Properness and Topological Degree for Nonlocal Reaction-Diffusion Operators

Narcisa Apreutesei, Vitaly Volpert

Properness and Topological Degree for Nonlocal Reaction-Diffusion Operators

N. Apreutesei[1] *, V. Volpert[2]

[1] Department of Mathematics, Technical University "Gh. Asachi" of Iasi
Bd. Carol. I, 700506 Iasi, Romania
E-mail: napreut@gmail.com
[2] Institut Camille Jordan, UMR 5208 CNRS, University Lyon 1
69622 Villeurbanne, France
E-mail: volpert@math.univ-lyon1.fr

Abstract. The paper is devoted to integro-differential operators, which correspond to nonlocal reaction-diffusion equations considered on the whole axis. Their Fredholm property and properness will be proved. This will allow one to define the topological degree.

Key words: nonlocal reaction-diffusion equations, Fredholm property, properness, topological degree, travelling waves

AMS subject classification: 35K57, 45K05, 47A53, 47H11

*Corresponding author. Tel.: +40 232 213737; fax: +40 232 217720.

1 Introduction

Consider the semi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F\left(u, J\left(u\right)\right), \tag{1.1}$$

where

$$J\left(u\right) = \int_{-\infty}^{\infty} \phi\left(x-y\right) u\left(y,t\right) dy.$$

Here $\phi : \mathbb{R} \to \mathbb{R}$ is a bounded function, not necessarily continuous, $\phi \geq 0$ on $\mathbb{R}$. The support of the function ϕ is supposed to be bounded, supp $\phi \subseteq [-N, N]$. We will also assume that $\int_{-\infty}^{\infty} \phi\left(y\right) dy = 1$. Conditions on the function F will be specified below.

Integro-differential equations of this type arise in population dynamics (see [10], [19] and references therein). They are referred to as nonlocal reaction-diffusion equations. A travelling wave solution of equation (1.1) is a solution of this equation of the particular form $u\left(x,t\right) = w\left(x - ct\right)$. It satisfies the equation

$$w'' + cw' + F\left(w, J\left(w\right)\right) = 0. \tag{1.2}$$

The constant $c \in \mathbb{R}$ is the wave speed. It is unknown and should be found together with the function $w(x)$. There are numerous works devoted to the existence [1]-[5], stability and nonlinear dynamics [10], [19], [6] - [15] of travelling wave solutions of some particular cases of equation (1.1). Properties of travelling waves are determined by the properties of the integro-differential operator in the left-hand side of (1.2). In this paper we will study the Fredholm property of this operator and its properness. We will use them to define the topological degree and will discuss some applications.

Let $E = C^{2+\alpha}\left(\mathbb{R}\right), E^0 = C^{\alpha}\left(\mathbb{R}\right), 0 < \alpha < 1$ the usual Holder spaces endowed with the norms

$$||u||_{E^0} = \sup_{x,y\in\mathbb{R}} \frac{\left|u\left(x\right) - u\left(y\right)\right|}{\left|x-y\right|^{\alpha}} + \sup_{x\in\mathbb{R}} |u(x)|, \;\; ||u||_E = ||u||_{E^0} + ||u'||_{E^0} + ||u''||_{E^0}.$$

We are interested in the solutions of equation (1.2) with the limits $w^{\pm}$ at $\pm\infty$, where the values $w^{\pm}$ are such that $F(w^{\pm}, w^{\pm}) = 0$. We are looking for the solutions w of (1.2) under the form $w = u + \psi$, where $\psi \in C^{\infty}\left(\mathbb{R}\right)$, such that $\psi\left(x\right) = w^+$ for $x \geq 1$ and $\psi\left(x\right) = w^-$ for $x \leq -1$. Thus equation (1.2) becomes

$$\left(u+\psi\right)'' + c\left(u+\psi\right)' + F\left(u+\psi, J\left(u+\psi\right)\right) = 0. \tag{1.3}$$

Denote by A the operator in the left-hand side of (1.3), that is $A : E \to E^0$,

$$Au = \left(u+\psi\right)'' + c\left(u+\psi\right)' + F\left(u+\psi, J\left(u+\psi\right)\right). \tag{1.4}$$

Suppose that F is differentiable with respect to both variables. The linearization of A about a function $u_1 \in E$ is the operator $L : E \to E^0$,

$$Lu \equiv A'(u_1)\,u = u'' + cu' + \frac{\partial F}{\partial u}\left(u_1+\psi, J\left(u_1+\psi\right)\right) u + \frac{\partial F}{\partial U}\left(u_1+\psi, J\left(u_1+\psi\right)\right) J(u)\,, \quad (1.5)$$

where $\partial F/\partial u$ and $\partial F/\partial U$ are the derivatives of $F(u,U)$ with respect to the first and to the second variable, respectively.

For the linearized operator L, we introduce the limiting operators. Since for $w_1 = u_1+\psi$, there exist the limits $\lim_{x\to\pm\infty} w_1(x) = w^{\pm}$, it follows that $J(w_1) = J(u_1+\psi) \to w^{\pm}$ as $x \to \pm\infty$ and the limiting operators are given by

$$L^{\pm}u = u'' + cu' + a^{\pm}u + b^{\pm}J(u)\,, \quad (1.6)$$

where

$$a^{\pm} = \frac{\partial F}{\partial u}\left(w^{\pm}, w^{\pm}\right), \quad b^{\pm} = \frac{\partial F}{\partial U}\left(w^{\pm}, w^{\pm}\right).$$

We will now recall the main definitions and results concerning the essential spectrum and Fredholm property for linear operators and the properness of nonlinear operators.

Essential spectrum and Fredholm property. Let us recall that a linear operator $M : E_1 \to E_2$ acting from a Banach space E_1 into another Banach space E_2 is called a Fredholm operator if its kernel has a finite dimension, its image is closed, and the codimension of the image is also finite. The last two conditions are equivalent to the following solvability condition: the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space E_2^*.

Suppose that $E_1 \subset E_2$. By definition, the essential spectrum of the operator L is the set of all complex λ for which the operator $L - \lambda I$, where I is the identity operator, does not satisfy the Fredholm property. The essential spectrum of general elliptic boundary value problems in unbounded domains can be determined in terms of limiting operators [16]. For the integro-differential operators under consideration, since they have constant coefficients at infinity, the essential spectrum can be found explicitly. It is proved [3], [4] that the operator $L - \lambda I$ is normally solvable † with a finite-dimensional kernel if and only if the equations $L^{\pm}u = \lambda u$ do not have nonzero bounded solutions. Applying the Fourier transform to the last equations, we obtain

$$\lambda_{\pm}(\xi) = -\xi^2 + ci\xi + a^{\pm} + b^{\pm}\tilde{\phi}(\xi), \quad \xi \in \mathbb{R},$$

where $\tilde{\phi}(\xi)$ is the Fourier transform of the function $\phi(x)$. Thus, the operator L is normally solvable with a finite-dimensional kernel if and only if the curves $\lambda_{\pm}(\xi)$ on the complex plane do not pass through the origin. Under some additional conditions, it can be also shown that

†By definition, an operator $A : E_1 \to E_2$ is normally solvable if the equation $Au = f$ has a solution if and only if $\phi(f) = 0$ for all functionals ϕ from some linear subspace in the dual space E_2^*. This is equivalent to the condition that the range of the operator is closed.

the codimension of the operator is finite, that is it satisfies the Fredholm property, and its index can be found.

A nonlinear operator $B : E_1 \to E_2$ is called Fredholm if the linearized operator B' satisfies this property. In what follows we will use the Fredholm property in some weighted spaces (see below).

Properness and topological degree. An operator $B : E_1 \to E_2$ is called proper on closed bounded sets if the intersection of the inverse image of a compact set $K \subset E_2$ with any closed bounded set in E_1 is compact. For the sake of brevity, we will call such operators proper. It is an important property because it implies that the set of solution of the operator equation $B(u) = 0$ is compact.

