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# A solution approach to the inventory routing problem in a three-level distribution system

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We consider the infinite horizon inventory routing problem in a three-level distribution system with a vendor, a warehouse and multiple geographically dispersed retailers. In this problem, each retailer faces a demand at a deterministic, retailer-specific rate for a single product. The demand of each retailer is replenished either from the vendor through the warehouse or directly from the vendor. Inventories are kept at both the retailers and the warehouse. The objective is to determine a combined transportation (routing) and inventory strategy minimizing a long-run average system-wide cost while meeting the demand of each retailer without shortage. We present a decomposition solution approach based on a fixed partition policy where the retailers are partitioned into disjoint and collectively exhaustive sets and each set of retailers is served on a separate route. Given a fixed partition, the original problem is decomposed into three sub-problems. Efficient algorithms are developed for the sub-problems by exploring important properties of their optimal solutions. A genetic algorithm is proposed to find a near-optimal fixed partition for the problem. Computational results show the performance of the solution approach.

## 1. Introduction

Nowadays, more and more companies are aware that great cost savings can be achieved by integrating inventory control and vehicle routing into a cost-effective strategy for their distribution systems, especially for Vendor Managed Inventory (VMI) systems (Kleywegt et al., 2002; Raa and Aghezzaf, 2009). Determining such a cost-effective distribution strategy is known as Inventory Routing Problem (IRP).

This paper considers an infinite-horizon deterministic IRP for a three-level distribution system with a single outside vendor, a single warehouse and multiple geographically dispersed retailers. In this problem, each retailer faces an external demand for a single product with a deterministic, retailer specific rate. The demand of each retailer is replenished either from the vendor through the warehouse or directly from the vendor by vehicles of limited capacity. Inventories are kept not only at the retailers but also at the warehouse. The objective is to determine an integrated inventory and routing strategy minimizing a long-run average system-wide cost while meeting each retailer's demand without shortage.

Our research is partially motivated by work done with a leading French automobile equipment company. This company delivers

products to its clients in two ways: direct shipping and multi-stop shipping through warehouse. In fact, IRPs arise in various industries, such as supermarket chain (Gaur and Fisher, 2004), gases company (Campbell and Savelsbergh, 2004), and vending machine chain (Huang and Lin, 2010). IRPs also have been attracting the attention of academic communities. Currently most papers in the IRP literature focus on two-level distribution systems, also called one-warehouse multi-retailer distribution systems, see e.g., Anily (1994), Anily and Bramel (2004), Anily and Federgruen (1990), Bramel and Simchi-Levi (1995), Burns et al. (1985), Campbell and Savelsbergh (2004), Chan et al. (1998), Gaur and Fisher (2004), Moin and Salhi (2007), Jung and Mathur (2007), and Zhao et al. (2007). The literature on IRPs for three-level distribution systems is rather limited. To the best of our knowledge, only Chan and Simchi-Levi (1998) and Zhao et al. (2008) are exceptional. Zhao et al. (2008) consider a three-level distribution system where the inventory of the warehouse is replenished from the vendor by a single train with a large capacity. They propose a solution strategy integrating a Fixed Partition Policy (FPP) for transportation and a Power-Of-Two (POT) policy for inventory replenishment. In their strategy, firstly an FPP is determined, in which the retailers are partitioned into disjoint and collectively exhaustive sets and each set of retailers is served on a separate route (Bramel and Simchi-Levi, 1995), and then a POT policy is determined, in which each set of retailers and the warehouse are restricted to be visited at a replenishment interval which is power of two times a basic planning period (Roundy,

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1985). A tabu search algorithm is developed to improve fixed partitions. The distribution system considered in Chan and Simchi-Levi (1998) is identical to the one considered in this paper although it includes multiple warehouses, because in their study the retailers served by each warehouse are determined with a pre-processing procedure. To solve the complex IRP in the three-level distribution system, they present a solution approach that decomposes the problem into two sub-problems: the warehouse-retailer transportation and inventory sub-problem, and the vendor-warehouse transportation and inventory sub-problem. The former is solved based on an FPP. The latter is then solved under a cross-docking strategy in which the warehouse acts as a coordinator of incoming orders from the vendor and outgoing orders to the retailers but does not keep inventory. The authors prove that the cross-docking strategy is asymptotically optimal.

As pointed out by Jung and Mathur (2007), however, if the number of retailers is finite and the inventory holding cost rate at the warehouse is relatively small compared with that at the retailers, it may be profitable to keep inventory also at the warehouse. Moreover, in Chan and Simchi-Levi (1998), it is assumed that all shipments are delivered from the vendor to the retailers through the warehouse. In Li et al. (2007), we prove that under certain conditions a strategy in which shipments are delivered directly from the vendor to the retailers has a higher asymptotic optimality, and conclude that a hybrid strategy, which consists of the direct shipping strategy and the multi-stop shipping strategy through the warehouse, should be used in three-level distribution systems with a limited number of retailers.

In this study, the restriction to the cross-docking strategy is relaxed, i.e., the warehouse is allowed to keep inventory. The restriction that all shipments are through the warehouse is also relaxed, i.e., each retailer is allowed to be delivered directly from the vendor bypassing the warehouse. Moreover, the POT restriction is relaxed, i.e., the replenishment intervals of each retailer set and the warehouse are not confined to power of two times a basic planning period.

We propose a decomposition solution approach based on a fixed partition policy. Given a fixed partition, the original problem is decomposed into three sub-problems: the first is to identify the retailers whose inventories are replenished directly from the vendor and to determine their corresponding direct shipping strategies; the second is to determine a combined routing and inventory strategy for the remaining retailers; the third is to determine an optimal plan for shipments from the vendor to the warehouse given the partition policy. Important properties are discovered for the optimal solutions of the sub-problems and efficient algorithms are developed for them based on the properties. A genetic algorithm is proposed to evolve fixed partitions for the problem.

The remainder of this paper is organized as follows. The infinite horizon inventory routing problem for the three-level distribution system is formally defined in Section 2, and a lower bound on the long-run average cost of any feasible distribution strategy of the system is proposed in Section 3. The FPP-based decomposition solution approach to the IRP is outlined in Section 4, and the resolution of the three sub-problems and the genetic algorithm used in the approach are discussed in Sections 5 and 6, respectively. Section 7 presents computational experiments and results of the approach, followed by the concluding Section 8.

