$. Let us denote the eigenvalues of the operator h located below $\mathcal{V}_+$ as $e_j,\ e_j < e_{j+1},\ j \geq 1$ and the corresponding elements of the orthonormal set of eigenfunctions as φ^k_j, such that $h\varphi^k_j = e_j\varphi^k_j,\ 1 \leq k \leq m_j,\ (\varphi^k_i, \varphi^l_j)_{L^2(\mathbb{R}^m)} = \delta_{i,j}\delta_{k,l}$, where m_j stands for the eigenvalue multiplicity, which is finite since the essential spectrum starts only at $\mathcal{V}_+$ and $\delta_{i,j}$ for the Kronecker symbol. We make the following key assumption on the discrete spectrum of the operator h relevant to the problem (1.3).

Assumption 1.3. *The eigenvalues $e_j < 0$ for all $1 \leq j \leq N - 1$ and $e_N = 0$.*

Thus under our assumptions the operator $\mathcal{L}$ is not Fredholm. Zero is the bottom of the essential spectrum of the operator $-\Delta_x$ and h has the square integrable zero modes. Moreover, the operator h has the negative eigenvalues $e_j,\ j = 1, ..., N-1$ and $-\Delta_x$ the Fourier harmonics $\dfrac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}$, such that $p \in S^n_{\sqrt{-e_j}}$. However, the equation (1.3) can be solved on the proper subspace and the orthogonality conditions will strongly depend on the dimensions of the problem.

Let us introduce the following subspace weighted in the first variable for the right side of the equation (1.3).

$$L^2_{\alpha,\ x} = \{g(x, y) : g(x, y) \in L^2(\mathbb{R}^{n+m}) \text{ and } |x|^{\frac{\alpha}{2}} g(x, y) \in L^2(\mathbb{R}^{n+m})\},\ \alpha > 0 \tag{1.4}$$

Our second main result is as follows.

Theorem 2. *Let the Assumptions 1.2 and 1.3 hold. Then for the equation (1.3):*

I) When $n = 1$ and $g(x, y) \in L^2_{\alpha,\ x}$ for some $\alpha > 5$ there exists a unique solution $u \in L^2(\mathbb{R}^{1+m})$ if and only if:

$(g(x, y), \varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} = 0,\ (g(x, y), x\varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} = 0,\ 1 \le k \le m_N$

and $(g(x, y), \dfrac{e^{\pm i\sqrt{-e_j}x}}{\sqrt{2\pi}}\varphi_j^k(y))_{L^2(\mathbb{R}^{1+m})} = 0,\ 1 \le j \le N-1,\ 1 \le k \le m_j$

II) When $n = 2$ such that $x = (x_1, x_2) \in \mathbb{R}^2$ and $g(x, y) \in L^2_{\alpha,\ x}$ for some $\alpha > 6$ there exists a unique solution $u \in L^2(\mathbb{R}^{2+m})$ if and only if:

$(g(x, y), \varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} = 0,\ (g(x, y), x_i\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} = 0,\ i = 1, 2,\ 1 \le k \le m_N$ *and*

$(g(x, y), \dfrac{e^{ipx}}{2\pi}\varphi_j^k(y))_{L^2(\mathbb{R}^{2+m})} = 0,\ a.e.\ p \in S^2_{\sqrt{-e_j}},\ 1 \le j \le N-1,\ 1 \le k \le m_j$

III) When $n = 3, 4$ and $g(x, y) \in L^2_{\alpha,\ x}$ for some $\alpha > n + 2$ there exists a unique solution $u \in L^2(\mathbb{R}^{n+m})$ if and only if:

$(g(x, y), \varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})} = 0,\ 1 \le k \le m_N$ *and*

$(g(x, y), \dfrac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\varphi_j^k(y))_{L^2(\mathbb{R}^{n+m})} = 0,\ a.e.\ p \in S^n_{\sqrt{-e_j}},\ 1 \le j \le N-1,\ 1 \le k \le m_j$

IV) When $n \ge 5$ and $g(x, y) \in L^2_{\alpha,\ x}$ for some $\alpha > n + 2$ there exists a unique solution $u \in L^2(\mathbb{R}^{n+m})$ if and only if:

$(g(x, y), \dfrac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\varphi_j^k(y))_{L^2(\mathbb{R}^{n+m})} = 0,\ a.e.\ p \in S^n_{\sqrt{-e_j}},\ 1 \le j \le N-1,\ 1 \le k \le m_j$

Proving solvability conditions for linear elliptic problems with non-Fredholm operators plays the crucial role in various applications including those to travelling wave solutions of reaction-diffusion systems (see [VKMP02]). Let us first establish several important properties for the functions of the spectrum of the Schrödinder operator in the left side of the equation (1.1) and for the related quantities.

2. Spectral properties of the operator H_0 and proof of Theorem 1

The functions of the continuous spectrum satisfy the Lippmann-Schwinger equation (see e.g. [RS79] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)\mathrm{d}y \tag{2.1}$$

and the orthogonality relations $(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3$. We define the integral operator

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)\mathrm{d}y, \quad \varphi \in L^\infty(\mathbb{R}^3)$$

Let us show that the norm of the operator $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ denoted as $\|Q\|_\infty$ is small when the potential $V(x)$ satisfies our assumptions. We prove the following lemma.

Lemma 2.1. *Let the Assumption 1.1 hold. Then* $||Q||_\infty < 1$.

Proof. Clearly

$$\|Q\|_\infty \leq \sup_{x\in\mathbb{R}^3} \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}dy$$

The expression involved in the right side of the inequality above can be written as

$$\frac{1}{4\pi}(\chi_{\{|x|\leq R\}}\frac{1}{|x|}) * |V(x)| + \frac{1}{4\pi}(\chi_{\{|x|>R\}}\frac{1}{|x|}) * |V(x)|$$

with some $R > 0$ and χ denoting the characteristic function of the correspondent set. This can be estimated above using the Young's inequality as

$$\frac{1}{4\pi}\|V\|_{L^\infty(\mathbb{R}^3)}\int_0^R 4\pi r dr + \frac{1}{4\pi}\|\chi_{\{|x|>R\}}\frac{1}{|x|}\|_{L^4(\mathbb{R}^3)}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} =$$

$$= \frac{1}{2}\|V\|_{L^\infty(\mathbb{R}^3)}R^2 + \frac{1}{(4\pi)^{\frac{3}{4}}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}R^{-\frac{1}{4}}$$

We optimize the right side of the equality above over R. The minimum occurs when $R = \left\{\frac{\|V\|_{L^\infty(\mathbb{R}^3)}(4\pi)^{\frac{3}{4}}4}{\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}\right\}^{-\frac{4}{9}}$, such that

$$\|Q\|_\infty \leq 4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)},$$

which is k-independent. The Assumption 1.1 yields the statement of the Lemma. Note that $V \in L^{\frac{4}{3}}(\mathbb{R}^3)$ which is guaranteed by its rate of decay given explicitly in the Assumption 1.1.

