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► **To cite this version:**

Philippe Angot, Pierre Fabrie. Convergence results for the vector penalty-projection and two-step artificial compressibility methods. *Discrete and Continuous Dynamical Systems - Series B*, 2012, 17 (5), pp. 1383–1405, communicated by Roger Temam. 10.3934/dcdsb.2012.17.1383 . hal-00653113v2

HAL Id: hal-00653113

<https://hal.science/hal-00653113v2>

Submitted on 19 Feb 2012

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CONVERGENCE RESULTS FOR THE VECTOR PENALTY-PROJECTION AND TWO-STEP ARTIFICIAL COMPRESSIBILITY METHODS

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(Communicated by Roger Temam)

ABSTRACT. In this paper, we propose and analyze a new artificial compressibility splitting method which is issued from the recent vector penalty-projection method for the numerical solution of unsteady incompressible viscous flows introduced in [1], [2] and [3]. This method may be viewed as an hybrid two-step prediction-correction method combining an artificial compressibility method and an augmented Lagrangian method without inner iteration. The perturbed system can be viewed as a new approximation to the incompressible Navier-Stokes equations. In the main result, we establish the convergence of solutions to the weak solutions of the Navier-Stokes equations when the penalty parameter tends to zero.

1. Introduction and setting of the problem. The artificial compressibility method was introduced by Chorin [6] and Temam [17] for the solution of the unsteady incompressible Stokes or Navier-Stokes equations; see also [20] for the theoretical analysis. Then, some other numerical schemes to efficiently compute the solutions of Navier-Stokes problems can be viewed as discretizations of perturbed systems of the type of penalization [14] or pseudo-compressibility. This is the case with the famous projection methods from Chorin [7] and Temam [18, 19] and their variants [10], see e.g. [15].

Here, we present a new approximation method for the Navier-Stokes equations modeling incompressible viscous flows in a bounded regular open set Ω endowed with Dirichlet boundary conditions on $\Gamma = \partial\Omega$ (Lipschitz-continuous). With a given source term f , the Navier-Stokes system reads:

2000 *Mathematics Subject Classification.* Primary: 35Q30, 76D05, 76N10, 35A35; Secondary: 65M12, 65N12.

Key words and phrases. Artificial compressibility, Navier-Stokes equations, vector penalty-projection, pseudo-compressibility, penalty method.

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{\mathcal{R}_e} \Delta v + \nabla p = f, \\ \operatorname{div} v = 0, \\ v(0) = v_0, v|_{\Gamma} = 0, \end{cases}$$

where \mathcal{R}_e denotes the Reynolds number.

According to the identity, $-\Delta \varphi = \operatorname{curl} \operatorname{curl} \varphi - \nabla \operatorname{div} \varphi$, we consider the following approximate method to obtain a solution of the above Navier-Stokes system, with the parameters $r \geq 0, \gamma > 0$ and, $\varepsilon > 0$

$$\begin{cases} \frac{\partial \tilde{v}_\varepsilon}{\partial t} + (v_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \tilde{v}_\varepsilon + \frac{1}{\mathcal{R}_e} \operatorname{curl} \operatorname{curl} \tilde{v}_\varepsilon - \frac{1}{\mathcal{R}_e} \nabla \operatorname{div} \tilde{v}_\varepsilon \\ \quad - r \nabla \operatorname{div} \tilde{v}_\varepsilon + \nabla p_\varepsilon = f \\ \frac{\partial \hat{v}_\varepsilon}{\partial t} + (v_\varepsilon \cdot \nabla) \hat{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \hat{v}_\varepsilon + \frac{1}{\mathcal{R}_e} \operatorname{curl} \operatorname{curl} \hat{v}_\varepsilon - \frac{1}{\mathcal{R}_e} \nabla \operatorname{div} \hat{v}_\varepsilon \\ \quad - r \nabla \operatorname{div} \hat{v}_\varepsilon - \frac{1}{\varepsilon \mathcal{R}_e} \nabla (\operatorname{div} \hat{v}_\varepsilon + \operatorname{div} \tilde{v}_\varepsilon) = 0 \\ v_\varepsilon = \tilde{v}_\varepsilon + \hat{v}_\varepsilon \\ \gamma \frac{\partial p_\varepsilon}{\partial t} + \gamma p_\varepsilon + \frac{1}{\varepsilon} \operatorname{div} v_\varepsilon + r \operatorname{div} \tilde{v}_\varepsilon = 0. \end{cases}$$

We associate to the previous system the following boundary conditions and initial data,

$$\begin{aligned} \tilde{v}_\varepsilon(0) &= v_0, \quad \hat{v}_\varepsilon(0) = 0, \quad p_\varepsilon(0) = p_0, \\ \tilde{v}_\varepsilon \cdot \nu|_{\Gamma} &= 0, \quad \hat{v}_\varepsilon \cdot \nu|_{\Gamma} = 0, \\ (\operatorname{curl} \tilde{v}_\varepsilon) \wedge \nu|_{\Gamma} &= 0, \quad (\operatorname{curl} \hat{v}_\varepsilon) \wedge \nu|_{\Gamma} = 0, \end{aligned}$$

where ν denotes the outward unit normal vector on Γ .

To vanish, at the limit process, the two tangential component of the velocity fields, $\tilde{v}_\varepsilon \wedge \nu$ and $\hat{v}_\varepsilon \wedge \nu$, we use a penalization method which will be detailed below.

This method is close to the artificial compressibility method of Chorin [6] and Temam [17], but presents one important difference. It is a two-step splitting method. The first equation of the previous system gives a predicted velocity \tilde{v}_ε and the second one is the approximate projection of \tilde{v}_ε on the free-divergence vector fields. This equation may be seen as an approximate method to solve the well-posed problem (see appendix A) :

$$\begin{aligned} \operatorname{div} \hat{v}_\varepsilon &= -\operatorname{div} \tilde{v}_\varepsilon, \\ \operatorname{curl} \hat{v}_\varepsilon &= 0, \\ \hat{v}_\varepsilon \cdot \nu|_{\Gamma} &= 0. \end{aligned}$$

Remark 1. This approximate method is issued from the Vector Penalty-Projection (VPP $_{r,\varepsilon}$) methods for the numerical solution of unsteady incompressible viscous flows introduced in [1] and [3]. A fast version of these methods, the so-called (VPP $_\varepsilon$) method, is recently proposed also for the numerical solution of the non-homogeneous Navier-Stokes equations in [2, 4]. It is shown to be very efficient to compute multiphase flows, *i.e.* fast, cheap, and robust whatever the density, viscosity or permeability jumps.

Even for $r = 0$, the resulting method which corresponds to a two-step pseudo-compressibility method, is different from the original artificial compressibility method of Chorin [6] and Temam [17, 20].

The new important point is the penalty term $\frac{1}{\varepsilon} (\nabla \operatorname{div} \widehat{v}_\varepsilon + \nabla \operatorname{div} \widetilde{v}_\varepsilon)$ that appears in the velocity correction step which allows us a direct estimate on the divergence of the velocity. Moreover, this system is quite easy to solve and presents good stability properties, see [1, 2, 3]. The velocity v_ε and the pressure p_ε satisfy the equations:

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial t} + ((v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon) + \frac{1}{\mathcal{R}_\varepsilon} \operatorname{curl} \operatorname{curl} v_\varepsilon - \frac{1}{\mathcal{R}_\varepsilon} \nabla \operatorname{div} v_\varepsilon \\ - r \nabla \operatorname{div} v_\varepsilon - \frac{1}{\varepsilon \mathcal{R}_\varepsilon} \nabla \operatorname{div} v_\varepsilon + \nabla p_\varepsilon = f \\ \gamma \frac{\partial p_\varepsilon}{\partial t} + \gamma p_\varepsilon + \frac{1}{\varepsilon} \operatorname{div} v_\varepsilon + r \operatorname{div} \widetilde{v}_\varepsilon = 0, \\ v_\varepsilon(0) = v_0, p_\varepsilon(0) = p_0, \\ v_\varepsilon \cdot \nu|_\Gamma = 0, (\operatorname{curl} v_\varepsilon) \wedge \nu|_\Gamma = 0. \end{cases}$$

The vanishing of the tangential component at the limit process, is fulfilled by a penalization method, which implies that this boundary condition is satisfied at the order ε for the approximate solution.

1.1. Notations. Let Ω be a regular bounded and connected open set of \mathbb{R}^d , for $d = 2$ or 3 . We note $H^s(\Omega)$ the classical Sobolev space, and $\|\cdot\|_{H^s}$ the associated norm. The norm of a function in $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p}$, and if B is a Banach space, we denote by $\|\cdot\|_{L^p, B}$ the norm in $L^p(]0, T[; B)$.

$$\begin{aligned} \mathbf{L}^p(\Omega) &= (L^p(\Omega))^d \\ \mathbf{H}_{\operatorname{div}}(\Omega) &= \{v \in (L^2(\Omega))^d, \operatorname{div} v \in L^2(\Omega)\} \\ \mathbf{H} &= \{v \in (L^2(\Omega))^d, \operatorname{div} v = 0, v \cdot \nu|_\Gamma = 0\} \\ \mathbf{H}_\nu^1(\Omega) &= \{v \in (H^1(\Omega))^d, v \cdot \nu|_\Gamma = 0\} \\ \mathbf{G} &= \{v \in (L^2(\Omega))^d, \exists q \in H^1(\Omega), v = \nabla q\} \end{aligned}$$

1.2. Mathematical recalls.

Proposition 1. *Under the previous hypothesis, one has the following properties:*

$$\begin{aligned} \mathbf{L}^2(\Omega) &= \mathbf{H} \oplus \mathbf{G}, \\ \operatorname{Ker}(\operatorname{curl}) &= \mathbf{G}. \end{aligned}$$

Moreover, there exists one constant $C > 0$ depending only on Ω such that:

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C (\|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2), \quad \forall u \in \mathbf{H}_\nu^1(\Omega). \quad (1)$$