It appears that elliptic (or ordinary differential) operators are not generally proper when considered in Hölder or Sobolev spaces in unbounded domains. We illustrate this situation with a simple example. Consider the equation

$$w'' + H(w) = 0, \quad x \in \mathbb{R}, \tag{1.7}$$

where $H(w) = w(w-1)$. It can be verified that this equation has a positive solution $w(x)$, which converges to zero at infinity. This convergence is exponential. So the solution belongs to Hölder and to Sobolev spaces. Along with the function $w(x)$, any shifted function $w(x+h)$, $h \in \mathbb{R}$ is also a solution. Hence there is a family of solutions, and the set of solutions is not compact. Similar examples can be constructed for the integro-differential equation.

In order to obtain proper operators, we introduce weighted spaces $C_\mu^{k+\alpha}(\mathbb{R})$ with a growing at infinity polynomial weight function $\mu(x)$. The norm in this space is given by the equality

$$\|u\|_{C_\mu^{k+\alpha}(\mathbb{R})} = \|\mu u\|_{C^{k+\alpha}(\mathbb{R})}.$$

Let us return to the previous example. The family of functions $w(x+h)$ is not uniformly bounded in the weighted space. If we take any bounded closed set in the function space, it can contain the solutions $w(x+h)$ only for a compact set of the values of h. Therefore the set of solutions is compact in any bounded closed set. This example shows the role of weighted spaces for the properness of the operators.

Properness of general nonlinear elliptic problems in unbounded domains and in weighted spaces is proved in [20]. In this work, we will prove properness of the integro-differential operators. After that, using the construction of the topological degree for Fredholm and proper operators with the zero index [20], we will define the degree for the integro-differential operators. We will finish this paper with some applications of these methods to travelling waves solutions.

2 Properness in weighted spaces

In this section we study the properness of the semilinear operator A.

Definition 2.1. If X, Y are Banach spaces, an operator $A : X \to Y$ is called *proper* if for any compact set $D \subset Y$ and any bounded closed set $B \subset X$, the intersection $A^{-1}(D) \cap B$ is a compact set in X.

Remark 2.2. The operator $A : E = C^{2+\alpha}(\mathbb{R}) \to E^0 = C^\alpha(\mathbb{R})$ may not be proper from E to E^0 (see the comments related to equation (1.7) from the introduction).

We will show in the sequel that A is proper in some weighted spaces associated to E and E^0.

Let $\mu : \mathbb{R} \to \mathbb{R}$ be the function given by $\mu(x) = 1 + x^2,\ x \in \mathbb{R}$. Denote $E = C^{2+\alpha}(\mathbb{R})$, $E^0 = C^\alpha(\mathbb{R})$ endowed with the usual norms $||\cdot||_E$ and $||\cdot||_{E^0}$. We will work in the weighted Holder spaces E_μ and E^0_μ, which are E and E^0, respectively, with the norms $||u||_\mu = ||\mu u||_E$ and $||u||_{0\mu} = ||\mu u||_{E^0}$.

We begin with the following estimate for the integral term $J(u)$.

Lemma 2.3. *Suppose that $\mu(x) = 1 + x^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a function such that $\phi \geq 0$ on $\mathbb{R}$, $supp\phi = [-N, N]$ is bounded, $\int_{-\infty}^{\infty} \phi(y)\,dy = 1$ and $\phi \in E^0$. Then*

$$||J(u)||_{0\mu} \leq K||\mu u||_{C(\mathbb{R})},\ (\forall)\, u \in E^0, \tag{2.1}$$

for some constant $K > 0$.

Proof. If supp $\phi = [-N, N]$, then supp $\phi(x - \cdot) = [x - N, x + N]$. First we write

$$\mu(x) J(u)(x) = \int_{x-N}^{x+N} \frac{\mu(x)}{\mu(y)} \phi(x - y)\mu(y) u(y)\,dy.$$

Since $\mu(x)/\mu(y)$ is bounded for $|x - y| \leq N$ and $\int_{-\infty}^{\infty} \phi(y)\,dy = 1$, we have $||\mu J(u)||_{C(\mathbb{R})} \leq K_1||\mu u||_{C(\mathbb{R})}$, for some positive constant K_1.

For every $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, denote

$$H(x_1, x_2) = \frac{|\mu(x_1) J(u)(x_1) - \mu(x_2) J(u)(x_2)|}{|x_1 - x_2|^\alpha}.$$

If $|x_1 - x_2| \geq 1$, we have $H(x_1, x_2) \leq 2K_1||\mu u||_{C(\mathbb{R})}$. If $|x_1 - x_2| < 1$, then

$$H(x_1, x_2) = \frac{1}{|x_1 - x_2|^\alpha} | \int_{x_1-N}^{x_1+N} [\mu(x_1)\phi(x_1 - y) - \mu(x_2)\phi(x_2 - y)] u(y)\,dy +$$

$$+ \int_{x_1-N}^{x_1+N} \mu(x_2)\phi(x_2 - y) u(y)\,dy - \int_{x_2-N}^{x_2+N} \mu(x_2)\phi(x_2 - y) u(y)\,dy| \leq$$

$$\leq | \int_{x_1-N}^{x_1+N} \{ \frac{[\mu(x_1) - \mu(x_2)]\phi(x_1 - y)}{\mu(y)|x_1 - x_2|^\alpha} + \frac{\mu(x_2)[\phi(x_1 - y) - \phi(x_2 - y)]}{\mu(y)|x_1 - x_2|^\alpha} \} \mu(y) u(y)\,dy| +$$

$$+ \frac{1}{|x_1 - x_2|^\alpha} \left| \int_{x_1-N}^{x_2-N} \frac{\mu(x_2)}{\mu(y)} \phi(x_2 - y)\mu(y) u(y)\,dy \right| +$$

$$+\frac{1}{|x_1-x_2|^{\alpha}}\left|\int_{x_2+N}^{x_1+N}\frac{\mu(x_2)}{\mu(y)}\phi(x_2-y)\,\mu(y)\,u(y)\,dy\right|.$$

Since $|x_1-x_2|<1$ and $|x_1-y|\le N$, then $|x_2-y|\le N+1$. In this case, the boundedness of ϕ and of $\dfrac{\mu(x_1)-\mu(x_2)}{\mu(y)\,|x_1-x_2|^{\alpha}}$, implies that

$$H(x_1,x_2)\le K_2||\mu u||_{C(\mathbb{R})}+K_3||\phi||_{E^0}||\mu u||_{C(\mathbb{R})}+2K_4\,|x_1-x_2|^{1-\alpha}\,||\mu u||_{C(\mathbb{R})},$$

for some $K_2,K_3,K_4>0$. Thus the desired estimate holds and the lemma is proved.

We study the operator A acting from E_μ into E_μ^0. In order to introduce a topological degree (in Section 4), we prove the properness of A in the more general case when the coefficient c and function F depend also on a parameter $\tau\in[0,1]$. Let $A_\tau: E_\mu\to E_\mu^0$, $\tau\in[0,1]$ be the operator defined through

$$A_\tau u=(u+\psi)''+c(\tau)(u+\psi)'+F_\tau(u+\psi,J(u+\psi)). \qquad (2.2)$$

We note that the linearization L_τ of A_τ about a function $u_1\in E_\mu$ is

$$L_\tau u=u''+c(\tau)u'+\frac{\partial F_\tau}{\partial u}(u_1+\psi,J(u_1+\psi))\,u+\frac{\partial F_\tau}{\partial U}(u_1+\psi,J(u_1+\psi))\,J(u),$$

while its limiting operators are given by

$$L_\tau^{\pm}u=u''+c(\tau)u'+\frac{\partial F_\tau}{\partial u}\left(w^{\pm},w^{\pm}\right)u+\frac{\partial F_\tau}{\partial U}\left(w^{\pm},w^{\pm}\right)J(u).$$

Assume that the following hypotheses are satisfied:

$(H1)$ For any $\tau\in[0,1]$, the function $F_\tau(u,U)$ and its derivatives with respect to u and U satisfy the Lipschitz condition: there exists $K>0$ such that

$$|F_\tau(u_1,U_1)-F_\tau(u_2,U_2)|\le K(|u_1-u_2|+|U_1-U_2|),$$

for any $(u_1,U_1),(u_2,U_2)\in\mathbb{R}^2$. Similarly for $\partial F_\tau/\partial u$ and $\partial F_\tau/\partial U$:

$$\left|\frac{\partial F_\tau(u_1,U_1)}{\partial u}-\frac{\partial F_\tau(u_2,U_2)}{\partial u}\right|\le K(|u_1-u_2|+|U_1-U_2|),$$
$$\left|\frac{\partial F_\tau(u_1,U_1)}{\partial U}-\frac{\partial F_\tau(u_2,U_2)}{\partial U}\right|\le K(|u_1-u_2|+|U_1-U_2|).$$

$(H2)$ $c(\tau)$, $F_\tau(u,U)$ and the derivatives of $F_\tau(u,U)$ are Lipschitz continuous in τ, i. e. there exists a constant $c>0$ such that

$$|c(\tau)-c(\tau_0)|\le c|\tau-\tau_0|,\ \ |F_\tau(u,U)-F_{\tau_0}(u,U)|\le c|\tau-\tau_0|,$$

$$\left|\frac{\partial F_\tau(u,U)}{\partial u}-\frac{\partial F_{\tau_0}(u,U)}{\partial u}\right|\le c|\tau-\tau_0|,\ \left|\frac{\partial F_\tau(u,U)}{\partial U}-\frac{\partial F_{\tau_0}(u,U)}{\partial U}\right|\le c|\tau-\tau_0|.$$

$(\forall)\, \tau, \tau_0 \in [0,1]$, for all (u,U) from any bounded set in $\mathbb{R}^2$.

$(H3)$ (**Condition** NS) For any $\tau \in [0,1]$, the limiting equations

$$u'' + c(\tau) u' + \frac{\partial F_\tau}{\partial u}\left(w^\pm, w^\pm\right) u + \frac{\partial F_\tau}{\partial U}\left(w^\pm, w^\pm\right) J(u) = 0$$

do not have nonzero solutions in E.

Lemma 2.4. *Suppose that conditions $(H1)-(H2)$ hold. If $\tau_n \to \tau_0$ and $\mu u_n \to \mu u_0$ in $C(\mathbb{R})$, then*

$$||F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi))||_{0\mu} \to 0.$$

Proof. We have the equality

$$F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi)) =$$
$$F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_n}(u_0+\psi, J(u_0+\psi)) + F_{\tau_n}(u_0+\psi, J(u_0+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi)).$$

Condition $(H1)$ leads to the estimate of the first difference

$$|F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_n}(u_0+\psi, J(u_0+\psi))| \le K\left(|u_n-u_0| + |J(u_n-u_0)|\right).$$

In view of hypothesis $\mu u_n \to \mu u_0$ *in* $C(\mathbb{R})$, the above inequality allows us to conclude that the weighted norm converges to zero.

In order to estimate the second difference, we begin with the following representation:

$$F_{\tau_n}(u_0+\psi, J(u_0+\psi)) - F_{\tau_n}(\psi, J(\psi)) =$$

$$J(u_0)\int_0^1 \frac{\partial F_{\tau_n}(u_0+\psi, J(\psi)+tJ(u_0))}{\partial U}\,dt + u_0\int_0^1 \frac{\partial F_{\tau_n}(tu_0+\psi, J(\psi))}{\partial u}\,dt.$$

Similarly,

$$F_{\tau_0}(u_0+\psi, J(u_0+\psi)) - F_{\tau_0}(\psi, J(\psi)) =$$

$$J(u_0)\int_0^1 \frac{\partial F_{\tau_0}(u_0+\psi, J(\psi)+tJ(u_0))}{\partial U}\,dt + u_0\int_0^1 \frac{\partial F_{\tau_0}(tu_0+\psi, J(\psi))}{\partial u}\,dt.$$

Therefore,

$$F_{\tau_n}(u_0+\psi, J(u_0+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi)) =$$

$$J(u_0)\left(\int_0^1 \frac{\partial F_{\tau_n}(u_0+\psi, J(\psi)+tJ(u_0))}{\partial U}\,dt - \int_0^1 \frac{\partial F_{\tau_0}(u_0+\psi, J(\psi)+tJ(u_0))}{\partial U}\,dt\right) +$$

$$+u_0\left(\int_0^1 \frac{\partial F_{\tau_n}(tu_0+\psi, J(\psi))}{\partial u}\, dt - \int_0^1 \frac{\partial F_{\tau_0}(tu_0+\psi, J(\psi))}{\partial u}\, dt\right)+$$

$$+F_{\tau_n}(\psi, J(\psi)) - F_{\tau_0}(\psi, J(\psi)) \equiv A+B+C$$

(A denotes the first line in the right-hand side, B - the second, C - the third). The expressions A and B converge to zero in the weighted norm of $E_\mu = C_\mu^\alpha(\mathbb{R})$, due to the Lipschitz condition with respect to τ of the derivatives of F_τ (see $(H2)$). The expression C is a function with a finite support. It also converges to zero in the weighted norm as $\tau_n \longrightarrow \tau_0$. This concludes the proof.

We can now prove the properness of the $\tau-$dependent operator A_τ. Denote by

$$E_{loc} = \left\{u \in C^{2+\alpha}(I),\ (\forall)\, I = \text{bounded interval}\right\},$$
$$E_{\mu,loc} = \left\{\mu u \in C^{2+\alpha}(I),\ (\forall)\, I = \text{bounded interval}\right\}$$

and similarly E_{loc}^0 and $E_{\mu,loc}^0$.

Theorem 2.5. *If $\phi \in E^0$, under assumptions $(H1)-(H3)$, the operator $A_\tau(u)$: $E_\mu \times [0,1] \to E_\mu^0$ from (2.2) is proper with respect to (u,τ) on $E_\mu \times [0,1]$.*

Proof. Consider a convergent sequence $f_n \in E_\mu^0$, say $f_n \longrightarrow f_0$ in E_μ^0. Let (u_n, τ_n) be a solution in $E_\mu \times [0,1]$ of the equation $A_{\tau_n}(u_n) = f_n$, such that

$$||u_n||_\mu \le M,\ (\forall) n \ge 1. \tag{2.3}$$

We prove that one can choose a convergent in E_μ subsequence of the sequence u_n. Without loss of generality we may assume that $\tau_n \longrightarrow \tau_0$ as $n \longrightarrow \infty$. Equation $A_{\tau_n}(u_n) = f_n$ can be written as

$$(u_n+\psi)'' + c(\tau_n)(u_n+\psi)' + F_{\tau_n}(u_n+\psi, J(u_n+\psi)) = f_n. \tag{2.4}$$

Multiplying the equation by μ and denoting $v_n(x) = \mu(x)\, u_n(x)$, $g_n(x) = \mu(x)\, f_n(x)$, we derive that

$$v_n'' + \left[-2\frac{\mu'}{\mu} + c(\tau_n)\right] v_n' + \left[-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau_n)\frac{\mu'}{\mu}\right] v_n +$$

$$+\mu F_{\tau_n}(u_n+\psi, J(u_n+\psi)) + \mu(\psi'' + c(\tau_n)\psi') = g_n. \tag{2.5}$$

Indeed, since $\mu u_n' = (\mu u_n)' - \mu' u_n = v_n' - \mu' v_n/\mu$ and $\mu u_n'' = (\mu u_n)'' - \mu'' u_n - 2\mu' u_n' = v_n'' - \mu'' v_n/\mu - 2\mu'\left(v_n'/\mu - \mu' v_n/\mu^2\right)$, by (2.4) one easily obtains (2.5).

The sequence $v_n = \mu u_n$ is uniformly bounded in E :

$$||v_n||_E = ||\mu u_n||_E = ||u_n||_\mu \le M, (\forall)\, n \ge 1. \tag{2.6}$$

Then it is locally convergent on a subsequence. More exactly, for every bounded interval $[-N,N]$ of x, there is a subsequence (denoted again v_n) converging in $C^{2+\alpha}[-N,N]$ to a limiting function $v_0 \in C^{2+\alpha}[-N,N]$. By a diagonalization process we can prolong v_0 to $\mathbb{R}$ such that $v_0 \in E$. Since $||v_n||_E \le M,\ (\forall) n \ge 1$, we can easily see that $||v_0||_E \le M$.

