HAL
open science

Orbit determination with the two-body integrals. II

G. F. Gronchi, D. Farnocchia, L. Dimare

To cite this version:

G. F. Gronchi, D. Farnocchia, L. Dimare. Orbit determination with the two-body integrals. II. Celestial Mechanics and Dynamical Astronomy, 2011, 110 (3), pp.257-270. 10.1007/s10569-011-9357-z . hal-00653185

HAL Id: hal-00653185

https://hal.science/hal-00653185

Submitted on 19 Dec 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Noname manuscript No.
(will be inserted by the editor)

Orbit Determination with the two-body Integrals. II

G. F. Gronchi · D. Farnocchia · L. Dimare

Received: date / Accepted: date

Abstract The first integrals of the Kepler problem are used to compute preliminary orbits starting from two short observed arcs of a celestial body, which may be obtained either by optical or by radar observations. We write polynomial equations for this problem, which can be solved using the powerful tools of computational Algebra. An algorithm to decide if the *linkage* of two short arcs is successful, i.e. if they belong to the same observed body, is proposed and tested numerically. This paper continues the research started in [6], where the angular momentum and the energy integrals were used. A suitable component of the Laplace-Lenz vector in place of the energy turns out to be convenient, in fact the degree of the resulting system is reduced to less than half.

1 Introduction

We present a new method, based on the first integrals of the Kepler problem, to compute a finite set of preliminary orbits of a celestial body from two short arcs of observations. We assume that the body moves on a Keplerian orbit with a known center of attraction O and is observed from a point P, whose motion is a known function of time. For asteroid orbits O corresponds to the center of the Sun, for space debris O is the center of the Earth. We deal with two different kinds of observations, optical and radar, and make use of the related attributables [8], [14].

In [6] the angular momentum and the energy integrals are used to solve the linkage problem for solar system bodies. This means to identify two attributables as related to

Giovanni F. Gronchi
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, Pisa, Italy
E-mail: gronchi@dm.unipi.it

Davide Farnocchia
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, Pisa, Italy
E-mail: farnocchia@mail.dm.unipi.it

Linda Dimare
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, Pisa, Italy
E-mail: dimare@mail.dm.unipi.it

the same observed object by computing (at least) one reliable orbit from the observations of both attributables. The equations of the problem are written in a polynomial form and the total degree of the system is 48. The use of these integrals for the linkage has been first proposed in [13], but without fully exploiting the algebraic character of the problem. The algorithm presented in [6] has been applied in [3] to the problem of *correlation*, that is the linkage in the context of space debris. There the authors have extended the method including the oblateness effect of the Earth. This method has been tested successfully with real data of geosynchronous satellites [10].

In this paper we propose different equations for the same problem: in particular we use a suitable projection of the Laplace-Lenz vector in place of the energy. The advantage of this approach is that there are several cancellations and the total degree is 20. Thus we do not need to use quadruple precision, as in our previous work.

The same equations can be written using different data, simply considering other quantities as unknowns. For example in Section 4 we deal with the case of an optical and a radar attributable. This case is interesting because we end up with a univariate polynomial of degree 4, hence the problem admits explicit solutions.

The solutions must fulfill compatibility conditions taking into account the other integrals of Kepler's problem, as in [6]. Here we propose a different strategy to select the solutions, based on the attribution algorithm of a very short arc to a known orbit [8], [7].

The structure of the paper is the following. After introducing preliminary definitions, in Sections 3, 4, 5 we study the linkage of two attributables, either optical or radar. The degenerate cases are shown in Section 6. Sections 7 and 8 explain the computation of the covariance matrix for each orbit and the selection of the solutions. Section 9 deals with the Earth oblateness effect. We conclude with numerical tests in Section 10.

2 Preliminaries

Let us fix an inertial reference frame, with the origin at the center of attraction O. The position $\mathbf{q}$ and velocity $\dot{\mathbf{q}}$ of the observer are known functions of time. We describe the position of the observed body as the vectorial sum

$$\mathbf{r} = \mathbf{q} + \rho \mathbf{e}^{\rho} \,, \tag{1}$$

with ρ the topocentric distance and $\mathbf{e}^{\rho}$ the *line of sight* unit vector. We choose spherical coordinates $(\alpha, \delta, \rho) \in [-\pi, \pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^{+}$, so that

$$\mathbf{e}^{\rho} = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta) \ .$$

A typical choice for α, δ is right ascension and declination. The velocity vector is

$$\dot{\mathbf{r}} = \dot{\mathbf{q}} + \dot{\rho}\mathbf{e}^{\rho} + \rho(\dot{\alpha}\cos\delta\mathbf{e}^{\alpha} + \dot{\delta}\mathbf{e}^{\delta}) \,, \qquad \dot{\rho}, \dot{\alpha}, \dot{\delta} \in \mathbb{R}, \rho \in \mathbb{R}^{+} \,, \tag{2}$$

where $\dot{\rho}$, $\rho\dot{\alpha}\cos\delta$, $\rho\dot{\delta}$ are the components of the velocity, relative to the observer in P, in the (positively oriented) orthonormal basis $\{\mathbf{e}^{\rho}, \mathbf{e}^{\alpha}, \mathbf{e}^{\delta}\}$, with

$$\mathbf{e}^{\alpha} = (\cos\delta)^{-1}\frac{\partial \mathbf{e}^{\rho}}{\partial \alpha} \,, \qquad \mathbf{e}^{\delta} = \frac{\partial \mathbf{e}^{\rho}}{\partial \delta} \ .$$

We recall the definitions of optical and radar attributables. Let $(t_i, \alpha_i, \delta_i)$ with $i = 1 \dots m$, $m \geq 2$, be a short arc of optical observations of a moving body, made from the same station. From these data it is possible to compute an **optical attributable**

$$\mathcal{A}_{opt} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}) \in [-\pi, \pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^2 \,,$$

representing the angular position and velocity of the body at a mean time $\bar{t}$ (see [8],[6]). In this case the radial distance and velocity $\rho, \dot{\rho}$ are completely undetermined and are the missing quantities to define an orbit for the body.
From a set of radar observations of a moving body $(t_i, \alpha_i, \delta_i, \rho_i)$, with $i = 1 \dots m$, $m \geq 2$, it is possible to compute a **radar attributable**, i.e. a vector

$$\mathcal{A}_{rad} = (\alpha, \delta, \rho, \dot{\rho}) \in [-\pi, \pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^+ \times \mathbb{R} \,,$$

at time $\bar{t}$ (see [14]). Here $\dot{\alpha}, \dot{\delta}$ are the unknowns needed to define an orbit.
We call **attributable coordinates** the vector $(\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho, \dot{\rho})$ representing the position and velocity of the body as seen from the observer at time $\bar{t}$.