Besides, if we suppose that the open set Ω is simply-connected, there exist two constants λ_0 and λ_1 depending only on Ω such that:

$$\begin{aligned} \|u\|_{L^2}^2 &\leq \lambda_0 (\|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2), \quad \forall u \in \mathbf{H}_\nu^1(\Omega), \\ \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &\leq \lambda_1 (\|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2), \quad \forall u \in \mathbf{H}_\nu^1(\Omega) \end{aligned}$$

and we have:

$$\operatorname{Ker}(\operatorname{curl}) \cap \mathbf{H} = \{0\}.$$

Proof. All these results may be found in [9] and [8]. □

For a Banach space E we introduce the Nikolskii space defined for $1 \leq q < +\infty$, $0 < \sigma < 1$:

$$N_q^\sigma([0, T[; E) = \left\{ f \in L^q([0, T[; E), \sup_{0 < h < T} \frac{\|f(\cdot + h) - f(\cdot)\|_{L^q([0, T-h[; E)}}{h^\sigma} < +\infty \right\},$$

endowed with the following norm:

$$\|f\|_{N_q^\sigma} = \left(\|f\|_{L^q([0, T[; E)}^q + \sup_{0 < h < T} \left(\frac{1}{h^\sigma} \|f(\cdot + h) - f(\cdot)\|_{L^q([0, T-h[; E)} \right)^q \right)^{\frac{1}{q}}.$$

Let us recall the following property see for example [5] page 105,

Proposition 2. *Let H be an Hilbert space and f a function given in $L^2([0, T[; H)$ such that, for some $0 < \sigma < 1$,*

$$\int_{\mathbb{R}} |\tau|^{2\sigma} \|\mathcal{F}(\tilde{f})(\tau)\|_H^2 d\tau \leq C^2,$$

where \tilde{f} denotes the extension by 0 of the function f outside $[0, T[$. Then $f \in N_2^\sigma([0, T[; H)$ and we have

$$\|f\|_{N_2^\sigma} \leq M_\sigma(1 + C),$$

where M_σ is a constant depending only on σ .

We now recall the important compactness theorem, see for example [16]

Theorem 1.1. Aubin-Lions-Simon

Let B_0, B_1, B_2 three Banach spaces with $B_0 \subset B_1 \subset B_2$ with continuous imbedding.

Suppose moreover that the injection of B_0 in B_1 is compact.

Then, for all $1 \leq q \leq +\infty$ and $0 < \sigma < 1$, the imbedding

$$L^q([0, T[; B_0) \cap N_q^\sigma([0, T[; B_2) \hookrightarrow L^q([0, T[; B_1)$$

is compact.

2. Main result. We associate to the previous approximate system, the variational problem where the tangential components of the velocities \tilde{v}_ε and \hat{v}_ε are penalized. This problem is studied in the next section.

Find $(\tilde{v}_\varepsilon, \widehat{v}_\varepsilon, p_\varepsilon)$ in $\left(L^\infty(]0, T[; \mathbf{L}^2(\Omega)) \cap L^2(]0, T[; \mathbf{H}_\nu^1(\Omega))\right)^2 \times L^\infty(]0, T[; L_0^2(\Omega))$ satisfying in $\mathcal{D}'(]0, T[)$,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{\partial \tilde{v}_\varepsilon}{\partial t} \cdot \varphi \, d\omega + \int_{\Omega} \left((v_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \tilde{v}_\varepsilon \right) \cdot \varphi \, d\omega \\ \quad + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{curl} \tilde{v}_\varepsilon \cdot \operatorname{curl} \varphi \, d\omega + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{div} \tilde{v}_\varepsilon \operatorname{div} \varphi \, d\omega \\ \quad + r \int_{\Omega} \operatorname{div} \tilde{v}_\varepsilon \operatorname{div} \varphi \, d\omega - \int_{\Omega} p_\varepsilon \operatorname{div} \varphi \, d\omega \\ \quad + \frac{1}{\varepsilon} \int_{\Gamma} (\tilde{v}_\varepsilon \wedge \nu) \cdot (\varphi \wedge \nu) \, d\sigma = \int_{\Omega} f \cdot \varphi \, dx, \\ \int_{\Omega} \frac{\partial \widehat{v}_\varepsilon}{\partial t} \cdot \psi \, d\omega + \int_{\Omega} \left((v_\varepsilon \cdot \nabla) \widehat{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \widehat{v}_\varepsilon \right) \cdot \psi \, d\omega \\ \quad + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{curl} \widehat{v}_\varepsilon \cdot \operatorname{curl} \psi \, d\omega + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{div} \widehat{v}_\varepsilon \operatorname{div} \psi \, d\omega \\ \quad + r \int_{\Omega} \operatorname{div} \widehat{v}_\varepsilon \operatorname{div} \psi \, d\omega + \frac{1}{\varepsilon \mathcal{R}_e} \int_{\Omega} \left(\operatorname{div} \tilde{v}_\varepsilon + \operatorname{div} \widehat{v}_\varepsilon \right) \operatorname{div} \psi \, d\omega \\ \quad + \frac{1}{\varepsilon} \int_{\Gamma} (\widehat{v}_\varepsilon \wedge \nu) \cdot (\psi \wedge \nu) \, d\sigma = 0, \\ \\ v_\varepsilon = \tilde{v}_\varepsilon + \widehat{v}_\varepsilon, \\ \\ \gamma \int_{\Omega} \frac{\partial p_\varepsilon}{\partial t} \pi \, d\omega + \gamma \int_{\Omega} p_\varepsilon \pi \, d\omega + \frac{1}{\varepsilon} \int_{\Omega} \pi \operatorname{div} v_\varepsilon \, d\omega + r \int_{\Omega} \pi \operatorname{div} \tilde{v}_\varepsilon \, d\omega = 0, \\ \\ \forall (\varphi, \psi, \pi) \in (\mathbf{H}_\nu^1(\Omega))^2 \times L_0^2(\Omega), \\ \\ \tilde{v}_\varepsilon(0) = v_0, \widehat{v}_\varepsilon(0) = 0, p_\varepsilon(0) = p_0. \end{array} \right. \quad (2)$$

Then the velocity v_ε and the pressure p_ε satisfy in $\mathcal{D}'(]0, T[)$,

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{\partial v_\varepsilon}{\partial t} \cdot \varphi \, d\omega + \int_{\Omega} \left((v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon \right) \cdot \varphi \, d\omega \\ \quad + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{curl} v_\varepsilon \cdot \operatorname{curl} \varphi \, d\omega + \frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{div} v_\varepsilon \operatorname{div} \varphi \, d\omega \\ \quad + r \int_{\Omega} \operatorname{div} v_\varepsilon \operatorname{div} \varphi \, d\omega + \frac{1}{\varepsilon \mathcal{R}_e} \int_{\Omega} \operatorname{div} v_\varepsilon \operatorname{div} \varphi \, d\omega \\ \quad - \int_{\Omega} p_\varepsilon \operatorname{div} \varphi \, d\omega + \frac{1}{\varepsilon} \int_{\Gamma} (v_\varepsilon \wedge \nu) \cdot (\varphi \wedge \nu) \, d\sigma \\ \quad = \int_{\Omega} f \cdot \varphi \, d\omega, \forall \varphi \in \mathbf{H}_\nu^1(\Omega), \\ \\ v_\varepsilon(0) = v_0 \end{array} \right. \quad (3)$$

Remark 2. In order to establish the strong convergence of the sequence $(v_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \rightarrow 0$, we use in Section 4 the Leray's orthogonal decomposition in the

bounded domain. The curl-free component vanishes with the penalty term introduced by our method, whereas the divergence-free component strongly converges thanks to an estimate of a fractional derivative in time, see [20]. However, this requires to consider velocity fields having only their normal component which is zero on the boundary. Since at the limit process, we aim at solving the Navier-Stokes problem with homogeneous Dirichlet boundary condition, we also penalize the tangential part of the velocity fields.

We prove in section 3 the following results.

Lemma 2.1. *Let us suppose that f belongs to $L^2(]0, T[; \mathbf{L}^2(\Omega))$. Then, there exists at least a solution to the system (2). This solution is unique in two space dimension. For the dimension $d \leq 3$, this solution satisfies the following energy inequality:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (r \varepsilon \|\tilde{v}_\varepsilon\|_{L^2}^2 + \varepsilon \|\widehat{v}_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2) \\ & + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2 + \frac{r \varepsilon}{2\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{2\mathcal{R}_e} \|\operatorname{curl} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{2\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 \\ & + \frac{1}{2\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \frac{r \varepsilon}{2\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{2\mathcal{R}_e} \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 \\ & + \varepsilon r^2 \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \varepsilon r \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{2\varepsilon\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 \\ & + r |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + |(\widehat{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\ & \leq \frac{\lambda \mathcal{R}_e}{2} (1 + r \varepsilon) \|f\|_{L^2}^2 \end{aligned}$$

For the dimension $d = 2$, one has the following energy equality:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (r \varepsilon \|\tilde{v}_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2) + \frac{r \varepsilon}{\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2 \\ & + \frac{r \varepsilon}{\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \varepsilon r^2 \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2\Gamma}^2 \\ & + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 = r \varepsilon \int_\Omega f \tilde{v}_\varepsilon \, d\omega + \int_\Omega f v_\varepsilon \, d\omega. \end{aligned}$$

This result is quite classical and we only give the sketch of proof in the section 3.

In fact, we can precise the previous energy inequality if we suppose that the data f belongs to $L^\infty(\mathbb{R}_+; \mathbf{L}^2(\Omega))$. This shows the absolute stability of the approximate method.