□

Corollary 2.2. *Let the Assumption 1.1 hold. Then the functions of the continuous spectrum of the operator H_0 are $\varphi_k(x) \in L^\infty(\mathbb{R}^3)$ for all $k \in \mathbb{R}^3$, such that*

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{1-\|Q\|_\infty} \frac{1}{(2\pi)^{\frac{3}{2}}}, \ k \in \mathbb{R}^3$$

Proof. By means of the Lippmann-Schwinger equation (2.1) and the fact that $\|Q\|_\infty < 1$ the functions can be expressed as

$$\varphi_k(x) = (I-Q)^{-1} \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}, \ k \in \mathbb{R}^3 \tag{2.2}$$

The Lemma 2.1 yields the bound on the operator norm $\|(I-Q)^{-1}\|_\infty \le \dfrac{1}{1-\|Q\|_\infty}$.

□

The following elementary lemma shows that in our problem the operator H_0 possesses the spectrum analogous to the one of the minus Laplacian and therefore only the functions $\varphi_k(x), \ k \in \mathbb{R}^3$ are needed to be taken into consideration.

Lemma 2.3. *Let the Assumption 1.1 be true. Then the operator H_0 is unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$.*

Proof. By means of the Hardy-Littlewood-Sobolev inequality (see e.g. p.98 [LL97]) and the Assumption 1.1 we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} dx dy \le c_{HLS} \|V\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)} < (4\pi)^2$$

The left side of the inequality above is usually referred to as the Rollnik norm (see e.g. [S71]) and the upper bound we obtained on it is the sufficient condition for the operator $H_0 = -\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ to be self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see e.g. [K65], also [RS04]) given by

$$\Omega^\pm := s - \lim_{t \to \mp\infty} e^{it(-\Delta+V)} e^{it\Delta}$$

where the limit is understood in the strong L^2 sense (see e.g. [RS79] p.34, [CFKS87] p.90).

□

By means of the spectral theorem for the self-adjoint operator H_0 any function $\psi(x) \in L^2(\mathbb{R}^3)$ can be expanded through the functions $\varphi_k(x), k \in \mathbb{R}^3$ forming the complete system in $L^2(\mathbb{R}^3)$. The generalized Fourier transform with respect to these functions is being denoted as

$$\tilde{\psi}(k) := (\psi(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3 \tag{2.3}$$

We prove the following technical estimate concerning the above mentioned generalized Fourier transform for the right side of the equations (1.1) and (1.2).

Lemma 2.4. *Let the Assumption 1.1 hold. Then*

$$\nabla_k \tilde{f}(k) \in L^\infty(\mathbb{R}^3)$$

Proof. Obviously $\nabla_k \tilde{f}(k) = (f(x), \nabla_k \varphi_k(x))_{L^2(\mathbb{R}^3)}$. ¿From the Lippmann-Schwinger equation (2.1) we easily obtain

$$\nabla_k \varphi_k = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix + (I-Q)^{-1} Q \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix + (I-Q)^{-1} (\nabla_k Q)(I-Q)^{-1} \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}\ , \tag{2.4}$$

where $\nabla_k Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3; \mathbb{C}^3)$ stands for the operator with the integral kernel

$$\nabla_k Q(x, y, k) = -\frac{i}{4\pi} e^{i|k||x-y|} \frac{k}{|k|} V(y)$$

An elementary computation shows that its norm

$$\|\nabla_k Q\|_\infty \leq \frac{1}{4\pi} \|V\|_{L^1(\mathbb{R}^3)} < \infty$$

due to the rate of decay of the potential $V(x)$ given explicitly by the Assumption 1.1. It is clear from the identity (2.4) that we need to show the boundedness in the k-space of the three terms. The first one is

$$T_1(k) := (f(x), \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix)_{L^2(\mathbb{R}^3)}\ ,$$

such that $|T_1(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \|xf\|_{L^1(\mathbb{R}^3)} < +\infty$ by the Assumption 1.1. The second term to be estimated is

$$T_2(k) := (f(x), (I-Q)^{-1} Q \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix)_{L^2(\mathbb{R}^3)}$$

Thus $|T_2(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \|f\|_{L^1(\mathbb{R}^3)} \frac{1}{1 - \|Q\|_\infty} \|Q e^{ikx} x\|_{L^\infty(\mathbb{R}^3)}$. Note that $f(x) \in L^1(\mathbb{R}^3)$ by means of the Assumption 1.1 and Fact 1 of the Appendix. Using the definition of the operator Q along with the Young's inequality we have the upper bound

$$|Q e^{ikx} x| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|V(y)||y|}{|x-y|} dy = \frac{1}{4\pi} \{ (\chi_{\{|x| \leq 1\}} \frac{1}{|x|}) * |V(x)||x| + (\chi_{\{|x|>1\}} \frac{1}{|x|}) * |V(x)||x| \} \leq$$

$$\leq \frac{1}{4\pi} \{ \|V(y) y\|_{L^\infty(\mathbb{R}^3)} \int_0^1 4\pi r dr + \|\chi_{\{|x|>1\}} \frac{1}{|x|}\|_{L^4(\mathbb{R}^3)} \|V(x) x\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \} < +\infty$$

and k-independent since $V(x)x \in L^\infty(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3)$ due to the explicit rate of decay of the potential $V(x)$ stated in the Assumption 1.1. Therefore, $T_2(k) \in L^\infty(\mathbb{R}^3)$. We complete the proof of the lemma with the estimate on the remaining term

$$T_3(k) := (f(x), (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1}\frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}})_{L^2(\mathbb{R}^3)} ,$$

such that we easily arrive at the k-independent upper bound

$$|T_3(k)| \leq \frac{1}{4\pi(2\pi)^{\frac{3}{2}}}\|f\|_{L^1(\mathbb{R}^3)}\frac{1}{(1-\|Q\|_\infty)^2}\|V\|_{L^1(\mathbb{R}^3)} < \infty$$

□

Armed with the auxiliary lemmas established above we proceed to prove the first theorem.

Proof of Theorem 1. First of all if the equation (1.1) admits two solutions $u_1(x), u_2(x) \in L^2(\mathbb{R}^3)$ their difference $v(x) := u_1(x) - u_2(x)$ would satisfy the homogeneous problem $H_a v = 0$. Since the operator H_a possesses no nontrivial square integrable zero modes, $v(x)$ will vanish a.e. The analogous argument holds for the solutions of the equation (1.2). ¿From the equation (1.1) by applying the transform (2.3) we obtain

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2 - a} \ , \ k \in \mathbb{R}^3 \ ,$$

which is convenient to write as the sum of the singular and the nonsingular parts

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma} + \frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma^c} \ , \tag{2.5}$$

where χ_{A_σ} is the characteristic function of the spherical layer

$$A_\sigma := \{k \in \mathbb{R}^3 : \sqrt{a} - \sigma \leq |k| \leq \sqrt{a} + \sigma\} \ , \ 0 < \sigma < \sqrt{a}$$

and $\chi_{A_\sigma^c}$ of the layer's complement in the three-dimensional k-space. For the second term in the right side of the identity (2.5)

$$\left|\frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma^c}\right| \leq \frac{|\tilde{f}(k)|}{\sqrt{a}\sigma} \in L^2(\mathbb{R}^3)$$