Let u_0 be the limit that corresponds to u_n. Then $\mu u_n \to \mu u_0$ in E_{loc} and $v_0 = \mu u_0$.

We now want to pass to the limit as $n \to \infty$ in (2.4) and (2.5). To this end observe that $(H2)$ implies that

$$||F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi))||_{0\mu} =$$

$$= ||F_{\tau_n}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_n+\psi, J(u_n+\psi))||_{0\mu} +$$

$$+||F_{\tau_0}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi))||_{0\mu} \le$$

$$\le c_1 |\tau_n - \tau_0| + ||F_{\tau_0}(u_n+\psi, J(u_n+\psi)) - F_{\tau_0}(u_0+\psi, J(u_0+\psi))||_{0\mu}.$$

Since F_{τ_0} is continuous from $E_\mu^0 \times E_\mu^0$ to E_μ^0 (see $(H1)$) and $J(u_n+\psi) \to J(u_0+\psi)$ in $E^0_{\mu,loc}$, we derive that

$$F_{\tau_n}(u_n+\psi, J(u_n+\psi)) \to F_{\tau_0}(u_0+\psi, J(u_0+\psi)) \text{ as } n\to\infty, \text{ in } E^0_{\mu,loc}. \tag{2.7}$$

Passing to the limit as $n \to \infty$, uniformly on bounded intervals of x in (2.4) and (2.5), one obtains that

$$(u_0+\psi)'' + c(\tau_0)(u_0+\psi)' + F_{\tau_0}(u_0+\psi, J(u_0+\psi)) = f_0, \tag{2.8}$$

$$v_0'' + \left[-2\frac{\mu'}{\mu} + c(\tau_0)\right] v_0' + \left[-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau_0)\frac{\mu'}{\mu}\right] v_0 +$$

$$+\mu F_{\tau_0}(u_0+\psi, J(u_0+\psi)) + \mu(\psi'' + c(\tau_0)\psi') = \mu f_0. \tag{2.9}$$

Subtracting (2.9) from (2.5) and denoting $V_n = v_n - v_0$, one finds

$$V_n'' + \left[-2\frac{\mu'}{\mu} + c(\tau_n)\right] V_n' + \left[-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau_n)\frac{\mu'}{\mu}\right] V_n + \mu[F_{\tau_n}(u_n+\psi, J(u_n+\psi)) -$$

$$-F_{\tau_0}(u_0+\psi, J(u_0+\psi))] + [c(\tau_n) - c(\tau_0)]\left(v_0' - \frac{\mu'}{\mu}v_0 + \mu\psi'\right) = \mu f_n - \mu f_0. \tag{2.10}$$

Recall that $V_n = v_n - v_0 \to 0$ as $n \to \infty$ in E_{loc}. We show that $V_n \to 0$ in $C(\mathbb{R})$. Suppose that it is not the case. Then, without any loss of generality, we can chose a sequence $x_n \to \infty$ such that $|V_n(x_n)| \ge \varepsilon > 0$. This means that $|v_n(x_n) - v_0(x_n)| \ge \varepsilon > 0$. Let

$$\widetilde{V}_n(x) = V_n(x+x_n) = v_n(x+x_n) - v_0(x+x_n) = \mu(x+x_n)[u_n(x+x_n) - u_0(x+x_n)]. \tag{2.11}$$

Then,

$$\left|\widetilde{V}_n(0)\right| = |V_n(x_n)| \ge \varepsilon > 0. \tag{2.12}$$

Writing (2.10) in $x + x_n$, one obtains

$$\widetilde{V}_n''(x) + \left[-2\frac{\mu'(x+x_n)}{\mu(x+x_n)} + c(\tau_n)\right]\widetilde{V}_n'(x) +$$

$$+\left[-\frac{\mu''(x+x_n)}{\mu(x+x_n)}+2\left(\frac{\mu'(x+x_n)}{\mu(x+x_n)}\right)^2-c(\tau_n)\frac{\mu'(x+x_n)}{\mu(x+x_n)}\right]\widetilde{V}_n(x)+$$

$$+\mu(x+x_n)\left[F_{\tau_n}(u_n+\psi,J(u_n+\psi))-F_{\tau_0}(u_0+\psi,J(u_0+\psi))\right](x+x_n)+$$

$$+\left[c(\tau_n)-c(\tau_0)\right]\left[v_0'-\frac{\mu'}{\mu}v_0+\mu\psi'\right](x+x_n)=(\mu f_n-\mu f_0)(x+x_n). \tag{2.13}$$

We will pass to the limit as $n\to\infty$ in (2.13). First we note that by (2.11) and (2.6), there exists $\widetilde{V}_0\in E$ such that $\widetilde{V}_n\to\widetilde{V}_0$ as $n\to\infty$ in E_{loc}. Next, it is obvious that

$$\frac{1}{\mu(x+x_n)}\to 0,\ \frac{\mu'(x+x_n)}{\mu(x+x_n)}\to 0,\ \frac{\mu''(x+x_n)}{\mu(x+x_n)}\to 0,\ n\to\infty,$$

while condition $f_n\to f_0$ in E_μ^0 leads to $(\mu f_n-\mu f_0)(x+x_n)\to 0$. Inequality (2.6) implies a similar estimate for v_0, so $v_0(x+x_n)$ and $v_0'(x+x_n)$ are bounded in E. We also have $\psi'(x+x_n)=0$ for $x+x_n>1$ and for $x+x_n<-1$ and

$$\mu(x+x_n)\left[F_{\tau_n}(u_n+\psi,J(u_n+\psi))-F_{\tau_0}(u_0+\psi,J((u_0+\psi)))\right](x+x_n)=$$

$$=\mu(x+x_n)\left[F_{\tau_n}(u_n+\psi,J(u_n+\psi))-F_{\tau_0}(u_n+\psi,J(u_n+\psi))\right](x+x_n)+$$

$$+\mu(x+x_n)\left[F_{\tau_0}(u_n+\psi,J(u_n+\psi))-F_{\tau_0}(u_0+\psi,J(u_n+\psi))\right](x+x_n)+$$

$$+\mu(x+x_n)\left[F_{\tau_0}(u_0+\psi,J(u_n+\psi))-F_{\tau_0}(u_0+\psi,J(u_0+\psi))\right](x+x_n). \tag{2.14}$$

Denote by T_1^n, T_2^n, T_3^n the three terms in the right-hand side. Hypothesis $(H2)$ for F_{τ_0} infers that

$$T_1^n\to 0,\ n\to\infty \text{ in } E_{loc}^0. \tag{2.15}$$

Next, (2.11) leads to

$$T_2^n=\widetilde{V}_n(x)\frac{\left[F_{\tau_0}(u_n+\psi,J(u_n+\psi))-F_{\tau_0}(u_0+\psi,J(u_n+\psi))\right](x+x_n)}{u_n(x+x_n)-u_0(x+x_n)}=$$

$$=\widetilde{V}_n(x)\frac{\partial F_{\tau_0}}{\partial u}\left(s(u_n+\psi)+(1-s)(u_0+\psi),J(u_n+\psi)\right)(x+x_n)=$$

$$=\widetilde{V}_n(x)\frac{\partial F_{\tau_0}}{\partial u}\left(su_n+(1-s)u_0+\psi,J(u_n+\psi)\right)(x+x_n),$$

for some $s\in[0,1]$. By (2.3) we obtain $|u_n(x+x_n)|\le M/\mu(x+x_n)$, $|u_0(x+x_n)|\le M/\mu(x+x_n)$, hence

$$(su_n+(1-s)u_0+\psi)(x+x_n)\to w^{\pm}$$

and

$$J(u_n+\psi)(x+x_n)=\int_{-\infty}^{\infty}\phi(x+x_n-y)u_n(y)\,dy+\int_{-\infty}^{\infty}\phi(x+x_n-y)\psi(y)\,dy.$$