Theorem 2.2. *Suppose that the data f satisfies*

$$f \in L^\infty(\mathbb{R}_+, \mathbf{L}^2(\Omega)),$$

then, there exists a constant α independent of the data, such that

$$\begin{aligned} & (r \varepsilon \|\tilde{v}_\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\widehat{v}_\varepsilon(t)\|_{L^2}^2 + \|v_\varepsilon(t)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(t)\|_{L^2}^2) \\ & \leq e^{-\alpha t} ((1 + r \varepsilon) \|v_0\|_{L^2}^2 + \gamma \varepsilon \|p_0\|_{L^2}^2) + \frac{\lambda \mathcal{R}_e}{\alpha} (1 + r \varepsilon) \|f\|_{L^\infty, L^2}^2, \quad \forall t \in \mathbb{R}_+. \end{aligned}$$

The main goal of this paper is to prove the following convergence theorem:

Theorem 2.3. *For $d \leq 3$, there exists a subsequence $(v_{\varepsilon_k}, p_{\varepsilon_k})_k$ solution of (3) that converges to a solution (v, p) to the system of Navier-Stokes equations with homogeneous Dirichlet boundary conditions.*

For $d = 2$, the solution (v, p) is unique, and for all sequences ε_k , $(v_{\varepsilon_k}, p_{\varepsilon_k})_{\varepsilon_k}$ converges to (v, p) . Moreover, for all sequences ε_k , $(v_{\varepsilon_k})_k$ converges strongly to v in $L^2(0, T; \mathbf{H}_v^1(\Omega))$.

We now give an interpretation of the pressure and precise its convergence. Let us define

$$q_\varepsilon = p_\varepsilon - \left(\frac{1 + \varepsilon}{\varepsilon \mathcal{R}_e} + r \right) \operatorname{div} v_\varepsilon.$$

The scalar function q_ε appears to be the effective approximate pressure, and we have

Theorem 2.4. *The function ∇q_{ε_k} satisfies*

- *if $d = 3$, ∇q_{ε_k} converges weakly to ∇p in $(H^{-1}((0, T) \times \Omega))^3$*
- *if $d = 2$, ∇q_{ε_k} converges strongly to ∇p in $(H^{-1}((0, T) \times \Omega))^2$*

These convergence results for both velocity and pressure are proved in Section 4.

3. Energy estimates. We first establish the following existence result.

Proposition 3. *For v_{ε_0}, p_0 given in $\mathbf{L}^2(\Omega) \times L_0^2(\Omega)$, there exists at least a solution of the system (2) satisfying for $d = 3$:*

$$\begin{aligned} \tilde{v}_\varepsilon &\in L^\infty(]0, T[; \mathbf{L}^2(\Omega) \cap L^2(]0, T[; \mathbf{H}_v^1(\Omega))), \quad \frac{\partial \tilde{v}_\varepsilon}{\partial t} \in L^{\frac{4}{3}}(]0, T[; (\mathbf{H}_v^1(\Omega))') \\ \widehat{v}_\varepsilon &\in L^\infty(]0, T[; \mathbf{L}^2(\Omega) \cap L^2(]0, T[; \mathbf{H}_v^1(\Omega))), \quad \frac{\partial \widehat{v}_\varepsilon}{\partial t} \in L^{\frac{4}{3}}(]0, T[; (\mathbf{H}_v^1(\Omega))') \\ p_\varepsilon &\in L^\infty(]0, T[; L_0^2(\Omega)), \quad \frac{\partial p_\varepsilon}{\partial t} \in L^2(]0, T[; L_0^2(\Omega)) \\ \tilde{v}_\varepsilon(0) &= v_{\varepsilon_0} \text{ in } (\mathbf{H}_v^1(\Omega))', \quad \widehat{v}_\varepsilon(0) = 0 \text{ in } (\mathbf{H}_v^1(\Omega))', \quad p_\varepsilon(0) = p_0 \text{ in } L_0^2(\Omega). \end{aligned}$$

If $d = 2$, the unique solution of (2) satisfies the following regularity results:

$$\begin{aligned} \tilde{v}_\varepsilon &\in L^\infty(]0, T[; L^2(\Omega) \cap L^2(]0, T[; \mathbf{H}_v^1(\Omega))), \\ \frac{\partial \tilde{v}_\varepsilon}{\partial t} &\in L^2(]0, T[; (\mathbf{H}_v^1(\Omega))') + L^{\frac{4}{3}}(]0, T[; \mathbf{L}^{\frac{4}{3}}(\Omega)) \\ \widehat{v}_\varepsilon &\in L^\infty(]0, T[; L^2(\Omega) \cap L^2(]0, T[; \mathbf{H}_v^1(\Omega))), \\ \frac{\partial \widehat{v}_\varepsilon}{\partial t} &\in L^2(]0, T[; (\mathbf{H}_v^1(\Omega))') + L^{\frac{4}{3}}(]0, T[; \mathbf{L}^{\frac{4}{3}}(\Omega)) \\ p_\varepsilon &\in L^\infty(]0, T[; L_0^2(\Omega)), \quad \frac{\partial p_\varepsilon}{\partial t} \in L^2(]0, T[; L_0^2(\Omega)) \\ \tilde{v}_\varepsilon(0) &= v_{\varepsilon_0} \text{ in } \mathbf{L}^2(\Omega), \quad \widehat{v}_\varepsilon(0) = 0 \text{ in } \mathbf{L}^2(\Omega), \quad p_\varepsilon(0) = p_0 \text{ in } L_0^2(\Omega). \end{aligned}$$

Remark 3. In the three-dimensional case, the equalities

$$\tilde{v}_\varepsilon(0) = v_{\varepsilon_0} \text{ in } (\mathbf{H}_v^1(\Omega))', \quad \widehat{v}_\varepsilon(0) = 0 \text{ in } (\mathbf{H}_v^1(\Omega))',$$

are valid in the trace sense.

Proof. For fixed parameters $\varepsilon > 0$, $r \geq 0$ and $\gamma > 0$, we build approximate solutions by a classical Galerkin process.

Let us introduce the self-adjoint operator $A = \text{curl curl} - \nabla \text{div}$ defined on the domain $\mathbf{H}_\nu^1(\Omega) \cap (H^2(\Omega))^d$. Then, for the approximation of the two fields of velocity, we use as special basis the eigenfunctions of this operator associated with the following boundary conditions:

$$\begin{aligned} u \cdot \nu|_\Gamma &= 0, \\ (\text{curl } u) \wedge \nu|_\Gamma &= 0. \end{aligned}$$

For the pressure, one can use as special basis the eigenfunctions of the self-adjoint operator $A = -\Delta$ with domain $H^2(\Omega)$ associated to the Neumann boundary conditions.

This approximate finite dimensional system is then a classical ordinary differential equation which has a unique solution. Next, to perform the limit we use the same strategy as for the classical Navier-Stokes equations *i.e.* *a priori* estimates and compactness results using an estimate on the temporal derivative, see for example [13], [20],[5].

Now, we will focus our attention on the estimates on the time derivative according to the dimension d . Let us begin with the three-dimensional case. We have to estimate the two nonlinear terms $v_\varepsilon \cdot \nabla w$, $w \text{div } v_\varepsilon$, with either $w = \tilde{v}_\varepsilon$ or $w = \hat{v}_\varepsilon$ and the pressure p_ε .

Suppose first that $d = 3$. By Sobolev embedding, the two nonlinear terms of the form $v_\varepsilon \cdot \nabla w$ and $w \text{div } v_\varepsilon$ belong to $L^{\frac{4}{3}}(0, T; \mathbf{H}_\nu^1(\Omega)')$ since we have, for example for all $\varphi \in \mathbf{H}_\nu^1(\Omega)$:

$$\begin{aligned} \left| \int_\Omega (v_\varepsilon \cdot \nabla) w \cdot \varphi \, d\omega \right| + \frac{1}{2} \left| \int_\Omega (\text{div } v_\varepsilon) w \cdot \varphi \, d\omega \right| \\ \leq C (\|v\|_{L^3} \|w\|_{H_\nu^1} \|\varphi\|_{L^6} + \|w\|_{L^3} \|v_\varepsilon\|_{H_\nu^1} \|\varphi\|_{L^6}) \\ \leq C \left(\|v_\varepsilon\|_{L^2}^{\frac{1}{2}} \|v_\varepsilon\|_{H_\nu^1}^{\frac{1}{2}} \|w\|_{H_\nu^1} + \|w\|_{L^2}^{\frac{1}{2}} \|w\|_{H_\nu^1}^{\frac{1}{2}} \|v_\varepsilon\|_{H_\nu^1} \right) \|\varphi\|_{H_\nu^1}. \end{aligned}$$

The bounds of the linear terms are straightforward and we have

$$\begin{aligned} \left| \int_\Omega \text{curl } \tilde{v}_\varepsilon \cdot \text{curl } \varphi \, d\omega \right| + \left| \int_\Omega \text{div } \tilde{v}_\varepsilon \text{div } \varphi \, d\omega \right| + r \left| \int_\Omega \text{div } \tilde{v}_\varepsilon \text{div } \varphi \, d\omega \right| \\ \leq C \|\tilde{v}_\varepsilon\|_{H_\nu^1} \|\varphi\|_{H_\nu^1} \\ \left| \int_\Omega p_\varepsilon \text{div } \varphi \, d\omega \right| \leq C \|p_\varepsilon\|_{L^2} \|\varphi\|_{H_\nu^1} \\ \left| \int_\Omega (\text{div } \tilde{v}_\varepsilon + \text{div } \hat{v}_\varepsilon) \text{div } \varphi \, d\omega \right| \leq C (\|\tilde{v}_\varepsilon\|_{H_\nu^1} + \|\hat{v}_\varepsilon\|_{H_\nu^1}) \|\varphi\|_{H_\nu^1}. \end{aligned}$$

and, with standard trace theorems

$$\left| \int_\Gamma (v_\varepsilon \wedge \nu) \cdot (\varphi \wedge \nu) \, d\omega \right| \leq C \|v_\varepsilon\|_{H_\nu^1} \|\varphi\|_{H_\nu^1}.$$

Thus it follows, from equation (2) that

$$\begin{aligned} \frac{\partial \tilde{v}_\varepsilon}{\partial t} &\in L^{\frac{4}{3}}(]0, T[; (\mathbf{H}_\nu^1(\Omega))'), \quad \frac{\partial \hat{v}_\varepsilon}{\partial t} \in L^{\frac{4}{3}}(]0, T[; (\mathbf{H}_\nu^1(\Omega))') \\ \frac{\partial p_\varepsilon}{\partial t} &\in L^2(]0, T[; L_0^2(\Omega)) \end{aligned}$$

These estimates show that the velocities $(\tilde{v}_\varepsilon, \widehat{v}_\varepsilon)$ are equal almost everywhere to continuous functions with values in $(\mathbf{H}_\nu^1(\Omega))'$. Besides, the pressure p_ε is equal almost everywhere to a continuous function with value in $L_0^2(\Omega)$.

