



**HAL**  
open science

# A Polynomial Dynamic Programming Algorithm for Crude Oil Transportation Planning

Chengbin Chu, Feng Chu, Mengchu Zhou, Haoxun Chen, Qingning Shen

► **To cite this version:**

Chengbin Chu, Feng Chu, Mengchu Zhou, Haoxun Chen, Qingning Shen. A Polynomial Dynamic Programming Algorithm for Crude Oil Transportation Planning. *IEEE Transactions on Automation Science and Engineering*, 2012, 9 (1), pp.42-55. <10.1109/TASE.2011.2164524>. <hal-00653649>

**HAL Id: hal-00653649**

**<https://hal.science/hal-00653649v1>**

Submitted on 13 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# A Polynomial Dynamic Programming Algorithm for Crude Oil Transportation Planning

Chengbin Chu, Feng Chu\*, MengChu Zhou, *Fellow, IEEE*, Haoxun Chen, and Qingning Shen

**Abstract**—Crude oil transportation is a central logistics operation in petrochemical industry because its cost represents a significant part in the cost of petrochemical products. In this paper, we consider the transportation by tankers or trucks. We show that under some realistic assumptions, this problem can be transformed into a single item lot sizing problem with limited production and inventory capacities. We develop a strongly polynomial dynamic programming algorithm to solve it. The problem of crude oil transportation is very difficult. There are few efficient methods in this domain. In the model considered in this paper, crude oil is directly shipped from a supplier port to  $n$  client ports to satisfy customer demands over  $T$  future periods. The supplier port disposes a fleet of identical tankers with limited capacity. The inventory capacities of customers are limited and time-varying. The backlogging is admitted. The objective is to find an optimal shipment plan minimizing the total cost over the  $T$ -period horizon. When the number of tankers is unlimited and customer demands are independent, shipment plans of different customers become independent. This problem can be considered as  $n$  independent problems. Each of them can be transformed into a single item lot sizing problem with limited production and inventory capacities, where tanker capacity corresponds to production capacity in classical lot sizing models. The main contributions of this paper are: 1) transformation of a transportation planning problem into a lot-sizing problem; 2) an  $O(T^3)$  algorithm is proposed to solve it; and 3) the results can also be applied to terrestrial transportation with direct deliveries.

**Note to Practitioners**—This paper addresses a real-life crude oil transportation planning problem with direct delivery. Based on mathematical properties, the problem is transformed into a lot sizing problem with limited production and inventory capacity. This latter problem is shown to be exactly solvable in polynomial time with dynamic programming. In other words, an optimal solution of this problem can be obtained within a very short amount

of computation time, even for large sized instances. Furthermore, the algorithm can be easily implemented with Excel tables and a case study with real-life data shows that the proposed algorithm yields a solution reducing very significantly (by more than 25%) the transportation cost, compared with the classical solution with immediate deliveries.

**Index Terms**—Complex system, crude oil transportation, dynamic programming, lot-sizing, scheduling.

## I. INTRODUCTION

WITH worldwide economic growth, the demand for energy, in particular, crude oil, is increasing very quickly and meeting it at the lowest cost becomes crucial for the economy of a country. The crude oil transportation problem recently attracted much attention in oil industry. Several modes are often used to transport crude oil: pipelines, tankers, trucks, and trains. The choice of transportation modes is made according to distance, crude oil type, cost, and available resources. We can use one or their combination in a crude oil supply chain. For example, in an upstream crude oil supply chain, pipeline and tankers are often used for large volume transportation over long distances. Trucks or trains are used in the downstream distribution of oil products. Typical unitary crude oil transportation cost ranges from 1.5 to 3.0 US\$ per barrel. Therefore, total yearly worldwide crude oil transportation cost for a large oil company can be as much as several billion dollars [4]. Hence, an appropriate plan can lead to significant transportation cost savings.

According to the United Nation Annual Review of Maritime Logistics [16], with an increasing worldwide demand for crude oil and petroleum products, tankers have become one of the principal crude oil transportation modes. However, there are few related works for oil transportation planning with tankers.

The problem studied in this paper can be summarized as follows. A type of crude oil is shipped from a supplier or hub port to  $n$  client ports to satisfy customer demands over  $T$  forthcoming periods. The supplier port disposes a fleet of identical tankers with limited capacity. The number of tankers, however, is assumed to be unlimited. This assumption is not restrictive since tankers are generally rented from third-party companies by anticipation, as done in China Petroleum. Furthermore, this is a common practice in industry, since companies seek for focusing on their own core business by outsourcing accessory activities such as logistic ones. The delivery lead time is assumed to be negligible in case where it is relatively short with respect to the length of each period. Customer demands are dynamic and estimated based on the forecast. The inventory capacities of customers are limited and time-varying. Backlogging is admitted

The work was supported in part by the National Natural Science Foundation of China under Grant 61034004. This paper was presented in part at the 2008 IEEE Conference on Automation Science and Engineering, Washington, DC, August 23–26, 2008. *Asterisk indicates corresponding author.*

C. Chu is with the School of Economics and Management, Tongji University, Shanghai, China, and also with the Laboratoire Génie Industriel, Ecole Centrale Paris, 92295 Châtenay-Malabry Cedex, France.

\*F. Chu is with the Laboratoire IBISC, Université d'Evry Val-Essonne, 91000 Evry Cedex, France (e-mail: Feng.chu@ibisc.univ-evry.fr).

M. Zhou is with The Key Laboratory of Embedded System and Service Computing, Ministry of Education, Tongji University, Shanghai 200092, China, and also with the Department of Electrical and Computer Engineering, New Jersey Institute of Technology, Newark, NJ 07102-1982 USA.

H. Chen is with the Institut Charles Delaunay and UMR CNRS STMR 6279, Université de Technologie de Troyes, 10010 Troyes Cedex, France.

Q. Shen is with the School of Business Administration, China University of Petroleum, Beijing, 102249, China.

but incurs a penalty. There is a setup cost for transportation, independent of the quantity, called fixed cost. This cost can represent the rental cost and is assumed to be constant over time. Transportation, inventory, and backlogging cost functions are linear. The objective is to find an optimal shipping plan over  $T$  periods minimizing the cost.

Iakovou [11] considers a single period, multicommodity, multisource-destination and multimodal maritime oil transportation problem. He develops a multiobjective (risk analysis and transportation costs) network flow model. In his linear programming model, the flow capacity for a type of vessels is limited and backlogging is not admitted. The model is then decomposed into two subproblems, one for the risk, and the other for the transportation. A commercial software package CPLEX is used to solve the subproblems. An interactive methodology is proposed based on a software tool known as Interactive Oil Transportation model.

Sherali *et al.* [14] study a  $T$ -period maritime transportation problem, in particular, the Kuwait Petroleum Corporation (KPC) problem. This problem involves multiple types of ships, multiple products, and time window constraints. A penalty is imposed on shipments that are not delivered within the time windows. A relatively complex mixed-integer linear programming (MILP) model for the KPC problem is proposed and solved by using CPLEX for small-sized test instances. Later on, an aggregate model (AP) that retains the principal features and is also an MILP model is formulated. Finally, a specific rolling horizon heuristic for AP is developed to solve practical sized instances.

Christiabsen and Nygreen [5] consider a single item pickup and delivery problem with time windows and inventory constraints. In this problem, inventory cost is not involved and backlogging is not admitted. A mathematical programming model is developed. The model is decomposed into subproblems respectively for ship routing and harbor inventory planning. The former consists of finding an approximately optimal route for each ship. The latter for each independent internal harbor is to find a sequence of visiting harbors with the least reduced cost. The subproblems are solved successfully by column generation. A solution for the overall problem is obtained by synchronizing the solutions of the subproblems.

Cheng and Duran [4] are the first to consider multimodal (pipelines and tankers), stochastic crude oil inventory/transportation problems. They developed a decision support system to investigate and improve a solution based on the discrete-event simulation and optimal control. An integrated simulation model using a commercial package Arena is constructed to simulate the dynamic and stochastic behavior of the system, and to evaluate various strategies for the design and operation of the system. For the optimal control problem, a Markov process mathematical model is formulated. The whole system is then decomposed into independent subsystems, one for each demand location. The cost-to-go function of each subsystem is linearly approximated. In their computational experiments, the MILP model of the subsystem is solved with Matlab. However, no computational results are reported.

The existing works cited before for oil inventory transportation problems provide constructive ideas and interesting

research directions. Their common characteristics are: Complete models are first constructed and then decomposed or simplified. Subproblems are solved by software packages. Finally, a solution of the original problem is obtained by synchronizing the solutions of the subproblems.

Our idea is to consider a relatively simple oil inventory transportation model and to develop an efficient optimization algorithm. Based on this algorithm, we can consider multimode or multiproduct oil inventory transportation problems. As a preliminary research for a long term project of developing an efficient tool for planning maritime crude oil transportation, we only consider tanker transportation problems in this paper.

We will first show that the shipment planning problem can be transformed into a single item lot sizing problem with production and inventory capacities and stationary unit production and setup costs, and linear inventory/backlogging cost functions. Even though it is widely studied in manufacturing environment, it is relatively new in transportation. To the best of our knowledge, there is not yet any polynomial algorithm for this problem. This paper shows, based on mathematical properties, that the problem can be solved in polynomial time.

Single item lot sizing problems have been widely investigated in the literature. For capacitated lot-sizing models without backlogging, Bitran and Yanasse [2] study the computational complexity with linear production and holding cost functions with a constant production capacity. Florian and Klein [9] obtain an  $O(T^4)$  algorithm with a constant production capacity, concave production and inventory holding cost functions. Chung and Lin [8] give an  $O(T^2)$  algorithm with production capacity that is nondecreasing over time, and where unit production and setup costs are nonincreasing over time and holding cost functions have arbitrary pattern.

For lot sizing models with inventory capacity and backlogging, Love [12] proposes an  $O(T^3)$  algorithm by assuming that the value of the concave production and holding cost functions can be computed in constant time. For the same models with linear production and holding cost functions and setup costs, Chu and Chu [6] propose an  $O(T^2)$  algorithm, and an  $O(T)$  algorithm if the production cost functions are linear with setup cost and nonsplicative holding/backlogging cost functions.

Backlogging models with inventory capacity, production capacity and general cost functions (with setup cost) are NP-hard, since the model without backlogging is NP-hard. For such problems, Swoveland [15], Shaw and Wagelmans [13], and Chu *et al.* [7] propose pseudopolynomial dynamic programming algorithms. Hoesl and Wagelmans [10] develop fully polynomial approximation schemes. For the problem with linear production and holding cost functions, which is a special case of minimum cost network flow problem, Ahuja and Hochnaum [1] propose an  $O(T \log T)$  algorithm. Although setup cost is not considered in their paper, this model has numerous applicable scenarios in today's manufacturing environment.

In Section II, the problem is described in detail. In Section III, its linear programming model is proposed. We prove mathematical properties of the model. Based on them, we show that the problem can be transformed into a single item lot sizing problem with limited inventory and production capacities. Section IV

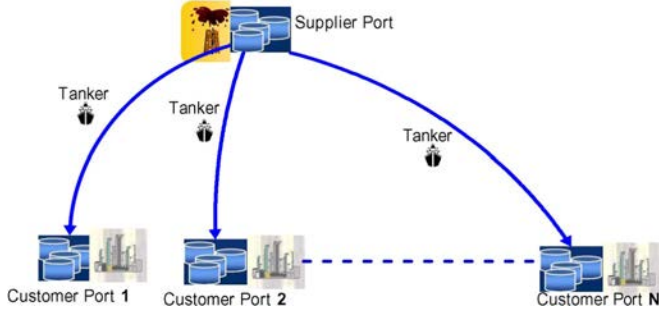


Fig. 1. A crude oil transportation problem.

presents a new polynomial dynamic programming algorithm to solve it. In Section V, a case study is presented to illustrate the algorithm. Section VI concludes this paper and indicates some future research directions.