To estimate the remaining term we will make use of the identity

$$\tilde{f}(k) = \tilde{f}(\sqrt{a}, \omega) + \int_{\sqrt{a}}^{|k|} \frac{\partial \tilde{f}(|s|, \omega)}{\partial |s|} d|s|$$

Here and further ω stands for the angle variables on the sphere and $d\omega$ will denote integration with respect to these variables. Thus we can split the first term in the right side of (2.5) as $\tilde{u}_1(k) + \tilde{u}_2(k)$, where

$$\tilde{u}_1(k) = \frac{\int_{\sqrt{a}}^{|k|} \frac{\partial \tilde{f}(|s|,\omega)}{\partial |s|} d|s|}{k^2 - a} \chi_{A_\sigma}, \quad \tilde{u}_2(k) = \frac{\tilde{f}(\sqrt{a}, \omega)}{k^2 - a} \chi_{A_\sigma} \tag{2.6}$$

Clearly, we have the bound

$$|\tilde{u}_1(k)| \leq \frac{\|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)}}{|k| + \sqrt{a}} \chi_{A_\sigma} \in L^2(\mathbb{R}^3)$$

by means of Lemma 2.4. We complete the proof of the Part a) of the theorem by estimating the norm

$$\|\tilde{u}_2(k)\|^2_{L^2(\mathbb{R}^3)} = \int_{\sqrt{a}-\sigma}^{\sqrt{a}+\sigma} d|k| \frac{|k|^2}{(|k| - \sqrt{a})^2 (|k| + \sqrt{a})^2} \int_{S^3} d\omega |\tilde{f}(\sqrt{a}, \omega)|^2 < \infty$$

if and only if $(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0$ for k a.e. on the sphere $S^3_{\sqrt{a}}$. Then we turn our attention to the equation (1.2) by applying to it the generalized Fourier transform with respect to the eigenfunctions of the continuous spectrum of the operator H_0, which yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2} = \frac{\tilde{f}(k)}{k^2} \chi_{\{|k| \leq 1\}} + \frac{\tilde{f}(k)}{k^2} \chi_{\{|k| > 1\}}$$

Clearly $\left| \frac{\tilde{f}(k)}{k^2} \chi_{\{|k|>1\}} \right| \leq |\tilde{f}(k)| \in L^2(\mathbb{R}^3)$. We use the formula

$$\tilde{f}(k) = \tilde{f}(0) + \int_0^{|k|} \frac{\partial \tilde{f}(|s|, \omega)}{\partial |s|} d|s|$$

with $\tilde{f}(0) = (f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}$ and $\varphi_0(x)$ is given by (2.2) with $k = 0$. Hence

$$\left| \frac{\int_0^{|k|} \frac{\partial \tilde{f}(|s|,\omega)}{\partial |s|} d|s|}{k^2} \chi_{\{|k| \leq 1\}} \right| \leq \|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)} \frac{\chi_{\{|k| \leq 1\}}}{|k|} \in L^2(\mathbb{R}^3)$$

via Lemma 2.4. Therefore it remains to estimate the norm

$$\left\| \frac{\tilde{f}(0)}{k^2} \chi_{\{|k| \leq 1\}} \right\|^2_{L^2(\mathbb{R}^3)} = 4\pi \int_0^1 d|k| \frac{|\tilde{f}(0)|^2}{|k|^2} < \infty$$

if and only if $(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0$ which completes the proof of the theorem.

$\square$

Note that if we let the potential function $V(x)$ in the statement of the Theorem 1 vanish, we obtain precisely the usual orthogonality conditions in terms of the Fourier harmonics.

In the following chapter we prove the Theorem 2. As distinct from the first example the dimensions of the problem are not fixed anymore and we show how robust the dependence of the solvability conditions on these dimensions can be.

3. Spectral properties of the operator $\mathcal{L}$ and proof of Theorem 2

Let $P_\pm$ and P_0 be the orthogonal projections onto the positive, negative and zero subspaces of the operator h. Applying these operators to both sides of the equation (1.3) via the spectral theorem we relate the problem to the equivalent system of the three following equations.

$$\mathcal{L}_+ u_+ = g_+ \ , \tag{3.1}$$

$$\mathcal{L}_- u_- = g_- \ , \tag{3.2}$$

and

$$\mathcal{L}_0 u_0 = g_0 \ , \tag{3.3}$$

where the operators $\mathcal{L}_\pm = P_\pm \mathcal{L} P_\pm$ and $\mathcal{L}_0 = P_0 \mathcal{L} P_0$ act on the functions $u_\pm = P_\pm u$ and $u_0 = P_0 u$ respectively and the right sides of the equations above are $g_\pm = P_\pm g$ and $g_0 = P_0 g$. Without loss of generality we can assume that

$$g_0(x, y) = v_0(x)\varphi_N^1(y) \ , \tag{3.4}$$

where $v_0(x) = (g_0, \varphi_N^1)_{L^2(\mathbb{R}^m)} = (g, \varphi_N^1)_{L^2(\mathbb{R}^m)}$. Let us first turn our attention to the equation (3.1). We have the following lemma.

Lemma 3.1 *The equation (3.1) possesses a solution $u_+ \in L^2(\mathbb{R}^{n+m}),\ n \in \mathbb{N},\ m \in \mathbb{N}$.*

Proof. By means of the orthogonal decomposition of the right side of the equation (1.3) $g = g_+ + g_0 + g_-$ we have the estimate

$$\|g_+\|_{L^2(\mathbb{R}^{n+m})} \le \|g\|_{L^2(\mathbb{R}^{n+m})}$$

The lower bound in the sense of the quadratic forms

$$\mathcal{L}_+ \ge P_+ h P_+ \ge e_{N+1} > 0 \ ,$$

where e_{N+1} is either the bottom of the essential spectrum $\mathcal{V}_+$ of the operator h or its lowest positive eigenvalue, whichever is smaller. Thus $\mathcal{L}_+$ is the self-adjoint operator on the product of spaces $L^2(\mathbb{R}^n)$ and the range $\mathrm{Ran}(P_+)$ such that the bottom of its spectrum is located above zero. Therefore it is invertible and the norm of the inverse $\mathcal{L}_+^{-1} : L^2(\mathbb{R}^n) \otimes \mathrm{Ran}(P_+) \to$

$L^2(\mathbb{R}^{n+m})$ is bounded above by $\dfrac{1}{e_{N+1}}$. Thus the equation (3.1) has the solution $u_+ = \mathcal{L}_+^{-1} g_+$ and its norm

$$\|u_+\|_{L^2(\mathbb{R}^{n+m})} \leq \frac{1}{e_{N+1}} \|g\|_{L^2(\mathbb{R}^{n+m})} < \infty$$

□

Let us turn our attention to the analysis of the solvability conditions for the equation (3.3) which is equivalent to

$$(-\Delta_x) u_0 = g_0 \tag{3.5}$$

The solution of this Poisson equation can be expressed as

$$\widehat{u}_0 = \frac{\widehat{g}_0}{p^2}\chi_1 + \frac{\widehat{g}_0}{p^2}\chi_{1^c} \ , \tag{3.6}$$

where χ_1 stands for the characteristic function of the unit ball in the Fourier space centered at the origin and χ_{1^c} for the characteristic function of its complement. Here and below the hat symbol stands for the Fourier transform in the first variable, such that

$$\widehat{\psi}(p) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \psi(x) e^{-ipx} dx$$

The second term in the right side of (3.6) is square integrable for all dimensions $n, m \in \mathbb{N}$ since $\widehat{g}_0 \in L^2(\mathbb{R}^{n+m})$ and $\frac{1}{p^2}$ is bounded away from the origin. Thus it remains to analyze the first term. We have the following lemma when the dimension $n = 1$.