For the two-dimensional case, the situation is quite different. We observe first that the velocity fields \widehat{v}_ε and \tilde{v}_ε belong to $L^4(]0, T[; L^4(\Omega))$, so that:

$$\begin{aligned} (v_\varepsilon \nabla) \tilde{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \tilde{v}_\varepsilon &\in L^{\frac{4}{3}}(]0, T[; \mathbf{L}^{\frac{4}{3}}(\Omega)), \\ (v_\varepsilon \nabla) \widehat{v}_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) \widehat{v}_\varepsilon &\in L^{\frac{4}{3}}(]0, T[; \mathbf{L}^{\frac{4}{3}}(\Omega)). \end{aligned}$$

So it follows from the equation (2) that

$$\begin{aligned} \frac{\partial \tilde{v}_\varepsilon}{\partial t} &\in L^2(]0, T[; (\mathbf{H}_\nu^1(\Omega))') + L^{\frac{4}{3}}(]0, T[; L^{\frac{4}{3}}(\Omega)) \\ \frac{\partial \widehat{v}_\varepsilon}{\partial t} &\in L^2(]0, T[; (\mathbf{H}_\nu^1(\Omega))') + L^{\frac{4}{3}}(]0, T[; L^{\frac{4}{3}}(\Omega)) \\ \frac{\partial p_\varepsilon}{\partial t} &\in L^2(]0, T[; L_0^2(\Omega)) \end{aligned}$$

We now observe that the two velocity fields \tilde{v}_ε and \widehat{v}_ε belong to

$$L^4(]0, T[; \mathbf{L}^4(\Omega)) \cap L^2(]0, T[; (\mathbf{H}_\nu^1(\Omega))')$$

which is the dual space of

$$L^2(]0, T[; (\mathbf{H}_\nu^1(\Omega))') + \mathbf{L}^{\frac{4}{3}}(]0, T[; \mathbf{L}^{\frac{4}{3}}(\Omega)).$$

Thus the functions $(\tilde{v}_\varepsilon, \widehat{v}_\varepsilon)$ are equal almost everywhere to continuous functions with values in $\mathbf{L}^2(\Omega)$.

This ends the proof of proposition 3. \square

3.1. Stability. In the case of three-dimensional vector spaces, we do not have an equality for the conservation of the energy, we have only an inequality. Nevertheless for two-dimensional vector spaces, the weak solutions satisfy the energy equality.

Proof. Through classical computations one obtains, with equations (2) and (3):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + r \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon} |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\ - \int_\Omega p_\varepsilon \operatorname{div} \tilde{v}_\varepsilon \, d\omega = \int_\Omega f \cdot \tilde{v}_\varepsilon \, d\omega, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + r \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon} |(\widehat{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\ + \frac{1}{\varepsilon \mathcal{R}_e} \int_0^\varepsilon \operatorname{div} v_\varepsilon \operatorname{div} \widehat{v}_\varepsilon \, d\omega = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + r \|\operatorname{div} v_\varepsilon\|_{L^2}^2 \\ + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 - \int_\Omega p_\varepsilon \operatorname{div} v_\varepsilon \, d\omega = \int_\Omega f \cdot v_\varepsilon \, d\omega, \end{aligned} \quad (6)$$

$$\frac{\gamma \varepsilon}{2} \frac{d}{dt} \|p_\varepsilon\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2 + \int_\Omega p_\varepsilon \operatorname{div} v_\varepsilon \, d\omega + r \varepsilon \int_\Omega p_\varepsilon \operatorname{div} \tilde{v}_\varepsilon \, d\omega = 0. \quad (7)$$

Multiplying (4) by $r\varepsilon$ and (5) by ε and summing with (6) and (7), one obtains:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (r\varepsilon \|\tilde{v}_\varepsilon\|_{L^2}^2 + \varepsilon \|\widehat{v}_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^2 + \gamma\varepsilon \|p_\varepsilon\|_{L^2}^2) + \frac{r\varepsilon}{\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{\mathcal{R}_e} \|\operatorname{curl} \widehat{v}_\varepsilon\|_{L^2}^2 \\
& + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 + \gamma\varepsilon \|p_\varepsilon\|_{L^2}^2 + \frac{r\varepsilon}{\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{\mathcal{R}_e} \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 \\
& + r |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + |(\widehat{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\
& + \varepsilon r^2 \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + r\varepsilon \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 \\
& = -\frac{1}{\mathcal{R}_e} \int_{\Omega} \operatorname{div} v_\varepsilon \operatorname{div} \widehat{v}_\varepsilon \, d\omega + r\varepsilon \int_{\Omega} f \cdot \tilde{v}_\varepsilon \, d\omega + \int_{\Omega} f \cdot v_\varepsilon \, d\omega,
\end{aligned}$$

Let us now give some bounds of the right-hand side terms.

The term $\frac{1}{\mathcal{R}_e} (\operatorname{div} v_\varepsilon, \operatorname{div} \widehat{v}_\varepsilon)$ is bounded by $\frac{1}{2\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{2\mathcal{R}_e} \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2$.

According to the estimate of the L^2 norm in $\mathbf{H}_\nu^1(\Omega)$ given by equation (1), we bound the source terms in the following way:

$$\begin{aligned}
\left| \int_{\Omega} f \cdot v \, d\omega \right| & \leq \|f\|_{L^2} \|v\|_{L^2} \leq \sqrt{\lambda} \|f\|_{L^2} (\|\operatorname{div} v\|_{L^2}^2 + \|\operatorname{curl} v\|_{L^2}^2)^{\frac{1}{2}} \\
& \leq \frac{1}{2\mathcal{R}_e} (\|\operatorname{div} v\|_{L^2}^2 + \|\operatorname{curl} v\|_{L^2}^2) + \frac{\mathcal{R}_e \lambda}{2} \|f\|_{L^2}^2.
\end{aligned}$$

Using these bounds, we get from the previous equation the following fundamental estimate:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (r\varepsilon \|\tilde{v}_\varepsilon\|_{L^2}^2 + \varepsilon \|\widehat{v}_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^2 + \gamma\varepsilon \|p_\varepsilon\|_{L^2}^2) \\
& + \gamma\varepsilon \|p_\varepsilon\|_{L^2}^2 + \frac{r\varepsilon}{2\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{\mathcal{R}_e} \|\operatorname{curl} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{2\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 \\
& + \frac{1}{2\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \frac{r\varepsilon}{2\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{2\mathcal{R}_e} \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \varepsilon r^2 \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 \quad (8) \\
& + \varepsilon r \|\operatorname{div} \widehat{v}_\varepsilon\|_{L^2}^2 + \frac{1}{2\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + r |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + |(\widehat{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\
& + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \leq \frac{\lambda \mathcal{R}_e}{2} (1 + \varepsilon r) \|f\|_{L^2}^2.
\end{aligned}$$

After integration in time, we deduce from the previous estimate that there exists a continuous function g defined on $[0, T]$ such that: for all $t > 0$,

$$\begin{aligned}
& \frac{1}{2} \left(r \varepsilon \|\tilde{v}_\varepsilon(t)\|^2 + \varepsilon \|\widehat{v}_\varepsilon(t)\|^2 + \|v_\varepsilon(t)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(t)\|_{L^2}^2 \right) \\
& + \gamma \varepsilon \int_0^t \|p_\varepsilon(s)\|_{L^2}^2 ds + \frac{r\varepsilon}{2\mathcal{R}_e} \int_0^t \|\operatorname{curl} \tilde{v}_\varepsilon(s)\|_{L^2}^2 ds + \frac{\varepsilon}{\mathcal{R}_e} \int_0^t \|\operatorname{curl} \widehat{v}_\varepsilon(s)\|_{L^2}^2 ds \\
& + \frac{1}{2\mathcal{R}_e} \int_0^t \|\operatorname{curl} v_\varepsilon(s)\|_{L^2}^2 ds + \frac{r\varepsilon}{2\mathcal{R}_e} \int_0^t \|\operatorname{div} \tilde{v}_\varepsilon(s)\|_{L^2}^2 ds \\
& + \frac{\varepsilon}{2\mathcal{R}_e} \int_0^t \|\operatorname{div} \widehat{v}_\varepsilon(s)\|_{L^2}^2 ds + \frac{1}{2\mathcal{R}_e} \int_0^t \|\operatorname{div} v_\varepsilon(s)\|_{L^2}^2 ds \\
& + r \int_0^t |(\tilde{v}_\varepsilon(s) \wedge \nu)|_{L^2(\Gamma)}^2 ds + \int_0^t |(\widehat{v}_\varepsilon(s) \wedge \nu)|_{L^2(\Gamma)}^2 ds \\
& + \frac{1}{\varepsilon} \int_0^t |(v_\varepsilon(s) \wedge \nu)|_{L^2(\Gamma)}^2 ds + \varepsilon r^2 \int_0^t \|\operatorname{div} \tilde{v}_\varepsilon(s)\|_{L^2}^2 ds \\
& + \varepsilon r \int_0^t \|\operatorname{div} \widehat{v}_\varepsilon(s)\|_{L^2}^2 ds + \frac{1}{2\varepsilon\mathcal{R}_e} \int_0^t \|\operatorname{div} v_\varepsilon(s)\|_{L^2}^2 ds \leq g(t),
\end{aligned} \tag{9}$$

with

$$g(t) = (r \varepsilon \|\tilde{v}_\varepsilon(0)\|^2 + \varepsilon \|\widehat{v}_\varepsilon(0)\|^2 + \|v_\varepsilon(0)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(0)\|_{L^2}^2) + \frac{\lambda \mathcal{R}_e}{2} (1+r\varepsilon) \int_0^t \|f(s)\|_{L^2}^2 ds.$$

This inequality is the key point to establish the convergence result.

To improve the convergence result in the two-dimensional case, we use the following energy equality derived as above without using (5).