## II. PROBLEM DESCRIPTION

In this section, we describe the crude oil transportation problem at hand. It can be presented as in Fig. 1 in which one unlimited capacitated hub port supplies multiple client ports by a fleet of identical tankers. The capacity of a tanker and inventory capacity of the client ports are limited.

The problem considered is to organize crude oil transportation from the supplier port to  $n$  client ports for the  $T$  forthcoming periods indexed from 1 to  $T$ . The demand of customer  $i$  ( $i = 1, \dots, n$ ) for period  $t$  is known and denoted  $D_{it}$ . In each client port, the inventory capacity is  $S_{it}$  and time-varying because several products may share the same storage area. Backlogging is admitted but incurs a penalty. Backlogging and holding cost functions are linear and stationary in the horizon. The supplier port sends out a fleet of identical tankers each with capacity  $C$ . The demand of a customer for a period may be greater than the tanker capacity. The transportation cost is stationary in the horizon and constituted by a fixed and a variable shipping cost. The stationariness of the cost generally holds, especially when the transportation is subcontracted. The fixed cost is charged to every tanker used, to represent the rental cost, for instance. The variable one is linear with the quantity transported and transportation distance. The number of tankers is considered to be unlimited. The delivery should be made directly from the supplier port to a client port. As a consequence, the delivery to a customer is independent from any other customer. Because of all these assumptions, the crude oil transportation problem with  $n$  customers can be considered to be  $n$  independent ones. Hence, if an optimal shipping plan for each customer can be determined, the optimal shipping plan for  $n$  customers is obtained. In the remainder of this paper, without loss of generality we consider one supplier, one customer shipping problem. The objective is to find a shipping plan for each customer to minimize the total transportation and inventory cost over the  $T$ -period planning horizon.

The single-product, single-supplier, and single-customer crude oil transportation problem has similarities with the single item lot sizing problem with limited production and inventory capacities in the literature, if we consider tanker capacity as production capacity, and shipping cost as production cost. The only difference is that in the model we consider, customer demand for some period may be greater than the tanker capacity

and more than one delivery is necessary in a period to satisfy the requirement of the customer. It means that in a lot sizing problem, demand exceeds the production capacity. Another point is that each delivery incurs a setup (fixed) cost in the oil transportation problem, while in a lot sizing problem, there is generally only one production setup cost for one period.

## III. MATHEMATICAL FORMULATION

The following notation will be used throughout the paper.

$T$	the number of periods in the planning horizon;
$C$	production (tanker) capacity;
$D_t$	demand of period $t$ ;
$K$	setup cost (fixed cost for each delivery);
$p$	unit transportation cost;
$h$	unit holding cost per period;
$g$	unit backlogging cost per period;
$n_t$	$n_t = \lfloor D_t/C \rfloor$ ;
$d_t$	actuated demand of period $t$ ( $0 \leq d_t < C, t = 1, \dots, T$ ), $d_t = D_t - n_t C$ ;
$S_{t,-1}$	lower limit of inventory level ( $S_{t,-1} \leq 0$ , $-S_{t,-1}$ is the upper limit of backlogging level) at the end of period $t$ ;
$S_{t,1}$	upper limit of inventory level ( $S_{t,1} \geq 0$ ) at the end of period $t$ ;
$z_t$	production (transportation) quantity of period $t$ ;
$I_t$	inventory level at the end of period $t$ ;
$P(\cdot)$	production (transportation) cost function that is period-independent, i.e., $P(z_t) = K \lceil z_t/C \rceil + p \cdot z_t$ , where $\lceil z_t/C \rceil$ gives the required number of tankers, the first term then gives the fixed cost related to the tankers and the second term gives the variable cost proportional to the quantity transported;
$H(\cdot)$	inventory holding/backlogging cost function which is period-independent, i.e.,

$$H(I_t) = \max(-gI_t, hI_t) = \begin{cases} h \cdot I_t, & \text{if } I_t \geq 0 \\ -g \cdot I_t, & \text{otherwise} \end{cases}.$$

Among these notations,  $z_t$ 's are decision variables and  $I_t$ 's are state variables. We assume that  $K \geq 0$ ,  $h > 0$ , and  $g > 0$ .

Without loss of generality, we assume that the inventory levels at the end of periods 0 and  $T$  are 0. We define

$$S_{t,0} = 0, \quad \forall t \in \{0, 1, \dots, T\}.$$

The corresponding lot sizing problem can be formulated as follows:

$$\text{Minimize } \sum_{t=1}^T [(Ky_t + p \cdot z_t + H(I_t))] \quad (1)$$

subject to

$$I_{t-1} + z_t = I_t + D_t \quad t = 1, \dots, T \quad (2)$$

TABLE I  
CUSTOMER DEMANDS

$T$	1	2	3	4
$D_t$	57	48	79	66

TABLE II  
ACTUATED CUSTOMER DEMANDS

$t$	1	2	3	4
$d_t = D_t - n_t C$	7	8	9	6

$$S_{t,-1} \leq I_t \leq S_{t,1} \quad t = 1, \dots, T \quad (3)$$

$$0 \leq z_t \leq C \cdot y_t \quad t = 1, \dots, T \quad (4)$$

$$I_0 = I_T = 0 \quad (5)$$

$$y_t \text{ is a non-negative integer} \quad t = 1, \dots, T \quad (6)$$

where  $y_t$  represents the number of tankers required in period  $t$ . In fact, due to constraint (4) and the minimization operation in (1), we must have  $y_t = \lceil z_t/C \rceil$  in optimal solutions. The constraints are linear functions. The objective function is not linear but it is well known that it can be transformed into a linear one by introducing an additional variable representing the inventory holding/backlogging cost for each period and additional constraints.

If demand  $D_t$  exceeds tanker capacity  $C$  and  $n_t C \leq D_t < (n_t + 1)C$ , it means that the demand of period  $t$  can fill  $n_t$  tankers. With stationary transportation cost, it is logical to immediately transport the partial demand that fills  $n_t$  tankers to avoid additional backlogging and/or holding costs. We have the following property.

*Property 1:* In an optimal plan,  $z_t \geq n_t C$  for any period  $t$  such that  $1 \leq t \leq T$ ; in other words, there are at least  $n_t$  immediate deliveries with full loaded tankers in any period  $t$ .

*Proof:* See Appendix I.

To illustrate the algorithm developed, the following example is used throughout this paper.

*Example 1:* We consider a hub port and one client port crude oil shipping problem. The forecast demands in the forthcoming periods are presented in Table I. The fixed and unit transportation costs are, respectively,  $K = 30$  and  $p = 1$ . The tanker's capacity is  $C = 10$ . For the customer, the holding and backlogging costs are respectively  $h = 1$  and  $g = 3$ . The upper limit of the inventory level at the end of periods are, respectively,  $S_{1,1} = 11$ ,  $S_{2,1} = 8$ ,  $S_{3,1} = 8$ ,  $S_{4,1} = 8$ . The lower limit of inventory level at the end of period are  $S_{t,-1} = -7$  for  $t = 1, \dots, 4$ .

According to Property 1, in an optimal solution, the numbers of full loaded tankers immediately delivered for four periods are, respectively, 5, 4, 7, and 6. The actuated customer's demands are given in Table II.  $\square$

*Property 2:* In an optimal solution, for any period  $t$ , if  $I_{t-1} \geq d_t$ , then  $z_t = n_t C$ . In other words, if the available inventory is enough to cover the actuated demand, then the quantity transported is exactly  $n_t$  full tankers.

*Proof:* Assume that there is an optimal plan in which there is some period  $t$  such that  $z_t > n_t C$  (note that the solutions in which  $z_t < n_t C$  can be discarded according to Property 1) and  $I_{t-1} \geq d_t$ .

TABLE III  
CUMULATIVE DEMANDS  $d_{i,j}$

$i \backslash j$	1	2	3	4
0	7	15	24	30
1	0	8	17	23
2	-	0	9	15
3	-	-	0	6

By the definition, we have  $I_t = z_t + I_{t-1} - D_t = z_t + I_{t-1} - n_t C - d_t \geq z_t - n_t C > 0$ . Due to the assumption that  $I_T = 0$ , we must have  $t < T$ . Construct another solution by shifting the production quantity  $z_t - n_t C$  from period  $t$  to period  $t + 1$  so that in the new solution, we have  $I'_t = I_{t-1} - d_t \geq 0$ . The holding cost of period  $t$  is reduced, while the other costs do not increase. Therefore, the new solution is strictly better than the initial solution, which is in contradiction with the assumption that the initial solution is optimal.  $\square$

*Property 3:* In an optimal plan, in addition to  $n_t$  tankers used for immediate deliveries, at most one tanker can be used in each period, i.e.,  $z_t \leq (n_t + 1)C$ , for any  $t$  such that  $1 \leq t \leq T$ .

*Proof:* See Appendix II.

From Properties 1 and 3, the remaining problem is to establish a transportation plan for the actuated demand. By defining  $x_t = z_t - n_t C$ , the two above properties imply

$$0 \leq x_t \leq C$$

which is identical to the capacity constraints in classical lot sizing problems, where the production capacity is given by  $C$ . With this in mind, the production cost function can be rewritten as

$$p(x) = \begin{cases} K + p \cdot x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and the range of  $y_t$  in the mathematical formulation can be rewritten as  $y_t \in \{0, 1\}$ . Furthermore, we have the following corollary from Property 2.

*Corollary 1:* In an optimal solution, for any period  $t$ , if  $I_{t-1} \geq d_t$ , then  $x_t = 0$ .

We introduce the notion of cumulative demand as follows:

$$d_{i,j} = \sum_{t=i+1}^j d_t$$

We have  $d_{i,j} = d_{0,j} - d_{0,i}$ , and  $d_{i,i} = 0$ , for any  $i$  such that  $0 \leq i \leq T$ .

*Example 1 (Continued):* The cumulative demands in the example 1 are given in Table III.  $\square$

In the remainder, we assume without loss of generality that the demands are actuated ones. For the actuated demands, we have  $0 \leq d_t < C$  and the actuated demand shipping problem corresponds to a lot sizing problem.

The problem at hand is a standard single item lot sizing problem with production and inventory capacities and stationary unit production and setup costs, and linear inventory/backlogging cost functions. In the next section, we show that the problem can be solved in polynomial time with a new dynamic programming algorithm.

IV. A NEW POLYNOMIAL DYNAMIC PROGRAMMING ALGORITHM

In this section, we develop some properties of optimal solutions, and then present a new dynamic programming algorithm for the lot sizing problem based on the properties developed.

A. Definitions and Properties of Optimal Solutions

We first recall some definitions derived from the result of Love [12].

*Definition 1:* If  $x_t \neq 0$ ,  $t$  is called a *production period*. In particular, if  $x_t = C$  (resp.  $0 < x_t < C$ ),  $t$  is called a full (resp. partial) production period.

*Definition 2:* If  $I_t = S_{t,v}$ ,  $v \in \{-1, 0, 1\}$ ,  $t$  is an *inventory point*.

For stationary transportation, holding and backlogging cost functions, we have the following properties.

*Property 4:* In an optimal solution,  $-C < I_t < C$ , for any  $t$  such that  $1 \leq t \leq T$ .

*Proof:* Assume that there is an optimal solution in which  $I_t \geq C$  for some  $t$  such that  $1 \leq t \leq T$ . Let  $t^*$  be the first one of such periods. This means that  $I_{t^*-1} < C$  which implies that  $x_{t^*} = I_{t^*} - I_{t^*-1} + d_{t^*} > C - C + d_{t^*} = d_{t^*} \geq 0$ . By considering the fact that  $x_{t^*} \leq C$  (Property 2), we have  $I_{t^*-1} = I_{t^*} - x_{t^*} + d_{t^*} \geq C - C + d_{t^*} = d_{t^*}$ . The facts that  $x_{t^*} > 0$  and  $I_{t^*-1} \geq d_{t^*}$  are in contradiction with the assumption that the solution is optimal according to Corollary 1.