## 2. The inventory routing problem and its formulation

In the three-level distribution system considered, there are a single outside vendor, a single warehouse and  $n$  geographically dispersed retailers (see Fig. 1 for illustration). Each retailer faces an external demand for a single product with a deterministic, retailer specific rate  $D_i$ . The vendor with an unlimited supply of the

product serves the warehouse using big vehicles of capacity  $Q$ . The warehouse serves the retailers using small vehicles of capacity  $q$ . The vendor can also directly serve the retailers using big vehicles. Each time a big (resp. small) vehicle is used to replenish the inventory of the warehouse or the inventories of a set of retailers, it incurs a fixed cost  $C$  (resp.  $c$ ) plus a variable cost proportional to the total distance travelled. A linear inventory holding cost at a constant rate  $h$  (resp.  $h_0$ ) is charged at each retailer (resp. the warehouse) whenever inventories are kept there. The frequency in which a given retailer can be visited is bounded from above by a real number  $f$ . For simplicity, we assume all retailers have the same upper bound for their delivery frequencies, but the results in this paper can be easily extended to the case with different upper bounds for the delivery frequencies. The objective of the problem is to determine a combined transportation (routing) and inventory strategy minimizing the long-run average system-wide cost including the transportation cost from the vendor to the warehouse, the transportation cost from the vendor to the retailers, the transportation cost from the warehouse to the retailers, the inventory cost at the warehouse and the inventory costs at the retailers, while meeting each retailer's demand without shortage or backlogging.

The following assumptions are made for the problem. Firstly, it is assumed that split deliveries are not allowed. This implies that: (1) the inventory of each retailer is replenished either through the warehouse or directly from the vendor; (2) the inventory replenishment of retailers through the warehouse applies a fixed partition policy; (3) each retailer is served by a single route. This assumption in fact has been widely adopted in the infinite horizon IRP literature. More discussion about split delivery can be found in Dror and Trudeau (1989) and Bolduc et al. (2010). Secondly, it is assumed that each direct shipment from the vendor to the retailers serves only one retailer. This assumption is reasonable when a retailer is far from the vendor and/or the retailer has a relatively high demand rate. In practice, direct shipping is usually used for long-distance, high-volume delivery whereas multi-stop shipping is used for short-distance, low-volume delivery.

For the three-level distribution system with constant demand rates at the retailers, when the shipments from the warehouse to

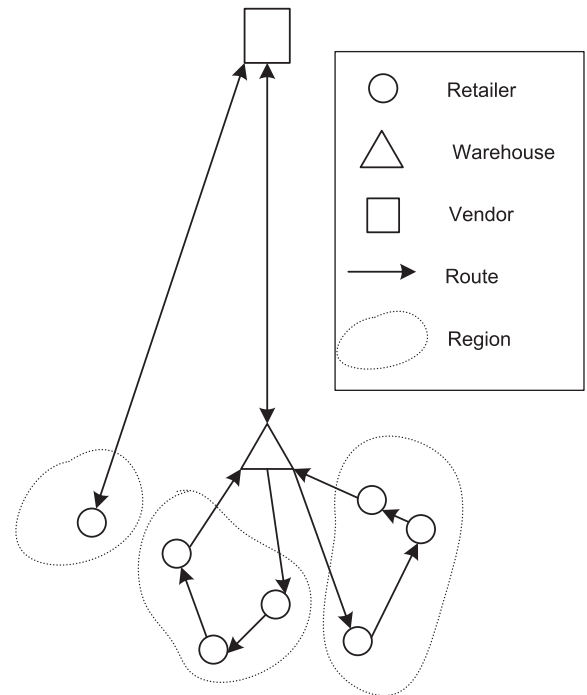


Fig. 1. The three-level distribution system.

the retailers use a fixed partition policy, it is a periodic system (cyclic system) whose behaviour is repeated after a certain number of periods corresponding to a cycle of the system. Let  $T$  denote the number of periods in each cycle of the system (the determination of  $T$  will be discussed later in Section 5.3.4), the IRP can be formulated only for one cycle since it is repetitive.

#### Indexes and parameters

$N$	set of all retailers, $N = \{1, 2, \dots, n\}$
$D_i$	demand rate of retailer $i$
$h_0$	inventory holding cost rate at the warehouse
$h$	inventory holding cost rate at each retailer
$d_0$	distance between the vendor and the warehouse
$d_i$	distance between the warehouse and retailer $i$
$\sigma_i$	distance between the vendor and retailer $i$
$Q$	capacity of each big vehicle
$q$	capacity of each small vehicle
$C$	fixed cost of using a big vehicle
$c$	fixed cost of using a small vehicle
$U$	variable transportation cost per unit distance of each big vehicle
$u$	variable transportation cost per unit distance of each small vehicle
$f$	maximal delivery frequency of each retailer
$l$	index of region in a fixed partitioning policy for all retailers, where each region is a subset of retailers
$t$	index of time period, $t = 0, 1, \dots, T - 1$

#### Variables

$S_w$	Set of retailers whose inventory is replenished by the warehouse
$S_p$	Set of retailers whose inventory is directly replenished by the vendor ( $S_w \cup S_p = N$ and $S_w \cap S_p = \emptyset$ )
$\chi$	A fixed partition of the retailers replenished from the warehouse under an FPP, $\chi = \{X_1, X_2, \dots, X_L\}$ , where $\cup_{i=1}^L X_i = S_w, X_i \cap X_j = \emptyset$ for $i \neq j$ ( $i, j = 1, 2, \dots, L$ )
$\theta_l$	The length of the TSP route for region $l$ , i.e., the 'shortest' route on which all retailers within the region are visited
$T_i$	Time interval of delivery from the vendor or the warehouse to retailer $i$
$T_l$	Time interval of delivery for region $l, T_i = T_l$ whenever $i \in X_l$
$D_l$	Total demand rate of the retailers within region $l$
$D(t)$	Demand of the warehouse in period $t$ (derived from the FPP for the retailers replenished by the warehouse)
$Q(t)$	Replenishment quantity from the vendor to the warehouse in period $t$
$I(t)$	Inventory level of the warehouse at the end of period $t$

With the above notations, the IRP can be formulated as the following mathematical programming model:

Model P:

$$\begin{aligned} \min z = & \frac{1}{T} \left\{ \sum_{t=0}^{T-1} \left[ \frac{Q(t)}{Q} \right] (C + 2d_0U) + \left[ \frac{I(0)}{2} + \sum_{t=1}^{T-1} I(t) \right] h_0 \right\} \\ & + \sum_{l \in S_w} \left( \frac{c + \theta_l u}{T_l} + \frac{1}{2} T_l \sum_{i \in X_l} D_i h_i \right) \\ & + \sum_{i \in S_p} \left( \frac{C + 2\sigma_i U}{T_i} + \frac{1}{2} T_i D_i h_i \right) \quad (1) \\ \text{s.t. } & T_l \geq 1/f, \quad \forall l = 1, 2, \dots, L \quad (2) \\ & T_l \sum_{i \in X_l} D_i \leq q, \quad \forall l = 1, 2, \dots, L \quad (3) \\ & T_i \geq 1/f, \quad \forall i \in S_p \quad (4) \\ & T_i D_i \leq Q, \quad \forall i \in S_p \quad (5) \\ & I(t) = \sum_{\tau=0}^t Q(\tau) - \sum_{i=1}^l \left( 1 + \left\lfloor \frac{t}{T_l} \right\rfloor \right) D_i T_l, \quad t = 0, 1, \dots, T - 1 \quad (6) \\ & Q(t), I(t) \geq 0, \quad t = 0, 1, \dots, T - 1, \quad (7) \end{aligned}$$

In the model, constraints (2) and (4) are the delivery frequency constraints ensuring that the frequency of delivery to each retailer does not exceed a given upper bound; constraints (3) and (5) are the vehicle capacity constraints ensuring that the load of each vehicle (big vehicle or small vehicle) is less than or equal to the vehicle capacity; constraints (6) ensure the flow balance at the warehouse, i.e., the inventory level of the warehouse equals the cumulative quantity delivered to the retailers minus the cumulative total demand of the retailers; constraints (7) are variable domain constraints.