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (r \varepsilon \|\tilde{v}_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2) + \frac{r\varepsilon}{\mathcal{R}_e} \|\operatorname{curl} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{curl} v_\varepsilon\|_{L^2}^2 \\
& + \gamma \varepsilon \|p_\varepsilon\|_{L^2}^2 + \frac{r\varepsilon}{\mathcal{R}_e} \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 + \frac{1}{\mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + \varepsilon r^2 \|\operatorname{div} \tilde{v}_\varepsilon\|_{L^2}^2 \\
& + \frac{1}{\varepsilon \mathcal{R}_e} \|\operatorname{div} v_\varepsilon\|_{L^2}^2 + r |(\tilde{v}_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 + \frac{1}{\varepsilon} |(v_\varepsilon \wedge \nu)|_{L^2(\Gamma)}^2 \\
& = r \varepsilon \int_\Omega f \cdot \tilde{v}_\varepsilon d\omega + \int_\Omega f \cdot v_\varepsilon d\omega.
\end{aligned} \tag{10}$$

This concludes the proof of lemma 2.1. \square

3.2. Uniform stability for the approximate solution. In this section, we deal with the stability of the proposed approximation method. We notice that this property is valid for all solutions satisfying the energy inequality (8), as it is the case when they are built by a finite dimensional approximation method such as the Galerkin method for example.

Proof. Let us write

$$\chi_\varepsilon(t) = \varepsilon r \|\tilde{v}_\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\widehat{v}_\varepsilon(t)\|_{L^2}^2 + \|v_\varepsilon(t)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(t)\|_{L^2}^2.$$

We note $\lambda > 0$ the smallest eigenvalue of the self-adjoint operator $A = \operatorname{curl} \operatorname{curl} - \nabla \operatorname{div}$ with the domain

$$\mathcal{D} = \mathbf{H}_\nu^1(\Omega) \cap (H^2(\Omega))^d,$$

and we introduce $\alpha = \min\left(\frac{\lambda}{\mathcal{R}_e}, 1\right)$. Classically, the inequality (8) leads to the differential inequality

$$\frac{d}{dt}\chi_\varepsilon(t) + \alpha\chi_\varepsilon(t) \leq \lambda\mathcal{R}_e(1+r\varepsilon)\|f(t)\|_{L^2}^2,$$

which implies the following uniform bound

$$\chi_\varepsilon(t) \leq e^{-\alpha t}\chi_\varepsilon(0) + \frac{\lambda\mathcal{R}_e}{\alpha}(1+r\varepsilon)\|f\|_{L^\infty, L^2}.$$

This concludes the proof of theorem 2.2. □

4. Convergence analysis and compactness results.

4.1. Compactness results for the velocity. Let us introduce the Leray projection w_ε of a velocity field $v_\varepsilon(t) \in \mathbf{H}_\nu^1(\Omega)$ defined as follows

$$\begin{aligned} v_\varepsilon &= w_\varepsilon + \nabla q_\varepsilon, \\ \operatorname{div} w_\varepsilon &= 0, \\ w_\varepsilon \cdot \nu|_\Gamma &= 0, \nabla q_\varepsilon \cdot \nu|_\Gamma = 0, \\ \int_\Omega q_\varepsilon d\omega &= 0. \end{aligned}$$

By the estimate (9), we see that the irrotational part of v_ε goes to zero with ε . Thus it remains to bring to the fore the behavior of the free divergence part w_ε and to obtain an estimate on a fractional time derivative of this term. We detail the different steps of this strategy.

From the regularity of the Leray projector (see R. Temam [20] page 18), one has:

$$\begin{aligned} \|w_\varepsilon\|_{L^\infty, L^2} &\leq c\|v_\varepsilon\|_{L^\infty, L^2}, \\ \|w_\varepsilon\|_{L^2, H^1} &\leq c\|v_\varepsilon\|_{L^2, H^1}. \end{aligned} \tag{11}$$

Moreover, we can easily prove the following lemma.

Lemma 4.1. *There exists two constants depending only on T and Ω such that:*

$$\begin{aligned} \|\nabla q_\varepsilon\|_{L^2, H^1} &\leq c\sqrt{\varepsilon}, \\ \|\nabla q_\varepsilon\|_{L^\infty, L^2} &\leq c \end{aligned} \tag{12}$$

Proof: The function q_ε belongs to $H^2(\Omega)$ and satisfies

$$\begin{aligned} -\Delta q_\varepsilon(t) &= -\operatorname{div} v_\varepsilon(t), \\ \nabla q_\varepsilon(t) \cdot \nu|_\Gamma &= 0. \end{aligned}$$

This implies using the estimate (9)

$$\|\Delta q_\varepsilon(t)\|_{L^2, L^2} = \|\operatorname{div} v_\varepsilon\|_{L^2, L^2} \leq C\sqrt{\varepsilon}.$$

Besides, we have

$$\int_\Omega \nabla q_\varepsilon \cdot \nabla q_\varepsilon d\omega = - \int_\Omega \Delta q_\varepsilon q_\varepsilon d\omega,$$

so that, with Poincaré-Neumann inequality, we get

$$\begin{aligned} \|\nabla q_\varepsilon(t)\|_{L^2, L^2}^2 &\leq C\|\Delta q_\varepsilon(t)\|_{L^2, L^2}\|q_\varepsilon\|_{L^2, L^2}, \\ &\leq C\sqrt{\varepsilon}\|\nabla q_\varepsilon(t)\|_{L^2, L^2}. \end{aligned}$$

The regularity properties of the Neumann problem give

$$\|\nabla q_\varepsilon\|_{L^2, H^1} \leq C \|q_\varepsilon\|_{L^2, H^2} \leq C \|\Delta q_\varepsilon\|_{L^2, L^2} \leq C\sqrt{\varepsilon}. \quad (13)$$

Moreover, by orthogonality of the Leray projector in L^2 , one has

$$\|\nabla q_\varepsilon\|_{L^\infty, L^2} \leq \|v_\varepsilon\|_{L^\infty, L^2} \leq C. \quad (14)$$

This concludes the proof of the lemma 4.1.

So by interpolation and using estimates (13)-(14), we have proved the result below.

Corollary 1. *The function q_ε satisfies:*

∇q_ε strongly converges to 0 in $(L^p([0, T]; L^2(\Omega)))$, $\forall p, 1 \leq p < +\infty$.

Now we have to write the equation satisfied by w_ε . As the Leray projection is orthogonal in $\mathbf{L}^2(\Omega)$, this equation reads

$$\left\{ \begin{array}{l} \forall \varphi \in \mathbf{H}_\nu^1(\Omega), \operatorname{div} \varphi = 0, \\ \int_\Omega \frac{\partial w_\varepsilon}{\partial t} \cdot \varphi \, d\omega + \int_\Omega \left((v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon \right) \cdot \varphi \, d\omega \\ + \frac{1}{\mathcal{R}_e} \int_\Omega \operatorname{curl} v_\varepsilon \cdot \operatorname{curl} \varphi \, d\omega + \frac{1}{\varepsilon} \int_\Gamma (v_\varepsilon \wedge \nu) \cdot (\varphi \wedge \nu) \, d\gamma \\ = \int_\Omega f \cdot \varphi \, d\omega \text{ in } L^1(0, T). \end{array} \right. \quad (15)$$

Now we introduce the extension by 0 of w_ε (resp. v_ε) outside $[0, T]$ denoted, only in this part, by $\widetilde{w}_\varepsilon$ (resp. $\widetilde{v}_\varepsilon$) and we take the Fourier transform in time of the equation (15) to obtain

$$\left\{ \begin{array}{l} \forall \varphi \in \mathbf{H}_\nu^1(\Omega), \operatorname{div} \varphi = 0, \\ i\tau \int_\Omega \mathcal{F}(\widetilde{w}_\varepsilon)(\tau) \cdot \varphi \, d\omega + \int_\Omega \mathcal{F} \left((\widetilde{v}_\varepsilon \cdot \nabla) \widetilde{v}_\varepsilon + \frac{1}{2} (\operatorname{div} \widetilde{v}_\varepsilon) \widetilde{v}_\varepsilon \right) (\tau) \cdot \varphi \, d\omega \\ + \frac{1}{\mathcal{R}_e} \int_\Omega \operatorname{curl} \mathcal{F}(\widetilde{v}_\varepsilon)(\tau) \cdot \operatorname{curl} \varphi \, d\omega + \frac{1}{\varepsilon} \int_\Gamma \mathcal{F}(\widetilde{v}_\varepsilon \wedge \nu)(\tau) \cdot (\varphi \wedge \nu) \, d\gamma \\ = \int_\Omega \mathcal{F}(\widetilde{f})(\tau) \cdot \varphi \, d\omega + \frac{1}{\sqrt{2\pi}} \int_\Omega v_\varepsilon(0) \cdot \varphi \, d\omega - \frac{e^{-i\tau T}}{\sqrt{2\pi}} \int_\Omega v_\varepsilon(T) \cdot \varphi \, d\omega. \end{array} \right.$$

Following Boyer-Fabrie [5, page 253], we take $\varphi = \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)}$ as test function in the previous equation to obtain for all $\tau \in \mathbb{R}$:

$$\begin{aligned} i\tau \int_\Omega |\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)|^2 \, d\omega &= - \int_\Omega (\mathcal{F}(\widetilde{v}_\varepsilon \cdot \nabla) \widetilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega \\ &\quad - \frac{1}{2} \int_\Omega (\mathcal{F}(\operatorname{div} \widetilde{v}_\varepsilon) \widetilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega \\ &\quad - \frac{1}{\mathcal{R}_e} \int_\Omega \operatorname{curl} \mathcal{F}(\widetilde{v}_\varepsilon)(\tau) \cdot \operatorname{curl} \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega \\ &\quad - \frac{1}{\varepsilon} \int_\Gamma \mathcal{F}(\widetilde{v}_\varepsilon \wedge \nu)(\tau) \cdot (\overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \wedge \nu) \, d\gamma \\ &\quad + \int_\Omega \mathcal{F}(\widetilde{f})(\tau) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_\Omega v_\varepsilon(0) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega - \frac{e^{-i\tau T}}{\sqrt{2\pi}} \int_\Omega v_\varepsilon(T) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)} \, d\omega. \end{aligned}$$