Assume that there is an optimal solution such that  $I_t \leq -C$  for some  $t$  such that  $1 \leq t \leq T$ . Let  $t^*$  be the first one of such periods. This implies that  $I_{t^*-1} > -C$  and  $x_{t^*} = I_{t^*} - I_{t^*-1} + d_{t^*} < -C + C + d_{t^*} = d_{t^*} < C$ . Let  $\tau$  be the next production period. We have  $I_i \leq I_{t^*} \leq -C < 0$  for any  $i$  such that  $t^* \leq i < \tau$ . Construct a new solution such that  $x'_{t^*} = \min(C, x_{t^*} + x_\tau) > x_{t^*}$ , and  $x'_\tau = x_\tau + x_{t^*} - x'_{t^*} < x_\tau$ . In the new solution, there is still backlogging in periods  $t^* to  $\tau - 1$ , but the backlogging cost is reduced, while the transportation cost does not increase, which is in contradiction with the assumption that the initial solution is optimal.  $\square$$

*Property 5:* In an optimal solution,  $I_t \geq 0$  for any *partial* production period  $t$ . In other words, if there is production in a period it is impossible in an optimal solution that the requirement of that period is not satisfied, while the production capacity is not fully utilized.

*Proof:* Assume now that there is an optimal solution in which there is a partial production period  $t$  such that  $I_t < 0$ . There must be at least one production period after  $t$  since otherwise we would have  $I_T < 0$  which is in contradiction with the assumption. Let  $t^*$  be the first production period after  $t$ . We must have  $I_\tau = I_t - d_{t,\tau} \leq I_t < 0$  for any  $\tau$  such that  $t \leq \tau < t^*$ . Reducing the production quantity by  $\Delta = \min(-I_t, x_{t^*}, C - x_t) > 0$  in period  $t^*$ , while increasing the production quantity of period  $t$  by the same quantity reduces the backlogging level of these periods and thus leads to a strictly better solution which is in contradiction with the assumption that the initial solution is optimal.  $\square$

*Example 2:* Note that Property 5 is valid only for partial production periods. It is possible that it does not hold for some full production periods as shown in the following four-period example with  $K = 40$ ,  $C = 10$ ,  $h = 3$ , and  $g = 5$ .  $S_{t,1} = 20$ ,  $S_{t,-1} = -15$  for  $t = 1, 2, 3$ , and  $4$ .

TABLE IV  
ACTUATED CUSTOMER DEMANDS

$T$	1	2	3	4
$d_t$	5	7	9	9

For this example, the only optimal solution is  $x_1 = x_3 = x_4 = 10$  and  $x_2 = 0$ . In this solution,  $t_3$  is a full production period but  $I_3 = -1$ .  $\square$

From Property 5, we have the following corollary.

*Corollary 2:* In an optimal solution, for any partial production period  $t$ ,  $-C < I_{t-1} - d_t < 0$ .

*Proof:* From Corollary 1, we know that  $I_{t-1} - d_t < 0$ . Furthermore, from Property 5, we have  $I_t \geq 0$ , which implies  $I_{t-1} - d_t = I_t - x_t > I_t - C \geq -C$ .  $\square$

*Property 6:* There is an optimal solution such that between every adjacent partial production periods  $j < j'$ , there is at least one inventory point  $k$ ,  $j \leq k < j'$ . Equivalently, between any two adjacent inventory points,  $i < k$ , there can be at most one partial production period  $j$ ,  $i < j \leq k$ .

*Proof:* It is obvious from [9] and [12], since the cost functions are linear or linear with fixed charges, therefore concave.  $\square$

From this property, an optimal plan can be decomposed into a series of subplans  $(i, u, k, v)$  as in [9] and [12], in which  $i < k$  and  $i$  and  $k$  are adjacent inventory points (the inventory levels of periods  $i$  and  $k$  are, respectively,  $S_{i,u}$  and  $S_{k,v}$ ) with  $u \in V_i$  and  $v \in V_k$  where  $V_t$  denotes the set of values of  $w$  such that the inventory level of period  $t$  can be equal to  $S_{t,w}$ . All values of  $S_{i,u}$  for  $u \in V_i$  and all values of  $S_{k,v}$  for  $v \in V_k$  are distinct; namely,  $S_{i,-1} < S_{i,0} = 0 < S_{i,1}$  and  $S_{k,-1} < S_{k,0} = 0 < S_{k,1}$ .

From Property 4, if  $S_{t,-1} \leq -C$  (resp.  $S_{t,1} \geq C$ ), the lower (resp. upper) limit will never be reached in an optimal solution. To be more specific, we have Corollary 3 according to Property 4.

*Corollary 3:* For each subplan  $(i, u, k, v)$  in an optimal solution,  $-C < S_{i,u}$  (resp.  $S_{k,v}$ )  $< C$ .

In the remainder, we will not consider the subplans  $(i, u, k, v)$  such that  $S_{i,u}$  (resp.  $S_{k,v}$ )  $\leq -C$  or  $S_{i,u}$  (resp.  $S_{k,v}$ )  $\geq C$ . In other words, we must have  $u \in V_i$  and  $v \in V_k$ .

*Example 1 (Continued):* For Example 1, the possible subplans are  $(0, 0, 1, v)$ ,  $(0, 0, 2, v)$ ,  $(0, 0, 3, v)$ ,  $(0, 0, 4, 0)$ ,  $(1, u, 2, v)$ ,  $(1, u, 3, v)$ ,  $(1, u, 4, 0)$ ,  $(2, u, 3, v)$ ,  $(2, u, 4, 0)$ ,  $(3, u, 4, 0)$  with  $I_0 = 0$  or  $I_4 = 0$ . For each subplan, we have  $u$  and  $v \in \{-1, 0, 1\}$ . In total, there are 46 possible subplans. In subplan  $(1, u, 4, v)$ , for instance, as  $S_{1,1} = 11 > C = 10$ , according to Property 4 or Corollary 3, the upper limit will never be reached in an optimal solution. Hence, the subplan  $(1, 1, k, v)$  for  $k = 2, 3, 4, v \in \{-1, 0, 1\}$  does not appear in an optimal solution.

Let  $c_{u,v}(i, k)$  denote the minimal cumulative production and inventory/backlogging cost of a subplan  $(i, u, k, v)$ , that is, from period  $i + 1$  to period  $k$ , such that the inventory levels at the end of periods  $i$  and  $k$  are  $S_{i,u}$  and  $S_{k,v}$ , respectively. In the remainder, we develop a new polynomial dynamic programming algorithm based on the properties obtained. For this purpose, we must calculate the minimal cumulative cost  $c_{u,v}(i, k)$  for each subplan  $(i, u, k, v)$ .

In a subplan  $(i, u, k, v)$ , the total production quantity is always  $S_{k,v} - S_{i,u} + d_{i,k}$ , independent of the positions of the

production periods, and there is at most one partial production period. We consider two cases. In the first case, there is an  $n$  such that  $nC < S_{k,v} - S_{i,u} + d_{i,k} < (n+1)C$ ; in other words, the total demand of the subplan is not a multiple of the production capacity, and there is exactly one partial production period  $j$  with a partial production quantity  $r(i, u, k, v) = (S_{k,v} - S_{i,u} + d_{i,k}) - nC > 0$ . In the second case, the total production quantity is  $S_{k,v} - S_{i,u} + d_{i,k} = nC$  and  $r(i, u, k, v) = 0$ .

In what follows, we will compute the minimal cumulative cost for each of these two cases. Note that in either of the cases, the production amount and the number of production periods in a given subplan are independent of the positions of the production periods. For a given subplan  $(i, u, k, v)$ , with stationary production and setup costs, the total setup and variable production cost is independent of the positions of the production periods and is always equal to  $K \lceil (S_{k,v} - S_{i,u} + d_{i,k})/C \rceil + p(S_{k,v} - S_{i,u} + d_{i,k})$ . That is why we do not take into account the production cost for the moment.

We first consider the case with one partial production period  $j$  such that  $i < j \leq k$ . First note the following corollary.

*Corollary 4:* If  $v = -1$ , then  $j \neq k$ .

*Proof:* If  $v = -1$  and  $j = k$ , we would have  $I_j = S_{k,v} = S_{k,-1} < 0$ . This is in contradiction with Property 5.  $\square$

This corollary means that in a subplan  $(i, u, k, v)$ , if  $S_{k,v} < 0$ , then  $k$  cannot be a partial production period. To obtain  $c_{u,v}(i, k)$ , the idea is to consider all possible positions for the partial production period  $j$ , if any, in a subplan  $(i, u, k, v)$  and the minimal cumulative inventory holding/backlogging cost before (resp. since) period  $j$ . In fact, let

- $A_{u,+}(i, t)$  (resp.  $A_{u,-}(i, t)$ ) with  $i \leq t \leq T$  be the minimal cumulative inventory/backlogging cost from  $i+1$  to  $t$  without any *partial* production period such that the inventory level at the end of period  $t$  is  $I_t > 0$  (resp.  $I_t \leq 0$ ). Without loss of generality, we assume that the full production periods will not create inventory points before  $t$ .
- $B_{v,+}(t, k)$  (resp.  $B_{v,-}(t, k)$ ) with  $1 \leq t \leq k$  be the minimal cumulative inventory/backlogging cost from  $t$  to  $k$  without any *partial* production period from  $t+1$  to  $k$  such that the inventory level at the end of period  $t$  is  $I_t > 0$  (resp.  $I_t \leq 0$ ). Without loss of generality, we assume that these full production periods will not create inventory points after  $t$ .

Then, we will have  $c_{u,v}(i, k) = A_{u,+}(i, j) + B_{v,+}(j, k) + K \lceil (S_{k,v} - S_{i,u} + d_{i,k})/C \rceil + p(S_{k,v} - S_{i,u} + d_{i,k})$ , if the partial production is at period  $j$ , where “\*” may be “+” or “-,” provided that  $I_{j-1} < d_j$ . We will come back to this point in Section IV-D.

In the following two subsections, we show how to compute  $A$ ’s and  $B$ ’s and then in Section IV-D, we show how to compute  $c_{u,v}(i, k)$  when all  $A$ ’s and  $B$ ’s are known. Note that, if  $r(i, u, k, v) = 0$  in a subplan  $(i, u, k, v)$ , then there is no partial production period  $j$  and  $c_{u,v}(i, k)$  is determined by  $A_{u,+}(i, k)$  and  $A_{u,-}(i, k)$ . In this case, it is not necessary to calculate  $B_{v,+}(t, k)$  and  $B_{v,-}(t, k)$ .