3 Linking two optical attributables

Given two optical attributables $\mathcal{A}_1, \mathcal{A}_2$ at epochs $\bar{t}_1, \bar{t}_2$, we assume they belong to the same observed body and write 4 scalar algebraic equations for the topocentric distances ρ_1, ρ_2 and the radial velocities $\dot{\rho}_1, \dot{\rho}_2$ at the two epochs.
We use some of the algebraic integrals of the Kepler problem, i.e. the angular momentum $\mathbf{c}$ and the Laplace-Lenz vector $\mathbf{L}$. The expressions of these integrals as functions of $\rho, \dot{\rho}$ are given below. The angular momentum is

$$\mathbf{c}(\rho, \dot{\rho}) = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{D}\dot{\rho} + \mathbf{E}\rho^2 + \mathbf{F}\rho + \mathbf{G} \,,$$

where

$$\begin{aligned}
&\mathbf{D} = \mathbf{q} \times \mathbf{e}^\rho \,, \\
&\mathbf{E} = \dot{\alpha}\cos\delta \mathbf{e}^\rho \times \mathbf{e}^\alpha + \dot{\delta}\mathbf{e}^\rho \times \mathbf{e}^\delta = \dot{\alpha}\cos\delta \mathbf{e}^\delta - \dot{\delta}\mathbf{e}^\alpha \,, \\
&\mathbf{F} = \dot{\alpha}\cos\delta \mathbf{q} \times \mathbf{e}^\alpha + \dot{\delta}\mathbf{q} \times \mathbf{e}^\delta + \mathbf{e}^\rho \times \dot{\mathbf{q}} \,, \\
&\mathbf{G} = \mathbf{q} \times \dot{\mathbf{q}} \,.
\end{aligned}$$

The Laplace-Lenz vector is given by

$$\mu\mathbf{L}(\rho, \dot{\rho}) = \dot{\mathbf{r}} \times \mathbf{c} - \mu\frac{\mathbf{r}}{|\mathbf{r}|} = \Big(|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}\Big)\mathbf{r} - (\dot{\mathbf{r}} \cdot \mathbf{r})\dot{\mathbf{r}} \,, \tag{3}$$

where

$$\begin{aligned}
|\mathbf{r}| &= (\rho^2 + |\mathbf{q}|^2 + 2\rho\mathbf{q} \cdot \mathbf{e}^\rho)^{1/2} \,, \\
|\dot{\mathbf{r}}|^2 &= \dot{\rho}^2 + (\dot{\alpha}^2\cos^2\delta + \dot{\delta}^2)\rho^2 + 2\dot{\mathbf{q}} \cdot \mathbf{e}^\rho\dot{\rho} + 2\dot{\mathbf{q}} \cdot (\dot{\alpha}\cos\delta\mathbf{e}^\alpha + \dot{\delta}\mathbf{e}^\delta)\rho + |\dot{\mathbf{q}}|^2 \,, \\
\dot{\mathbf{r}} \cdot \mathbf{r} &= \rho\dot{\rho} + \mathbf{q} \cdot \mathbf{e}^\rho\dot{\rho} + (\dot{\mathbf{q}} \cdot \mathbf{e}^\rho + \mathbf{q} \cdot \mathbf{e}^\alpha\dot{\alpha}\cos\delta + \mathbf{q} \cdot \mathbf{e}^\delta\dot{\delta})\rho + \dot{\mathbf{q}} \cdot \mathbf{q} \,.
\end{aligned}$$

For objects orbiting around the Sun $\mu = Gm_\odot$, for satellites of the Earth $\mu = Gm_\oplus$.

Remark 1 If O corresponds to the center of the Sun, then we use interpolated values for $\mathbf{q}, \dot{\mathbf{q}}$, as suggested by Poincaré [11] and discussed in [9].

These dynamical quantities give 6 scalar integrals of the motions: only 5 are mutually independent, in fact $\mathbf{L} \cdot \mathbf{c} = 0$. Since we have 4 unknowns, generically we need 4 scalar conservation laws to define a finite number of solutions. We select the angular momentum and a particular component of the Laplace-Lenz vector. The choice of the latter integral presents a substantial advantage with respect to the use of the energy, as in [6]: this will be discussed later.

3.1 The polynomial equations

We use the notation above, with index 1 or 2 referring to the epoch. If $\mathcal{A}_1$, $\mathcal{A}_2$ correspond to the same observed object, by assuming that the latter moves on a Keplerian orbit,[1] the angular momentum vectors at the two epochs must coincide:

$$\mathbf{c}_1(\rho_1, \dot\rho_1) = \mathbf{c}_2(\rho_2, \dot\rho_2) \ . \tag{4}$$

Equation (4) can be written as

$$\mathbf{D}_1 \dot\rho_1 - \mathbf{D}_2 \dot\rho_2 = \mathbf{J}(\rho_1, \rho_2) \, , \tag{5}$$

where

$$\mathbf{J}(\rho_1, \rho_2) = \mathbf{E}_2 \rho_2^2 - \mathbf{E}_1 \rho_1^2 + \mathbf{F}_2 \rho_2 - \mathbf{F}_1 \rho_1 + \mathbf{G}_2 - \mathbf{G}_1 \ .$$

Following [6] we eliminate the variables $\dot\rho_1, \dot\rho_2$ and obtain

$$\mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{J}(\rho_1, \rho_2) = 0 \ . \tag{6}$$

We can write the left-hand side of (6) as

$$q(\rho_1, \rho_2) \stackrel{\text{def}}{=} q_{20} \rho_1^2 + q_{10} \rho_1 + q_{02} \rho_2^2 + q_{01} \rho_2 + q_{00} \, , \tag{7}$$

with

$$\begin{aligned} q_{20} &= -\mathbf{E}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2 \, , & q_{02} &= \mathbf{E}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2 \, , \\ q_{10} &= -\mathbf{F}_1 \cdot \mathbf{D}_1 \times \mathbf{D}_2 \, , & q_{01} &= \mathbf{F}_2 \cdot \mathbf{D}_1 \times \mathbf{D}_2 \, , \end{aligned}$$

$$q_{00} = (\mathbf{G}_2 - \mathbf{G}_1) \cdot \mathbf{D}_1 \times \mathbf{D}_2 \ .$$

The radial velocities are given by

$$\dot\rho_1(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_2) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2} \, , \qquad \dot\rho_2(\rho_1, \rho_2) = \frac{(\mathbf{J} \times \mathbf{D}_1) \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{|\mathbf{D}_1 \times \mathbf{D}_2|^2} \ . \tag{8}$$

Also the Laplace-Lenz vectors at the two epochs must coincide. We equate the projection of both vectors along $\mathbf{v} = \mathbf{e}_2^\rho \times \mathbf{q}_2$:

$$\mathbf{L}_1(\rho_1, \dot\rho_1) \cdot \mathbf{v} = \mathbf{L}_2(\rho_2, \dot\rho_2) \cdot \mathbf{v} \ . \tag{9}$$

The projection of $\mathbf{L}_2$ along $\mathbf{v}$ is particularly simple:

$$\mu \mathbf{L}_2 \cdot \mathbf{v} = -(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\dot{\mathbf{r}}_2 \cdot \mathbf{v}) \, ,$$

[1] This two-body approximation may fail in some cases, e.g. when a close approach with a third body occurs at an intermediate time.