By the change of variable $x_n - y = -z$, it follows that

$$J\left(u_n+\psi\right)\left(x+x_n\right)=\int_{-\infty}^{\infty}\phi\left(x-z\right)u_n\left(x_n+z\right)dz+\int_{-\infty}^{\infty}\phi\left(x-z\right)\psi\left(x_n+z\right)dz\rightarrow w^{\pm},$$

uniformly on bounded intervals of x. Hypothesis $(H1)$ shows that

$$T_2^n\rightarrow\frac{\partial F_{\tau_0}}{\partial u}\left(w^{\pm},w^{\pm}\right)\widetilde{V}_0,\ \text{as } n\rightarrow\infty \text{ in } E^0_{loc}. \tag{2.16}$$

On the other hand,

$$T_3^n=\mu\left(J\left(u_n+\psi\right)-J\left(u_0+\psi\right)\right)\times$$

$$\frac{F_{\tau_0}\left(u_0+\psi,J\left(u_n+\psi\right)\right)-F_{\tau_0}\left(u_0+\psi,J\left(u_0+\psi\right)\right)}{J(u_n+\psi)-J(u_0+\psi)}\left(x+x_n\right)=$$

$$I_n\left(x\right)\cdot\frac{\partial F_{\tau_0}}{\partial U}\left(u_0+\psi,sJ(u_n+\psi)+\left(1-s\right)J(u_0+\psi)\right)\left(x+x_n\right),$$

for some $s\in\left[0,1\right]$, where $I_n\left(x\right)=\mu\left(J\left(u_n+\psi\right)-J\left(u_0+\psi\right)\right)$. For $\mu\left(x\right)=1+x^2$, $x\in\mathbb{R}$, with the aid of (2.11), we arrive at

$$I_n\left(x\right)=\int_{-\infty}^{\infty}\frac{\mu\left(x+x_n\right)}{\mu\left(z+x_n\right)}\phi\left(x-z\right)\widetilde{V}_n\left(z\right)dz\rightarrow\int_{-\infty}^{\infty}\phi\left(x-z\right)\widetilde{V}_0\left(z\right)dz=J\left(\widetilde{V}_0\right),$$

in E^0_{loc}. As above, since $J\left(u_n\right)\left(x+x_n\right)\rightarrow 0$, $J\left(u_0\right)\left(x+x_n\right)\rightarrow 0$, $J\left(\psi\right)\left(x+x_n\right)\rightarrow w^{\pm}$ uniformly on bounded intervals of x, we deduce that

$$T_3^n\rightarrow\frac{\partial F_{\tau_0}}{\partial U}\left(w^{\pm},w^{\pm}\right)J(\widetilde{V}_0),\ \text{as } n\rightarrow\infty \text{ in } E^0_{loc}. \tag{2.17}$$

Now we may pass to the limit in (2.13). With the aid of $(2.14)-(2.17)$ and $(H2)$, one arrives at

$$\widetilde{V}_0''+c\left(\tau_0\right)\widetilde{V}_0'+\frac{\partial F_{\tau_0}}{\partial u}\left(w^{\pm},w^{\pm}\right)\widetilde{V}_0+\frac{\partial F_{\tau_0}}{\partial U}\left(w^{\pm},w^{\pm}\right)J\left(\widetilde{V}_0\right)=0,$$

which contradicts $(H3)$. Therefore we have proved that $V_n\rightarrow 0$ in $C\left(\mathbb{R}\right)$.

Now we have to show that $V_n\rightarrow 0$ in $E=C^{2+\alpha}\left(\mathbb{R}\right)$. To this end, we write equation (2.10) in the form $S\left(V_n\right)=h_n$, where $S\left(V_n\right)$ is the linear part from the left-hand side and

$$h_n=\left(\mu f_n-\mu f_0\right)-\mu[F_{\tau_n}\left(u_n+\psi,J\left(u_n+\psi\right)\right)-$$

$$-F_{\tau_0}\left(u_0+\psi,J\left(u_0+\psi\right)\right)]-\left[c\left(\tau_n\right)-c\left(\tau_0\right)\right]\left(v_0'-\frac{\mu'}{\mu}v_0+\mu\psi'\right).$$

Using Lemma 2.1 from [4] for the linear operator S, we can write

$$||V_n||_E\le C\left(||S\left(V_n\right)||_{E^0}+||V_n||_{C(\mathbb{R})}\right).$$

We make use of Lemma 2.4, hypothesis (H_2) for $c\left(\tau_n\right)-c\left(\tau_0\right)$, and of the convergence $f_n\rightarrow f_0$ in E^0_μ, to deduce that $S\left(V_n\right)=h_n\rightarrow 0$ in $E^0=C^\alpha\left(\mathbb{R}\right)$. Since $V_n\rightarrow 0$ in $C\left(\mathbb{R}\right)$, we conclude that $u_n\rightarrow u_0$ in E_μ. The theorem is proved.

3 Fredholm property in weighted spaces

Consider the operator $L_\tau : E_\mu = C^{2+\alpha}_\mu \to E^0_\mu = C^\alpha_\mu$,

$$L_\tau u = u'' + c(\tau)\, u' + \frac{\partial F_\tau}{\partial u}\left(u_1 + \psi, J(u_1 + \psi)\right) u + \frac{\partial F_\tau}{\partial U}\left(u_1 + \psi, J(u_1 + \psi)\right) J(u), \quad (3.1)$$

and its limiting operators

$$L^\pm_\tau u = u'' + c(\tau)\, u' + \frac{\partial F_\tau}{\partial u}\left(w^\pm, w^\pm\right) u + \frac{\partial F_\tau}{\partial U}\left(w^\pm, w^\pm\right) J(u). \quad (3.2)$$

Recall here Condition NS for L_τ, i. e. hypothesis $(H3)$: *For each* $\tau \in [0,1]$, *the limiting equations* $L^\pm_\tau u = 0$ *do not have nonzero solutions.*

We prove now the Fredholm property of L_τ as an operator acting between the above weighted Holder spaces.

Theorem 3.1. *If condition NS is satisfied, then the operator* $L_\tau : E_\mu = C^{2+\alpha}_\mu \to E^0_\mu = C^\alpha_\mu$ *(acting between weighted spaces) is normally solvable with a finite dimensional kernel.*

Proof. Like in Theorem 2.2 from [4], we can prove that L_τ from E to E^0 is normally solvable with a finite dimensional kernel. To verify the property in the weighted spaces, we use Lemma 2.24 in [20]: if $L_\tau : E \to E^0$ is normally solvable with a finite dimensional kernel and the operator $K : E_\mu \to E^0$, $Ku = \mu L_\tau u - L_\tau(\mu u)$ is compact, then $L_\tau : E_\mu \to E^0_\mu$ is normally solvable with a finite dimensional kernel.

Let $\{u_i\}$ be a sequence such that $||u_i||_{E_\mu} \leq M$. We prove the existence of a subsequence of $\{Ku_i\}$ which converges in $E^0 = C^\alpha(\mathbb{R})$. Consider the sequence $v_i = \mu u_i$. Since $||v_i|| = ||u_i||_{E_\mu} \leq M$, one can find a subsequence, denoted again $\{v_i\}$, which converges locally in C^2 to a function v_0, which can be prolonged to $\mathbb{R}$ by a diagonalization process. We have $v_0 \in E$, $||v_0||_E \leq M$ and $v_i \to v_0$ in E_{loc} (in $C^{2+\alpha}(I)$, for every bounded interval I).