As we look for an estimate independent of ε , we have to pay a special attention to the imaginary part of the penalty term:

$$A_\varepsilon = -\frac{1}{\varepsilon} \int_{\Gamma} \mathcal{F}(\tilde{v}_\varepsilon \wedge \nu)(\tau) \cdot \overline{(\mathcal{F}(\tilde{w}_\varepsilon)(\tau) \wedge \nu)} d\gamma. \quad (16)$$

By writing $w_\varepsilon = v_\varepsilon - \nabla q_\varepsilon$, we have:

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\Gamma} \mathcal{F}(\tilde{v}_\varepsilon \wedge \nu)(\tau) \cdot \overline{(\mathcal{F}(\tilde{w}_\varepsilon)(\tau) \wedge \nu)} d\gamma &= \frac{1}{\varepsilon} \int_{\Gamma} |\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu|^2 d\gamma \\ &\quad - \frac{1}{\varepsilon} \int_{\Gamma} \overline{(\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau) \wedge \nu)} \cdot (\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu) d\gamma. \end{aligned}$$

So, the imaginary part of A_ε is bounded as follows:

$$\begin{aligned} \frac{1}{\varepsilon} \left| \int_{\Gamma} \overline{(\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau) \wedge \nu)} \cdot (\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu) d\gamma \right| \\ \leq \frac{1}{\varepsilon} |\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau)|_{L^2(\Gamma)} |\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu|_{L^2(\Gamma)}, \quad (17) \\ \leq C \frac{1}{\varepsilon} \|\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau)\|_{H^1} |\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu|_{L^2(\Gamma)}. \end{aligned}$$

From estimates (12) and (9), we have

$$\begin{aligned} \|\nabla q_\varepsilon\|_{L^2, H^1} &\leq C\sqrt{\varepsilon}, \\ |v_\varepsilon \wedge \nu|_{L^2, L^2(\Gamma)} &\leq C\sqrt{\varepsilon}. \end{aligned}$$

So there exists a function $f_\varepsilon^4(\tau) \in L^1(\mathbb{R})$ bounded independently on ε such that:

$$\frac{1}{\varepsilon} \left| \int_{\Gamma} \overline{(\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau) \wedge \nu)} \cdot (\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu) d\gamma \right| \leq f_\varepsilon^4(\tau) \quad (18)$$

Now we can derive the estimate of $|\tau| \int_{\Omega} |\mathcal{F}(\tilde{w}_\varepsilon)(\tau)|^2 d\omega$, and we have

$$\begin{aligned} |\tau| \int_{\Omega} |\mathcal{F}(\tilde{w}_\varepsilon)(\tau)|^2 d\omega &\leq \left| \int_{\Omega} (\mathcal{F}(\tilde{v}_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| \\ &\quad + \frac{1}{2} \left| \int_{\Omega} (\mathcal{F}(\operatorname{div} \tilde{v}_\varepsilon) \tilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| \\ &\quad + \frac{1}{\mathcal{R}_e} \left| \int_{\Omega} \operatorname{curl} \mathcal{F}(\tilde{v}_\varepsilon)(\tau) \cdot \operatorname{curl} \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| \\ &\quad + \frac{1}{\varepsilon} \left| \int_{\Gamma} \overline{(\mathcal{F}(\nabla \tilde{q}_\varepsilon)(\tau) \wedge \nu)} \cdot (\mathcal{F}(\tilde{v}_\varepsilon)(\tau) \wedge \nu) d\gamma \right| \quad (19) \\ &\quad + \left| \int_{\Omega} \mathcal{F}(f)(\tau) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| + \frac{1}{\sqrt{2\pi}} \left| \int_{\Omega} v_\varepsilon(0) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| \\ &\quad + \frac{1}{\sqrt{2\pi}} \left| \int_{\Omega} v_\varepsilon(T) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right| \\ &\leq f_\varepsilon^1(\tau) + f_\varepsilon^2(\tau) + f_\varepsilon^3(\tau) + f_\varepsilon^4(\tau) + f_\varepsilon^5(\tau) + f_\varepsilon^6(\tau) + f_\varepsilon^7(\tau). \end{aligned}$$

We now estimate each term of the right-hand side of the previous inequality for $d \leq 3$.

$$\mathbf{Term} f_\varepsilon^1 = \left| \int_{\Omega} (\mathcal{F}(\tilde{v}_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\tilde{w}_\varepsilon)(\tau)} d\omega \right|$$

According to the energy estimate (9), the function v_ε is bounded in $L^2(0, T; \mathbf{H}_\nu^1(\Omega))$ and hence, by Sobolev injection, it is bounded in $L^{\frac{6}{5}}(0, T; \mathbf{L}^6(\Omega))$. So by Hausdorff-Young theorem,

$$(\overline{\mathcal{F}(\widetilde{w}_\varepsilon)})_\varepsilon \text{ is bounded in } L^6(\mathbb{R}; \mathbf{L}^6(\Omega)). \quad (20)$$

We also have the inequality:

$$\|\widetilde{v}_\varepsilon \cdot \nabla \widetilde{v}_\varepsilon\|_{L^{\frac{6}{5}}} \leq \|\nabla v_\varepsilon\|_{L^2} \|v_\varepsilon\|_{L^3} \leq C \|\nabla v_\varepsilon\|_{L^2}^{\frac{3}{2}} \|v_\varepsilon\|_{L^2}^{\frac{1}{2}},$$

which implies, according to (9), that $v_\varepsilon \cdot \nabla v_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0, T; \mathbf{L}^{\frac{6}{5}}(\Omega))$, and necessarily in $L^{\frac{6}{5}}(0, T; \mathbf{L}^{\frac{6}{5}}(\Omega))$. So, by Hausdorff-Young theorem, the family of functions $(\mathcal{F}(\widetilde{v}_\varepsilon \cdot \nabla \widetilde{v}_\varepsilon))_\varepsilon$ is bounded in $L^6(\mathbb{R}; \mathbf{L}^{\frac{6}{5}}(\Omega))$. Then with Hölder inequality,

$$(f_\varepsilon^1)_\varepsilon \text{ is bounded in } L^3(\mathbb{R}). \quad (21)$$

$$\text{Term } f_\varepsilon^2 = \frac{1}{2} \left| \int_{\Omega} (\mathcal{F}(\operatorname{div} \widetilde{v}_\varepsilon) \widetilde{v}_\varepsilon)(\tau) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)}(\tau) d\omega \right|$$

The same arguments show that

$$(f_\varepsilon^2)_\varepsilon \text{ is bounded in } L^3(\mathbb{R}). \quad (22)$$

$$\text{Term } f_\varepsilon^3 = \frac{1}{\mathcal{R}_e} \left| \int_{\Omega} \operatorname{curl} \mathcal{F}(\widetilde{v}_\varepsilon)(\tau) \cdot \operatorname{curl} \overline{\mathcal{F}(\widetilde{w}_\varepsilon)}(\tau) d\omega \right|$$

According to the regularity of the Leray projection recalled above and estimate (9), one has:

$$(f_\varepsilon^3)_\varepsilon \text{ is bounded in } L^1(\mathbb{R}). \quad (23)$$

$$\text{Terms } f_\varepsilon^4 = \frac{1}{\varepsilon} \left| \int_{\Gamma} (\overline{\mathcal{F}(\nabla \widetilde{q}_\varepsilon)}(\tau) \wedge \nu) \cdot (\mathcal{F}(\widetilde{v}_\varepsilon)(\tau) \wedge \nu) d\gamma \right|$$

As we have seen by the estimate (18),

$$(f_\varepsilon^4)_\varepsilon \text{ is bounded in } L^1(\mathbb{R}). \quad (24)$$

$$\text{Term } f_\varepsilon^5 = \left| \int_{\Omega} \mathcal{F}(\tilde{f})(\tau) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)}(\tau) d\omega \right|$$

By hypothesis, \tilde{f} is a given function in $L^2(\mathbb{R}; \mathbf{L}^2(\Omega))$ and from estimate (9), we get that $\widetilde{v}_\varepsilon$ is bounded in $L^2(\mathbb{R}; \mathbf{L}^2(\Omega))$, so as $\widetilde{w}_\varepsilon$ is the Leray projection of $\widetilde{v}_\varepsilon$, the function $\widetilde{w}_\varepsilon$ is also bounded in $L^2(\mathbb{R}; \mathbf{L}^2(\Omega))$. Finally, we obtain

$$(f_\varepsilon^5)_\varepsilon \text{ is bounded in } L^1(\mathbb{R}). \quad (25)$$

$$\text{Terms } f_\varepsilon^6 = \frac{1}{\sqrt{2\pi}} \left| \int_{\Omega} v_\varepsilon(0) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)}(\tau) d\omega \right| \text{ and } f_\varepsilon^7 = \frac{1}{\sqrt{2\pi}} \left| \int_{\Omega} v_\varepsilon(T) \cdot \overline{\mathcal{F}(\widetilde{w}_\varepsilon)}(\tau) d\omega \right|$$

These two terms come from the Dirac measure when we derive discontinuous functions. Let us consider f_ε^7 .

$$f_\varepsilon^7(\tau) \leq \frac{1}{\sqrt{2\pi}} \|v_\varepsilon(T)\|_{L^2} \|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}.$$

According to (9), this term is bounded in $L^\infty(\mathbb{R})$. Moreover, the set of functions $(\mathcal{F}(\widetilde{w}_\varepsilon)(\tau))_\varepsilon$ is bounded in $L^2(\mathbb{R}; \mathbf{L}^2(\Omega))$ so, $(f_\varepsilon^7)_\varepsilon$ is bounded in $L^2(\mathbb{R})$.

We treat in the same way the term f_ε^6 , and thus:

$$(f_\varepsilon^6)_\varepsilon \text{ and } (f_\varepsilon^7)_\varepsilon \text{ are bounded in } L^2(\mathbb{R}) \quad (26)$$

We are now able to show that the set of functions $(\widetilde{w}_\varepsilon)_\varepsilon$ is bounded in an appropriate Nikolskii space.