thus (9) becomes

$$\Big(|\dot{\mathbf{r}}_1|^2 - \frac{\mu}{|\mathbf{r}_1|}\Big)(\mathbf{r}_1 \cdot \mathbf{v}) - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)(\dot{\mathbf{r}}_1 \cdot \mathbf{v}) = -(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\dot{\mathbf{r}}_2 \cdot \mathbf{v}) \ . \tag{10}$$

After substituting (8), this is an algebraic equation in ρ_1, ρ_2. Rearranging the terms in (10) and squaring we obtain

$$p(\rho_1, \rho_2) \stackrel{def}{=} \mu^2(\mathbf{r}_1 \cdot \mathbf{v})^2 - |\mathbf{r}_1|^2 \left\{ \left[|\dot{\mathbf{r}}_1|^2 \mathbf{r}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\dot{\mathbf{r}}_1 + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)\dot{\mathbf{r}}_2\right] \cdot \mathbf{v} \right\}^2 = 0 \ . \tag{11}$$

This is a polynomial equation of degree 10 in ρ_1, ρ_2: in fact, the projection

$$\begin{aligned} \dot{\mathbf{r}}_2 \cdot \mathbf{v} &= \mathbf{q}_2 \cdot (\rho_2(\dot{\alpha}_2 \cos\delta_2 \mathbf{e}_2^\alpha + \dot{\delta}_2 \mathbf{e}_2^\delta) + \dot{\mathbf{q}}_2) \times \mathbf{e}_2^\rho = \\ &= \rho_2(-\dot{\alpha}_2 \cos\delta_2 \mathbf{q}_2 \cdot \mathbf{e}_2^\delta - \dot{\delta}_2 \mathbf{q}_2 \cdot \mathbf{e}_2^\alpha) + \mathbf{e}_2^\rho \cdot \mathbf{q}_2 \times \dot{\mathbf{q}}_2 \end{aligned} \tag{12}$$

does not depend on $\dot{\rho}_2$ and, in the difference $|\dot{\mathbf{r}}_1|^2 \mathbf{r}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\dot{\mathbf{r}}_1$, the second degree term in $\dot{\rho}_1$ (i.e. $\dot{\rho}_1^2 \mathbf{r}_1$) cancels out.
Therefore, to solve the linkage problem, we can consider the polynomial system

$$\begin{cases} p(\rho_1, \rho_2) = 0 \\ q(\rho_1, \rho_2) = 0 \end{cases}, \qquad \rho_1, \rho_2 > 0 \tag{13}$$

with total degree 20. This shows the advantage of this method compared with the one in [6], which gives total degree 48.

3.2 Computation of the solutions

To compute the solutions of (13) we define an algorithm similar to the one in [5], [6]. By grouping the monomials with the same power of ρ_2 we write

$$p(\rho_1, \rho_2) = \sum_{j=0}^{8} a_j(\rho_1)\, \rho_2^j \,, \qquad \text{where} \tag{14}$$

$$\deg(a_j) = \begin{cases} 10 & \text{for } j = 0 \\ 10 - (j+1) & \text{for } j = 2k-1 \quad \text{with } k \geq 1 \\ 10 - j & \text{for } j = 2k \quad \text{with } k \geq 1 \end{cases}$$

and

$$q(\rho_1, \rho_2) = b_2\, \rho_2^2 + b_1\, \rho_2 + b_0(\rho_1) \tag{15}$$

for some univariate polynomial coefficients a_j, b_0 and constants b_1, b_2.
We consider the resultant $Res(\rho_1)$ of p, q with respect to ρ_2, which is the determinant of the 10×10 Sylvester matrix

$$\mathtt{S}(\rho_1) = \begin{pmatrix} a_8 & 0 & b_2 & 0 & \ldots\ldots & & 0 \\ a_7 & a_8 & b_1 & b_2 & 0 & \ldots & 0 \\ \vdots & \vdots & b_0 & b_1 & b_2 & \ldots & \vdots \\ \vdots & \vdots & 0 & b_0 & b_1 & \ldots & \vdots \\ a_0 & a_1 & \vdots & \vdots & \vdots & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix} \tag{16}$$

and is generically a degree 20 polynomial. The positive real roots of $Res(\rho_1)$ are the only possible values of ρ_1 for a solution (ρ_1, ρ_2) of (13).

Remark 2 The resultant of p, q with respect to ρ_1 leads to compute the determinant of a 12×12 matrix, thus the elimination of ρ_2 is more convenient. On the other hand, if we project the Laplace-Lenz vectors on $\mathbf{e}_1^\rho \times \mathbf{q}_1$, it is better to eliminate ρ_1.

The coefficients of $Res(\rho_1)$ are computed by an evaluation/interpolation method based on the FFT, and the roots $\rho_1(k)$ of $Res(\rho_1)$ by the algorithm described in [1]. The computation of the preliminary orbits is concluded as follows:

1) solve the equation $q(\rho_1(k), \rho_2) = 0$;
2) discard spurious solutions, i.e. pairs (ρ_1, ρ_2) solving (11) but not (10);
3) compute the values of $\dot\rho_1(k), \dot\rho_2(k)$ by (8);
4) write the corresponding orbital elements. The related epochs, corrected by aberration, are $t_i = \bar t_i - \rho_i(k)/c$, $i = 1, 2$ where c is the velocity of light.

4 Linking radar and optical attributables

Assume we have a radar attributable $\mathcal{A}_{rad} = (\alpha, \delta, \rho, \dot\rho)$ at epoch $\bar t$. We introduce the variables

$$\xi = \rho\dot\alpha\cos\delta\,, \qquad \zeta = \rho\dot\delta\ .$$

The angular momentum as a function of ξ, ζ is

$$\mathbf{c}_{rad}(\xi, \zeta) = \mathbf{A}\xi + \mathbf{B}\zeta + \mathbf{C}\,, \tag{17}$$

where

$$\mathbf{A} = \mathbf{r} \times \mathbf{e}^\alpha\,, \qquad \mathbf{B} = \mathbf{r} \times \mathbf{e}^\delta\,, \qquad \mathbf{C} = \mathbf{r} \times \dot{\mathbf{q}} + \dot\rho\,\mathbf{q} \times \mathbf{e}^\rho\ .$$

The Laplace-Lenz vector is given by (3) with

$$\begin{aligned}
\dot{\mathbf{r}} &= \xi\mathbf{e}^\alpha + \zeta\mathbf{e}^\delta + (\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}})\,,\\
|\dot{\mathbf{r}}|^2 &= \xi^2 + \zeta^2 + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\alpha\xi + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\delta\zeta + |\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}}|^2\,,\\
\dot{\mathbf{r}}\cdot\mathbf{r} &= \mathbf{q}\cdot\mathbf{e}^\alpha\xi + \mathbf{q}\cdot\mathbf{e}^\delta\zeta + (\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}})\cdot\mathbf{r}\ .
\end{aligned}$$