Let u_0 be such that $v_0 = \mu u_0$. Then

$$||Ku_i - Ku_0||_{E^0} = ||K\left(\frac{z_i}{\mu}\right)||_{E^0},\ z_i = v_i - v_0 = \mu(u_i - u_0).$$

Observe that $||z_i||_E \leq 2M$ and $z_i \to 0$ in E_{loc}. Now we can write

$$K\left(\frac{z_i}{\mu}\right) = \mu L_\tau\left(\frac{z_i}{\mu}\right) - L_\tau(z_i) = \left(-\frac{\mu''}{\mu} + 2\left(\frac{\mu'}{\mu}\right)^2 - c(\tau)\frac{\mu'}{\mu}\right) z_i - 2\frac{\mu'}{\mu} z_i' +$$

$$+ \frac{\partial F_\tau}{\partial U}\left(u_1 + \psi, J(u_1 + \psi)\right)\left[\mu J\left(\frac{z_i}{\mu}\right) - J(z_i)\right]. \quad (3.3)$$

But

$$\mu J\left(\frac{z_i}{\mu}\right) - J(z_i) = \int_{-\infty}^{\infty} \phi(x-y)\, z_i(y)\left[\frac{\mu(x)}{\mu(y)} - 1\right] dy =$$

$$= \int_{-\infty}^{\infty} \phi(\xi) z_i(x-\xi) \left[\frac{\mu(x)}{\mu(x-\xi)} - 1 \right] d\xi.$$

Since $\mu(x)/\mu(x-\xi) - 1 \leq h(x)$, where $h(x) \to 0$ as $x \to \pm\infty$, $z_i(x)$ is uniformly bounded and $z_i(x) \to 0$ as $i \to \infty$ locally with respect to x, it follows that $\mu J(z_i/\mu) - J(z_i) \to 0$ as $i \to \infty$, uniformly with respect to x on $\mathbb{R}$. Similarly, z_i, z_i' are uniformly bounded, $z_i \to 0$, $z_i' \to 0$ as $i \to \infty$ locally and $\mu''/\mu \to 0$, $\mu'/\mu \to 0$ as $x \to \pm\infty$, so the first two terms from (3.3) tend to zero uniformly with respect to $x \in \mathbb{R}$, as $i \to \infty$. This implies that $K(z_i/\mu) \to 0$ as $i \to \infty$ in $C(\mathbb{R})$. Therefore, with the aid of the local convergence $z_i \to 0$ in C^2, we conclude that $Ku_i \to Ku_0$ as $i \to \infty$ in $E^0 = C^\alpha(\mathbb{R})$. The theorem is proved.

We prove now the Fredholm property for $L_\tau : E_\mu \to E_\mu^0$, under an additional hypothesis. To this end, let I be the identity operator on E_μ.

Condition NS(λ). *For each $\tau \in [0,1]$, the limiting equations $L_\tau^\pm u - \lambda u = 0$ associated to the operator $L_\tau - \lambda I$ do not have nonzero solutions in E_μ, for any $\lambda \geq 0$.*

We recall an auxiliary result from [4] which will be employed below.

Lemma 3.2. *Consider the operators $L^0, L^1 : C^{2+\alpha}(\mathbb{R}) \to C^\alpha(\mathbb{R})$ defined by $L^0 u = L_\tau u - \rho u$, $L^1 u = u'' - \rho u$ $(\rho \geq 0)$ and the homotopy $L^s : C^{2+\alpha}(\mathbb{R}) \to C^\alpha(\mathbb{R})$, $L^s = (1-s) L^0 + s L^1$, $s \in [0,1]$. Then there exists $\rho \geq 0$ large enough such that the limiting equations $(L^s)^\pm u = 0$ do not have nonzero solutions for any $s \in [0,1]$.*

Theorem 3.3. *If Condition NS(λ) is satisfied, then L_τ, regarded as an operator from E_μ to E_μ^0, has the Fredholm property and its index is zero.*

Proof. We put $L^0 u = L_\tau u - \lambda u$, $L^1 u = u'' - \lambda u$ and $L^s : E_\mu = C_\mu^{2+\alpha}(\mathbb{R}) \to E_\mu^0 = C_\mu^\alpha(\mathbb{R})$, $L^s = (1-s) L^0 + s L^1$, $s \in [0,1]$. Condition NS(λ) for L_τ implies Condition NS for $L^0 = L_\tau - \lambda I$. Then, Theorem 3.1 ensures that $L^0 = L_\tau - \lambda I$, regarded from E_μ to E_μ^0, is normally solvable with a finite dimensional kernel. For operator L^1, we have $\ker L^1 = \{0\}$, $\mathrm{Im} L^1 = E_\mu^0$, hence L^1 is a Fredholm operator and its index is $\mathrm{ind} L^1 = 0$.

By Lemma 3.2 applied for L^s, there exists $\lambda \geq 0$ large enough such that Condition NS holds for all L^s, $s \in [0,1]$. In view of Theorem 3.1, it follows that the operators L^s are normally solvable with a finite dimensional kernel. In other words, the homotopy L^s gives a continuous deformation from the operator L^0 to the operator L^1, in the class of the normally solvable operators with finite dimensional kernels. Such deformation preserves the Fredholm property and the index. Since the index of L^1 is zero, we derive that the index of all L^s is zero. In particular, for $s = 0$ and $\lambda = 0$, one arrives at the conclusion that L_τ has the Fredholm property and its index is zero. This completes the proof.

4 Topological degree

In this section we apply the topological degree construction for Fredholm and proper operators with the zero index constructed in [20] to the integro-differential operators.

Definitions. Recall in the beginning the definition of the topological degree. Consider two Banach spaces $E_1,\ E_2$, a class Φ of operators acting from E_1 to E_2 and a class of homotopies

$$H = \{A_\tau(u)\colon E_1 \times [0,1] \to E_2,\ \text{such that } A_\tau(u) \in \Phi,\ (\forall)\, \tau \in [0,1]\}.$$

Let $D \subset E_1$ be an open bounded set and $A \in \Phi$ such that $A(u) \neq 0$, $u \in \partial D$, where ∂D is the boundary of D. Suppose that for such a pair (D, A), there exists an integer $\gamma(A, D)$ with the following properties:

(i) *Homotopy invariance.* If $A_\tau(u) \in H$ and $A_\tau(u) \neq 0$, for $u \in \partial D$, $\tau \in [0,1]$, then

$$\gamma(A_0, D) = \gamma(A_1, D).$$

(ii) *Additivity.* If $A \in \Phi$, $\overline{D}$ is the closure of D and $D_1,\ D_2 \subset D$ are open sets, such that $D_1 \cap D_2 = \oslash$ and $A(u) \neq 0$, for all $u \in \overline{D} \setminus (D_1 \cup D_2)$, then

$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

(iii) *Normalization.* There exists a bounded linear operator $J : E_1 \to E_2$ with a bounded inverse defined on all E_2 such that, for every bounded set $D \subset E_1$ with $0 \in D$,

$$\gamma(J, D) = 1.$$

The integer $\gamma(A, D)$ is called a *topological degree*.

Degree for Fredholm and proper operators. We now recall a general result concerning the existence of a topological degree which was proved in [21], [20].

Let E_1 and E_2 be Banach spaces, $E_1 \subseteq E_2$ algebraically and topologically and let $G \subset E_1$ be an open bounded set.

Denote by $I : E_1 \to E_2$ the imbedding operator, $Iu = u$, and by Φ a class of bounded linear operators $L\colon E_1 \to E_2$ satisfying the following conditions:

(a) The operator $L - \lambda I\colon E_1 \to E_2$ is Fredholm for all $\lambda \geq 0$,

(b) For every operator $L \in \Phi$, there is $\lambda_0 = \lambda_0(L)$ such that $L - \lambda I$ has a uniformly bounded inverse for all $\lambda > \lambda_0$.

Denote by $\mathcal{F}$ the class

$$\mathcal{F} = \{B \in C^1(G, E_2),\ B \text{ proper},\ B'(x) \in \Phi,\ (\forall)\, x \in G\}, \tag{4.1}$$

where $B'(x)$ is the Fréchet derivative of the operator B.

Finally, one introduces the class $\mathcal{H}$ of homotopies given by

$$\mathcal{H} = \{B(x,\tau) \in C^1(G \times [0,1], E_2),\ B \text{ proper},\ B(.,\tau) \in \mathcal{F},\ (\forall)\, \tau \in [0,1]\}. \tag{4.2}$$

Here the properness of B is understood in both variables $x \in G$ and $\tau \in [0,1]$.