Lemma 3.2 *Let the assumptions of the Theorem 2 hold. Then the equation (3.5) possesses a solution* $u_0 \in L^2(\mathbb{R}^{1+m}),\ m \in \mathbb{N}$ *if and only if*

$$(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} = 0, \quad (g(x,y), \varphi_N^k(y)x)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \leq k \leq m_N$$

Proof. We will make use of the following representation

$$\widehat{g}_0(p,y) = \widehat{g}_0(0,y) + \frac{\partial}{\partial p}\widehat{g}_0(0,y)p + \int_0^p \Big(\int_0^s \frac{\partial^2}{\partial q^2}\widehat{g}_0(q,y) dq \Big) ds \ ,$$

where

$$\frac{\partial^2}{\partial p^2}\widehat{g}_0(p,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(x,y) e^{-ipx} x^2 dx$$

Hence the first term in the right side of (3.6) in our case equals to

$$\frac{\widehat{g}_0(0,y)}{p^2}\chi_1 + \frac{\partial}{\partial p}\widehat{g}_0(0,y)\frac{\chi_1}{p} + \int_0^p \Big(\int_0^s \frac{\partial^2}{\partial q^2}\widehat{g}_0(q,y) dq \Big) ds \frac{\chi_1}{p^2} \tag{3.7}$$

Clearly we have the upper bound

$$\Big|\frac{\partial^2}{\partial q^2}\widehat{g}_0(q,y)\Big| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|g_0(x,y)|x^2dx$$

By means of the Schwarz inequality and (3.4) we have the estimate valid in a space of arbitrary dimensions

$$|g_0(x,y)| \leq \sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2dz}|\varphi_N^1(y)|\ ,\ x\in\mathbb{R}^n,\ y\in\mathbb{R}^m,\ n,m\geq 1 \tag{3.8}$$

which yields

$$\Big|\frac{\partial^2}{\partial q^2}\widehat{g}_0(q,y)\Big| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx\frac{x^2}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,s)|^2ds}|\varphi_N^1(y)|$$

with $\alpha>5$ such that $g(x,y)\in L^2_{\alpha,\ x}$. The Schwarz inequality yields the upper bound

$$\frac{1}{\sqrt{2\pi}}\sqrt{\int_{-\infty}^{+\infty}dx\frac{x^4}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{1+m})}+\||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{1+m})}}|\varphi_N^1(y)| = C|\varphi_N^1(y)|$$

Therefore, for the last term in (3.7) we obtain

$$\Big|\int_0^p\Big(\int_0^s\frac{\partial^2}{\partial q^2}\widehat{g}_0(q,y)dq\Big)ds\frac{\chi_1}{p^2}\Big| \leq \frac{C}{2}|\varphi_N^1(y)|\chi_1\in L^2(\mathbb{R}^{1+m})$$

Due to the behavior in the Fourier space of the first two terms, the expression (3.7) belongs to $L^2(\mathbb{R}^{1+m})$ if and only if

$$\widehat{g}_0(0,y)=0,\quad \frac{\partial}{\partial p}\widehat{g}_0(0,y)=0\quad a.e.\ ,$$

which is equivalent to

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})}=0,\quad (g(x,y),\varphi_N^k(y)x)_{L^2(\mathbb{R}^{1+m})}=0,\ 1\leq k\leq m_N$$

□

When the dimension $n=2$ we come up with the following analogous statement.

Lemma 3.3 *Let the assumptions of the Theorem 2 hold. Then the equation (3.5) possesses a solution* $u_0\in L^2(\mathbb{R}^{2+m}),\ m\in\mathbb{N}$ *if and only if*

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}=0,\quad (g(x,y),\varphi_N^k(y)x_i)_{L^2(\mathbb{R}^{2+m})}=0,\quad i=1,2,\quad 1\leq k\leq m_N$$

Proof. Let us use the expansion analogous to the one we had for proving the previous lemma.

$$\widehat{g}_0(p,y)=\widehat{g}_0(0,y)+\frac{\partial}{\partial|p|}\widehat{g}_0(0,\theta_p,y)|p|+\int_0^{|p|}\Big(\int_0^s\frac{\partial^2}{\partial|q|^2}\widehat{g}_0(|q|,\theta_p,y)d|q|\Big)ds\ ,$$

with

$$\widehat{g}_0(|p|,\theta_p,y)=\frac{1}{2\pi}\int_{\mathbb{R}^2}g_0(x,y)e^{-i|p||x|cos\theta}dx$$

and the angle between the $p=(|p|,\theta_p)$ and $x=(|x|,\theta_x)$ vectors on the plane is $\theta=\theta_p-\theta_x$. Therefore the first term in the right side of (3.6) when $n=2$ equals to

$$\frac{\widehat{g}_0(0,y)}{p^2}\chi_1+\frac{\partial}{\partial|p|}\widehat{g}_0(0,\theta_p,y)\frac{\chi_1}{|p|}+\int_0^{|p|}\Big(\int_0^s\frac{\partial^2}{\partial|q|^2}\widehat{g}_0(|q|,\theta_p,y)d|q|\Big)ds\frac{\chi_1}{p^2}\tag{3.9}$$

Obviously

$$\Big|\frac{\partial^2}{\partial|q|^2}\widehat{g}_0\Big|\le\frac{1}{2\pi}\int_{\mathbb{R}^2}|g_0(x,y)||x|^2dx$$

Using the estimate (3.8) we arrive at the upper bound for the right side of this inequality

$$\frac{1}{2\pi}|\varphi_N^1(y)|\int_{\mathbb{R}^2}dx\frac{|x|^2}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2dz}$$

with $\alpha>6$ such that $g(x,y)\in L^2_{\alpha,\ x}$. By means of the Schwarz inequality we estimate this from above as

$$\frac{1}{\sqrt{2\pi}}|\varphi_N^1(y)|\sqrt{\int_0^\infty d|x|\frac{|x|^5}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{2+m})}+\||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{2+m})}}=C|\varphi_N^1(y)|$$