#### B. Minimal Cumulative Costs $A_{u,+}(i, t)$ and $A_{u,-}(i, t)$

Before presenting the cumulative costs  $A_{u,+}(i, t)$  and  $A_{u,-}(i, t)$ , let  $f_u(i, t)$  denote the number of full production periods from period  $i+1$  until  $t$ . For any period  $t$  such

TABLE V  
NUMBER OF FULL PRODUCTION PERIODS  $\bar{f}_u(i, t)$

$\begin{matrix} (i,t) \\ u \end{matrix}$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
1	*	*	*	*	•	•	•	1	1	0
0	1	2	3	3	1	2	3	1	2	1
-1	*	*	*	*	-	-	3	-	-	-

TABLE VI  
NUMBER OF FULL PRODUCTION PERIODS  $\underline{f}_u(i, t)$

$\begin{matrix} (i,t) \\ u \end{matrix}$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
1	*	*	*	*	•	•	•	0	0	-1
0	0	1	2	3	0	1	2	0	1	0
-1	*	*	*	*	1	2	3	1	2	1

that  $0 \leq i \leq t < T$ , the inventory level at period  $t$  is  $I_t = S_{i,u} - d_{i,t} + f_u(i, t)C$ . Due to Property 4,  $f_u(i, t)$  is either  $\underline{f}_u(i, t) = \lfloor (d_{i,t} - S_{i,u})/C \rfloor$  with  $I_t < 0$  or  $\bar{f}_u(i, t) = \lceil (d_{i,t} - S_{i,u})/C \rceil$  with  $I_t > 0$ . Otherwise, if  $\underline{f}_u(i, t) = \bar{f}_u(i, t)$ , the inventory level at period  $t$  is 0. Furthermore, if there is some period  $t$  such that  $i < t < k$  and  $t - i < \bar{f}_u(i, t)$ ,  $\bar{f}_u(i, t)$  is infeasible according to Property 3. Thus,  $f_u(i, t) = \underline{f}_u(i, t)$ . If  $S_{i,u} - d_{i,t} > 0$ , it means that the inventory level at period  $i$  can satisfy the demands until period  $t$  without any production period. Naturally, we must have  $f_u(i, t) = 0$ , while  $\underline{f}_u(i, t) = -1$  and  $\bar{f}_u(i, t) = 0$  according to Property 4. To sum up

$$f_u(i, t) = \begin{cases} \underline{f}_u(i, t) = \bar{f}_u(i, t), & \text{if } \lfloor (d_{i,t} - S_{i,u})/C \rfloor \\ & = \lceil (d_{i,t} - S_{i,u})/C \rceil \\ \underline{f}_u(i, t), & \text{if } t - i < \lceil (d_{i,t} - S_{i,u})/C \rceil \\ \bar{f}_u(i, t), & \text{if } \lfloor (d_{i,t} - S_{i,u})/C \rfloor < 0 \\ \underline{f}_u(i, t) \text{ or } \bar{f}_u(i, t), & \text{otherwise.} \end{cases}$$

Therefore, since  $0 \leq d_t < C$ , we have

$$\begin{aligned} 0 &\leq \underline{f}_u(i, t) - \underline{f}_u(i, t-1) \leq 1 \\ 0 &\leq \bar{f}_u(i, t) - \bar{f}_u(i, t-1) \leq 1 \\ 0 &\leq \bar{f}_u(i, t) - \underline{f}_u(i, t) \leq 1 \\ \min(-u, 0) &\leq \underline{f}_u(i, t) \leq t - i \\ 0 &\leq \bar{f}_u(i, t) \leq t - i - \min(u, 0) \end{aligned}$$

*Example 1 (Continued):* The bounds on the numbers of full production periods  $\bar{f}_u(i, t)$  and  $\underline{f}_u(i, t)$  in Example 1 are given in Tables V and VI. In these tables, “\*” indicates that the corresponding subplan with  $I_0 = 0$  does not exist; “•” means that it is impossible that the corresponding subplan appears in an optimal solution and therefore can be ignored, and “-” means that  $\bar{f}_u(i, t)$  is incompatible with  $t - i < \bar{f}_u(i, t)$ .  $\square$

We can write  $I_t$  as follows:

$$I_t = \begin{cases} 0, & \text{if } f_u(i, t) = \underline{f}_u(i, t) \\ & = \bar{f}_u(i, t) \\ S_{i,u} + \underline{f}_u(i, t)C - d_{i,t} < 0, & \text{if } f_u(i, t) = \underline{f}_u(i, t) \\ S_{i,u} + \bar{f}_u(i, t)C - d_{i,t} > 0, & \text{if } f_u(i, t) = \bar{f}_u(i, t). \end{cases}$$

We distinguish two cases according to whether the inventory level of period  $t$  is positive or negative. These two cases also respectively correspond to the cases where the number of full production periods is  $\bar{f}_u(i, t)$  or  $\underline{f}_u(i, t)$ . In fact,  $A_{u,+}(i, t)$  (resp.  $A_{u,-}(i, t)$ ) is the minimal cumulative inventory and backloging cost from  $i + 1$  to  $t$  with  $\bar{f}_u(i, t)$  (resp.  $\underline{f}_u(i, t)$ ) full production periods and *without any partial production* from  $i + 1$  until  $t$ , such that  $I_t \in (0, C)$  (resp.  $(-C, 0)$ ).

According to the definition, there must be no inventory points before  $t$ . If  $\bar{f}_u(i, t') = \underline{f}_u(i, t')$  for some  $t'$  such that  $0 \leq i < t' < T$ , we will have  $I_{t'} = 0$ . This means that  $t'$  is an inventory point. We have naturally  $A_{u,+}(i, t) = A_{u,-}(i, t) = +\infty$  for any  $t$  such that  $t' < t \leq T$ .

By the definition of  $\bar{f}_u(i, t)$ ,  $A_{u,+}(i, t)$ , and  $A_{u,-}(i, t)$ , these quantities can be computed recursively.

From the definitions of the number of full production periods, we have  $\bar{f}_1(i, i) = 0$  and  $\underline{f}_1(i, i) = -1$ ;  $\bar{f}_0(i, i) = \underline{f}_0(i, i) = 0$ ;  $\bar{f}_{-1}(i, i) = 1$  and  $\underline{f}_{-1}(i, i) = 0$ .

By the definition, we can initialize the minimal cost from  $i + 1$  to  $i$

$$A_{u,+}(i, i) = \begin{cases} +\infty, & \text{if } u = -1 \\ 0, & \text{otherwise.} \end{cases}$$

For any  $t$  such that  $i < t \leq T$ , we have the first equation shown at the bottom of the page.

By the definition,  $A_{u,+}(i, t)$  corresponds to the case where  $I_t > 0$  and the number of full production periods from  $i + 1$  to  $t$  must be  $\bar{f}_u(i, t)$ .

If  $S_{i,u} + \bar{f}_u(i, t)C - d_{i,t} > S_{t,1}$ , we have  $I_t > S_{t,1}$  and the solution is infeasible. Hence, the minimal cumulative cost  $A_{u,+}(i, t)$  is  $+\infty$ .

For the second and third cases of  $A_{u,+}(i, t)$ , its first term is  $hI_t$ , the holding cost of period  $t$ . Its second term corresponds to the minimal cumulative cost from  $i + 1$  to  $t - 1$  without any partial production.

If  $\bar{f}_u(i, t) = \bar{f}_u(i, t - 1)$ , period  $t$  may or may not be a full production period. If so, the number of full production periods from  $i + 1$  to  $t - 1$  will be  $\bar{f}_u(i, t) - 1 = \bar{f}_u(i, t - 1) - 1 = \underline{f}_u(i, t - 1)$ . This leads to  $I_{t-1} \leq 0$  according to the definition of  $\underline{f}_u(i, t - 1)$  and the cumulative inventory/backloging cost of these periods is  $A_{u,-}(i, t - 1)$ . If  $t$  is not a full production

TABLE VII  
MINIMAL CUMULATIVE COSTS  $A_{u,+}(i, t)$

$\begin{matrix} (i,t) \\ u \end{matrix}$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
1	*	*	*	*	•	•	•	$+\infty$	6	2
0	3	8	14	14	2	5	12	1	6	4
-1	*	*	*	*	$+\infty$	$+\infty$	27	$+\infty$	$+\infty$	$+\infty$

period, the number of full production periods from  $i + 1$  to  $t - 1$  is  $\bar{f}_u(i, t) = \bar{f}_u(i, t - 1)$ . This implies  $I_{t-1} = I_t + d_t > 0$ . The cumulative inventory/backloging cost of these periods is  $A_{u,+}(i, t - 1)$ . In an optimal solution, the decision corresponds to the best one of these situations.

If  $\bar{f}_u(i, t) > \bar{f}_u(i, t - 1)$ ,  $t$  must be a full production period, since otherwise the number of full production periods from  $i + 1$  to  $t - 1$  would be  $\bar{f}_u(i, t) > \bar{f}_u(i, t - 1)$  which is infeasible. Therefore, the number of full production periods from  $i + 1$  to  $t - 1$  must be  $\bar{f}_u(i, t - 1)$ , which implies  $I_{t-1} > 0$  and the cumulative inventory/backloging cost of these periods is  $A_{u,+}(i, t - 1)$ . In the case that  $t - i < \bar{f}_u(i, t)$ , we must have  $u = -1$ , and  $\bar{f}_u(i, i) < \bar{f}_u(i, i + 1) < \dots < \bar{f}_u(i, t)$ . From the recursive equation, by considering the third case, we have  $A_{-1,+}(i, \tau) = +\infty$  for any  $\tau$  such that  $i \leq \tau \leq t$ , since  $A_{-1,+}(i, i) = +\infty$ .

For the case that  $-C < I_t \leq 0$ , by the definition, the minimal cost from  $i + 1$  to  $i$  is given by

$$A_{u,-}(i, i) = \begin{cases} +\infty, & \text{if } u = 1 \\ 0, & \text{otherwise} \end{cases}$$

which is the initial condition of the dynamic programming algorithm. For any  $t$  such that  $i < t \leq T$ , we have the following recursive equations (second equation) shown at the bottom of the page.

The explanation of expression  $A_{u,-}(i, t)$  can be found in Appendix III. From the recursive formulations, we can see that all  $A_{u,+}(i, t)$ ,  $A_{u,-}(i, t)$ ,  $\bar{f}_u(i, t)$  and  $\underline{f}_u(i, t)$  can be computed in  $O(T^2)$  time.

*Example 1 (Continued):* All minimal cumulative costs  $A_{u,+}(i, t)$  and  $A_{u,-}(i, t)$  are presented, respectively, in Tables VII and VIII.

For subplans  $(1, u, 4, v)$  with  $I_4 = 0$ , there are three possible subplans  $(1, 1, 4, 0)$   $(1, 0, 4, 0)$  and  $(1, -1, 4, 0)$ . Subplan  $(1,$

---


$$A_{u,+}(i, t) = \begin{cases} +\infty, & \text{if } S_{i,u} + \bar{f}_u(i, t)C - d_{i,t} > S_{t,1} \\ h [S_{i,u} + \bar{f}_u(i, t)C - d_{i,t}] \\ \quad + \min \{A_{u,+}(i, t - 1), A_{u,-}(i, t - 1)\}, & \text{if } \bar{f}_u(i, t) = \bar{f}_u(i, t - 1) \\ h [S_{i,u} + \bar{f}_u(i, t)C - d_{i,t}] + A_{u,+}(i, t - 1), & \text{if } \bar{f}_u(i, t) > \bar{f}_u(i, t - 1) \end{cases}$$


---

$$A_{u,-}(i, t) = \begin{cases} +\infty, & \text{if } S_{i,u} + \underline{f}_u(i, t)C - d_{i,t} < S_{t,-1} \\ g [d_{i,t} - S_{i,u} - \underline{f}_u(i, t)C] \\ \quad + \min \{A_{u,+}(i, t - 1), A_{u,-}(i, t - 1)\}, & \text{if } \underline{f}_u(i, t) > \underline{f}_u(i, t - 1) \\ g [d_{i,t} - S_{i,u} - \underline{f}_u(i, t)C] + A_{u,-}(i, t - 1), & \text{if } \underline{f}_u(i, t) = \underline{f}_u(i, t - 1) \end{cases}$$

TABLE VIII  
MINIMAL CUMULATIVE COSTS  $A_{u,-}(i, t)$

$(i,t)$ $u$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
1	*	*	*	*	•	•	•	3	24	$+\infty$
0	21	18	20	14	$+\infty$	23	14	$+\infty$	16	18
-1	*	*	*	*	15	27	27	18	24	9