Theorem 4.1. ([20]) *For every $B \in \mathcal{H}$ and every open set D, with $\overline{D} \subset G$, there exists a topological degree $\gamma(B, D)$.*

Remark 4.2. Condition (b) can be weakened. Let E_1' and E_2' be two Banach spaces such that $E_i \subset E_i'$, i=1,2 where the inclusion is understood in the algebraic and topological sense. In the case of the Hölder space $C^{k+\alpha}(\mathbb{R})$, this can be the space $C^k(\mathbb{R})$ with an integer nonnegative k. We can also consider some integral spaces $W_\infty^{k,p}(\mathbb{R})$ [16]. Instead of (b) above we can impose the following condition [18]:

(b') For every operator $L : E_1' \to E_2'$, there is $\lambda_0 = \lambda_0(L)$ such that $L - \lambda I$ has a uniformly bounded inverse for all $\lambda > \lambda_0$.

Degree for the integro-differential operators. Now, let $E_1 = E_\mu$ and $E_2 = E_\mu^0$ be the weighted spaces introduced in the previous section, with $\mu(x) = 1 + x^2$, $x \in \mathbb{R}$. We will apply Theorem 4.1 for the integro-differential operator A of the form (1.4), where function $\psi \in C^\infty(\mathbb{R})$, $\psi(x) = w^+$, for $x \geq 1$, $\psi(x) = w^-$ for $x \leq -1$ and

$(H4)$ $F(u, U)$ and its derivatives with respect to u and U are Lipschitz continuous in (u, U);

$(H5)$ The limiting equations

$$u'' + cu' + \frac{\partial F}{\partial u}\left(w^\pm, w^\pm\right) u + \frac{\partial F}{\partial U}\left(w^\pm, w^\pm\right) J(u) - \lambda u = 0$$

do not have nonzero solutions in E, $(\forall)\, \lambda \geq 0$.

Under these hypotheses, Theorem 2.5 assures that operator A is proper. Moreover, its Fréchet derivative is $A' = L$ from (1.5) and it is a Fredholm operator with the index zero (Theorem 3.3).

Consider $\mathcal{F}$ the class of operators A defined through (1.4), such that $(H4) - (H5)$ are satisfied. Consider also the class $\mathcal{H}$ of homotopies $A_\tau : E_\mu \to E_\mu^0$, $\tau \in [0, 1]$, of the form (2.2), satisfying $(H1) - (H2)$ and

$(H6)$ For every $\tau \in [0, 1]$, the equations

$$u'' + c(\tau) u' + \frac{\partial F_\tau}{\partial u}\left(w^\pm, w^\pm\right) u + \frac{\partial F_\tau}{\partial U}\left(w^\pm, w^\pm\right) J(u) - \lambda u = 0$$

do not have nonzero solutions in E, $(\forall)\, \lambda \geq 0$. By Theorem 2.5 and Theorem 3.3, we infer that operators $A_\tau(u)$ are Fréchet differentiable, proper with respect to (u, τ) and their Fréchet derivatives $A_\tau' = L_\tau$ verify condition (a) above. Condition (b') follows from the lemma in the appendix. Hence $\mathcal{H}$ has the form (4.2). Applying Theorem 4.1 for the class of operators $\mathcal{F}$ and the class of homotopies $\mathcal{H}$, we are led to the following result.

Theorem 4.3. *Suppose that functions F_τ and $c(\tau)$ satisfy conditions $(H1) - (H2)$ and $(H4) - (H6)$. Then a topological degree exists for the class $\mathcal{F}$ of operators and the class $\mathcal{H}$ of homotopies.*

5 Applications to travelling waves

In this section we will discuss some applications of the Fredholm property, properness and topological degree to study travelling wave solutions of equation (1.1). Let us begin with the classification of the nonlinearities. Denote

$$F_0(w) = F(w, w).$$

We obtain this function from $F(w, J(w))$ if we formally replace the kernel $\phi(x)$ of the integral by the δ-function. The corresponding reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F_0(u) \tag{5.1}$$

is called bistable if $F_0'(w^\pm) < 0$, monostable if one of these derivatives is positive and another one negative and, finally, unstable if $F_0'(w^\pm) > 0$. As it is well-known, it can have travelling wave solutions, that is solutions, which satisfy the problem

$$w'' + cw' + F_0(w) = 0, \quad w(\pm\infty) = w^\pm. \tag{5.2}$$

Let $w_0(x)$ be a solution of (5.2) with some $c = c_0$. The operator L_0 linearized about this solution,

$$L_0 u = u'' + c_0 u' + F_0'(w_0)u$$

has the essential spectrum given by two parabolas:

$$\lambda_\pm^0(\xi) = -\xi^2 + c_0 i\xi + F_0'(w^\pm), \quad \xi \in \mathbb{R}.$$

Therefore the operator L_0 satisfies the Fredholm property if and only if $F_0'(w^\pm) \neq 0$. If this condition is satisfied, then the index of the operator is well defined. In the bistable case it equals 0, in the monostable case 1, in the unstable case 0 [7].

In the case of the integro-differential operator

$$Lu = u'' + c_0 u' + F_u'(w, J(w))u + F_U'(w, J(w))J(u),$$

the essential spectrum is given by the curves

$$\lambda_\pm(\xi) = -\xi^2 + c_0 i\xi + F_u'(w^\pm, w^\pm) + F_U'(w^\pm, w^\pm)\tilde{\phi}(\xi), \quad \xi \in \mathbb{R},$$

where $\tilde{\phi}(\xi)$ is the Fourier transform of the function $\phi(x)$. If we replace $J(u)$ by u, that is $\phi(x)$ by the δ-function, then the spectrum of the integro-differential operator coincides with the spectrum of the reaction-diffusion operator.

We note that

$$F_u'(w^\pm, w^\pm) + F_U'(w^\pm, w^\pm)\tilde{\phi}(0) = F_0'(w^\pm)$$

and

$$\text{Re } \tilde{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) \cos(\xi x) dx < \int_{-\infty}^{\infty} \phi(x) dx = 1.$$

Fredholm property. Bistable case. Let

$$F_0'(w^\pm) = F_u'(w^\pm, w^\pm) + F_U'(w^\pm, w^\pm) < 0$$

(we recall that $\tilde{\phi}(0) = 1$). Suppose that $F_u'(w_+, w_+) < 0$ and $F_U'(w_+, w_+) > 0$. Then Re $\lambda_+(\xi) < 0$ for all $\xi \in \mathbb{R}$ since

$$F_u'(w^\pm, w^\pm) + \text{Re } F_U'(w^\pm, w^\pm)\tilde{\phi}(\xi) \le F_u'(w^\pm, w^\pm) + F_U'(w^\pm, w^\pm) = F_0'(w^\pm).$$

Hence the essential spectrum is completely in the left-half plane. This allows us to prove properness of the corresponding operators in weighted spaces and to define the topological degree.

Consider now the case where $F_u'(w_+, w_+) > 0$ and $F_U'(w_+, w_+) < 0$. The principal difference with the previous case is that the essential spectrum of the integro-differential operator may not be completely in the left-half plane (Figure 1) though this is the case for the reaction-diffusion operator. Depending on the parameters, the essential spectrum can cross the imaginary axis for some pure imaginary values. However the linear operator remains Fredholm since the essential spectrum does not cross the origin; the nonlinear operator remains proper in the corresponding weighted spaces.

Thus, the bistable case for the reaction-diffusion equation gives rise to two different cases for the integro-differential equation. We will call both of them bistable but will distinguish them when necessary.

Monostable case. Suppose that $F_0'(w_+) > 0$ and $F_0'(w_-) < 0$. Then $\lambda_-^0(\xi)$ is in the left-half plane for all $\xi \in \mathbb{R}$; $\lambda_+^0(\xi)$ is partially in the right-half plane, $\lambda_+^0(0) > 0$. The essential spectrum of the integro-differential operator L given by the curves $\lambda_\pm(\xi)$ has a similar structure. It does not cross the origin, so that the operator satisfies the Fredholm property. The curve $\lambda_+(\xi)$ is partially in the right-half plane, $\lambda_+(0) = \lambda_+^0(0) > 0$. The curve $\lambda_-(\xi)$ can be completely in the left-half plane or partially in the right-half plane (Figure 1). Similar to the bistable case, there are two subcases in the monostable case.

Index. In order to find the index of the operator L, we consider the operator L_τ which depends on the parameter τ characterizing the width of the support of the function ϕ_τ, supp $\phi_\tau = [-N_\tau, N_\tau]$. We recall that $\int_{-\infty}^{\infty} \phi_\tau(x) dx = 1$. Let $L_1 = L$, that is the value $\tau = 1$ corresponds to the function ϕ in the operator L.