Therefore, for the last term in (3.9) we arrive at

$$\frac{\chi_1}{p^2}\Big|\int_0^{|p|}\Big(\int_0^s\frac{\partial^2}{\partial|q|^2}\widehat{g}_0(|q|,\theta_p,y)d|q|\Big)ds\Big|\le\frac{C}{2}\chi_1|\varphi_N^1(y)|\in L^2(\mathbb{R}^{2+m})$$

A simple computation using the Fourier transform yields

$$\frac{\partial}{\partial|p|}\widehat{g}_0(0,\theta_p,y)=-\frac{i}{2\pi}\int_{\mathbb{R}^2}g_0(x,y)|x|cos\theta dx=Q_1(y)cos\theta_p+Q_2(y)sin\theta_p\ ,$$

where

$$Q_1(y):=-\frac{i}{2\pi}\int_{\mathbb{R}^2}g_0(x,y)x_1dx\ ,\quad Q_2(y):=-\frac{i}{2\pi}\int_{\mathbb{R}^2}g_0(x,y)x_2dx$$

and $x=(x_1,x_2)\in\mathbb{R}^2$. Computing the square of the $L^2(\mathbb{R}^{2+m})$ norm of the first two terms of (3.9) we arrive at

$$2\pi\int_0^1\frac{d|p|}{|p|^3}\int_{\mathbb{R}^m}dy|\widehat{g}_0(0,y)|^2+\pi\int_0^1\frac{d|p|}{|p|}\int_{\mathbb{R}^m}(|Q_1(y)|^2+|Q_2(y)|^2)dy\ ,$$

which is finite if and only if the quantities $\widehat{g}_0(0,y)$, $Q_1(y)$ and $Q_2(y)$ vanish a.e. This is equivalent to the orthogonality conditions

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}=0, \quad (g(x,y),x_1\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}=0,$$

$$(g(x,y),x_2\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}=0,$$

with $1\le k\le m_N$.

$\square$

Let us investigate how the situation with solvability conditions differs in dimensions $n=3,4$.

Lemma 3.4 *Let the assumptions of the Theorem 2 hold. Then the equation (3.5) possesses a solution* $u_0\in L^2(\mathbb{R}^{n+m})$, $n=3,4$, $m\in\mathbb{N}$ *if and only if*

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})}=0,\ n=3,4,\ 1\le k\le m_N$$

Proof. Let us expand as

$$\widehat{g}_0(p,y)=\widehat{g}_0(0,y)+\int_0^{|p|}\frac{\partial}{\partial|s|}\widehat{g}_0(|s|,\omega,y)d|s|$$

Thus by means of (3.6) we need to estimate

$$\frac{\chi_1}{p^2}[\widehat{g}_0(0,y)+\int_0^{|p|}\frac{\partial}{\partial|s|}\widehat{g}_0(|s|,\omega,y)d|s|] \tag{3.10}$$

By means of the Fourier transform

$$\frac{\partial}{\partial|p|}\widehat{g}_0(|p|,\omega,y)=\frac{-i}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}g_0(x,y)e^{-i|p||x|cos\theta}|x|cos\theta dx\ ,$$

where θ is the angle between p and x in $\mathbb{R}^n$. Using (3.8) along with the Schwarz inequality and $\alpha>n+2$ such that $g(x,y)\in L^2_{\alpha,\ x}$ we easily obtain

$$\Big|\frac{\partial}{\partial|s|}\widehat{g}_0\Big|\le\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}dx|x|\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2dz}|\varphi_N^1(y)|\le$$

$$\le\frac{1}{(2\pi)^{\frac{n}{2}}}\sqrt{\int_0^\infty|S^n|\frac{|x|^{n+1}}{1+|x|^\alpha}d|x|}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})}+\||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{n+m})}}|\varphi_N^1(y)|=C|\varphi_N^1(y)|\ ,$$

which implies the bound

$$\Big|\frac{\chi_1}{p^2}\int_0^{|p|}\frac{\partial}{\partial|s|}\widehat{g}_0(|s|,\omega,y)d|s|\Big|\le C\frac{\chi_1}{|p|}|\varphi_N^1(y)|\in L^2(\mathbb{R}^{n+m}),\ n=3,4$$

We finalize the proof of the lemma by estimating the square of the L^2 norm of the first term in (3.10).

$$|S^n| \int_{\mathbb{R}^m} dy |\widehat{g}_0(0,y)|^2 \int_0^1 d|p||p|^{n-5} < \infty, \quad n = 3, 4$$

if and only if $\widehat{g}_0(0,y) = 0$ a.e., which is equivalent to

$$(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})} = 0, \quad n = 3, 4, \quad 1 \le k \le m_N$$

□

Thus it remains only to establish the orthogonality conditions in dimensions five and higher in the x-variable under which the equation (3.5) admits a square integrable solution.

Lemma 3.5 *Let the assumptions of the Theorem 2 hold. Then the equation (3.5) possesses a solution* $u_0 \in L^2(\mathbb{R}^{n+m}),\ n \ge 5,\ m \in \mathbb{N}$.

Proof. We estimate the Fourier transform using the bound (3.8) along with the Schwarz inequality and $\alpha > n+2$ such that $g(x,y) \in L^2_{\alpha,\ x}$.

$$|\widehat{g}_0(p,y)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g_0(x,y)| dx \le \frac{|\varphi_N^1(y)|}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dx \sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2 dz} \le$$

$$\le \frac{|\varphi_N^1(y)|}{(2\pi)^{\frac{n}{2}}} \sqrt{\int_{\mathbb{R}^n} \frac{dx}{1+|x|^\alpha}} \sqrt{\int_{\mathbb{R}^n} dx (1+|x|^\alpha) \int_{\mathbb{R}^m} |g(x,z)|^2 dz} =$$

$$= \frac{|\varphi_N^1(y)|}{(2\pi)^{\frac{n}{2}}} \sqrt{\int_0^\infty d|x| \frac{|x|^{n-1}}{1+|x|^\alpha} |S^n|} \sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})} + \||x|^{\frac{\alpha}{2}} g\|^2_{L^2(\mathbb{R}^{n+m})}} =$$

$$= C|\varphi_N^1(y)|, \quad n \ge 5, \quad m \in \mathbb{N}$$

This enables us to obtain the bound on the square of the L^2 norm of the first term in the right side of (3.6).

$$\int_{\mathbb{R}^n} dp \int_{\mathbb{R}^m} dy \frac{|\widehat{g}_0|^2}{|p|^4} \chi_1 \le C \int_0^1 d|p||p|^{n-5} |S^n| \int_{\mathbb{R}^m} |\varphi_N^1(y)|^2 dy < \infty$$

which completes the proof of the lemma.