1, 4, 0) is not in an optimal solution, because  $S_{1,1} = 11 > C = 10$  according to Property 4. For subplan (1, -1, 4, 0),  $r(1, -1, 4, 0) = 0$ , there is no partial production period. For subplan (1, 0, 4, 0), the partial quantity  $r(1, 0, 4, 0) = 3$ , or the partial production period is  $j \in \{2, 3, 4\}$ . If  $j = 2$ ,  $f_0(1, 1) = 0$ , then  $I_2 = r(1, 0, 4, 0) + S_{1,0} - d_{1,2} = 3 - 8 = -5$ . Thus,  $j = 2$  is not in an optimal solution according to Property 5. Similarly, if  $j = 3$ ,  $f_0(1, 2) \in \{0, 1\}$ , then  $I_3 = S_{1,0} + f_0(1, 2) \cdot C + r(1, 0, 4, 0) - d_{1,3} \leq 0 + 10 + 3 - 17 < 0$ . Thus,  $j = 3$  is not either. For subplan (1, 0, 4, 0) and  $j = 4$  with  $\bar{f}_0(1, 3) > \bar{f}_0(1, 2) > \bar{f}_0(1, 1)$  and  $\underline{f}_0(1, 3) > \underline{f}_0(1, 2) = \underline{f}_0(1, 1)$ , we have

$$\begin{aligned} A_{0,+}(1, 3) &= h [S_{1,0} + \bar{f}_0(1, 3)C - d_{1,3}] + A_{0,+}(1, 2) \\ &= h [S_{1,0} + \bar{f}_0(1, 3)C - d_{1,3}] \\ &\quad + h [S_{1,0} + \bar{f}_0(1, 2)C - d_{1,2}] + A_{0,+}(1, 1) \\ &= 1(0 + 2 \times 10 - 17) + 1(0 + 1 \times 10 - 8) = 5 \end{aligned}$$

and

$$\begin{aligned} A_{0,-}(1, 3) &= g [d_{1,3} - S_{1,0} - \underline{f}_0(1, 3)C] \\ &\quad + \min \{A_{0,+}(1, 2), A_{0,-}(1, 2)\} \\ &= g [d_{1,3} - \underline{f}_0(1, 3)C] \\ &\quad + \min \{h [\bar{f}_0(1, 2)C - d_{1,2}] \\ &\quad\quad + A_{0,+}(1, 1), g [d_{1,2} - \underline{f}_0(1, 2)C] \\ &\quad\quad + A_{0,-}(1, 1)\} \\ &= 3[17 - 10] + \min\{2 + 0, 3 \times 8 + 0\} = 23. \end{aligned}$$

□

### C. Minimal Cumulative Costs $B_{v,+}(t, k)$ and $B_{v,-}(t, k)$

Similarly to the previous subsection, recall that  $B_{v,+}(t, k)$  (resp.  $B_{v,-}(t, k)$ ) for any  $t$  such that  $0 \leq i < t \leq k \leq T$ , is the minimal cumulative inventory and backlogging cost from  $t$  to  $k$  without any partial production from  $t + 1$  to  $k$  such that the inventory level at  $t$  is  $I_t \in (0, C)$  (resp.  $(-C, 0)$ ).

To calculate  $B_{v,+}(t, k)$  and  $B_{v,-}(t, k)$ , we can make the same observations for any period  $t$  such that  $1 \leq t \leq k$ , as those for  $A_{u,+}(i, t)$  and  $A_{u,-}(i, t)$ . In the remainder, let  $q_v(t, k)$  be the number of full production periods from  $t + 1$  to  $k$ , and  $\underline{q}_v(t, k) = \lfloor (d_{t,k} + S_{k,v})/C \rfloor$ ,  $\bar{q}_v(t, k) = \lceil (d_{t,k} + S_{k,v})/C \rceil$  which are respectively the lower and upper bounds of the number of full production periods from  $t + 1$  to  $k$ . Furthermore, if there is some  $t$  such that  $1 \leq t \leq k$  and  $k - t < \bar{q}_v(t, k)$ ,  $\bar{q}_v(t, k)$  is infeasible according to Property 3. Thus,  $q_v(t, k) = \underline{q}_v(t, k)$ . If  $d_{t,k} + S_{k,v} < 0$ , it means that no production period is needed after  $t$ . Naturally,  $q_v(t, k) = 0$  and we have  $\underline{q}_v(t, k) = -1$  and  $\bar{q}_v(t, k) = 0$  according to Property 4.

TABLE IX  
NUMBER OF FULL PRODUCTION PERIODS  $\bar{q}_v(t, k)$

$\bar{q}_v(t, k)$	$\bar{q}_v(1,2)$	$\bar{q}_v(1,3)$	$\bar{q}_v(1,4)$	$\bar{q}_v(2,3)$	$\bar{q}_v(2,4)$	$\bar{q}_v(3,4)$
$v=1$	—	—	*	—	*	*
$v=0$	1	2	3	1	2	1
$v=-1$	1	1	*	1	*	*

TABLE X  
NUMBER OF FULL PRODUCTION PERIODS  $\underline{q}_v(t, k)$

$\underline{q}_v(t, k)$	$\underline{q}_v(1,2)$	$\underline{q}_v(1,3)$	$\underline{q}_v(1,4)$	$\underline{q}_v(2,3)$	$\underline{q}_v(2,4)$	$\underline{q}_v(3,4)$
$v=1$	1	2	*	1	*	*
$v=0$	0	1	2	0	1	0
$v=-1$	0	1	*	0	*	*

Similarly to  $f_u(i, t)$ , we have

$$q_v(t, k) = \begin{cases} \underline{q}_v(t, k) = \bar{q}_v(t, k), & \text{if } \lfloor (d_{t,k} + S_{k,v})/C \rfloor \\ & = \lceil (d_{t,k} + S_{k,v})/C \rceil \\ \underline{q}_v(t, k), & \text{if } k - t < (d_{t,k} + S_{k,v})/C \\ \bar{q}_v(t, k) & \text{if } d_{t,k} + S_{k,v} < 0 \\ \underline{q}_v(t, k) \text{ or } \bar{q}_v(t, k), & \text{otherwise.} \end{cases}$$

We can observe that

$$\begin{aligned} 0 &\leq \underline{q}_v(t, k) - \underline{q}_v(t + 1, k) \leq 1 \\ 0 &\leq \bar{q}_v(t, k) - \bar{q}_v(t + 1, k) \leq 1 \\ 0 &\leq \bar{q}_v(t, k) - \underline{q}_v(t, k) \leq 1 \\ \min(v, 0) &\leq \underline{q}_v(t, k) \leq k - t \\ 0 &\leq \bar{q}_v(t, k) \leq k - t + \max(v, 0). \end{aligned}$$

*Example 1 (Continued):* The numbers of full production periods  $\bar{q}_v(t, k)$  and  $\underline{q}_v(t, k)$  in Example 1 are given in Tables IX and X. Note that “\*” means that the corresponding subplan with  $I_4 = 0$  does not exist, “•” means that it is not in an optimal solution and “—” means that  $\bar{q}_v(t, k)$  is infeasible with  $k - t < \bar{q}_v(t, k)$ . □

To summarize, we have

$$I_t = \begin{cases} 0, & \text{if } q_v(t, k)C = \underline{q}_v(t, k)C \\ & = \bar{q}_v(t, k)C \\ S_{k,v} - \underline{q}_v(t, k)C + d_{t,k}C > 0, & \text{if } q_v(t, k)C = \underline{q}_v(t, k)C \\ S_{k,v} - \bar{q}_v(t, k)C + d_{t,k}C < 0, & \text{if } q_v(t, k)C = \bar{q}_v(t, k)C. \end{cases}$$

By the definition, the number of full production periods from  $t + 1$  to  $k$  corresponding to  $B_{v,+}(t, k)$  (resp.  $B_{v,-}(t, k)$ ) is  $\underline{q}_v(t, k)$  (resp.  $\bar{q}_v(t, k)$ ).

Recall that we assume that there is no inventory point between  $t$  and  $k$ . As a consequence, if  $\bar{q}_v(t', k) = \underline{q}_v(t', k)$  for some  $t'$  such that  $1 < t' < k \leq T$ , then  $B_{v,+}(t, k) = B_{v,-}(t, k) = +\infty$  for all  $t$  such that  $1 \leq t < t' < k \leq T$ .

If  $v = 1$  (resp.  $v \leq 0$ ), the holding cost of period  $k$  is  $hS_{k,v}$  (resp.  $-gS_{k,v}$ ).

From the definitions of the number of full production periods, we have  $\bar{q}_1(k, k) = 1$  and  $\underline{q}_1(k, k) = 0$ ;  $\bar{q}_0(k, k) = \underline{q}_0(k, k) = 0$ ;  $\bar{q}_{-1}(k, k) = 0$  and  $\underline{q}_{-1}(k, k) = -1$ .

TABLE XI  
MINIMAL CUMULATIVE COSTS  $B_{v,+}(t, k)$

$B_{v,+}(t,k)$	$B_{v,+}(1,2)$	$B_{v,+}(1,3)$	$B_{v,+}(1,4)$	$B_{v,+}(2,3)$	$B_{v,+}(2,4)$	$B_{v,+}(3,4)$
$v=1$	14	20	*	15	*	*
$v=0$	8	10	14	$+\infty$	11	6
$v=-1$	22	23	*	23	*	*

TABLE XII  
MINIMAL CUMULATIVE COSTS  $B_{v,-}(t, k)$

$B_{v,-}(t,k)$	$B_{v,-}(1,2)$	$B_{v,-}(1,3)$	$B_{v,-}(1,4)$	$B_{v,-}(2,3)$	$B_{v,-}(2,4)$	$B_{v,-}(3,4)$
$v=1$	$+\infty$	$+\infty$	*	$+\infty$	*	*
$v=0$	6	12	48	3	27	12
$v=-1$	$+\infty$	23	*	$+\infty$	*	*

For  $0 < I_k < C$ , we initialize the minimal cost from  $k$  to  $k$

$$B_{v,+}(k, k) = \begin{cases} +\infty, & \text{if } v = -1 \\ hS_{k,v}, & \text{otherwise} \end{cases}$$

For any  $t$  such that  $0 < t < k \leq T$ , we have the first equation shown at the bottom of the page. The explanation of expression  $B_{v,+}(t, k)$  can be found in Appendix IV.

For  $-C < I_k \leq 0$ , we initialize the minimal cost from  $k$  to  $k$

$$B_{v,-}(k, k) = \begin{cases} +\infty, & \text{if } v = 1 \\ -gS_{k,v}, & \text{otherwise.} \end{cases}$$

For any  $t$  such that  $0 < t < k \leq T$ , we have the second equation shown at the bottom of the page.

The explanation of expression  $B_{v,-}(t, k)$  can be found in Appendix V.

From the recursive formulations, we can see that all these quantities can be computed in  $O(T^2)$  time in a recursive manner.

*Example 1 (Continued):* All minimal cumulative costs  $B_{v,+}(t, k)$  and  $B_{v,-}(t, k)$  are presented, respectively, in Tables XI and XII.

For subplan  $(1, 0, 4, 0)$  and  $j = 4$ , the minimal cumulative inventory and backlogging cost  $B_{0,+}(4, 4) = 0$  and  $B_{0,-}(4, 4) = 0$ .  $\square$

#### D. Minimal Cumulative Cost $c_{u,v}(i, k)$ and Polynomial Algorithm

We establish now the conditions on the partial production period  $j$  to calculate  $c_{u,v}(i, k)$ . By the definition, the quantity that should be produced at period  $j$  must be  $x_j = r(i, u, k, v)$ . Note that  $j$  is a partial production period. From Corollary 4, in an optimal solution,  $j \neq k$  if  $v = -1$ . As a consequence, we have either  $j \neq k$  or  $v \geq 0$ .