For all $\gamma < 1$, there exists a constant d such that:

$$|\tau|^{1-\gamma} \leq d \left(1 + \frac{|\tau|}{1+|\tau|^\gamma} \right),$$

so that,

$$|\tau|^{1-\gamma} \|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}^2 \leq d \left(\|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}^2 + \frac{|\tau|}{1+|\tau|^\gamma} \|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}^2 \right)$$

Let us denote $f_\varepsilon^0(\tau) = \|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}^2$, which belongs to $L^1(\mathbb{R})$, the previous inequality reads with (19):

$$\begin{aligned} & |\tau|^{1-\gamma} \|\mathcal{F}(\widetilde{w}_\varepsilon)(\tau)\|_{L^2}^2 \\ & \leq d f_\varepsilon^0(\tau) + \frac{d}{1+|\tau|^\gamma} \left(f_\varepsilon^1(\tau) + f_\varepsilon^2(\tau) + f_\varepsilon^3(\tau) + f_\varepsilon^4(\tau) + f_\varepsilon^5(\tau) + f_\varepsilon^6(\tau) + f_\varepsilon^7(\tau) \right) \\ & \leq h(\tau) \end{aligned}$$

If we suppose that the function $\tau \mapsto \frac{1}{1+|\tau|^\gamma}$ belongs to $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, then the function $\tau \mapsto h(\tau)$ belongs to $L^1(\mathbb{R})$. This condition is satisfied for $\gamma \in]\frac{2}{3}, 1[$. So we have proved :

Lemma 4.2. *Let us suppose that $\sigma \in]0, \frac{1}{6}[$, then there exists a constant C such that*

$$\int_{\mathbb{R}} |\tau|^{2\sigma} \|\mathcal{F}(w_\varepsilon)(\tau)\|_{L^2}^2 d\tau \leq C. \quad (27)$$

Then, from lemma 4.1 and 4.2 we deduce the following key result:

Theorem 4.3. *There exists a sequence $(\varepsilon_k)_k$ which converges to zero and a function $v \in L^2(]0, T[; \mathbf{L}^2(\Omega))$ satisfying $\operatorname{div} v = 0$ such that:*

$$(v_{\varepsilon_k})_k \rightarrow v \text{ in } L^2(]0, T[; \mathbf{L}^2(\Omega)) \text{ strongly.}$$

Proof. The function v_ε is the sum of two terms ∇q_ε and w_ε . From corollary 1 the first term converges strongly to 0 in $L^2(]0, T[; \mathbf{L}^2(\Omega))$. Now, from Aubin-Lions-Simon Theorem, it follows from lemma 4.2, that there exists a sequence $(\varepsilon_k)_k$ such that:

$$(w_{\varepsilon_k})_k \rightarrow v \text{ in } L^2(]0, T[; \mathbf{L}^2(\Omega)) \text{ strongly.}$$

Moreover, since $\operatorname{div} w_{\varepsilon_k} = 0$, we have $\operatorname{div} v = 0$. \square

4.2. Convergence of the method. We first give a general convergence theorem for a subsequence solution of the approximate scheme (3), to a weak solution of the initial Navier-Stokes problem, in the case $d \leq 3$. For the two-dimensional case, since the weak solution of the Navier-Stokes equation is unique, the whole sequence of approximate solution v_ε converges to v . Moreover, in this case, we prove that the convergence is strong.

4.2.1. *The general case $d \leq 3$.* Let θ an element of $\mathcal{C}^\infty(0, T)$, satisfying $\theta(T) = 0$, and φ a free-divergence vector field in $(H_0^1(\Omega))^d \cap (H^2(\Omega))^d$. An integration by parts gives from the equation (3)

$$\begin{aligned} & - \int_0^T \int_\Omega v_\varepsilon \cdot \varphi \, d\omega \, \theta'(\tau) \, d\tau + \int_0^T \int_\Omega \left((v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon \right) \cdot \varphi \, d\omega \, \theta(\tau) \, d\tau \\ & + \frac{1}{\mathcal{R}_e} \int_0^T \int_\Omega \operatorname{curl} v_\varepsilon \cdot \operatorname{curl} \varphi \, d\omega \, \theta(\tau) \, d\tau \\ & = \int_0^T \int_\Omega f \varphi \, d\omega \, \theta(\tau) \, d\tau + \int_\Omega v_0 \varphi \, d\omega \, \theta(0). \end{aligned} \quad (28)$$

According to estimate (9), there exists a sequence ε_k such that

$$\begin{aligned} v_{\varepsilon_k} & \rightarrow v \text{ in } L^2(0, T; \mathbf{L}^2(\Omega)) \text{ strongly,} \\ \operatorname{div} v_{\varepsilon_k} & \rightarrow \operatorname{div} v = 0 \text{ in } L^2(0, T; \mathbf{L}^2(\Omega)) \text{ strongly,} \\ \operatorname{curl} v_{\varepsilon_k} & \rightharpoonup \operatorname{curl} v \text{ in } L^2(0, T; \mathbf{L}^2(\Omega)) \text{ weakly.} \end{aligned}$$

Since $(\|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2)^{\frac{1}{2}}$ is a norm equivalent to the H^1 -norm on $\mathbf{H}_\nu^1(\Omega)$, we have

$$\nabla v_{\varepsilon_k} \rightharpoonup \nabla v \text{ in } L^2(0, T; \mathbf{L}^2(\Omega)) \text{ weakly.}$$

and so, we can take the limit on the term $(v_\varepsilon \cdot \nabla) v_\varepsilon$ as

$$\begin{aligned} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} & \rightarrow v \cdot \nabla v \text{ in } L^1(0, T; \mathbf{L}^1(\Omega)) \\ (\operatorname{div} v_{\varepsilon_k}) v_{\varepsilon_k} & \rightarrow 0 \text{ in } L^1(0, T; \mathbf{L}^1(\Omega)). \end{aligned}$$

From estimate (9), we have for the tangential traces

$$\int_0^t |(v_\varepsilon \wedge \nu)(\tau)|_{L^2(\Gamma)}^2 \, d\tau \leq \varepsilon g(t),$$

and since for any function v in $(H^1(\Omega))^d$,

$$|v|_{L^2(\Gamma)} \leq C \|v\|_{L^2}^{\frac{1}{2}} \|v\|_{H^1}^{\frac{1}{2}},$$

we obtain that

$$v_{\varepsilon_k} \rightarrow v \text{ in } L^2(\]0, T[; (L^2(\Gamma))^d) \text{ strongly.}$$

This implies $(v \wedge \nu)|_\Gamma = 0$, and so, since by construction $(v \cdot \nu)|_\Gamma = 0$, v belongs to $(H_0^1(\Omega))^d$

Finally, at the limit process we obtain

$$\left\{ \begin{array}{l} - \int_0^T \int_\Omega v \cdot \varphi \, d\omega \, \theta'(\tau) \, d\tau + \int_0^T \int_\Omega (v \cdot \nabla) v \cdot \varphi \, d\omega \, \theta(\tau) \, d\tau \\ + \frac{1}{\mathcal{R}_e} \int_0^T \int_\Omega \operatorname{curl} v \cdot \operatorname{curl} \varphi \, d\omega \, \theta(\tau) \, d\tau = \int_0^T \int_\Omega f \cdot \varphi \, d\omega \, \theta(\tau) \, d\tau + \int_\Omega v_0 \varphi \, d\omega \, \theta(0), \\ \operatorname{div} v = 0, \\ v|_\Gamma = 0. \end{array} \right.$$

From the identity

$$\int_\Omega \nabla v : \nabla \varphi \, d\omega = \int_\Omega \operatorname{curl} v \cdot \operatorname{curl} \varphi \, d\omega + \int_\Omega \operatorname{div} v \cdot \operatorname{div} \varphi \, d\omega,$$

which is valid for all functions $(v, \varphi) \in (H_0^1(\Omega))^d \times (H_0^1(\Omega))^d$, the previous equality shows that the limit function v satisfies the classical Navier-Stokes equations in a weak sense.

4.2.2. *The special case $d = 2$.* The key point to establish the strong convergence in the two-dimensional case, lies on an idea of R. Temam [20]. It is based on the fact that in this case, the solution of the approximate problem and the solution of the Navier-Stokes equation verify the equality of energy. The idea is to bring to the fore an energy equation satisfied by the difference between the approximate solution and the exact solution.

We first observe that, according to the equality $-\Delta = \text{curl curl} - \nabla \text{div}$ the classical weak solution v of the Navier-Stokes equation satisfies for any test function in $(H_0^1(\Omega))^d$, with free-divergence:

$$\int_{\Omega} \frac{\partial v}{\partial t} \cdot \varphi \, d\omega + \int_{\Omega} ((v \cdot \nabla)v) \cdot \varphi \, d\omega + \frac{1}{\mathcal{R}_e} \int_{\Omega} \text{curl } v \cdot \text{curl } \varphi \, d\omega = \int_{\Omega} f \cdot \varphi \, d\omega.$$

The equation satisfied by the error $v_\varepsilon - v = u_\varepsilon$ with a free-divergence test function φ in $(H_0^1(\Omega))^d \subset \mathbf{H}_v^1(\Omega)$.

$$\begin{cases} \int_{\Omega} \frac{\partial u_\varepsilon}{\partial t} \cdot \varphi \, d\omega + \int_{\Omega} ((u_\varepsilon \cdot \nabla)u_\varepsilon + \frac{1}{2}(\text{div } u_\varepsilon)u_\varepsilon) \cdot \varphi \, d\omega + \int_{\Omega} (v \cdot \nabla)u_\varepsilon \cdot \varphi \, d\omega \\ + \int_{\Omega} ((u_\varepsilon \cdot \nabla)v + \frac{1}{2}(\text{div } u_\varepsilon)v) \cdot \varphi \, d\omega + \frac{1}{\mathcal{R}_e} \int_{\Omega} \text{curl } u_\varepsilon \cdot \text{curl } \varphi \, d\omega = 0. \end{cases}$$