The objective function (1) is to minimize the total cost of the system which includes three parts: the first part is the transportation cost from the vendor to the warehouse and the inventory cost of the warehouse, the second part is the transportation cost for shipments from the warehouse to retailers and the inventory cost of the retailers associated with this type of shipping, the third part is the transportation cost for direct shipments from the vendor to retailers and the inventory cost of the retailers associated with this type of shipping.

### 3. A lower bound on the long-run average cost of any feasible distribution strategy

To evaluate the performance of the solution found by our FPP-based decomposition approach to be presented in next section, a lower bound on the long-run average cost of any feasible distribution strategy is required so that the performance can be evaluated by the relative gap between the cost associated with the solution (the upper bound) and the lower bound.

#### Theorem 1.

$$B^{**} = \sum_{i=1}^n \left\{ \frac{\min(2\sigma_i U + C, 2d_0 U + C + 2d_i U + c) D_i}{Q} + \frac{D_i h}{2f} \right\}, \quad (8)$$

is a lower bound on the long-run average cost of the three level distribution system over all feasible distribution strategies.

**Proof.** The inventory of a retailer is replenished either directly from the vendor or from the vendor via the warehouse. For those retailers whose inventories are replenished directly from the vendor, i.e.,  $\forall i \in S_p$ , the total transportation and inventory holding costs  $B_1 \geq \sum_{i \in S_p} \left[ \frac{(2\sigma_i U + C) D_i}{Q} + \frac{D_i h}{2f} \right]$ . For those retailers whose inventories are replenished from the vendor via the warehouse, i.e.,  $\forall i \in S_w, B_2 \geq \sum_{i \in S_w} \left[ \frac{(2d_0 U + C) D_i}{Q} + \frac{(2d_i U + c) D_i}{Q} + \frac{D_i h}{2f} \right]$  according to the proof of Lemma 3.1 in Chan and Simchi-Levi (1998). Therefore,

$$B_1 + B_2 \geq \sum_{i=1}^n \left\{ \frac{\min(2\sigma_i U + C, 2d_0 U + C + 2d_i U + c) D_i}{Q} + \frac{D_i h}{2f} \right\}.$$

Let  $B^{**} = \sum_{i=1}^n \left\{ \frac{\min(2\sigma_i U + C, 2d_0 U + C + 2d_i U + c) D_i}{Q} + \frac{D_i h}{2f} \right\}$ , clearly  $B^{**}$  is a lower bound on the long run average cost over all feasible strategies.  $\square$

Observe that the lower bound  $B^{**}$  is less than

$$B^* = \sum_{i=1}^n \left[ \frac{D_i (2d_0 U + C + 2d_i U + c)}{Q} + \frac{D_i h}{2f} \right], \quad (9)$$

which is a lower bound on the average cost over all feasible cross-docking strategies developed in Chan and Simchi-Levi (1998).

### 4. A solution approach to the IRP

The IRP studied is NP-hard since one of its sub-problems – Vehicle Routing Problem (VRP) – is NP-hard. It is therefore impossible to find an algorithm that can solve the problem to optimality in

a reasonable computation time. As an alternative, we propose a decomposition solution approach based on a fixed partition policy because of its relative easiness for implementation and practical popularity. Given a fixed partition, the original problem is decomposed into three sub-problems: (1) sub-problem 1: the direct shipping sub-problem which determines which retailers are replenished directly from the vendor and their corresponding direct shipping strategy; (2) sub-problem 2: the warehouse-retailer transportation and inventory sub-problem which determines a routing strategy for the shipments from the warehouse to the remaining retailers and the inventories at these retailers; (3) sub-problem 3: the vendor-warehouse transportation and inventory sub-problem which determines the shipment plan from the vendor to the warehouse. The three sub-problems are solved in order. Sub-problem 1 tells sub-problem 2 the set of the retailers to be replenished from the warehouse. Sub-problem 2 tells sub-problem 3 the demand mode of the warehouse.

A genetic algorithm (GA) is proposed to find a near-optimal fixed partition for the IRP. In the GA, at the first iteration, multiple fixed partitions are generated and the fixed partitions in the next iterations are obtained by crossover and/or mutation operations. The solution procedure of the GA is repeated until no further improvement is possible or a specified number of iteration has been achieved. In the GA, for each new solution (fixed partition) generated, three sub-problems are solved to calculate the cost of the system and then evaluate the fitness of the solution. The framework of the FPP-based decomposition approach is illustrated in Fig. 2. The approach will be presented in detail in the rest of this paper.

It should be noted that we can also use meta-heuristics other than the genetic algorithm to optimize the fixed partition policy in the framework of our proposed solution approach. For example, we can use a local search based meta-heuristic, e.g., a tabu search algorithm like the one designed in Zhao et al. (2008), to find a near-optimal partition starting from an initial partition by applying local move operators such as exchanging two retailers between two regions or moving a retailer from one region to another.

## 5. Resolution of the sub-problems

This section discusses the three sub-problems and their solution algorithms by exploring important properties of their optimal solutions.

### 5.1. The direct shipping sub-problem (sub-problem 1)

The task of this sub-problem is to determine whether or not a retailer is replenished directly from the vendor bypassing the warehouse. If a retailer is not replenished directly from the vendor, then it is replenished from the vendor through the warehouse. Based on the findings in Li et al. (2010), it is reasonable to specify that a retailer is replenished directly from the vendor whenever its corresponding region in a given fixed partition contains only itself and its demand rate is large enough, for example, when its demand rate is larger than  $qf$  or close to  $Qf$ . If the demand rate of a retailer exceeds  $qf$ , it has to be replenished by direct shipping. If the demand rate of the retailer is larger than  $0.85 Qf$ , according to Li et al. (2010), the effectiveness of the direct shipping exceeds 92%, and hence the direct shipping is good enough. Particularly when  $Q \geq 2q$ , using the formulas developed by Li et al. (2010) we can also determine whether or not a retailer is replenished directly from the vendor. In the case, for any retailer with a demand rate larger than  $qf$ , its demand must be satisfied by direct shipping from the vendor, whereas for any retailer with a demand rate less than or equal to  $qf$ , its demand must be replenished from the warehouse