□

We proceed with establishing the conditions under which the equation (3.2) admits a square integrable solution. Let $\{P_{-,j}\}_{j=1}^{N-1}$ be the orthogonal projections onto the subspaces correspondent to $\{e_j\}_{j=1}^{N-1}$, the negative eigenvalues of the operator h, such that

$$P_- = \sum_{j=1}^{N-1} P_{-,j}, \quad P_{-,j} P_{-,k} = P_{-,j} \delta_{j,k}, \quad 1 \le j,k \le N-1$$

Applying these projection operators to both sides of the equation (3.2) and using the orthogonal decompositions $u_- = \sum_{j=1}^{N-1} u_{-,j}$ and $g_- = \sum_{j=1}^{N-1} g_{-,j}$ with $P_{-,j}u_- = u_{-,j}$ and $P_{-,j}g_- = g_{-,j}$ we easily obtain the system of equations equivalent to (3.2).

$$[-\Delta_x - \Delta_y + \mathcal{V}(y)]u_{-,j} = g_{-,j}, \quad 1 \le j \le N-1 \tag{3.11}$$

Without loss of generality we can assume that

$$g_{-,j}(x,y) = v_j(x)\varphi_j^1(y), \quad 1 \le j \le N-1 \quad , \tag{3.12}$$

where $v_j(x) := (g_{-,j}, \varphi_j^1)_{L^2(\mathbb{R}^m)} = (g, \varphi_j^1)_{L^2(\mathbb{R}^m)}$. By means of the Schwarz inequality

$$|v_j(x)| \le \sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2 dz}, \quad x \in \mathbb{R}^n \tag{3.13}$$

Hence the goal is to establish the conditions under which such an equation as (3.11) possesses a square integrable solution. We make the Fourier transform in the x-variable and using the fact that the operator $-\Delta_x$ does not have positive eigenvalues on $L^2(\mathbb{R}^n)$ obtain the expression for a solution of (3.11) as

$$\widehat{u}_{-,j}(p,y) = \frac{\widehat{v}_j(p)}{p^2 + e_j}\varphi_j^1(y), \quad 1 \le j \le N-1$$

We distinguish the two cases dependent upon the dimension of the problem in the first variable.

Lemma 3.6 *Let the assumptions of the Theorem 2 hold. Then the equation (3.11) possesses a solution $u_{-,j}(x,y) \in L^2(\mathbb{R}^{1+m})$, $m \in \mathbb{N}$ if and only if*

$$(g(x,y), \frac{e^{\pm i\sqrt{-e_j}x}}{\sqrt{2\pi}}\varphi_j^k(y))_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \le k \le m_j, \quad 1 \le j \le N-1$$

Proof. We express a solution of (3.11) as the sum of its regular and singular components

$$\widehat{u}_{-,j}(p,y) = \frac{\widehat{v}_j(p)\chi_{\Omega_\delta^c}}{p^2 + e_j}\varphi_j^1(y) + \frac{\widehat{v}_j(p)\chi_{\Omega_\delta}}{p^2 + e_j}\varphi_j^1(y) \ , \tag{3.14}$$

where the set in the Fourier space $\Omega_\delta := [\sqrt{-e_j} - \delta, \ \sqrt{-e_j} + \delta] \cup [-\sqrt{-e_j} - \delta, \ -\sqrt{-e_j} + \delta] = \Omega_\delta^+ \cup \Omega_\delta^-$ with $0 < \delta < \sqrt{-e_j}$ and Ω_δ^c is its complement, χ_{Ω_δ} and $\chi_{\Omega_\delta^c}$ are their characteristic functions. It is trivial to estimate the first term in the right side of (3.14) since we are away from the positive and negative singularities $\pm\sqrt{-e_j}$. Thus

$$\left|\frac{\widehat{v}_j(p)\chi_{\Omega_\delta^c}}{p^2 + e_j}\varphi_j^1(y)\right| \le C|\varphi_j^1(y)||\widehat{v}_j(p)|\chi_{\Omega_\delta^c} \ ,$$

which along with (3.13) enables us to estimate the square of its L^2 norm.

$$\int_{-\infty}^{+\infty} dp \int_{\mathbb{R}^m} dy |\varphi_j^1(y)|^2 |\widehat{v}_j(p)|^2 \chi_{\Omega_\delta^c} \leq \|v_j\|^2_{L^2(\mathbb{R})} \leq \|g\|^2_{L^2(\mathbb{R}^{1+m})} < \infty$$

To obtain the conditions under which the remaining term in (3.14) is square integrable we first study its behavior near its negative singularity using the formula

$$\widehat{v}_j(p) = \int_{-\sqrt{-e_j}}^{p} \frac{d\widehat{v}_j(s)}{ds} ds + \widehat{v}_j(-\sqrt{-e_j})$$

Thus one needs to estimate

$$\frac{\widehat{v}_j(-\sqrt{-e_j}) + \int_{-\sqrt{-e_j}}^{p} \frac{d\widehat{v}_j(s)}{ds} ds}{p^2 + e_j} \chi_{\Omega_\delta^-} \varphi_j^1(y) \tag{3.15}$$

We derive the upper bound for the derivative using (3.13) along with the Schwarz inequality with $\alpha > 5$ such that $g(x,y) \in L^2_{\alpha,\ x}$.

$$\Big|\frac{d\widehat{v}_j(p)}{dp}\Big| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx |x||v_j(x)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{|x|}{\sqrt{1+|x|^\alpha}} \sqrt{1+|x|^\alpha} \sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2 dz} \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \sqrt{\int_{-\infty}^{+\infty} dx \frac{x^2}{1+|x|^\alpha}} \sqrt{\int_{-\infty}^{\infty} dx (1+|x|^\alpha) \int_{\mathbb{R}^m} |g(x,z)|^2 dz} =$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\int_{-\infty}^{+\infty} dx \frac{x^2}{1+|x|^\alpha}} \sqrt{\|g\|^2_{L^2(\mathbb{R}^{1+m})} + \||x|^{\frac{\alpha}{2}} g\|^2_{L^2(\mathbb{R}^{1+m})}} = C < \infty$$

This enables us to prove the square integrability for the second term in (3.15).

$$\left|\frac{\int_{-\sqrt{-e_j}}^{p} \frac{d\widehat{v}_j(s)}{ds} ds}{p^2 + e_j} \chi_{\Omega_\delta^-} \varphi_j^1(y)\right| \leq \frac{C}{|p - \sqrt{-e_j}|} \chi_{\Omega_\delta^-} |\varphi_j^1(y)| \leq \frac{C}{2\sqrt{-e_j} - \delta} \chi_{\Omega_\delta^-} |\varphi_j^1(y)| \in L^2(\mathbb{R}^{1+m})$$

Near the positive singularity we use the identity

$$\widehat{v}_j(p) = \int_{\sqrt{-e_j}}^{p} \frac{d\widehat{v}_j(s)}{ds} ds + \widehat{v}_j(\sqrt{-e_j})$$

to study the conditions of the square integrability of the term

$$\frac{\widehat{v}_j(\sqrt{-e_j}) + \int_{\sqrt{-e_j}}^{p} \frac{d\widehat{v}_j(s)}{ds} ds}{p^2 + e_j} \chi_{\Omega_\delta^+} \varphi_j^1(y) \tag{3.16}$$