The number of full production periods from  $i + 1$  to  $j - 1$ , i.e.,  $f_u(i, j - 1)$  and  $f_u(i, j - 1) = f_u(i, j)$  according to the definition of the number of full production periods. From Corollary 2, we must make sure that  $-C < I_{j-1} - d_j = S_{i,u} + f_u(i, j - 1)C - d_{i,j-1} - d_j = S_{i,u} + f_u(i, j)C - d_{i,j} < 0$ . In other words, we must have  $f_u(i, j) = \underline{f}_u(i, j)$ .

If  $\underline{f}_u(i, j) = \underline{f}_u(i, j - 1)$ , we have  $I_{j-1} \leq 0$ . If  $\underline{f}_u(i, j) > \underline{f}_u(i, j - 1)$ , we must have  $\underline{f}_u(i, j) = \underline{f}_u(i, j - 1) + 1 = \underline{f}_u(i, j - 1)$ , which implies  $I_{j-1} > 0$ .

On the other hand, the number of full production periods from  $j + 1$  to  $k$  must be  $q_v(j, k) = q_v(j + 1, k) = \underline{q}_v(j, k)$  in order that  $I_j = S_{k,v} - q_v(j, k) + d_{j,k} > 0$  according to Property 5.

By definition, the total number of full production periods in a subplan must be  $\lfloor S_{k,v} - S_{i,u} + d_{i,k} \rfloor$ . This means that we must have  $\underline{f}_u(i, j) + \underline{q}_v(j, k) = \lfloor (S_{k,v} - S_{i,u} + d_{i,k})/C \rfloor$ . Let  $J_{u,v}(i, k)$  be the set of  $j$ 's such that this condition is fulfilled.

To facilitate the expression of  $c_{u,v}(i, k)$ , the minimal total cost of a subplan  $(i, u, k, v)$ , we distinguish two cases  $v \geq 0$  and  $v = -1$ . Let

$$J_{u,v}(i, k) = \left\{ j \mid i < j \leq k, \underline{f}_u(i, j) \geq 0, \underline{q}_v(j, k) \geq 0, \right. \\ \left. \underline{f}_u(i, j) + \underline{q}_v(j, k) = \lfloor (S_{k,v} - S_{i,u} + d_{i,k})/C \rfloor \right\} \\ \text{if } v \geq 0, \text{ and} \\ J_{u,-1}(i, k) = \left\{ j \mid i < j < k, \underline{f}_u(i, j) \geq 0, \underline{f}_u(i, j) + \underline{q}_{-1}(j, k) \right. \\ \left. = \lfloor (S_{k,-1} - S_{i,u} + d_{i,k})/C \rfloor \right\} \text{ if } v = -1.$$

$$B_{v,+}(t, k) = \begin{cases} +\infty, & \text{if } S_{k,v} + d_{t,k} - \underline{q}_v(t, k)C > S_{t,1} \\ h [S_{k,v} + d_{t,k} - \underline{q}_v(t, k)C] \\ \quad + B_{v,+}(t + 1, k), & \text{if } \underline{q}_v(t, k) = \underline{q}_v(t + 1, k) \\ h [S_{k,v} + d_{t,k} - \underline{q}_v(t, k)C] \\ \quad + \min [B_{v,-}(t + 1, k), B_{v,+}(t + 1, k)], & \text{if } \underline{q}_v(t, k) > \underline{q}_v(t + 1, k) \end{cases}$$

$$B_{v,-}(t, k) = \begin{cases} +\infty, & \text{if } d_{t,k} + S_{k,v} - \bar{q}_v(t, k)C < S_{t,-1} \\ g [\bar{q}_v(t, k)C - d_{t,k} - S_{k,v}] \\ \quad + \min \{B_{v,-}(t + 1, k), B_{v,+}(t + 1, k)\}, & \text{if } \bar{q}_v(t, k) = \bar{q}_v(t + 1, k) \\ g [\bar{q}_v(t, k)C - d_{t,k} - S_{k,v}] \\ \quad + B_{v,-}(t + 1, k), & \text{if } \bar{q}_v(t, k) > \bar{q}_v(t + 1, k) \end{cases}$$

TABLE XIII  
MINIMAL CUMULATIVE COSTS  $c_{u,v}(i, k)$

$(i,k) \backslash (u,v)$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
(1,1)	*	*	*	*	$+\infty$	$+\infty$	$+\infty$	38	*	*
(1,0)	*	*	*	*	$+\infty$	$+\infty$	$+\infty$	30	33	$+\infty$
(1,-1)	*	*	*	*	$+\infty$	$+\infty$	$+\infty$	$+\infty$	*	*
(0,1)	$+\infty$	$+\infty$	$+\infty$	*	$+\infty$	$+\infty$	*	$+\infty$	*	*
(0,0)	30	63	98	104	30	62	95	30	61	30
(0,-1)	21	52	83	*	$+\infty$	53	*	$+\infty$	*	*
(-1,1)	*	*	*	*	$+\infty$	$+\infty$	*	$+\infty$	*	*
(-1,0)	*	*	*	*	$+\infty$	$+\infty$	117	$+\infty$	$+\infty$	$+\infty$
(-1,-1)	*	*	*	*	$+\infty$	$+\infty$	*	$+\infty$	*	*

TABLE XIV  
CORRESPONDING  $j$  OR  $r$  OF  $c_{u,v}(i, k)$

$(i,k) \backslash (u,v)$	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
(1,1)	*	*	*	*	•	•	•	$j=3$	*	*
(1,0)	*	*	*	*	•	•	•	$j=3$	$j=4$	-
(1,-1)	*	*	*	*	•	•	•	-	*	*
(0,1)	•	•	•	*	•	•	*	•	*	*
(0,0)	$j=1$	$j=2$	$j=3$	$r=0$	$j=2$	$j=3$	$j=4$	$j=3$	$j=4$	$j=4$
(0,-1)	$r=0$	$j=1$	$j=1$	*	•	$r=0$	*	•	*	*
(-1,1)	*	*	*	*	•	•	*	•	*	*
(-1,0)	*	*	*	*	•	•	$r=0$	•	•	•
(-1,-1)	*	*	*	*	•	•	*	•	*	*

For each given  $(i, u, k, v)$ , this set can be computed in  $O(T)$  time. All  $J_{u,v}(i, k)$ 's can then be computed in  $O(T^3)$  time.

By definition, we have the following relation according to Property 6:

$$c_{u,v}(i, k) = \min_{j \in J_{u,v}(i, k)} \{A_{u,*}(i, j-1) + B_{v,+}(j, k)\} + K \lceil (S_{k,v} - S_{i,u} + d_{i,k})/C \rceil$$

where

$$A_{u,*}(i, j-1) = \begin{cases} A_{u,-}(i, j-1), & \text{if } \underline{f}(i, j) = \underline{f}(i, j-1) \\ A_{u,+}(i, j-1), & \text{if } \underline{f}(i, j) = \overline{f}(i, j-1). \end{cases}$$

Note that if  $J_{u,v}(i, k)$  is empty, we have  $c_{u,v}(i, k) = +\infty$ .

If there is no partial production period; i.e.,  $r(i, u, k, v) = 0$ , we have, by definition

$$c_{u,v}(i, k) = K \lceil (S_{k,v} - S_{i,u} + d_{i,k})/C \rceil + \begin{cases} A_{u,+}(i, k), & \text{if } v = 1 \\ A_{u,-}(i, k), & \text{otherwise.} \end{cases}$$

When all  $A$ 's and  $B$ 's are known, all  $c_{u,v}(i, k)$ 's can be compute in  $O(T^3)$  time.

*Example 1 (Continued):* The minimal cumulative cost  $c_{u,v}(i, k)$  and corresponding  $j$  or  $r(i, u, k, v)$  are given in Tables XIII and XIV, respectively. In which, "\*" means that the corresponding subplan with  $I_0 = 0$  or  $I_4 = 0$  does not exist. "•" means that it is not in an optimal solution and the corresponding value of  $c_{u,v}(i, k)$  is set to  $+\infty$ . "-" means that  $J_{u,v}(i, k)$  is empty.

TABLE XV  
MINIMAL CUMULATIVE COST  $E_v(k)$

$E_v(k)$	$E_v(1)$	$E_v(2)$	$E_v(3)$	$E_v(4)$
$v=1$	$+\infty$	$+\infty$	$+\infty$	*
$v=0$	30	60	90	104
$v=-1$	21	52	83	*

For subplan (1, 0, 4, 0), we have  $J_{0,0}(1, 4) = \{4\}$ . For instance, we have

$$\begin{aligned} c_{0,0}(1, 4) &= K \lceil (S_{4,0} - S_{1,0} + d_{1,4})/C \rceil \\ &\quad + A_{0,*}(1, 3) + B_{0,+}(4, 4) \\ &= K \lceil (S_{4,0} - S_{1,0} + d_{1,4})/C \rceil \\ &\quad + \min(A_{0,+}(1, 3), A_{0,-}(1, 3)) + B_{0,+}(4, 4) \\ &= 30 \times \lceil 23/10 \rceil + \min\{5, 23\} + 0 = 95. \end{aligned}$$

The corresponding production quantities are

$$\begin{aligned} x_3 &= C (\overline{f}_0(1, 3) - \overline{f}_0(1, 2)) = 10(2 - 1) = 10, \\ x_2 &= C (\overline{f}_0(1, 2) - \overline{f}_0(1, 1)) = 10(1 - 0) = 10 \text{ and } x_4 = 3. \end{aligned}$$

For subplan (1, -1, 4, 0), there is not partial production. We have

$$\begin{aligned} c_{-1,0}(1, 4) &= K \lceil (S_{4,0} - S_{1,-1} + d_{1,4})/C \rceil + A_{-1,-}(1, 4) \\ &= 30 \times \lceil 30/10 \rceil + 27 = 117. \end{aligned}$$

The corresponding production quantities are  $x_2 = x_3 = x_4 = 10$ .

The subplan (1, 1, 4, 0) is not in an optimal solution, or  $c_{1,0}(1, 4) = +\infty$ .  $\square$

We can now give an algorithm to compute an optimal production plan. Let  $E_w(t)$ , with  $0 \leq t \leq T$  and  $w \in V_t$ , be the minimal cumulative (production, inventory and backloging) cost from period 0 to period  $t$  such that the inventory level at the end of period  $t$  is  $S_{t,w}$ . According to Property 6, we have the following relations that respectively give the starting condition and the recursive equation for the dynamic programming approach

$$E_0(0) = 0.$$

For any  $k$  and  $v$  such that  $0 < k \leq T$  and  $v \in V_k$ , we have

$$E_v(k) = \min_{\substack{0 \leq i < k \\ u \in V_i}} \{E_u(i) + c_{u,v}(i, k)\}.$$

The optimal solution corresponds to  $E_0(T)$ . The optimal solution can be obtained in a backward manner, and the total optimal cost (production, inventory and backloging) equal to  $E_0(T) + d_{0,T} \cdot p$ .

As a consequence, for each  $k$  and  $v$ , in order to compute  $E_v(k)$ , we need to consider all values of  $i$ . The number of these values is  $O(T)$ , and the number of  $k$  is  $O(T)$ . Therefore,  $O(T^2)$  additional time is needed to compute all  $E_v(k)$ 's after the computation of all  $c_{u,v}(i, k)$ 's which needs  $O(T^3)$  time. Therefore, the overall complexity to find an optimal solution is  $O(T^3)$ .

*Example 1 (Continued):* The minimal cumulative cost  $E_v(k)$  is given in Table XV.

The optimal total cost is  $E_0(4) + d_{0,4} \cdot p = E_0(0) + c_{0,0}(0, 4) + d_{0,4} \cdot p = 104 + 30 = 134$ .