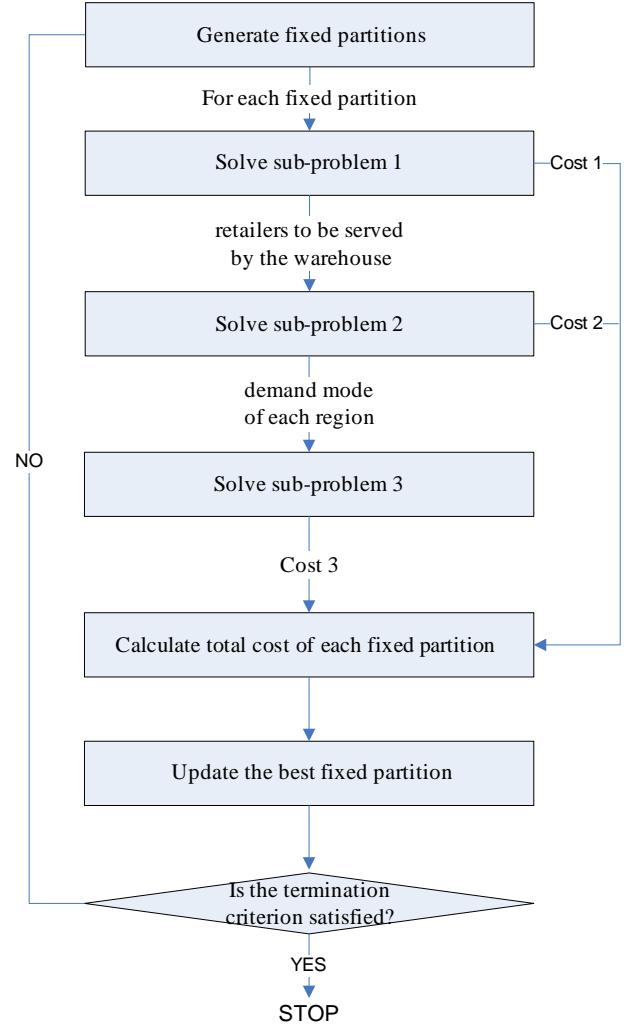


Fig. 2. Framework of the proposed solution approach.

because the effectiveness of direct shipping is less than 70%. In summary, with little loss of accuracy, sub-problem 1 can be simply integrated into the generation of fixed partitions. That is, for each fixed partition generated by the GA, if a region in the partition contains only one retailer, whether or not the retailer is replenished directly from the vendor is determined by the following standard: if and only if the demand rate of the retailer is larger than  $qf$  or  $0.85Qf$ , direct shipping is adopted.

Once the retailers to be replenished directly from the vendor are determined, the optimal replenishment interval  $T_i$  and replenishment quantity  $T_i D_i$  for each of the retailers can be easily computed using the following Model P1, so can be the long-run average transportation and inventory cost  $Z1$  associated with sub-problem.

Model P1:

$$\min Z1 = \sum_{i \in S_p} \left( \frac{C + 2\sigma_i U}{T_i} + \frac{1}{2} T_i D_i h_i \right), \quad (10)$$

s.t. (4), (5)

The computation procedure for  $T_i$  and  $Z1$  is as follows. By setting  $dZ1/dT_i = -(C + 2\sigma_i U)/(T_i)^2 + D_i h_i/2 = 0$ , we have  $T_i = \sqrt{2(C + 2\sigma_i U)/D_i h_i}$ . If  $T_i \geq 1/f$ , then the right value of  $T_i$  is obtained already, otherwise  $T_i = 1/f$ . Substitute each  $T_i$  into the objective function (10), we can obtain the value of  $Z1$ .

### 5.2. The warehouse-retailer transportation and inventory sub-problem (sub-problem 2)

Under a given fixed partition, this sub-problem is to determine the optimal replenishment interval  $T_l$  (and replenishment quantity  $T_l D_l$ ) for each region  $l$  to be replenished from the warehouse, and the optimal route within the region. It can thus be described as the following model.

Model P2:

$$\min Z2 = \sum_{l \in S_w} \frac{c + \theta_l u}{T_l} + \frac{1}{2} T_l \sum_{i \in X_l} D_i h_i, \quad (11)$$

s.t. (2), (3)

where  $\theta_l$  is calculated by solving the corresponding Travelling Salesman Problem (TSP). Various effective algorithms for the TSP are available in the literature, e.g., Lin and Kernighan (1973) and Gendreau et al. (1992). In this paper, rather than finding an optimal route for the sub-problem by using an exact algorithm, we use the nearest insertion heuristic plus 2-opt heuristic to find a high quality near-optimal route. In our computational experiments, with the combined heuristics, optimal solutions for multiple TSP benchmark problems were obtained within 1 second. This proves the quality of the heuristics and verifies our choice. Once  $\theta_l$  is determined,  $T_l$  and Z2 can be calculated in the same way as provided in Section 5.1.

### 5.3. The vendor-warehouse transportation and inventory sub-problem (sub-problem 3)

This sub-problem is to determine simultaneously the shipment plan from the vendor to the warehouse and the inventory decision at the warehouse, i.e., to determine the delivery quantity from the vendor to the warehouse and the inventory level at the warehouse in each period, with the objective to minimize the long-run average transportation and inventory costs while meeting the retailers' demands without shortage. The results of sub-problem 2 are input parameters of this sub-problem. That is, when the replenishment interval and the replenishment quantity in each replenishment interval for each retailer region are known, the demand of the warehouse in each period  $t$ , i.e.,  $D(t)$ , is determined.  $D(t)$  can be computed by Eq. (12).

$$D(t) = \sum_{l=1}^L \left( 1 + \left\lfloor \frac{t}{T_l} \right\rfloor \right) D_l T_l - \sum_{l=1}^L \left( 1 + \left\lfloor \frac{t-1}{T_l} \right\rfloor \right) D_l T_l, \quad (12)$$

$$t = 0, 1, \dots, T-1.$$

To solve this sub-problem, Chan and Simchi-Levi (1998) and Zhao et al. (2008) consider a specific strategy, cross-docking strategy and power-of-two policy respectively, and thus obtain only a sub-optimal solution of the sub-problem. In this paper, we solve the sub-problem to optimality based on the following important finding: the sub-problem can be reduced to a C/C/C/Z capacitated dynamic lot sizing problem under certain condition. The notation  $\alpha/\beta/\gamma/\delta$  introduced by Bitran and Yanasse (1982) represents a specific family of dynamic lot sizing problems, where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  specify respectively the time structure of the setup costs, holding costs, production costs, and production capacities, and may take the following letters: G, C, ND, NI, Z to indicate arbitrary pattern, constant, nondecreasing, nonincreasing, and zero. The detailed derivation of this result is described as follows.