Analogously to the situation at the negative singularity we prove the square integrability of the second term in (3.16) using the bound on the derivative involved in it. Hence

$$\left|\frac{\int_{\sqrt{-e_j}}^{p}\frac{d\widehat{v}_j(s)}{ds}ds}{p^2+e_j}\chi_{\Omega_\delta^+}\varphi_j^1(y)\right| \leq \frac{C}{|p+\sqrt{-e_j}|}\chi_{\Omega_\delta^+}|\varphi_j^1(y)| \leq$$

$$\leq \frac{C}{2\sqrt{-e_j}-\delta}\chi_{\Omega_\delta^+}|\varphi_j^1(y)| \in L^2(\mathbb{R}^{1+m})$$

Thus it remains to derive the conditions under which the first term in (3.15) and the first term in (3.16) are square integrable. Estimating the square of the $L^2(\mathbb{R}^{1+m})$ norm of $\dfrac{\widehat{v}_j(-\sqrt{-e_j})}{p^2+e_j}\chi_{\Omega_\delta^-}\varphi_j^1(y)+\dfrac{\widehat{v}_j(\sqrt{-e_j})}{p^2+e_j}\chi_{\Omega_\delta^+}\varphi_j^1(y)$ we easily arrive at

$$\int_{-\sqrt{-e_j}-\delta}^{-\sqrt{-e_j}+\delta}dp\frac{|\widehat{v}_j(-\sqrt{-e_j})|^2}{(p^2+e_j)^2}+\int_{\sqrt{-e_j}-\delta}^{\sqrt{-e_j}+\delta}dp\frac{|\widehat{v}_j(\sqrt{-e_j})|^2}{(p^2+e_j)^2}\ ,$$

which can be bounded below by

$$\frac{|\widehat{v}_j(-\sqrt{-e_j})|^2}{(2\sqrt{-e_j}+\delta)^2}\int_{-\delta}^{\delta}\frac{ds}{s^2}+\frac{|\widehat{v}_j(\sqrt{-e_j})|^2}{(2\sqrt{-e_j}+\delta)^2}\int_{-\delta}^{\delta}\frac{ds}{s^2}$$

This bound implies that the necessary and sufficient conditions for the existence of $u_{-,j}(x,y) \in L^2(\mathbb{R}^{1+m})$ solving the equation (3.11) are

$$\widehat{v}_j(\sqrt{-e_j})=0, \quad \widehat{v}_j(-\sqrt{-e_j})=0$$

which by means of the definition of the functions $v_j(x)$ is equivalent to

$$(g(x,y),\frac{e^{\pm i\sqrt{-e_j}x}}{\sqrt{2\pi}}\varphi_j^k(y))_{L^2(\mathbb{R}^{1+m})}=0, \quad 1\leq k\leq m_j, \quad 1\leq j\leq N-1$$

$\square$

After establishing the solvability conditions for the equation (3.11) when the situation is one dimensional in the first variable we turn our attention to the cases of dimensions two and higher.

Lemma 3.7 *Let the assumptions of the Theorem 2 hold. Then the equation (3.11) possesses a solution* $u_{-,j}(x,y) \in L^2(\mathbb{R}^{n+m}),\ n\geq 2,\ m\in\mathbb{N}$ *if and only if*

$$(g(x,y),\frac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\varphi_j^k(y))_{L^2(\mathbb{R}^{n+m})}=0,\ a.e.\ p\in S^n_{\sqrt{-e_j}}, \quad 1\leq k\leq m_j, \quad 1\leq j\leq N-1$$

Proof. It is convenient to represent a solution of (3.11) as the sum of the singular and the regular parts

$$\widehat{u}_{-,j}(p,y) = \frac{\widehat{v}_j(p)\chi_{A_\delta}}{p^2+e_j}\varphi_j^1(y) + \frac{\widehat{v}_j(p)\chi_{A_\delta^c}}{p^2+e_j}\varphi_j^1(y) \ , \tag{3.17}$$

where the spherical layer in Fourier space $A_\delta := \{p \in \mathbb{R}^n : \sqrt{-e_j} - \delta \leq |p| \leq \sqrt{-e_j} + \delta\}$, its complement in $\mathbb{R}^n$ is A_δ^c, their characteristic functions are χ_{A_δ} and $\chi_{A_\delta^c}$ respectively and $0 < \delta < \sqrt{-e_j}$. Clearly for the second term in the right side of (3.17) we have the upper bound

$$\Big|\frac{\widehat{v}_j(p)\chi_{A_\delta^c}\varphi_j^1(y)}{p^2+e_j}\Big| \leq \frac{|\widehat{v}_j(p)||\varphi_j^1(y)|}{\delta\sqrt{-e_j}} \ ,$$

such that via (3.13) $\int_{\mathbb{R}^n} |\widehat{v}_j(p)|^2 dp \int_{\mathbb{R}^m} |\varphi_j^1(y)|^2 dy = \|v_j\|^2_{L^2(\mathbb{R}^n)} \leq \|g\|^2_{L^2(\mathbb{R}^{n+m})} < \infty$. Hence the first term in the right side of (3.17) will play the crucial role for establishing the solvability conditions for the equation (3.11). We will make use of the formula

$$\widehat{v}_j(p) = \int_{\sqrt{-e_j}}^{|p|} \frac{\partial \widehat{v}_j}{\partial |s|}(|s|,\omega)d|s| + \widehat{v}_j(\sqrt{-e_j},\omega)$$

to get the estimate for

$$\frac{\int_{\sqrt{-e_j}}^{|p|} \frac{\partial \widehat{v}_j}{\partial |s|}(|s|,\omega)d|s| + \widehat{v}_j(\sqrt{-e_j},\omega)}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y)$$

Let us derive the upper bound for the derivative of the Fourier transform involved in it using (3.13) along with the Schwarz inequality, $\alpha > 6$ for $n = 2$ and $\alpha > n+2$ for $n \geq 3$ such that $g(x,y) \in L^2_{\alpha,\ x}$.

$$\Big|\frac{\partial \widehat{v}_j}{\partial |p|}\Big| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |v_j(x)||x|dx \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} dx|x|\sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2 dz} =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} dx \frac{|x|}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2 dz} \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\sqrt{\int_{\mathbb{R}^n} dx \frac{|x|^2}{1+|x|^\alpha}}\sqrt{\int_{\mathbb{R}^n} dx(1+|x|^\alpha)\int_{\mathbb{R}^m} |g(x,z)|^2 dz} =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\sqrt{\int_0^\infty d|x||S^n|\frac{|x|^{n+1}}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})} + \||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{n+m})}} = C < \infty$$

Therefore

$$\Big|\frac{\int_{\sqrt{-e_j}}^{|p|} \frac{\partial \widehat{v}_j}{\partial |s|}(|s|,\omega)d|s|}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y)\Big| \leq \frac{C}{\sqrt{-e_j}}\chi_{A_\delta}|\varphi_j^1(y)| \in L^2(\mathbb{R}^{n+m})$$

and it remains to estimate from below the square of the L^2 norm of the term

$$\frac{\widehat{v}_j(\sqrt{-e_j},\omega)}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y)$$