Suppose we have a radar attributable $\mathcal{A}_{rad}$ at time $\bar t_1$ and an optical attributable $\mathcal{A}_{opt}$ at time $\bar t_2$. Equating the angular momentum vectors $\mathbf{c}_{rad}$ and $\mathbf{c}_{opt}$ at the two epochs we obtain a polynomial system of 3 equations in the 4 unknowns $\xi_1, \zeta_1, \rho_2, \dot\rho_2$:

$$\mathbf{A}_1\xi_1 + \mathbf{B}_1\zeta_1 + \mathbf{C}_1 = \mathbf{D}_2\dot\rho_2 + \mathbf{E}_2\rho_2^2 + \mathbf{F}_2\rho_2 + \mathbf{G}_2\ . \tag{18}$$

The system is linear in $\xi_1, \zeta_1, \dot\rho_2$. By solving for these variables we obtain

$$\begin{cases}
\xi_1(\rho_2) = \mathsf{X}_2\rho_2^2 + \mathsf{X}_1\rho_2 + \mathsf{X}_0\\
\zeta_1(\rho_2) = \mathsf{Z}_2\rho_2^2 + \mathsf{Z}_1\rho_2 + \mathsf{Z}_0\\
\dot\rho_2(\rho_2) = \mathsf{R}_2\rho_2^2 + \mathsf{R}_1\rho_2 + \mathsf{R}_0
\end{cases}, \tag{19}$$

where

$$\begin{aligned}
&\mathsf{X}_2 = \gamma\,\mathbf{E}_2\cdot\mathbf{B}_1\times\mathbf{D}_2\,, && \mathsf{X}_1 = \gamma\,\mathbf{F}_2\cdot\mathbf{B}_1\times\mathbf{D}_2\,, && \mathsf{X}_0 = \gamma\,(\mathbf{G}_2 - \mathbf{C}_1)\cdot\mathbf{B}_1\times\mathbf{D}_2\,,\\
&\mathsf{Z}_2 = -\gamma\,\mathbf{E}_2\cdot\mathbf{A}_1\times\mathbf{D}_2\,, && \mathsf{Z}_1 = -\gamma\,\mathbf{F}_2\cdot\mathbf{A}_1\times\mathbf{D}_2\,, && \mathsf{Z}_0 = -\gamma\,(\mathbf{G}_2 - \mathbf{C}_1)\cdot\mathbf{A}_1\times\mathbf{D}_2\,,\\
&\mathsf{R}_2 = -\gamma\,\mathbf{E}_2\cdot\mathbf{A}_1\times\mathbf{B}_1\,, && \mathsf{R}_1 = -\gamma\,\mathbf{F}_2\cdot\mathbf{A}_1\times\mathbf{B}_1\,, && \mathsf{R}_0 = -\gamma\,(\mathbf{G}_2 - \mathbf{C}_1)\cdot\mathbf{A}_1\times\mathbf{B}_1\,,
\end{aligned}$$

and $\gamma = 1/(\mathbf{A}_1\cdot\mathbf{B}_1\times\mathbf{D}_2)$.

Equating the expressions of the Laplace-Lenz vectors at the two epochs, and projecting along $\mathbf{v} = \mathbf{e}_2^\rho \times \mathbf{q}_2$, yields

$$\begin{aligned} &\mu[\mathbf{L}_{rad}(\xi_1,\zeta_1) - \mathbf{L}_{opt}(\rho_2,\dot\rho_2)]\cdot\mathbf{v} = \\ &= \left[\left(|\dot{\mathbf{r}}_1|^2 - \tfrac{\mu}{|\mathbf{r}_1|}\right)\mathbf{r}_1 - (\dot{\mathbf{r}}_1\cdot\mathbf{r}_1)\dot{\mathbf{r}}_1\right]\cdot\mathbf{v} + (\dot{\mathbf{r}}_2\cdot\mathbf{r}_2)(\dot{\mathbf{r}}_2\cdot\mathbf{v}) = 0\ . \end{aligned} \tag{20}$$

The term $\dot{\mathbf{r}}_2\cdot\mathbf{v}$ does not depend on $\dot\rho_2$ and is linear in ρ_2, see (12). Thus, after substituting (19), the terms $(\dot{\mathbf{r}}_2\cdot\mathbf{r}_2)(\dot{\mathbf{r}}_2\cdot\mathbf{v})$ and $\left[|\dot{\mathbf{r}}_1|^2\mathbf{r}_1 - (\dot{\mathbf{r}}_1\cdot\mathbf{r}_1)\dot{\mathbf{r}}_1\right]\cdot\mathbf{v}$ are polynomials of degree 4 in ρ_2. We obtain a univariate polynomial equation with degree 4 in ρ_2, which admits explicit solutions. For each positive root $\rho_2(k)$ we can compute orbital elements at epochs $t_1 = \bar t_1 - \rho_1/c, t_2 = \bar t_2 - \rho_2(k)/c$ using (19).

By replacing equation (20) with the conservation of energy we would obtain a polynomial equation of degree 10. Hence (20) yields a simpler polynomial problem, to the point that the solutions can be explicitly computed.

5 Linking two radar attributables

Given two radar attributables $\mathcal{A}_1, \mathcal{A}_2$ at epochs $\bar t_1, \bar t_2$ for the same observed body, we can write polynomial equations for the variables ξ, ζ at the two times. The energy as a function of ξ, ζ is given by

$$2\mathcal{E}(\xi,\zeta) = \xi^2 + \zeta^2 + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\alpha\xi + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\delta\zeta + |\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}}|^2 - \frac{2\mu}{|\mathbf{r}|}\ .$$

Equating angular momentum and energy at epochs $\bar t_1$ and $\bar t_2$ yields a polynomial system with total degree 2, see [13], [3]. In this case the Laplace-Lenz vector does not allow us to achieve a smaller degree. Moreover, when dealing with the Earth oblateness effect, the use of the energy conservation is preferable (see Section 9).

6 Degenerate cases

In case of two optical attributables the quadratic form (7) is completely degenerate if

$$\mathbf{E}_1\cdot\mathbf{D}_1\times\mathbf{D}_2 = \mathbf{E}_2\cdot\mathbf{D}_1\times\mathbf{D}_2 = 0\ .$$

For a discussion on the geometric meaning of these conditions see [6]. For the radar-optical case system (18) degenerates if

$$\mathbf{A}_1\times\mathbf{B}_1\cdot\mathbf{D}_2 = (\mathbf{r}_1\cdot\mathbf{e}_1^\rho)(\mathbf{r}_1\cdot\mathbf{D}_2) = 0\ .$$

This occurs when $\mathbf{r}_1\cdot\mathbf{e}_1^\rho = 0$, or $\mathbf{r}_1\times\mathbf{r}_2 = \mathbf{0}$, or when $\mathbf{e}_2^\rho$ is in the orbital plane (orthogonal to $\mathbf{r}_1\times\mathbf{r}_2$). Moreover, in both cases the projection of $\mathbf{L}_1, \mathbf{L}_2$ identically vanishes when $\mathbf{v} = 0$. For space debris this corresponds to a zenith observation. The degenerate cases for two radar attributables are discussed in [3].