Let  $T$  be the number of periods in a cycle, i.e.,  $T$  is the minimum positive integer number such that  $D(t+T) = D(t)$ ,  $Q(t+T) = Q(t)$ ,  $I(t+T) = I(t)$  for any  $t$ . Without loss of generality, only the states of the distribution system during the first  $T$  periods are needed to be considered for the vendor-warehouse transportation and inventory sub-problem. Given a fixed partition  $\chi = \{X_1, X_2, \dots, X_L\}$ ,

the vendor-warehouse transportation and inventory sub-problem can be formulated as the following model:

Model P3:

$$\min Z3 = \left[ \sum_{t=0}^{T-1} \left\lceil \frac{Q(t)}{Q} \right\rceil (C + 2d_0 U) + \sum_{t=0}^{T-1} I(t) h_0 \right] / T. \quad (13)$$

S.t. (6), (7)

Note that constraints (6) can be written as

$$I(t+1) = I(t) + Q(t+1) - D(t+1), \quad t = 0, 1, \dots, T-1, \quad (6')$$

where  $D(t+1)$  is the demand of the warehouse in period  $t+1$ ,  $Q(t+1)$  is the quantity delivered from the vendor to the warehouse in period  $t+1$ ,  $I(t+1)$  is the inventory level of the warehouse at the end of period  $t+1$ .

In what follows we explore properties of optimal solutions of the model and prove that the problem (P3) can be reduced to a C/C/C/Z capacitated dynamic lot sizing problem.

#### 5.3.1. Properties of optimal solutions

Without loss of generality, let  $D(t) = k_t Q + q_t$ ,  $k_t \geq 0$  and integer,  $0 \leq q_t < Q$ ,  $\forall t$ .

**Proposition 1.**  $I(t-1) * (Q(t) \% Q) = 0$ ,  $\forall t$ .

**Proof** (By contradiction). Assume there is an optimal solution such that  $I(\tau-1) > 0$  for some  $\tau$  and  $Q(\tau) = kQ + q$ , where  $k \geq 0$  and integer,  $0 < q < Q$ . It is clear that a part of the demand after period  $\tau-1$  is replenished before period  $\tau$ . Let  $y = \min\{I(\tau-1), Q - x\} > 0$ . We can construct a new solution with  $I(\tau-1) = I(\tau-1) - y$ , which means that  $y$  units of  $I(\tau-1)$  is no longer delivered before period  $\tau$ , instead they are delivered in period  $\tau$ , i.e.,  $Q'(\tau) = Q(\tau) + y \leq (k+1)Q$ . This solution change does not incur additional transportation costs but does reduce the inventory cost by at least  $y h_0$ . This implies that the original solution is not an optimal solution.  $\square$

**Proposition 2.**  $0 \leq I(t) < Q$ ,  $\forall t$ .

**Proof** (By contradiction). Assume there is an optimal solution such that  $I(\tau) = kQ + x$  for some  $\tau$ , where  $k \geq 1$  and integer,  $0 \leq x < Q$ . We can construct a new solution:  $S'(\tau) = S(\tau) - kQ$  and thus  $I'(\tau) = I(\tau) - kQ = x$ , i.e.,  $0 \leq I'(\tau) < Q$ ;  $Q'(\eta) = Q(\eta) + kQ$  and thus  $I'(\eta) = I(\eta)$ , where  $\eta(>\tau)$  is the earliest period when the delivery quantity of the vendor is larger than zero. This solution change does not incur additional transportation costs but does reduce the inventory cost by  $(\eta - \tau)kQ h_0$ . This implies that the original solution is not an optimal solution.  $\square$

**Proposition 3.**  $k_t Q \leq Q(t) \leq (k_t + 1)Q$ ,  $\forall t$ , where  $D(t) = k_t Q + q_t$ ,  $k_t \geq 0$  and integer,  $0 \leq q_t < Q$ ,  $\forall t$

**Proof.** (i). Firstly, we prove  $Q(t) \geq k_t Q$ . Case 1:  $k_t = 0$ . The conclusion is obvious. Case 2:  $k_t \geq 1$ . (By contradiction) Assume there is an optimal solution such that  $Q(\tau) < k_t Q$  for some  $\tau$ . Without loss of generality, let  $k_t Q - Q(\tau) = nQ + x > 0$ ,  $n \geq 0$  and integer,  $n \leq k_t - 1$ ,  $0 \leq x < Q$ , we have  $D(t) - Q(\tau) \geq nQ + x$ . This implies that the demand in period  $\tau$  has at least  $nQ + x$  to be satisfied before period  $\tau$ , i.e.,  $I(\tau-1) \geq nQ + x$ . If  $n \geq 1$ , we can construct a new solution:  $I'(\tau-1) = I(\tau-1) - nQ$ ,  $Q'(\tau) = Q(\tau) + nQ$ . This solution change does not incur additional transportation costs but does reduce the inventory cost by at least  $nQ h_0$ . This implies that the original solution is not an optimal solution. If  $n = 0$ , i.e.,  $Q(\tau) = k_t Q - x$  and  $0 < x < Q$ , i.e.,  $Q(t) \% Q = Q - x > 0$ . At the same time we have  $I(\tau-1) \geq x > 0$ . According to Proposition 1, it is impossible. (ii). Now we prove  $Q(t) \leq (k_t + 1)Q$ . (By contradiction) The proof of (i) implies that we need to consider only  $q_t$  for  $D(t)$ ,  $\forall t$ . That is, the original proposition turns to prove  $Q(t) \leq Q$  whenever  $D(t) = q_t$ ,  $0 \leq q_t < Q$ ,  $\forall t$ . Assume there is an optimal solution,  $Q(\tau) > Q$  for some  $\tau$ . It is clear that a part

of the demand after period  $\tau$  is satisfied in period  $\tau$ . Therefore,  $q'_\tau + q''_\tau = nQ + x = Q(\tau)$ ,  $n \geq 1, 0 \leq x < Q$ , where  $q'_\tau$  is the whole or part of the demand in period  $\tau$  to be satisfied in period  $\tau$ , and  $q'_\tau \leq q_\tau$  because a part of the demand in period  $\tau$  may be satisfied before period  $\tau$ ;  $q''_\tau$  is the total demand after period  $\tau$  that is satisfied in period  $\tau$ . Observe that  $q_t + q_{t'} < 2Q$  for  $\forall t, t'$ . This implies that the demands of at least  $n - 1$  periods after period  $\tau$  are satisfied in period  $\tau$ . Without loss of generality, let  $\eta(>\tau)$  is the earliest period in these periods. If  $n = 1$ ,  $q'_\tau + q''_\tau = Q + x$  and  $0 < x < Q$ , and then  $q''_\tau > x$  since  $q'_\tau \leq q_\tau < Q$ . We can construct a new solution:  $Q'(\tau) = Q(\tau) - x = Q$  (decreasing the number of vehicles by one);  $Q'(\eta) = Q(\eta) + x$  (increasing the number of vehicles by at most one). This solution change does not incur additional transportation costs but does reduce the inventory cost by at least  $(\eta - \tau)xh_0$ . If  $n \geq 2$ ,  $Q(\tau) = q'_\tau + q''_\tau = nQ + x$  and  $0 \leq x < Q$ , and then  $q''_\tau > (n - 1)Q + x$  since  $q'_\tau \leq q_\tau < Q$ . We can construct a new solution:  $Q'(\tau) = Q(\tau) - (n - 1)Q$ ,  $Q'(\eta) = Q(\eta) + (n - 1)Q$ . This solution change does not incur additional transportation cost and does reduce the inventory cost by at least  $(n - 1)(\eta - \tau)Qh_0$ . This implies that the original solution is not an optimal solution.  $\square$