TABLE XVI  
DEMAND DATA

Month	1	2	3	4	5	6	7	8	9	10	11	12
$D_t$	43	41	26	25	24	28	40	45	37	27	22	39

TABLE XVII  
ACTUATED CUSTOMER DEMANDS

Month	1	2	3	4	5	6	7	8	9	10	11	12
$d_t$	13	11	26	25	24	28	10	15	7	27	22	9

## V. A CASE STUDY

To illustrate the applicability of our model and algorithm to real problems, we devise a case originating from crude oil transportation at a Chinese oil company. This company imports every year more than three million tons of crude oil from Venezuela, and the import quantity per month is variable and affected by oil price and demand. Generally, when the marine distance is more than 6 000 miles, supertankers with tonnage over 200 thousand tons (mainly, very large crude career (VLCC) with tonnage of 300 thousand tons) are used to transport crude oil in order to save transportation cost. Venezuela is about ten thousand miles away from China. Hence, the company adopts VLCC as its transportation means. If the number of tankers is large enough, the problem can be regarded as a single item lot sizing problem with limited production (tanker) capacity and inventory capacity. Based on the business data of the oil company, the monthly demands in 2007 are given in Table XVI and expressed in 10 thousand tons.

According to Property 2 in Section III, in an optimal transportation plan, when the customer demand in one period is more than the capacity of a tanker, the most part of the demand will be immediately delivered by fully loaded tankers except for the actuated customer demands, which are calculated and shown in Table XVII.

Other parameters of the problem are given as follows:  $S_{t,1} = 22$ ,  $S_{t,-1} = -15$  (in 10 thousand tons) for  $t = 1, 2, \dots, 12$ , and we suppose  $I_0 = I_{12} = 0$ ;  $K = 350$  (in 10 thousand dollars) per tanker,  $p = 4$  (in 10 thousand dollars) per 10 thousand tons,  $C = 30$  (in 10 thousand tons),  $g = 10$  (in 10 thousand dollars) per 10 thousand tons and  $h = 2$  (in 10 thousand dollars) per 10 thousand tons. Note that at the destination harbour crude oil is unloaded from a tanker to the depot rented by the oil company for a long period of time, or parameter  $h$  is estimated based on the inventory holding cost and the rental cost to take both costs into account.

If these actuated customer demands are satisfied by immediate delivery, the corresponding total transportation and inventory cost is 50.68 million dollars.

Our dynamic programming algorithm is implemented with Excel tables on a PC with Intel Core DUO CPU of 2.0 GHz with 2.0 GB RAM. An optimal solution is computed simultaneously when the data are input. The computation time is therefore 0. It is given in Table XVIII with a total cost of 37.64 million dollars.

TABLE XVIII  
NUMERICAL RESULTS

Month	1	2	3	4	5	6	7	8	9	10	11	12
Transportation Quantity	24	0	26	27	30	30	0	22	0	28	30	0

In the optimal transportation schedule, only eight tankers are needed to satisfy all the actuated demands over 12 months, and total transportation and inventory cost is 37.64 million dollars. Compared with the immediate delivery cost, the proposed algorithm reduces the transportation cost by  $(50.68 - 37.64)/50.68 = 25.7\%$ . This clearly represents very significant cost saving.

Since the problem can also be modeled as a mixed integer linear program, we also used CPLEX to optimally solve it. The computation time is 0.2 s CPU for this case study. Since the method used in CPLEX is based on branch and bound approach, the computation time exponentially increases with the size of the problem while our method is polynomial. The difference of computation times will certainly increase with instances of larger size. Furthermore, Excel is a common software, while very few industrial people are familiar with CPLEX. Therefore, the algorithm proposed in this paper is much more appropriate to handle industrial problems.

## VI. CONCLUSION

In this paper, we investigate the transportation planning of crude oil from a supplier port to  $n$  client ports to satisfy customer forecast demands by tankers. Under some realistic assumptions, the problem can be considered to be  $n$  independent one-customer shipping planning problems. We show that each shipping planning problem can be transformed into a capacitated lot sizing problem with limited production and inventory capacities. So far, this problem has not been studied neither in manufacturing environment nor in transportation. Then, we develop a new polynomial algorithm that can be used in transportation and production environments.

For the obtained lot sizing model, we can cite some prospective research directions. At first, if the number of tankers in the supplier port is limited, the problem cannot be decomposed into independent one-customer problems. One has to coordinate the decisions between these client ports. Second, an outsourcing policy with production and inventory capacities is still an open problem in a lot sizing model. Third, we can further integrate time-varying purchase cost in our transportation model.

This work is a preliminary one for a long-term project to develop efficient tools for the planning of crude oil transportation. In the near future, we intend to extend our current research by considering multimodes and multi-items in crude oil transportation and other related operations [17]–[23].

## APPENDIX I PROOF OF PROPERTY 1

Assume that there is an optimal solution in which  $z_t < n_t C$  for some period  $t$  such that  $1 \leq t \leq T$ . Since  $z_t \geq 0$ , we

should have  $n_t \geq 1$ . Without loss of generality, we can write  $z_t = a_t C - b_t$  with  $a_t \leq n_t$  and  $0 < b_t \leq C$ . We have two cases.

If  $I_t < -hb_t/(g+h) < 0$ , there must be at least one transportation period after  $t$  due to the assumption that  $I_T = 0$ . Let  $\alpha$  be the first transportation period after  $t$ . Construct a new solution by increasing the transportation quantity of period  $t$  by  $\Delta = \min(b_t, z_\alpha) > 0$  and decreasing the transportation quantity of period  $\alpha$  by  $\Delta$ . In this new solution, the transportation cost does not increase, while the backloging level between  $t$  and  $\alpha - 1$  is reduced. For each period  $i$  such that  $t \leq i < \alpha$ , the inventory/backloging cost varies from  $-g(I_t - \sum_{\tau=t+1}^i D_\tau)$  to  $\max[-g(I_t - \sum_{\tau=t+1}^i D_\tau + \Delta), h(I_t - \sum_{\tau=t+1}^i D_\tau + \Delta)]$ ; i.e., decreases by  $-g(I_t - \sum_{\tau=t+1}^i D_\tau) - \max[-g(I_t - \sum_{\tau=t+1}^i D_\tau + \Delta), h(I_t - \sum_{\tau=t+1}^i D_\tau + \Delta)] = \min[g\Delta, (g+h)(\sum_{\tau=t+1}^i D_\tau - I_t) - h\Delta] > 0$  since  $g\Delta > 0$  and  $(g+h)(\sum_{\tau=t+1}^i D_\tau - I_t) - h\Delta \geq -(g+h)I_t - h\Delta \geq -(g+h)I_t - hb_t > 0$  according to the assumption. This is in contradiction with the assumption that the initial solution is optimal.

If  $I_t \geq -hb_t/(g+h)$ , we have

$$\begin{aligned} I_{t-1} &= I_t + D_t - z_t = I_t + (n_t - a_t)C + d_t + b_t \\ &\geq I_t + b_t \geq gb_t/(g+h) > 0. \end{aligned} \quad (\text{A1})$$

There must be at least one transportation period before  $t$  due to the assumption that  $I_0 = 0$ . Let  $\beta$  be the last transportation period before  $t$ . Construct a new solution by increasing the transportation quantity of period  $t$  by  $\Delta = \min(b_t, z_\beta) > 0$  and decreasing the transportation quantity of period  $\beta$  by  $\Delta$ . In this new solution, the transportation cost does not increase, while the inventory level between  $\beta$  and  $t - 1$  is reduced. For each period  $i$  such that  $\beta \leq i < t$ , the inventory/backloging cost varies from  $h(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau)$  to  $\max[-g(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau - \Delta), h(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau - \Delta)]$ ; i.e., decreases by  $h(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau) - \max[-g(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau - \Delta), h(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau - \Delta)] = \min[h\Delta, (g+h)(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau) - g\Delta] > 0$  since  $h\Delta > 0$  and  $(g+h)(I_{t-1} + \sum_{\tau=i+1}^{t-1} D_\tau) - g\Delta \geq (g+h)I_{t-1} - g\Delta \geq (g+h)I_{t-1} - gb_t > 0$  according to (A1). This means that the new solution is strictly better than the initial one which is a contradiction.  $\square$

## APPENDIX II PROOF OF PROPERTY 3

Assume that there is an optimal solution in which more than one tanker is used in some period  $t$ , in addition to  $n_t$  tankers used for immediate deliveries, i.e.,  $z_t > (n_t + 1)C$ . We can always write  $z_t = a_t C + b_t$ , where  $a_t$  is an integer such that  $a_t > n_t$  and  $0 < b_t \leq C$ . Then, one tanker is needed to transport quantity  $b_t$ . Thus,  $I_t$  can be distinguished into the following two cases.

If  $I_t > gb_t/(g+h) > 0$ , construct another solution by increasing the transportation quantity of  $t + 1$  by  $b_t$  and by reducing that of period  $t$  by  $b_t$ . In this new solution, the transportation cost does not increase while the inventory/backloging cost of period  $t$  changes from  $hI_t$  to  $\max[-g(I_t - b_t), h(I_t - b_t)]$ .

The total cost is decreased by  $hI_t - \max[-g(I_t - b_t), h(I_t - b_t)] = \min[(g+h)I_t - gb_t, hb_t] > 0$  by considering the fact that  $hb_t > 0$  and the assumption that  $I_t > gb_t/(g+h)$ , which is equivalent to  $(g+h)I_t - gb_t > 0$ . This is in contradiction with the assumption that the initial solution is optimal.

If  $I_t \leq gb_t/(g+h)$ , we have

$$\begin{aligned} I_{t-1} &= I_t - z_t + D_t = I_t - a_t C - b_t + n_t C + d_t \\ &\geq n_t C + d_t - a_t C - hb_t/(g+h) \\ &\geq d_t - C - hb_t/(g+h) < -hb_t/(g+h) < 0. \end{aligned} \quad (\text{A2})$$

Construct another solution by increasing the transportation quantity of  $t - 1$  by  $b_t$  and by reducing that of period  $t$  by  $b_t$ . In this new solution, the transportation cost does not increase while the inventory/backloging cost of period  $t - 1$  changes from  $-gI_{t-1}$  to  $\max[-g(I_{t-1} + b_t), h(I_{t-1} + b_t)]$ . The total cost is decreased by at least  $-gI_{t-1} - \max[-g(I_{t-1} + b_t), h(I_{t-1} + b_t)] = \min[gb_t, -(g+h)I_{t-1} - hb_t]$ . Since  $gb_t > 0$  and  $I_{t-1} < -hb_t/(g+h)$  or equivalently  $-(g+h)I_{t-1} - hb_t > 0$  [see (A2)], we have  $\min\{gb_t, -(g+h)I_{t-1} - hb_t\} > 0$ ; namely, the new solution is strictly better than the initial one, which is a contradiction.  $\square$

## APPENDIX III EXPLANATION OF EXPRESSION $A_{u,-}(i, t)$

By definition,  $A_{u,-}(i, t)$  corresponds to  $I_t \leq 0$  and the number of full production periods from  $i + 1$  to  $t$  must be  $\underline{f}_u(i, t)$ . The first case of  $A_{u,-}(i, t)$  is similar to that of  $A_{u,+}(i, t)$ .

For the second and third cases of  $A_{u,-}(i, t)$ , its first term corresponds to  $-gI_t$ , the backloging cost of period  $t$ . Its second term corresponds to the minimal cumulative inventory and backloging cost from  $i + 1$  to  $t - 1$  without any partial production.

If  $\underline{f}_u(i, t) = \underline{f}_u(i, t - 1)$ , period  $t$  cannot be a full production period since otherwise the number of full production periods from  $i + 1$  to  $t - 1$  would be  $\underline{f}_u(i, t) - 1 = \underline{f}_u(i, t - 1) - 1 < \underline{f}_u(i, t - 1)$ , which is infeasible. As a consequence, the number of full production periods from  $i + 1$  to  $t - 1$  is  $\underline{f}_u(i, t - 1)$  which implies that  $I_{t-1} \leq 0$  and the cumulative inventory/backloging cost for these periods is  $A_{u,-}(i, t - 1)$ .