7 Covariance of the solutions

Let $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2)$ be the vector of two optical attributables and $\Gamma_{\mathbf{A}}$ its covariance matrix. For each solution $\mathbf{Y} = (\rho_1, \dot\rho_1, \rho_2, \dot\rho_2)$ of the linkage problem

$$\begin{cases} \mathbf{c}_1(\rho_1, \dot\rho_1) = \mathbf{c}_2(\rho_2, \dot\rho_2) \\ \mathbf{L}_1(\rho_1, \dot\rho_1) \cdot \mathbf{v} = \mathbf{L}_2(\rho_2, \dot\rho_2) \cdot \mathbf{v} \end{cases} \qquad \rho_1, \rho_2 > 0\,, \tag{21}$$

we can compute the Cartesian coordinates $E^{(1)}_{car}$, $E^{(2)}_{car}$ at epochs t_1, t_2, and their covariance matrices $\Gamma^{(1)}_{car}$, $\Gamma^{(2)}_{car}$. We introduce the following notation:

1) $\mathbf{E}_{car} = (E^{(1)}_{car}, E^{(2)}_{car})$ is the 2 epochs Cartesian coordinates vector;
2) $\mathbf{E}_{att} = (E^{(1)}_{att}, E^{(2)}_{att})$, where

$$E^{(i)}_{att} = (\alpha_i, \delta_i, \dot\alpha_i, \dot\delta_i, \rho_i, \dot\rho_i)\,, \quad i = 1, 2\ .$$

If we use interpolated values for $\mathbf{q}, \dot{\mathbf{q}}$, as suggested in [11], then $E^{(i)}_{att}$ are not the attributable coordinates corresponding to $E^{(i)}_{car}$.

Define the map $\boldsymbol{\Phi} : \mathbb{R}^{12} \to \mathbb{R}^4$ by

$$\mathbf{E}_{att} \overset{\boldsymbol{\Phi}}{\mapsto} \begin{bmatrix} \mathbf{c}_1 - \mathbf{c}_2 \\ \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{w} \end{bmatrix}\ , \qquad \mathbf{w} = \mathbf{r}_2 \times \mathbf{q}_2\ .$$

Moreover, introduce the transformation $\mathcal{T}^{car}_{att} : \mathbf{E}_{att} \to \mathbf{E}_{car}$ by (1), (2) for both epochs, and consider the map $\boldsymbol{\Psi}$ defined by $\boldsymbol{\Phi} = \boldsymbol{\Psi} \circ \mathcal{T}^{car}_{att}$. Note that $\boldsymbol{\Phi} = \mathbf{0}$ is equivalent to (21). We use $\mathbf{w}$ instead of $\mathbf{v}$ to obtain simpler expressions for the derivatives of $\boldsymbol{\Phi}$.
The covariance matrix of the Cartesian coordinates at epoch t_1 is

$$\Gamma^{(1)}_{car} = \frac{\partial E^{(1)}_{car}}{\partial \mathbf{A}} \Gamma_{\mathbf{A}} \left[\frac{\partial E^{(1)}_{car}}{\partial \mathbf{A}} \right]^T\ ,$$

with

$$\frac{\partial E^{(1)}_{car}}{\partial \mathbf{A}} = \frac{\partial E^{(1)}_{car}}{\partial E^{(1)}_{att}} \frac{\partial E^{(1)}_{att}}{\partial \mathbf{A}}\,, \qquad \frac{\partial E^{(1)}_{att}}{\partial \mathbf{A}} = \begin{bmatrix} I_4 \ O_4 \\ \dfrac{\partial(\rho_1, \dot\rho_1)}{\partial \mathbf{A}} \end{bmatrix}\ .$$

From the implicit function theorem

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{A}}(\mathbf{A}) = -\left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{Y}}(\mathbf{E}_{att}) \right]^{-1} \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{A}}(\mathbf{E}_{att})\,,$$

where

$$\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{Y}} = \left(\frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{E}_{car}} \circ \mathcal{T}^{car}_{att} \right) \frac{\partial \mathcal{T}^{car}_{att}}{\partial \mathbf{Y}}\,, \qquad \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{A}} = \left(\frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{E}_{car}} \circ \mathcal{T}^{car}_{att} \right) \frac{\partial \mathcal{T}^{car}_{att}}{\partial \mathbf{A}}\ .$$

The matrices $\partial \mathcal{T}^{car}_{att} / \partial \mathbf{Y}$ and $\partial \mathcal{T}^{car}_{att} / \partial \mathbf{A}$ are respectively made by columns 5,6,11,12 and by columns 1,2,3,4,7,8,9,10 of $\partial \mathbf{E}_{car} / \partial \mathbf{E}_{att}$.
For a given vector $\mathbf{u} \in \mathbb{R}^3$ define the **hat map**

$$\mathbb{R}^3 \ni (u_1, u_2, u_3) = \mathbf{u} \mapsto \widehat{\mathbf{u}} \overset{def}{=} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in so(3)\ .$$

Then we have, using $\hat{\mathbf{u}}^T = -\hat{\mathbf{u}}$,

$$\frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{E}_{car}} = \begin{bmatrix} -\widehat{\dot{\mathbf{r}}_1} & \widehat{\mathbf{r}_1} & \widehat{\dot{\mathbf{r}}_2} & -\widehat{\mathbf{r}_2} \\ \frac{\partial \Delta_{\mathcal{L}}}{\partial \mathbf{r}_1} & \frac{\partial \Delta_{\mathcal{L}}}{\partial \dot{\mathbf{r}}_1} & \frac{\partial \Delta_{\mathcal{L}}}{\partial \mathbf{r}_2} & \frac{\partial \Delta_{\mathcal{L}}}{\partial \dot{\mathbf{r}}_2} \end{bmatrix} ,$$

where $\Delta_{\mathcal{L}} = \mu(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{w}$, and

$$\begin{aligned}
\frac{\partial \Delta_{\mathcal{L}}}{\partial \mathbf{r}_1} &= \left(|\dot{\mathbf{r}}_1|^2 - \mu \frac{1}{|\mathbf{r}_1|} \right) \mathbf{w}^T + \mu \frac{(\mathbf{r}_1 \cdot \mathbf{w})}{|\mathbf{r}_1|^3} \mathbf{r}_1^T - (\dot{\mathbf{r}}_1 \cdot \mathbf{w}) \dot{\mathbf{r}}_1^T , \\
\frac{\partial \Delta_{\mathcal{L}}}{\partial \dot{\mathbf{r}}_1} &= 2(\mathbf{r}_1 \cdot \mathbf{w}) \dot{\mathbf{r}}_1^T - (\dot{\mathbf{r}}_1 \cdot \mathbf{w}) \mathbf{r}_1^T - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1) \mathbf{w}^T , \\
\frac{\partial \Delta_{\mathcal{L}}}{\partial \mathbf{r}_2} &= \left(|\dot{\mathbf{r}}_1|^2 - \frac{\mu}{|\mathbf{r}_1|} \right) [\mathbf{q}_2 \times \mathbf{r}_1]^T - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)[\mathbf{q}_2 \times \dot{\mathbf{r}}_1]^T + (\dot{\mathbf{r}}_2 \cdot \mathbf{w}) \dot{\mathbf{r}}_2^T + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)[\mathbf{q}_2 \times \dot{\mathbf{r}}_2]^T , \\
\frac{\partial \Delta_{\mathcal{L}}}{\partial \dot{\mathbf{r}}_2} &= (\dot{\mathbf{r}}_2 \cdot \mathbf{w}) \mathbf{r}_2^T + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2) \mathbf{w}^T .
\end{aligned}$$