**Proposition 4.**  $Q(t) = k_t Q$  whenever  $D(t) = k_t Q$ ,  $k_t \geq 0$  and integer.

**Proof** (By contradiction). Assume there is an optimal solution such that  $Q(\tau) = k_\tau Q + x$  for some  $\tau$ , where  $k_\tau \geq 0$  and integer,  $0 < x \leq Q$ . Note that  $k_\tau Q \leq Q(\tau) \leq (k_\tau + 1)Q$  according to Proposition 3. Similar to the proof of Proposition 3, we can construct a new solution:  $Q'(\tau) = Q(\tau) - x = k_\tau Q$  (decreasing the number of vehicles by one),  $Q'(\eta) = Q(\eta) + x$  (increasing the number of vehicles by at most one). This solution change does not incur additional transportation costs but does reduce the inventory cost by at least  $(\eta - \tau)xh_0$ .  $\square$

It is important to note that, according to Proposition 4,  $Q(t) = 0$  whenever  $D(t) = 0$ ,  $\forall t$ .

### 5.3.2. The reduced problem P4

Let P4 be a special case of P3 with  $D(t) = q_t$ ,  $0 \leq q_t < Q$ ,  $\forall t$ , i.e., the following condition holds: the demand of the warehouse at any time is less than the capacity of a big vehicle.

**Theorem 2.** P3 can be reduced to P4.

**Proof.** According to Propositions 3 and 4, the portion  $k_t Q$  of  $D(t)$  is always replenished by  $k_t$  vehicles in period  $t$ . Therefore, only the replenishment of the portion  $q_t$  of  $D(t)$  needs to be decided.  $\square$

**Proposition 5.** Given a planning horizon  $T$ , P4 is a C/C/C/Z capacitated dynamic lot sizing problem.

**Proof.** In P4, if  $Q(t)$  is viewed as the production (or order) quantity in period  $t$  to be decided,  $Q$  corresponds to the production capacity bound,  $C + 2dU$  corresponds to the setup cost,  $h_0$  is the inventory holding cost per unit item per unit time, and the production cost per unit item is zero and hence is omitted. P4 is therefore a C/C/C/Z capacitated dynamic lot sizing problem.  $\square$

### 5.3.3. Resolution of P4

According to Proposition 5, given a planning horizon, P4 is a C/C/C/Z capacitated dynamic lot sizing problem. Consequently, all algorithms for the C/C/C/Z capacitated dynamic lot sizing problem without backlog can be used to solve P4.

**Theorem 3.** Given a planning horizon  $T$ , there is an algorithm to solve P4 in  $O(T^2)$  time.

**Proof.** In Chung and Lin (1988), the authors propose a dynamic programming algorithm to solve the NI/G/NI/ND capacitated dynamic lot sizing problem in  $O(T^2)$  time. It is clear that the C/C/C/Z problem is a special case of the NI/G/NI/ND, therefore the algorithm can also be used to solve the C/C/C/Z problem in no more than  $O(T^2)$  time.  $\square$

### 5.3.4. Determination of the cycle time $T$

For solving P4, the remaining problem is to determine its planning horizon  $T$  which is the number of periods in each cycle of the IRP, referred to as its cycle time. It is clear that  $T$  is the minimum positive integer number such that  $D(t + T) = D(t)$ ,  $Q(t + T) = Q(t)$ ,  $I(t + T) = I(t)$  for any  $t$ . Without loss of generality, we assume that  $D(t)$ ,  $Q(t)$ ,  $\forall t$  and  $Q$  are integers, see also Chan et al. (1998) for this assumption. Let  $M$  be the least common multiple of  $T_1, T_2, \dots, T_L$ , where  $T_i, 1 \leq i \leq L$ , is the time interval of delivery from the vendor or the warehouse to retailer  $i$ , then  $M$  is the minimum positive integer number satisfying  $D(t + M) = D(t)$  for any  $t$ .

**Theorem 4.**  $M \leq T \leq QM$ .

**Proof.** According to Proposition 2,  $0 \leq I(t) < Q$ ,  $\forall t$ . This implies that  $I(t)$  has at most  $Q$  different possible values since  $I(t)$  is an integer. Therefore, for any  $\tau$  with  $0 \leq \tau \leq M - 1$ , there exists two periods, denoted by  $iM + \tau$  and  $jM + \tau$  respectively, in  $\tau, M + \tau, 2M + \tau, \dots, QM + \tau$  within the planning horizon of length  $(Q+1)M$ , such that  $I(iM + \tau) = I(jM + \tau)$  is satisfied,  $0 \leq i < j \leq Q$ . In addition,  $D(iM + \tau + 1) = D(jM + \tau + 1)$  for any  $\tau$  with  $0 \leq \tau \leq M - 1$  from the definition of  $M$ . For the distribution system considered, its behavior when it starts from period  $iM + \tau + 1$  with the initial inventory level  $I(iM + \tau)$  and its behavior when it starts from period  $jM + \tau + 1$  with the initial inventory level  $I(jM + \tau)$  should be the same in its optimal solution, since  $I(iM + \tau) = I(jM + \tau)$  and  $D(iM + \tau + 1) = D(jM + \tau + 1)$  for any  $\tau$  with  $0 \leq \tau \leq M - 1$ . This implies that  $Q(iM + \tau + 1) = Q(jM + \tau + 1)$  for any  $\tau$  with  $0 \leq \tau \leq M - 1$ . From (8'), we have  $I(iM + \tau + 1) = I(jM + \tau + 1)$  for any  $\tau$  with  $0 \leq \tau \leq M - 1$ . This implies that  $T = (j - i)M$  satisfying  $D(t + T) = D(t)$ ,  $Q(t + T) = Q(t)$ ,  $I(t + T) = I(t)$ . Consequently,

$$M \leq T = (j - i)M \leq QM. \quad \square$$

To obtain the exact value of  $T$ , one way is to solve P4 and obtain its optimal solution for  $T = M, 2M, \dots, QM$  respectively, and then select the best one from them.

## 6. Genetic algorithm

One key issue for our solution approach is to find a fixed partition (of the retailers) that minimizes the long run average total cost of the distribution system studied. As this problem is NP-hard, it is impossible to develop an exact algorithm that can solve the problem to optimality in a reasonable time. Instead, a genetic algorithm is developed in this section to find a near-optimal fixed partition.