If  $\underline{f}_u(i, t) > \underline{f}_u(i, t - 1)$ , period  $t$  may or may not be a full production period. If so, the number of full production periods from  $i + 1$  to  $t - 1$  is  $\underline{f}_u(i, t) - 1 = \underline{f}_u(i, t - 1)$ . This implies  $I_{t-1} \leq 0$  and the cumulative inventory/backloging cost of these periods is  $A_{u,-}(i, t - 1)$ . If  $t$  is not a full production period, the number of full production periods from  $i + 1$  to  $t - 1$  is  $\underline{f}_u(i, t - 1) + 1 = \bar{f}_u(i, t - 1)$ , which implies  $I_{t-1} > 0$  and the cumulative inventory/backloging cost of these periods is  $A_{u,+}(i, t - 1)$ . In an optimal solution, the decision corresponds to the best one of these situations.

In the case of  $\underline{f}_u(i, t) < 0$ , we must have  $u = 1$ , and  $\underline{f}_u(i, i) = \underline{f}_u(i, i + 1) = \dots = \underline{f}_u(i, t) = -1$ . From the recursive equation, by considering the third case, we have

$A_{1,-}(i, \tau) = +\infty$  for any  $\tau$  such that  $i \leq \tau \leq t$ , since  $A_{1,-}(i, i) = +\infty$ .  $\square$

#### APPENDIX IV

##### EXPLANATION OF EXPRESSION $B_{v,+}(t, k)$

The first case of  $B_{v,+}(t, k)$  is similar to the corresponding case of  $A_{u,+}(i, t)$ .

For the second and third cases of  $B_{v,+}(t, k)$ , its first term corresponds to  $hI_t$ , the holding cost of period  $t$ . Its second term corresponds to the minimal cumulative cost from  $t + 1$  to  $k$  without any partial production from period  $t + 2$  to  $k$ . Note that there must be  $\underline{q}_v(t, k)$  full production periods from  $t + 1$  to  $k$ .

If  $\underline{q}_v(t, k) = \underline{q}_v(t + 1, k)$ ,  $t + 1$  must not be a full production period, since otherwise the number of full production periods from  $t + 2$  to  $k$  would have to be  $\underline{q}_v(t, k) - 1 = \underline{q}_v(t + 1, k) - 1 < \underline{q}_v(t + 1, k)$  which would be impossible. Therefore, the number of full production periods from  $t + 2$  to  $k$  must be  $\underline{q}_v(t + 1, k)$ , which implies that  $I_{t+1} > 0$ , and the cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $B_{v,+}(t + 1, k)$ . Note that if  $\underline{q}_v(t, k) < 0$ , we must have  $v = -1$  and  $\underline{q}_v(t, k) = \underline{q}_v(t + 1, k) = \dots = \underline{q}_v(k, k)$ . By the recursion, we have  $B_{v,+}(t, k) = +\infty$ .

If  $\underline{q}_v(t, k) > \underline{q}_v(t + 1, k)$ ,  $t + 1$  may or may not be a production period. If so, the number of production periods from  $t + 2$  to  $k$  is  $\underline{q}_v(t, k) - 1 = \underline{q}_v(t + 1, k)$ , which implies that  $I_{t+1} > 0$ . The cumulative inventory/backlogging cost is  $B_{v,+}(t + 1, k)$ . If  $t + 1$  is not a production period, the number of full production periods from  $t + 2$  to  $k$  is  $\underline{q}_v(t, k) = \underline{q}_v(t + 1, k) + 1 = \bar{q}_v(t + 1, k)$ , which implies that  $I_{t+1} \leq 0$ . The cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $B_{v,-}(t + 1, k)$ . The minimal cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $\min(B_{v,-}(t + 1, k), B_{v,+}(t + 1, k))$ .

#### APPENDIX V

##### EXPLANATION OF EXPRESSION $B_{v,-}(t, k)$

The first case of  $B_{v,-}(t, k)$  is similar to the corresponding case of  $A_{u,+}(i, t)$ .

For the second and third cases of  $B_{v,-}(t, k)$ , its first term corresponds to  $-gI_t$ , the backlogging cost of period  $t$ . Its second term corresponds to the minimal cumulative cost from  $t + 1$  to  $k$  without any partial production. Note that the number of full production periods from  $t + 1$  to  $k$  must be  $\bar{q}_v(t, k)$  in this case.

If  $\bar{q}_v(t, k) = \bar{q}_v(t + 1, k)$ , period  $t + 1$  may or may not be a full production period. If so, the number of full production periods from  $t + 2$  to  $k$  is  $\bar{q}_v(t, k) - 1 = \bar{q}_v(t + 1, k) - 1 < \bar{q}_v(t + 1, k)$ , which implies, we will have  $I_{t+1} > 0$  and the cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $B_{v,+}(t + 1, k)$ . If  $t + 1$  is not a full production period, the number of full production periods from  $t + 2$  to  $k$  is  $\bar{q}_v(t, k) = \bar{q}_v(t + 1, k)$ , which leads to  $I_{t+1} \leq 0$  and the cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $B_{v,-}(t + 1, k)$ . In an optimal solution, the minimal cumulative inventory/backlogging cost from  $t + 1$  to  $k$  must be  $\min[B_{v,-}(t + 1, k), B_{v,+}(t + 1, k)]$ .

If  $\bar{q}_v(t, k) > \bar{q}_v(t + 1, k)$ ,  $t + 1$  must be a full production period, since otherwise the number of full production periods from  $t + 2$  to  $k$  would be  $\bar{q}_v(t, k) > \bar{q}_v(t + 1, k)$  which is infeasible. Therefore, the number of full production periods from  $t + 2$  to  $k$  must be  $\bar{q}_v(t + 1, k)$ , which implies  $I_{t+1} \leq 0$  and

the cumulative inventory/backlogging cost from  $t + 1$  to  $k$  is  $B_{v,-}(t + 1, k)$ . Note that if  $\underline{q}_v(t, k) > k - t$ , we must have  $v = 1$  and  $\bar{q}_v(t, k) > \bar{q}_v(t + 1, k) > \dots > \bar{q}_v(k, k)$ . By the recursion, we have  $B_{v,-}(t, k) = +\infty$ .

#### REFERENCES

- [1] R. Ahuja and D. Hochbaum, "Solving linear cost dynamic lot sizing problem in  $O(n \log n)$  time," University of California, Berkeley, CA, Tech. Rep., working paper, 2004.
- [2] G. Bitran and H. Yanasse, "Computational complexity of the capacitated lot size problem," *Manage. Sci.*, vol. 28, no. 10, pp. 1174–1186, October 1982.
- [3] G. G. Brown, G. W. Graves, and D. Ronen, "Scheduling ocean transportation of crude oil," *Manage. Sci.*, vol. 33, no. 3, pp. 335–346, 1987.
- [4] L. Cheng and M. A. Duran, "Logistics for world-wide crude oil transportation using discrete event simulation and optimal control," *Comput. Chem. Eng.*, vol. 28, no. 6–7, pp. 897–911, Jun. 15, 2004.
- [5] M. Christiansen and B. Nygreen, "A method for solving ship routing problems with inventory constraints," *Ann. Oper. Res.*, vol. 39, pp. 357–378, 1998.
- [6] F. Chu and C. Chu, "Polynomial algorithms for single item lot sizing models with bounded inventory and backlogging or outsourcing," *IEEE Trans. Autom. Sci. Eng.*, vol. 4, no. 2, pp. 233–251, Apr. 2007.
- [7] F. Chu, C. Chu, and X. Liu, "Lot sizing models with backlog or outsourcing," in *Proc. IEEE Int. Conf. Syst., Man, Cybern.*, The Hague, The Netherlands, Oct. 10–13, 2004, pp. 4342–4347.
- [8] C. Chung and C. Lin, "An  $O(T^2)$  algorithm for the NI/G/NI/ND capacitated lot size problem," *Manage. Sci.*, vol. 34, pp. 420–426, 1988.
- [9] M. Florian and M. Klein, "Deterministic production planning with concave cost and capacity constraints," *Manage. Sci.*, vol. 1, no. 1, pp. 12–20, 1971.
- [10] C. Van Hoesel and A. Wagelmans, "Fully polynomial approximation schemes for single-item capacitated economic lot-sizing problems," *Math Oper. Res.*, vol. 26, no. 2, pp. 339–357, 2001.
- [11] E. T. Iakovou, "An interactive multiobjective model for the strategic maritime transportation of petroleum products: Risk analysis and routing," *Safety Sci.*, vol. 39, no. 1–2, pp. 19–29, 2001.
- [12] S. F. Love, "Bounded production and inventory models with piecewise concave costs," *Manage. Sci.*, vol. 20, no. 3, pp. 313–318, 1973.
- [13] D. Shaw and A. Wagelmans, "An algorithm for single-item capacitated economic lot sizing with piecewise linear production costs and general holding costs," *Manage. Sci.*, vol. 44, pp. 831–838, 1998.
- [14] S. D. Hanif, M. A. Salem, and H. M. Merza, "Fleet management models and algorithms for an oil-tanker routing and scheduling problem," *IIE Trans.*, vol. 31, no. 5, pp. 395–406, 1999.
- [15] C. Swoveland, "A deterministic multi-period production planning model with piecewise concave production and holding-backorder costs," *Manage. Sci.*, vol. 21, no. 9, pp. 1007–1013, 1975.
- [16] *United Nations Annual Review of Maritime Logistics 1998*. Geneva, Switzerland: United Nation Press, 1999, United Nations.
- [17] N. Wu, M. C. Zhou, and F. Chu, "Short-term scheduling for refinery process: Bridging the gap between theory and applications," *Int. J. Intell. Control Syst.*, vol. 10, no. 2, pp. 162–174, Jun. 2005.
- [18] N. Wu, M. C. Zhou, and F. Chu, "A Petri net based heuristic algorithm for realizability of target refining schedules in oil refinery," *Trans. Autom. Sci. Eng.*, vol. 5, no. 4, pp. 661–676, Oct. 2008.
- [19] N. Wu, F. Chu, C. Chu, and M. C. Zhou, "Short-term schedulability analysis of crude oil operations in refinery with residency time constraint using Petri nets," *IEEE Trans. Syst., Man, Cybern.: Part C*, vol. 38, no. 6, pp. 765–778, Nov. 2008.
- [20] N. Wu, F. Chu, C. Chu, and M. C. Zhou, "Short-term schedulability analysis of multiple distiller crude oil operations in refinery with oil residency time constraint," *IEEE Trans. Syst., Man, Cybern.: Part C*, vol. 39, no. 1, pp. 1–16, Jan. 2009.
- [21] N. Wu, F. Chu, C. Chu, and M. C. Zhou, "Hybrid Petri Net Modeling and schedulability analysis of high fusion point oil transportation under tank grouping strategy for crude oil operations in refinery," *IEEE Trans. Syst., Man, Cybern.: Part C*, vol. 40, no. 2, pp. 159–175, Mar. 2010.

- [22] N. Wu, F. Chu, C. Chu, and M. C. Zhou, "Tank cycling and scheduling analysis of high fusion point oil transportation for crude oil operations in refinery," *Comput. Chem. eng.*, vol. 34, no. 4, pp. 529–543, Apr. 2010.
- [23] N. Wu, F. Chu, C. Chu, and M. C. Zhou, "Schedulability analysis of short-term scheduling for crude oil operations in refinery with oil residency time and charging-tank-switch-overlap constraints," *IEEE Trans. Autom. Sci. Eng.*, vol. 8, no. 1, pp. 190–204, Jan. 2011.