In the case of one radar and one optical attributable the covariance of the solutions can be computed in a similar way, with the following differences: a) the vector of the attributables is $\mathbf{A} = (\mathcal{A}_{rad}, \mathcal{A}_{opt})$; b) the vector of unknowns is $\mathbf{Y} = (\dot{\alpha}_1, \dot{\delta}_1, \rho_2, \dot{\rho}_2)$; c) the matrix of the derivatives of $E_{att}^{(1)}$ with respect to $\mathbf{A}$ is

$$\frac{\partial E_{att}^{(1)}}{\partial \mathbf{A}} = \begin{bmatrix} I_2 \; O_2 \; O_2 \; O_2 \\ \frac{\partial(\dot{\alpha}_1, \dot{\delta}_1)}{\partial \mathbf{A}} \\ O_2 \; I_2 \; O_2 \; O_2 \end{bmatrix} ; \tag{22}$$

d) the matrices $\partial \mathcal{T}_{att}^{car} / \partial \mathbf{Y}$ and $\partial \mathcal{T}_{att}^{car} / \partial \mathbf{A}$ are respectively made by columns 3,4,11,12 and by columns 1,2,5,6,7,8,9,10 of $\partial \mathbf{E}_{car} / \partial \mathbf{E}_{att}$.

In the case of two radar attributables we use the map

$$\mathbf{E}_{att} \overset{\boldsymbol{\Phi}}{\mapsto} \begin{bmatrix} \mathbf{c}_1 - \mathbf{c}_2 \\ \mathcal{E}_1 - \mathcal{E}_2 \end{bmatrix} ,$$

and the derivatives

$$\frac{\partial \Delta_{\mathcal{E}}}{\partial \mathbf{r}_1} = \mu \frac{1}{|\mathbf{r}_1|^3} \mathbf{r}_1^T , \quad \frac{\partial \Delta_{\mathcal{E}}}{\partial \dot{\mathbf{r}}_1} = \dot{\mathbf{r}}_1^T , \quad \frac{\partial \Delta_{\mathcal{E}}}{\partial \mathbf{r}_2} = -\mu \frac{1}{|\mathbf{r}_2|^3} \mathbf{r}_2^T , \quad \frac{\partial \Delta_{\mathcal{E}}}{\partial \dot{\mathbf{r}}_2} = -\dot{\mathbf{r}}_2^T$$

of $\Delta_{\mathcal{E}} = \mathcal{E}_1 - \mathcal{E}_2$. Moreover: a) the vector of the attributables is $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2)$; b) the vector of unknowns is $\mathbf{Y} = (\dot{\alpha}_1, \dot{\delta}_1, \dot{\alpha}_2, \dot{\delta}_2)$; c) the matrix of the derivatives of $E_{att}^{(1)}$ with respect to $\mathbf{A}$ has the same structure as (22); d) the matrices $\partial \mathcal{T}_{att}^{car} / \partial \mathbf{Y}$ and $\partial \mathcal{T}_{att}^{car} / \partial \mathbf{A}$ are respectively made by columns 3,4,9,10 and by columns 1,2,5,6,7,8,11,12 of $\partial \mathbf{E}_{car} / \partial \mathbf{E}_{att}$.

8 Selecting solutions

The solutions of (21) are defined by using only four conservation laws. Thus $E_{car}^{(1)}$, $E_{car}^{(2)}$ may not correspond to the same orbit. We select the solutions of the linkage problem by means of the **attribution** algorithm [8], [7]. In these references only optical

attributables are used, but the algorithm can be easily extended to radar ones. Here we recall briefly the procedure.

Let E_1 be a set of orbital elements for the observed body at time t_1, with 6×6 covariance matrix Γ_1. We can propagate the orbit with covariance to the epoch t_2 of an attributable $\mathcal{A}_2$, with a given 4×4 covariance matrix $\Gamma_{\mathcal{A}_2}$, by the formula

$$\Gamma_p = \frac{\partial \Phi(E_1, \bar{t}_2)}{\partial E_1} \Gamma_1 \left[\frac{\partial \Phi(E_1, \bar{t}_2)}{\partial E_1} \right]^T$$

where $\Phi(E_1, t)$ is the integral flow associated to the dynamical model. Then we can extract a predicted attributable $\mathcal{A}_p$, at time $\bar{t}_2$, with covariance matrix $\Gamma_{\mathcal{A}_p}$. Let $C_{\mathcal{A}_p} = (\Gamma_{\mathcal{A}_p})^{-1}$ and $C_{\mathcal{A}_2} = (\Gamma_{\mathcal{A}_2})^{-1}$. We define

$$C_0 = C_{\mathcal{A}_p} + C_{\mathcal{A}_2} \,, \qquad \Gamma_0 = C_0^{-1} \,.$$

The **identification penalty** is given by

$$\chi_4^2 = (\mathcal{A}_2 - \mathcal{A}_p) \cdot [C_{\mathcal{A}_p} - C_{\mathcal{A}_p} \Gamma_0 C_{\mathcal{A}_p}](\mathcal{A}_2 - \mathcal{A}_p) \,.$$

If the value of χ_4 is within a fixed threshold, we can accept the orbit E_1.

Remark 3 To select solutions we could also use compatibility conditions, as in [6]. When using $(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v} = 0$ the conditions could be

$$(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{e}_2^\rho = 0 \,, \qquad \ell_1 - \ell_2 = n_1(t_1 - t_2) \,. \tag{23}$$

9 Including J_2 oblateness effect

When we study the linkage problem for attributables of satellites of the Earth, the non-spherical shape of our planet plays an important role. In general the problem of the motion of a satellite around an oblate body is not integrable [2]. Including in the model the effect of the J_2 coefficient (i.e. the second zonal spherical harmonic of the gravity field) we obtain an integrable problem by keeping only first order terms [12]. The Keplerian elements a, e, I of the body remain constant, while the pericenter and the node evolve according to

$$\begin{aligned} \dot{\omega} &= \frac{3}{4} J_2 n \left(\frac{R_\oplus}{a} \right)^2 \frac{(4 - 5\sin^2 I)}{(1-e^2)^2} \,, \\ \dot{\Omega} &= -\frac{3}{2} J_2 n \left(\frac{R_\oplus}{a} \right)^2 \frac{\cos I}{(1-e^2)^2} \,. \end{aligned} \tag{24}$$

Thus the lengths of $\mathbf{c}$ and $\mathbf{L}$ remain constant, while their directions change. We write the vector equations

$$\begin{cases} R_c \mathbf{c}_1 = \mathbf{c}_2 \\ R_L \mathbf{L}_1 = \mathbf{L}_2 \end{cases} \tag{25}$$

where

$$R_c = R^{\hat{z}}_{\Delta\Omega} \,, \qquad R_L = R^{\mathbf{c}_2}_{\omega_1 + \Delta\omega} R^{\hat{z}}_{\Delta\Omega} R^{\mathbf{c}_1}_{-\omega_1} \,, \qquad \Delta\omega = \omega_2 - \omega_1 \,, \qquad \Delta\Omega = \Omega_2 - \Omega_1 \,.$$

Given a first guess for the orbital elements, we substitute them in the expressions of R_c, R_L, obtaining equations with the same algebraic structure as the system $\mathbf{c}_1 - \mathbf{c}_2 = \mathbf{0}\,, (\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v} = 0$. Then we start an iterative procedure and take the solutions at convergence, if any.