### 6.1. Coding and decoding of each individual in the population

Each solution, called an individual, in the population of the GA at each iteration, is represented by a chromosome whose length is equal to the number of retailers. Each gene of the chromosome is related to a retailer and is assigned to an integer number between 1 and the number of available vehicles. If the  $i$ th gene is assigned to integer  $m$ , for instance, then it means that retailer  $i$  is served by vehicle  $m$  (the  $m$ th vehicle). To get the corresponding fixed partition from a chromosome, it must be decoded. If multiple genes in the chromosome are assigned to the same integer, the corresponding retailers form a fixed region, and their inventories are replenished from the warehouse by using a small vehicle. The optimal replenishment interval, replenishment quantity in each interval and long-run average cost of this region can be calculated according to Model P2 as discussed in Section 5.2. If only one gene has a particular integer value, that is, only one retailer is assigned to a particular vehicle, we have to determine whether the inventory of this retailer is replenished directly from the vendor or through the warehouse. As discussed in Section 5.1, if the demand

rate of the retailer is larger than  $qf$  or  $0.85Qf$ , then it is replenished directly from the vendor by using a big vehicle. Otherwise, the retailer demand rate is not large enough to justify the replenishment directly from the vendor. The retailer itself then forms a fixed region and is replenished from the warehouse by using a small vehicle. For any direct shipping, the optimal replenishment interval, replenishment quantity in each interval and the corresponding long-run average cost can be calculated according to Model P1 as discussed in Section 5.1.

## 6.2. Generation of the initial population

70% individuals (solutions) in the initial population are produced in random, i.e., each gene of each chromosome is assigned to a random integer number between 1 and the number of available vehicles. The remaining 30% individuals are produced as structural chromosomes using a heuristic method. The heuristic to produce the structural chromosomes is described as follows. First, take the warehouse as the centre of the coordinates of the plane on which all retailers are situated and sort the retailers in terms of their polar angles with the x-axis of the plane; open the first vehicle; pack retailers one by one to the currently opened vehicle following their sorted order until the capacity of the current packet (current vehicle) is used up. A new vehicle is opened if there are retailers which cannot be packed to the current vehicle. Let the capacity of each packet is  $qf$ . If the difference between the total demand rate of the retailers that have been packed to the current packet and  $qf$  is larger than a designated tightness factor multiplied by  $qf$ , then the next retailer is still packed into the packet. Otherwise pack the retailer to a new opened vehicle. Here the tightness factor defines how much percentage of the packet capacity can be exceeded. If a retailer that is going to be packed into a vehicle has a demand rate larger than  $qf$ , then the vehicle serves only the retailer. In this way, a structural chromosome (solution) is produced. With different tightness factors, different structural chromosomes are generated. Note that some initial solutions generated may be infeasible because they violate the capacity constraint of a vehicle. We allow some degree of infeasibility (capacity violation) in generating the initial solutions in order to diversify the search space of our GA. The readers can refer to Barrie et al. (2003) for a more detailed introduction of the generation of structural chromosomes.

## 6.3. Genetic operations

Standard two-point crossover is adopted as the crossover operator of our GA. For the mutation, the value of each gene is changed with a given probability to an arbitrary integer number between 1 and the number of available vehicles.

### 6.3.1. Selection and replacement

Fitness proportionate selection (roulette-wheel selection) based on the value of every chromosome is used to generate a new generation. For the replacement, the best 25% chromosomes in the pool comprising both the parent and child chromosomes are first selected; they then replace the best 50% of the parent chromosomes. In this way, good chromosomes are obtained, and at the same time premature convergence is avoided.

### 6.3.2. Evaluation of the fitness of each individual in the population

The fitness of each individual (chromosome) in the current population is defined as a big number minus the long-run average cost of the distributed system considered minus the infeasibility penalty of the individual (solution), where the long-run average cost is calculated without considering the vehicle capacity constraints, the infeasibility penalty is defined as a penalty rate multiplied by the amount by which the vehicle capacity is exceeded. In our GA,

the penalty rate is taken as  $2h(C + 2Ud_0)$ . Since the fitness of each chromosome is positive, the big number in its definition must be larger than the sum of the long-run average cost and the infeasibility penalty of any chromosome. To make the fitness proportionate selection work well, however, the big number should not be too large. In our GA, the big number is set to a very big number initially, e.g.  $100 \times \text{RAND\_MAX}$ , where  $\text{RAND\_MAX}$  is a built-in-number of C++ programs. In the next generation, the big number is set to 3 times the sum of the long-run average cost and the infeasibility penalty of the worst chromosome. As crossover and mutation operations may produce a very bad chromosome, to ensure that no negative fitness will be produced for any chromosome, our GA program includes a check statement. Once a negative fitness happens, our program will be interrupted and a new multiple large than 3 will be manually set. In our numerical experiments, the multiple 3 works well.

### 6.3.3. Parameter settings of the GA

After testing a few groups of their values, the parameters of our GA are chosen as follows. Population size = 100, the probability of crossover = 0.7, the probability of mutation = 0.02, and the maximum number of generations (iterations) = 2000.

## 7. Computation results

The performance of the FPP-based decomposition approach proposed and the corresponding genetic algorithm is evaluated on randomly generated instances by comparing the lower bound found in Section 3 for the long-run average cost of any feasible distribution strategy with the cost of the distribution strategy (an upper bound) found by the approach. We generate the instances based on the parameters provided by Chan and Simchi-Levi (1998) but with necessary modifications.  $f$  is larger than 1 in Chan and Simchi-Levi (1998). In this paper  $f$  is defined as the maximal number that each retailer can be visited per period. Since in each period at most one delivery is performed to each retailer, it is naturally assumed that  $f \leq 1$  as in Li et al. (2010) and Zhao et al. (2008). In order to keep the scale among the modified parameters as same as that among the original parameters, the demand rates are scaled down 10 times when  $f$  is reduced 10 times. When  $f$  is scaled down only 5 times, then  $q$  and  $Q$  are scaled up two times at the same time. The rest can be deduced by analogy. Moreover, in the implementation of the FPP-based decomposition approach, all time intervals of delivery involved are rounded to their nearest positive integers as they would be in practice.

All numerical experiments were done with a PC of RAM 2G, CPU 1.7G. We randomly generate a few instances in a square of size  $100 \times 100$ . The number of retailers is respectively 75, 100, 120 and 150 while their locations are uniformly distributed in the square. The locations of the warehouse and the vendor are also randomly generated in the square. The retailer demand rates are uniformly distributed on the integer set  $\{3, 4, 5, 6, 7, 8\}$ . The inventory holding cost at the warehouse and at each retailer is taken respectively as  $h_0 = 3$ ,  $h = 6$ . The unit transportation costs of vehicles are taken as  $u = U = 1$ . For each instance, we generate 11 problems with different parameters  $f$ ,  $q$ ,  $Q$ ,  $c$  and  $C$ , which are given in Table 1. Different from Chan and Simchi-Levi (1998), we do not consider the case that  $q > Q$ , i.e., the parameter set 3 provided by them, since the case rarely happens in practice. Usually the vehicles with bigger capacities are used to longer distance transportation.