Since the essential spectrum of the operator L_τ can be determined explicitly, then we can affirm that it converges to the essential spectrum of the operator L_0 as $\tau \to 0$. Moreover, L_τ converges to L_0 in the operator norm. The essential spectrum of the operator L_τ does not

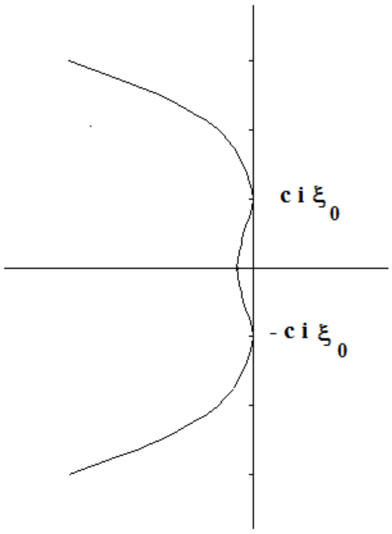

Figure 1: Schematic representation of a possible location of the essential spectrum of the operator L.

cross the origin. Therefore it is normally solvable with a finite dimensional kernel. Hence the index of the operator L equals the index of the operator L_0 [11]. It is 0 in the bistable case and 1 in the monostable case (cf. [7]).

Topological degree and existence of solutions. In the bistable case we can define the topological degree for the integro-differential operator and use the Leray-Schauder method to prove existence of solutions. In order to use this method we need to obtain a priori estimates of solutions. In [8], a priori estimates are obtained in the case where

$$F(u, J(u)) = J(u)u(1-u) - \alpha u.$$

Thus, we can now conclude about the existence of waves for this particular form of the nonlinearity. More general functions will be considered in the subsequent works.

Local bifurcations and branches of solutions. Other conventional applications of the degree are related to local bifurcations and global branches of solutions (see, e.g., [13]). We can now use the corresponding results for the integro-differential operator in the bistable case. Let us emphasize that these results apply in particular for the case where the essential spectrum of the linearized operator crosses the imaginary axis (see above). Therefore the wave persists in this case unless a priori estimates are lost.

Acknowledgements

This work was partially supported by the LEA Math Mode between CNRS France and Romanian Academy through the joint project "Existence of travelling waves for nonlocal reaction-diffusion equations".

References

[1] S. Ai. Traveling wave fronts for generalized Fisher equations with spatio-temporal delays. J. Diff. Equations, 232 (2007), 104—133.

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter. Spatial structures and generalized travelling waves for an integro-differential equation. Discrete Cont. Dynam. Systems B, 13 (2010), no. 3, 537–557.

[3] N. Apreutesei, A. Ducrot, V. Volpert. Competition of species with intra-specific competition. Math. Model. Nat. Phenom., 3 (2008), no. 4, 1–27.

[4] N. Apreutesei, A. Ducrot, V. Volpert. Travelling waves for integro-differential equations in population dynamics. Discrete Cont. Dynam. Systems Ser. B, 11 (2009), no. 3, 541–561.

[5] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik. The non-local Fisher-KPP equation: travelling waves and steady states. Nonlinearity, 22 (2009), No. 12, 2813–2844.

[6] N.F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. SIAM J. Appl. Math., 6 (1990), 1663–1688.

[7] J.F. Collet, V. Volpert. Computation of the index of linear elliptic operators in unbounded cylinders. J. Funct. Analysis, 164 (1999), 34–59.

[8] I. Demin, V. Volpert. Existence of waves for a nonlocal reaction-diffusion equation. Math. Model. Nat. Phenom., 5 (2010), No. 5, 80-101.

[9] A. Ducrot. Travelling wave solutions for a scalar age-structured equation. Discrete Contin. Dyn. Syst. Ser. B, 7 (2007), 251—273.

[10] S. Genieys, V. Volpert, P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. Math. Model. Nat. Phenom., 1 (2006), no. 1, 65–82.

[11] I. Gohberg, M. Krein. The basic propositions on defect numbers and indices of linear operators. Amer. Math. Soc. Transl., 13 (1969), No. 2, 185–264.

[12] S. A. Gourley. Travelling front solutions of a nonlocal Fisher equation. J. Math. Biol., 41 (2000), 272–284.

[13] M.A. Krasnoselskii, P.P. Zabreiko. Geometrical methods of nonlinear analysis. Springer-Verlag, New York, 1984.

[14] R. Lefever, O. Lejeune. On the origin of tiger bush. Bul. Math. Biol., 59 (1997), No. 2, 263–294.

[15] B. Perthame, S. Genieys. Concentration in the nonlocal Fisher equation: The Hamilton-Jacobi limit. Math. Model. Nat. Phenom., 4 (2007), 135–151.

[16] A. Volpert, V. Volpert. Fredholm property of elliptic operators in unbounded domains. Trans. Moscow Math. Soc., 67 (2006), 127–197.

[17] A. Volpert, Vit. Volpert, Vl. Volpert. Traveling wave solutions of parabolic systems. Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.

[18] V. Volpert. Elliptic partial differential equations. Vol. 1. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011.

[19] V. Volpert, S. Petrovskii. Reaction-diffusion waves in biology. Physics of Life Reviews, 6 (2009), 267—310.

[20] V. Volpert, A. Volpert. Properness and topological degree for general elliptic operators, Abstract and Applied Analysis (2003), No. 3, 129–182.

[21] V. Volpert, A. Volpert, J.F. Collet. Topological degree for elliptic operators in unbounded cylinders. Adv. Diff. Eq., 4 (1999), No. 6, 777–812.

Appendix

Sectorial property of an operator implies certain location of its essential spectrum and an estimate of the resolvent. For general elliptic problems in unbounded domains it is proved in [18]. A simple particular case of second-order operators on the axis is considered in [17]. In the lemma below we prove an estimate of the resolvent using this last result.

Lemma. *Let* $M_0 : C^2(R) \to C(R)$*,*

$$M_0 u = u'' + b(x)u'(x) + c(x)u + d(x)J(u),$$

where the coefficients of this operator are sufficiently smooth bounded functions. Then the operator $M_\lambda u = M_0 u - \lambda u$*, considered as acting in the same spaces, has a bounded inverse with the norm independent of* λ *for* $\lambda \geq \lambda_0 > 0$*, where* λ_0 *is sufficiently large.*

Proof. Consider the equation

$$M_\lambda u = f. \tag{5.3}$$

We need to obtain the estimate

$$\|u\|_{C^2(\mathbb{R})} \le K\|f\|_{C(\mathbb{R})} \tag{5.4}$$

of this equation where K is independent of λ for all λ sufficiently large. Here and below we denote by K the constants independent of u, f and λ.

We first prove the estimate

$$\|u\|_{C(\mathbb{R})} \le K\frac{\|f\|_{C(\mathbb{R})}}{\lambda} \ . \tag{5.5}$$

Since the operator

$$\widehat{M}u = u'' + b(x)u'(x) + c(x)u$$

is sectorial [17], then

$$\|u\|_{C(\mathbb{R})} \le K\frac{\|\widehat{M}u - \lambda u\|_{C(\mathbb{R})}}{\lambda} = K\frac{\|f - d(x)J(u)\|_{C(\mathbb{R})}}{\lambda} \le \frac{K}{\lambda}(\|f\|_{C(\mathbb{R})} + \|u\|_{C(\mathbb{R})}).$$

Estimate (5.5) follows from the last one for λ sufficiently large.

We can write equation (5.3) in the form

$$M_0u - \sigma u = f + \lambda u - \sigma u.$$

We can choose $\sigma > 0$ such that the operator in the left-hand side is invertible. Hence

$$\|u\|_{C^2(\mathbb{R})} \le K(\|f\|_{C(\mathbb{R})} + \lambda\|u\|_{C(\mathbb{R})} + \sigma\|u\|_{C(\mathbb{R})}).$$

This estimate and (5.5) give (5.4). The lemma is proved.

This lemma remains valid for the operators acting in the weighted spaces.