We have three ways to select a first guess. One is trying to compute orbits without the oblateness effect. The other two ways are to compute a circular orbit from an attributable, either the first or the second one. For optical attributables we refer to [4], while the procedure for radar attributables is discussed in Section A. With circular orbits the elements ω, Ω are not defined, but we can write

$$\Delta\omega = \dot{\omega}(t_2 - t_1), \qquad \Delta\Omega = \dot{\Omega}(t_2 - t_1)\,, \tag{26}$$

with $\dot{\omega}$, $\dot{\Omega}$ given by (24). If we employ the attributable at time $\bar{t}_1$, we set

$$R_L = R^{\mathbf{c}_2}_{\Delta\omega} R^{\hat{z}}_{\Delta\Omega}\,, \quad \text{with } \mathbf{c}_2 = R_c \mathbf{c}_1\,;$$

while for the attributable at time $\bar{t}_2$ we use

$$R_L = R^{\hat{z}}_{\Delta\Omega} R^{\mathbf{c}_1}_{\Delta\omega}\,, \quad \text{with } \mathbf{c}_1 = R_c^T \mathbf{c}_2\ .$$

In the case of two radar attributables equating the energies is preferable, since energy is invariant under the J_2 effect, according to our approximated model.

10 Two test cases

First we test the method explained in Section 3 with asteroid (99942) Apophis. We take two sets of 13 and 12 real observations respectively with mean epochs $\bar{t}_1 = 53175.59$, $\bar{t}_2 = 53357.45$ (time in MJD). After removing duplicate and spurious solutions we obtain

	ρ_1	ρ_2
1	0.78987	0.04345
2	1.13777	0.09569

Table 1 Solutions in AU of the system (13) for (99942).

The two solutions give respectively $\chi_4(1) = 1797.5$, $\chi_4(2) = 1.51$, therefore we select the second one, with Keplerian elements

$$a = 0.9230\,,\ e = 0.189\,,\ I = 3.287\,,\ \Omega = 204.912\,,\ \omega = 124.778\,,\ \ell = 249.003$$

at epoch $t_1 = 53175.59$ (distances in AU, angles in degrees). The related standard deviations are

$$\sigma_a = 4.8{\times}10^{-3}\,,\ \sigma_e = 5.6{\times}10^{-3}\,,\ \sigma_I = 0.112\,,\ \sigma_\Omega = 2.646\,,\ \sigma_\omega = 10.431\,,\ \sigma_\ell = 13.304\,.$$

We can compare the results with the known orbit propagated at epoch t_1:

$$a = 0.9219\,,\ e = 0.191\,,\ I = 3.333\,,\ \Omega = 204.575\,,\ \omega = 126.176\,,\ \ell = 247.500\ .$$

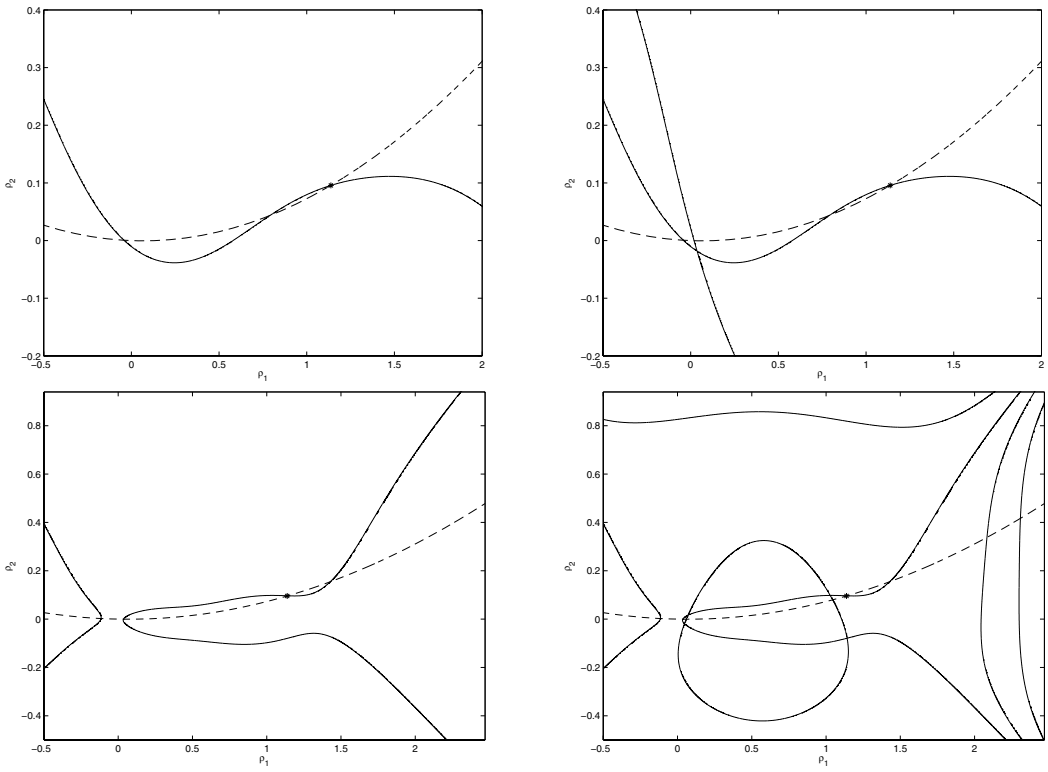

Fig. 1 For the test case of (99942) Apophis, this figure shows the advantage of using equation (9) instead of the conservation of the energy $\mathcal{E}$. Top left: $\mathbf{c}$, $\mathbf{L} \cdot \mathbf{v}$ integrals. Top right: $\mathbf{c}$, $\mathbf{L} \cdot \mathbf{v}$ integrals, polynomial form. Bottom left: $\mathbf{c}$, $\mathcal{E}$ integrals. Bottom right: $\mathbf{c}$, $\mathcal{E}$ integrals, polynomial form. The asterisk corresponds to the known orbit.

Figure 1 shows for this test case the intersections of the curves defined in this paper compared with the ones obtained by the conservation of the energy. In the four pictures the hyphened curve corresponds to equation (6). We also draw the curve defined by (10) on top left, and the one by (11), in polynomial form, on top right. The conservation of the energy defines the curve drawn on bottom left, its polynomial form (obtained by rearranging terms and squaring twice) defines the one on bottom right. The orbit determination method introduced in this paper, searching for the intersections shown on top right, is clearly convenient with respect to the method investigated in [6], related to the figure on bottom right.