The computational results are reported in Table 2. From the table, we can see that the minimal, maximal, and average value of  $Z^H/B^*$  for all instances are 101.3%, 164.0% and 120.3%, respectively, whereas the minimal, maximal, and average value of  $Z^H/B^{**}$  are



**Table 1**  
List of parameter values for each instance.

Parameter set	$f$	$q$	$Q$	$c$	$C$
1	0.24	54	120	2	10
2	0.24	54	120	2	500
3	0.24	54	400	2	10
4	0.16	54	120	2	10
5	0.20	54	120	2	10
6	0.46	54	120	2	10
7	1.00	54	120	2	10
8	1.00	108	240	2	10
9	1.00	270	600	2	10
10	0.24	454	4500	2	10
11	0.24	1129	1200	2	10

101.5%, 222.3% and 139.0%, respectively. It indicates that the average performance of our decomposition approach in terms of  $Z^H/B^*$  is better than that reported in Chan and Simchi-Levi (1998). Also our algorithm is more robust in respect of the solution quality, since the performance measure  $Z^H/B^*$  of the algorithm is within 164.0% for any instance, whereas  $Z^H/B^*$  reported in the reference is more than 300% for some cases. The maximal percentage deviation of the cost found by our approach from the lower bound  $B^{**}$  is 120.8%, which is relatively large. Therefore, more precise evaluation of the performance of our decomposition approach and its further improvement is necessary in future research. It should be noted that it is impossible to make a fair comparison between our results and those of Chan and Simchi-Levi (1998), since their approach does not allow inventory at the warehouse and our definition of the maximal delivery frequency  $f$  is different from theirs.

Note that Chan and Simchi-Levi (1998) did not mention the locations of the warehouses and vendor and the values of  $u$  and  $U$  in their experimental report. For the instances in Table 2, the locations of the warehouse and the vendor are generated randomly

in a square of size  $100 \times 100$  as the retailers and  $u = U = 1$ . In practice, however,  $U$  should be larger than  $u$  and the vendor is usually far from the warehouse and retailers. It is therefore reasonable to expect that the transportation cost from the vendor to the warehouse and retailers tends to have a bigger proportion in the total cost. As a result, it is reasonable to conclude that relaxing the restrictions made to the problem by the existing studies and optimally solving sub-problem 3 contribute greatly to total cost savings.

The results in Table 3 are obtained on a set of instances with more reasonable parameters, where the warehouse is located at the centre of the square with the coordinates  $[0,0]$  and the coordinates of the vendor are  $[-200,0]$ . The demand retailer rates are distributed uniformly on the integers between 1 and 10 instead of between 3 and 8. It can be seen that our algorithm also performs well in these cases.

Regarding the computational time, for some instances in Tables 2 and 3 our GA algorithm is time consuming. This is because the running time of the dynamic programming algorithm for sub-problem 3 depends on the vehicle capacity  $Q$  and the least common multiple  $M$  of the delivery time intervals of all regions in a fixed partition. Sometimes  $M$  is so big that the dynamic programming involves a quite large number of iterations. In this case, one possible way to significantly reduce the computational time without sacrificing much the solution quality is, instead of using the real value of  $M$  to set the planning horizon (time cycle)  $T$  in the algorithm, a reasonable large number, for instance 10000, can be used as the planning horizon for the dynamic programming. It is important to note that for almost all time-consuming instances in Tables 2 and 3, a good solution with little deviation of the cost from that found after the termination of our GA algorithm can be found by the algorithm in 100 seconds. For example, a solution of  $106.7\%B^*$  is found within 100 seconds for the instance with parameter set 11 and  $n = 75$  in Table 2.

**Table 2**  
Computation results 1.

Parameter set	$n = 75$			$n = 100$			$n = 120$			$n = 150$		
	$Z^H/B^*$	$Z^H/B^{**}$	Time	$Z^H/B^*$	$Z^H/B^{**}$	Time	$Z^H/B^*$	$Z^H/B^{**}$	Time	$Z^H/B^*$	$Z^H/B^{**}$	Time
1	1.084	1.221	91	1.068	1.191	115	1.203	1.309	135	1.080	1.222	180
2	1.306	1.438	103	1.289	1.407	122	1.427	1.523	138	1.302	1.439	201
3	1.140	1.289	99	1.104	1.235	113	1.215	1.370	138	1.132	1.296	183
4	1.069	1.167	85	1.051	1.138	101	1.137	1.205	183	1.061	1.163	146
5	1.114	1.254	155	1.097	1.223	163	1.058	1.136	182	1.249	1.623	86
6	1.126	1.354	3055	1.096	1.300	2554	1.315	1.524	772	1.089	1.319	405
7	1.474	2.223	81	1.402	2.074	107	1.256	1.637	179	1.371	2.082	160
8	1.459	1.882	175	1.384	1.754	200	1.313	1.537	374	1.460	1.897	347
9	1.640	1.847	685	1.493	1.665	785	1.351	1.452	1318	1.364	1.544	1492
10	1.037	1.053	2372	1.026	1.041	2098	1.016	1.029	2512	1.029	1.047	3653
11	1.027	1.033	13751	1.018	1.024	8407	1.014	1.015	9940	1.013	1.019	16119

**Table 3**  
Computation results 2.

Parameter set	$n = 120, h = 6$										
	$f$	$q$	$Q$	$c$	$C$	$u$	$U$	$h_0$	$Z^H/B^*$	$Z^H/B^{**}$	Time
1	0.25	54	120	2	10	1	1.2	3	1.322	1.435	6673
2	0.25	54	120	2	10	1	1.5	3	1.371	1.480	5273
3	0.25	54	120	2	10	1	2	3	1.446	1.548	5949
4	0.25	54	54	2	2	1	1	3	1.034	1.103	6546
5	0.50	120	120	2	2	1	1	3	1.094	1.163	12987
6	0.75	54	54	2	2	1	1	3	1.018	1.116	4275
7	1.00	54	54	2	2	1	1	3	1.120	1.259	300
8	1.00	30	30	2	2	1	1	3	1.109	1.226	486
9	1.00	30	30	2	2	1	1	1	1.069	1.221	103
10	1.00	30	30	2	2	1	1	6	1.067	1.219	104

## 8. Conclusion

In this paper we investigate the three-level inventory routing problem in which each retailer can either get its inventory replenishment from the vendor directly or through the warehouse. We relax a couple of restrictions made to the problem in the existing studies by allowing inventory at the warehouse, direct shipment from the vendor, and general replenishment interval of each retailer. For this more general problem, we have proposed an FPP-based decomposition solution approach which decomposes the problem into three sub-problems under a given fixed partition. Efficient algorithms are developed for the sub-problems by exploring important properties of their optimal solutions. In particular, the vendor-warehouse transportation sub-problem, which is reduced to a C/C/C/Z capacitated dynamic lot sizing problem based on some important properties of the sub-problem, is optimally solved by using a dynamic programming algorithm.

For evaluating the performance of the proposed approach and its corresponding genetic algorithm, we develop a lower bound of the long-run average cost of any feasible distribution strategy for the three-level distribution system. Like other lower bounds for infinite horizon inventory routing problems, however, the lower bound is only asymptotically optimal. Therefore, our approach should have a better performance than that reported in our computational experiments. To more precisely evaluate the performance of the approach, a tighter lower bound on the long-run average cost of any feasible distribution strategy must be found, this is one of our future research directions. In addition, we only use typical operations in our genetic algorithm. It is an interesting direction to apply more sophisticated GA operations in our GA algorithm in future research.

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