Thus

$$\int_{\mathbb{R}^n} dp \int_{\mathbb{R}^m} dy \frac{|\widehat{v}_j(\sqrt{-e_j},\omega)|^2}{(p^2+e_j)^2}\chi_{A_\delta}|\varphi_j^1(y)|^2 \geq$$

$$\geq \int_{\sqrt{-e_j}-\delta}^{\sqrt{-e_j}+\delta} \frac{d|p||p|^{n-1}}{(|p|-\sqrt{-e_j})^2(2\sqrt{-e_j}+\delta)^2}\int_{S^n} d\omega |\widehat{v}_j(\sqrt{-e_j},\omega)|^2 \geq$$

$$\geq \frac{(\sqrt{-e_j}-\delta)^{n-1}}{(2\sqrt{-e_j}+\delta)^2}\int_{S^n} d\omega|\widehat{v}_j(\sqrt{-e_j},\omega)|^2 \int_{-\delta}^{\delta}\frac{ds}{s^2}\ ,$$

which yields the necessary and sufficient conditions of solvability of the equation (3.11) in $L^2(\mathbb{R}^{n+m}),\ n\geq 2$, namely $\widehat{v}_j(\sqrt{-e_j},\omega)=0$ a.e. on the sphere $S^n_{\sqrt{-e_j}}$. Using the definition of the functions $v_j(x)$ we easily arrive at

$$(g(x,y), \frac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\varphi_j^k(y))_{L^2(\mathbb{R}^{n+m})}=0,\ a.e.\ p\in S^n_{\sqrt{-e_j}},\ 1\leq k\leq m_j,\ 1\leq j\leq N-1$$

□

Having established the orthogonality conditions in the lemmas above which guarantee the existence of square integrable solutions for our equations we conclude the proof of Theorem 2.

Proof of Theorem 2. We construct the solution of the equation (1.3) as $u:=u_+ + u_0+\sum_{j=1}^{N-1}u_{-,j}$, where the existence of $u_+\in L^2(\mathbb{R}^{n+m})$ is guaranteed by Lemma 3.1, of $u_0\in L^2(\mathbb{R}^{n+m})$ by Lemmas 3.2–3.5, of $\{u_{-,j}\}_{j=1}^{N-1}\in L^2(\mathbb{R}^{n+m})$ by Lemmas 3.6 and 3.7.

Suppose the equation (1.3) admits two solutions $u_1,u_2\in L^2(\mathbb{R}^{n+m})$. Then their difference $w:=u_1-u_2\in L^2(\mathbb{R}^{n+m})$ solves the homogeneous problem with separation of variables

$$\mathcal{L}w=0$$

which admits two types of solutions. The first ones are of the form $\gamma(x)\varphi_N^k(y),\ 1\leq k\leq m_N$ with $\gamma(x)$ harmonic. The second ones are of the kind $\frac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\varphi_j^k(y)$ with $p\in S^n_{\sqrt{-e_j}},\ 1\leq j\leq N-1,\ 1\leq k\leq m_j$. In both cases to vanish is the only possibility for them to belong to the space $L^2(\mathbb{R}^{n+m})$.

□

Appendix

Fact 1 *Let $f(x) \in L^2(\mathbb{R}^3)$ and $|x|f(x) \in L^1(\mathbb{R}^3)$. Then $f(x) \in L^1(\mathbb{R}^3)$.*

Proof. The norm $\|f\|_{L^1(\mathbb{R}^3)}$ is being estimated from above by means of the Schwarz inequality as

$$\sqrt{\int_{|x|\leq 1} |f(x)|^2 dx}\sqrt{\int_{|x|\leq 1} dx} + \int_{|x|>1} |x||f(x)|dx \leq \|f\|_{L^2(\mathbb{R}^3)}\sqrt{\frac{4\pi}{3}} + \||x|f\|_{L^1(\mathbb{R}^3)} < \infty$$

$\square$

Acknowledgement *The first author thanks I.M.Sigal for partial support via NSERC grant No. 7901.*

References

[AB] C. Amrouche, F. Bonzom: "Mixed exterior Laplace's problem", J. Math. Anal. Appl., **338**, 124-140 (2008).

[B] N. Benkirane: "Propriete d'indice en theorie Holderienne pour des operateurs elliptiques dans R^n" CRAS, **307**, serie I 577-580 (1988).

[BP] P. Bolley, T.L. Pham: "Propriété d'indice en théorie Holderienne pour des opérateurs différentiels elliptiques dans R^n", J. Math. Pures Appl., **72**, 105-119 (1993).

[CFKS87] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon: "Schrödinger Operators with Application to Quantum Mechanics and Global Geometry", Berlin-Heidelberg-New York: Springer-Verlag, (1987)

[DMV05] A. Ducrot, M. Marion, V. Volpert: "Systemes de réaction-diffusion sans propriété de Fredholm", CRAS, **340**, 659–664, (2005).

[DMV08] A. Ducrot, M. Marion, V. Volpert: "Reaction-diffusion problems with non Fredholm operators", Advances Diff. Equations, to appear.

[JMST] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting: "Applied Analysis", 349 pages (in preparation)

[K65] T. Kato: "Wave operators and similarity for some non-selfadjoint operators", Math.Ann., **162**, 258-279, (1965/1966)

[KV06] S. Kryzhevich, V. Volpert: "Different types of solvability conditions for differential operators", Electron. J. Diff. Eqns., **100**, 1–24, (2006).

[LL97] E. Lieb, M. Loss: "Analysis", Graduate studies in Mathematics, 14, American Mathematical Society, Providence, RI, (1997).

[L] R.B. Lockhart: "Fredholm property of a class of elliptic operators on non-compact manifolds", Duke Math. J. **48**, 289-312 (1981).

[LM] R.B. Lockhart, R.C. McOwen: "On elliptic systems in R^n", Acta Mmathematica, **150**, 125-135 (1983).

[NW] L. Nirenberg, H.F. Walker: "The null spaces of elliptic partial differential operators in R^n", J. Math. Analysis and Appl., **42**, 271-301 (1973).

[RS79] M. Reed, B. Simon: "Methods of Modern Mathematical Physics, III. Scattering Theory", Academic Press, (1979).

[RS04] I. Rodnianski, W. Schlag: "Time decay for solutions of Schrödinger equations with rough and time-dependent potentials", Invent.Math., **155, 3**, 451–513, (2004).

[S71] B. Simon: "Quantum mechanics for Hamiltonians defined as quadratic forms", Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1971.

[VV06] A. Volpert, V. Volpert: "Fredholm property of elliptic operators in unbounded domains", Trans. Moscow Math. Soc., **67**, 127–197, (2006).

[VKMP02] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski: "Solvability conditions for elliptic problems with non-Fredholm operators", Appl. Math., **29, 2**, 219–238, (2002).

[W1] H.F. Walker: "On the null-space of first-order elliptic partial differential operators in R^n", Proc. of AMS, **30**, No. 2, 278-286 (1971).

[W2] H.F. Walker: "A Fredholm theory for a class of first-order elliptic partial differential operators in R^n", Trans. Amer. Math. Soc., **165**, 75-86 (1972).