Now we show the results of the method explained in Section 4 in the case of space debris, for a Low Earth Orbit (LEO)[2], including the oblateness effect of the Earth as described in Section 9. By propagation of the known orbit we produce a radar attributable at time $\bar{t}_1 = 54468.23$ and an optical attributable at time $\bar{t}_2 = 54468.57$. For the radar attributable we assume the following standard deviations: 10^{-4} rad in the angles, 10 m in range, 10 km/d in range rate. For the optical one we use 1 arcsec in the

[2] generated with the ESA software Master (`http://www.master-2005.net/`).

angles, 1 arcsec/s in the angle rates. All the variables are assumed to be uncorrelated. We add a random error to the attributables according to our error model.
In Table 2 we write the orbits found at time t_1 corresponding to different starting guesses, the orbit obtained at convergence and the known one.

	a	e	I	Ω	ω	ℓ	χ_4
no J_2	7731.784	0.010	2.034	64.783	279.204	58.763	114.41
C-R	7723.628	0.009	2.022	64.907	273.205	64.619	68.77
C-O	7724.015	0.009	2.023	64.901	273.497	64.334	71.04
found	7723.731	0.009	2.022	64.905	273.283	64.543	69.37
known	7724.227	0.010	1.998	65.144	272.143	65.381	/

Table 2 (no J_2): solution without including the oblateness of the Earth. (C-R)/(C-O): solution of (25) using the circular orbit from the radar/optical attributable as first guess. (found): orbit at convergence of the iterative procedure. (known): known orbit at time $t_1 = 54468.23$. The values of the identification penalty are given in the last column.

11 Conclusions

We have introduced a new orbit determination method, based on the first integrals of the Kepler problem. A main difference with respect to the method of [6] is the use of the Laplace-Lenz vector in place of the energy, which allows us to write polynomial equations with much lower degree. The method is extended to include either radar or optical observations, and the oblateness effect of the Earth. Two numerical tests have been presented, one with asteroid (99942) Apophis, the other with a piece of space debris on LEO orbit. We plan to perform large scale tests in the next future to validate our algorithms.

12 Acknowledgments

We wish to thank A. Milani for his useful suggestions during the development of this work.

A Appendix. Circular orbits from a single attributable

As shown in [4], for an object orbiting around the Earth it is always possible to compute a circular orbit starting from an optical attributable $\mathcal{A} = (\alpha, \delta, \dot\alpha, \dot\delta)$. Moreover, the circular solutions are at most 3. Here we search for circular solutions in the case of a radar attributable $\mathcal{A}_{rad} = (\alpha, \delta, \rho, \dot\rho)$.

An orbit is circular when the Laplace-Lenz vector is zero, $\mathbf{L} = \mathbf{0}$. This is equivalent to the equations

$$\mathbf{r} \cdot \dot{\mathbf{r}} = 0 \,, \qquad |\dot{\mathbf{r}}|^2 = \mu/|\mathbf{r}| \ . \tag{27}$$

In the variables ξ, ζ introduced in Section 4 relations (27) are the equations of a straight line and a circumference with center $C = -(\mathbf{e}^\alpha \cdot \dot{\mathbf{q}},\, \mathbf{e}^\delta \cdot \dot{\mathbf{q}})$ and radius

$$R = \sqrt{\mu/|\mathbf{r}| - |\dot{\mathbf{q}} + \dot\rho \mathbf{e}^\rho|^2 + (\dot{\mathbf{q}} \cdot \mathbf{e}^\alpha)^2 + (\dot{\mathbf{q}} \cdot \mathbf{e}^\delta)^2} = \sqrt{\mu/|\mathbf{r}| - (\dot{\mathbf{r}} \cdot \mathbf{e}^\rho)^2} \ .$$

The first equation degenerates when $\mathbf{r} \times \mathbf{e}^\rho = \mathbf{q} \times \mathbf{e}^\rho = 0$, which corresponds to a zenith observation. In this case if $\dot{\rho} \neq 0$ there are no circular solutions. Otherwise, if $\dot{\rho} = 0$ all the points of the circumference are acceptable solutions. If $\mathbf{q} \times \mathbf{e}^\rho \neq 0$ the condition for the existence of a solution is

$$(\dot{\mathbf{q}} \cdot \mathbf{e}^\rho + \dot{\rho})^2 \leq \frac{\mu}{|\mathbf{r}|^3} \left[(\mathbf{r} \cdot \mathbf{e}^\alpha)^2 + (\mathbf{r} \cdot \mathbf{e}^\delta)^2 \right]$$

which means that the circumference intersects the straight line in two points (counted with multiplicity), corresponding to the two circular solutions.

References

1. Bini, D. A.: 1997, *Numerical computation of polynomial zeros by means of Aberth method*, Numer. Algorithms **13**, no. 3-4, 179–200.
2. Celletti, A., Negrini, P.: 1995, *Non-integrability of the problem of motion around an oblate planet*, CMDA **61**, 253–260
3. Farnocchia, D., Tommei, G., Milani, A., Rossi, A.: 2010, *Innovative methods of correlation and orbit determination for space debris*, CMDA **107/1-2**, 169–185
4. Fujimoto, K., Maruskin, J. M., Scheeres, D. J.: 2010, *Circular and zero-inclination solutions for optical observations of Earth-orbiting objects*, CMDA **106**, 157–182
5. Gronchi, G. F.: 2002, *On the stationary points of the squared distance function between two ellipses with a common focus*, SIAM Journ. Sci. Comp. **24/1**, 61–80
6. Gronchi, G. F., Dimare, L. and Milani, A.: 2010, *Orbit determination with the two-body integrals*, CMDA **107/3**, 299–318
7. Milani, A., Gronchi, G. F.: 2009, *Theory of Orbit Determination*, Cambridge University Press
8. Milani, A., Sansaturio, M. E., Chesley, S. R.: 2001, *The Asteroid Identification Problem IV: Attributions*, Icarus **151**, 150–159.
9. Milani, A., Gronchi, G. F., Farnocchia, D., Knežević, Z., Jedicke, R., Denneau, L. and Pierfederici, F.: 2008, *Topocentric Orbit Determination: algorithms for the next generation surveys*, Icarus **195**, 474–492.
10. Milani, A., Tommei, G., Farnocchia, D., Rossi, A., Schildknecht, T., Jehn, R.: 2011, *Orbit determination of space objects based on sparse optical data*, arXiv:1012.5232
11. Poincaré, H.: 1906, *Sur la détermination des orbites par la méthode de Laplace*, Bulletin astronomique **23**, 161–187.
12. Roy, A. E.: Orbital Motion. Institute of Physics Publishing (2005).
13. Taff, L. G., Hall, D. L.: 1977, *The use of angles and angular rates. I - Initial orbit determination*, CMDA **16**, 481–488
14. Tommei, G., Milani, A. and Rossi, A.: 2007, *Orbit Determination of Space Debris: Admissible Regions*, CMDA **97/4**, 289–304