After integration in time, this equation gives

$$\begin{cases} \int_{\Omega} u_\varepsilon(t) \cdot \varphi \, d\omega + \int_0^t \int_{\Omega} ((u_\varepsilon \cdot \nabla)u_\varepsilon + \frac{1}{2}(\text{div } u_\varepsilon)u_\varepsilon) \cdot \varphi \, d\omega \, d\tau \\ + \int_0^t \int_{\Omega} (v \cdot \nabla)u_\varepsilon \cdot \varphi \, d\omega \, d\tau + \int_0^t \int_{\Omega} ((u_\varepsilon \cdot \nabla)v + \frac{1}{2}(\text{div } u_\varepsilon)v) \cdot \varphi \, d\omega \, d\tau \\ + \frac{1}{\mathcal{R}_e} \int_0^t \int_{\Omega} \text{curl } u_\varepsilon \cdot \text{curl } \varphi \, d\omega \, d\tau = \int_{\Omega} u_\varepsilon(0) \cdot \varphi \, d\omega. \end{cases}$$

Taking the limit when ε goes to 0, one obtains with the convergences properties stated in the previous section:

Lemma 4.4.

$$\forall \varphi \in (H_0^1(\Omega))^d, \text{div } \varphi = 0, \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(t) \cdot \varphi \, d\omega = 0. \quad (29)$$

Following R. Temam [20], we introduce

$$\begin{aligned} \chi_\varepsilon(t) &= \frac{1}{2} (\varepsilon r \|\tilde{v}_\varepsilon(t)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(t)\|_{L^2}^2 + \|(v_\varepsilon - v)(t)\|_{L^2}^2) \\ &+ \frac{r \varepsilon}{\mathcal{R}_e} \int_0^t \|\text{curl } \tilde{v}_\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \gamma \varepsilon \int_0^t \|p_\varepsilon(\tau)\|_{L^2}^2 \, d\tau \\ &+ \frac{1}{\mathcal{R}_e} \int_0^t \|\text{curl } (v_\varepsilon - v)(\tau)\|_{L^2}^2 \, d\tau + \frac{1}{\mathcal{R}_e} \int_0^t \|\text{div } (v_\varepsilon - v)(\tau)\|_{L^2}^2 \, d\tau \\ &+ r \int_0^t |(\tilde{v}_\varepsilon \wedge \nu)(\tau)|_{L^2(\Gamma)}^2 \, d\tau + \frac{1}{\varepsilon} \int_0^t |(v_\varepsilon \wedge \nu)(\tau)|_{L^2(\Gamma)}^2 \, d\tau \\ &+ \frac{\varepsilon r}{\mathcal{R}_e} \int_0^t \|\text{div } \tilde{v}_\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \frac{1}{\varepsilon \mathcal{R}_e} \int_0^t \|\text{div } v_\varepsilon(\tau)\|_{L^2}^2 \, d\tau. \end{aligned}$$

Here v is the unique solution of the two-dimensional Navier-Stokes equations. Due to the energy equality (10), $\chi_\varepsilon(t)$ satisfies

$$\begin{aligned} \chi_\varepsilon(t) &= \frac{1}{2} (\varepsilon r \|\tilde{v}_\varepsilon(0)\|_{L^2}^2 + \|v_\varepsilon(0)\|_{L^2}^2 + \gamma \varepsilon \|p_\varepsilon(0)\|_{L^2}^2) + r\varepsilon \int_0^t \int_\Omega f(\tau) \cdot \tilde{v}_\varepsilon(\tau) d\omega d\tau \\ &+ \int_0^t \int_\Omega f(\tau) \cdot v_\varepsilon(\tau) d\omega d\tau - \int_\Omega v_\varepsilon(t) \cdot v(t) d\omega - \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \int_\Omega \operatorname{curl} v_\varepsilon(\tau) \cdot \operatorname{curl} v(\tau) d\omega d\tau \\ &- \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \int_\Omega \operatorname{div} v_\varepsilon(\tau) \operatorname{div} v(\tau) d\omega d\tau + \frac{1}{2} \|v(t)\|_{L^2}^2 + \frac{1}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{curl} v(\tau)\|_{L^2}^2 d\tau \\ &+ \frac{1}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{div} v(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

By weak convergence in $L^2([0, t]; H_0^1(\Omega))$ of the sequence $(v_\varepsilon)_\varepsilon$ and from the lemma 4.4, we observe that :

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} &\left(\frac{1}{2} \|v_\varepsilon(0)\|_{L^2}^2 - \int_\Omega v_\varepsilon(t) \cdot v(t) d\omega + \int_0^t \int_\Omega f(\tau) \cdot v_\varepsilon(\tau) d\omega d\tau \right. \\ &\quad \left. - \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \int_\Omega \operatorname{curl} v_\varepsilon(\tau) \cdot \operatorname{curl} v(\tau) d\omega d\tau - \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \int_\Omega \operatorname{div} v_\varepsilon(\tau) \cdot \operatorname{div} v(\tau) d\omega d\tau \right) \\ &= \frac{1}{2} \|v(0)\|_{L^2}^2 + \int_0^t \int_\Omega f(\tau) \cdot v(\tau) d\omega d\tau - \|v(t)\|_{L^2}^2 - \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{curl} v(\tau)\|_{L^2}^2 d\tau \\ &\quad - \frac{2}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{div} v(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

In the two-dimensional case, the unique solution of the Navier-Stokes equation satisfies the following energy equality

$$\begin{aligned} \frac{1}{2} \|v(t)\|_{L^2}^2 + \frac{1}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{curl} u(\tau)\|_{L^2}^2 d\tau + \frac{1}{\mathcal{R}_\varepsilon} \int_0^t \|\operatorname{div} u(\tau)\|_{L^2}^2 d\tau \\ = \frac{1}{2} \|v(0)\|_{L^2}^2 + \int_0^t \int_\Omega f(\tau) \cdot v(\tau) d\omega d\tau. \end{aligned}$$

Moreover, from estimate (10),

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^t \int_\Omega f(\tau) \cdot \tilde{v}_\varepsilon(\tau) d\omega d\tau = 0.$$

So, we have proved that

$$\lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(t) = 0.$$

In other words, we have established the following result

$$\begin{aligned} v_\varepsilon &\rightarrow v \text{ in } C^0([0, T]; L^2(\Omega)) \cap L^2([0, T]; \mathbf{H}_v^1(\Omega)) \\ v &\in C^0([0, T]; H) \cap L^2([0, T]; H_0^1(\Omega)) \end{aligned}$$

This concludes the proof of theorem 2.3.

We are now able to precise the convergence for the effective pressure and establish the theorem 2.4

Let us write the equation satisfied by the velocity v_ε and the pressure p_ε in the distribution sense. We have

$$\frac{\partial v_\varepsilon}{\partial t} + (v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon + \frac{1}{\mathcal{R}_\varepsilon} \operatorname{curl} \operatorname{curl} v_\varepsilon + \nabla \left(p_\varepsilon - \left(\frac{1+\varepsilon}{\varepsilon \mathcal{R}_\varepsilon} + r \right) \operatorname{div} v_\varepsilon \right) = f.$$

Introducing the effective pressure

$$q_\varepsilon = p_\varepsilon - \left(\frac{1 + \varepsilon}{\varepsilon \mathcal{R}_e} + r \right) \operatorname{div} v_\varepsilon,$$

this equation reads

$$\nabla q_\varepsilon = f - \left(\frac{\partial v_\varepsilon}{\partial t} + (v_\varepsilon \cdot \nabla) v_\varepsilon + \frac{1}{2} (\operatorname{div} v_\varepsilon) v_\varepsilon + \frac{1}{\mathcal{R}_e} \operatorname{curl} \operatorname{curl} v_\varepsilon \right)$$

and the proof follows, from the previous steps.

5. Appendix. Let us consider the following problem:

Proposition 4. *For a fixed $\varepsilon > 0$ and a couple of functions (f, g) given in $L_0^2(\Omega) \times L^2(\Omega)$, there exists a unique solution $v_\varepsilon \in \{w \in \mathbf{H}_{div}(\Omega), w \cdot \nu|_\Gamma = 0\}$ solution of:*

$$\varepsilon v_\varepsilon - \nabla \operatorname{div} v_\varepsilon = \nabla f + \varepsilon g. \quad (30)$$

Moreover, if $(u, u_1) \in \mathbf{H}_{div}(\Omega) \times H$ is solution of

$$\begin{aligned} \operatorname{div} u &= -f, \operatorname{curl} u = 0, \\ u \cdot \nu|_\Gamma &= 0, \end{aligned} \quad (31)$$

$$\operatorname{div} u_1 = 0, \operatorname{curl} u_1 = \operatorname{curl} g,$$

$$u_1 \cdot \nu|_\Gamma = 0,$$

and we have the following estimate

$$\|v_\varepsilon - u - u_1\|_{H^1} \leq \varepsilon \|u + u_1 - g\|_{L^2}.$$

Proof. Step 1: Existence of v_ε

Let us note $\mathbf{H}_{div,0}(\Omega) = \{v \in L^2(\Omega), \operatorname{div} v \in \mathbf{L}^2, v \cdot \nu|_\Gamma = 0\}$. The existence of a unique solution to the equation (30) is obtained by a straightforward application of the Lax-Milgram theorem with the bilinear form defined on $\mathbf{H}_{div,0}(\Omega)$ by

$$\varepsilon(u, v) + (\operatorname{div} u, \operatorname{div} v),$$

and the right-hand side: $-(f, \operatorname{div} v) + \varepsilon(g, v)$.

Step 2: existence of u and u_1

The existence of u satisfying (31) comes from the resolution of the following Neumann problem

$$\begin{aligned} -\Delta q &= f, \\ \nabla q \cdot \nu|_\Gamma &= 0, \end{aligned}$$

and we set $u = \nabla q$, with $q \in (H^1(\Omega)/\mathbb{R}) \cap H^2(\Omega)$.

The existence of $u_1 \in \mathbf{H}$ is the consequence of the Leray projection applied to g by writing

$$\begin{aligned} g &= u_1 + \nabla p, \\ \operatorname{div} u_1 &= 0, \\ u_1 \cdot \nu|_\Gamma &= 0. \end{aligned}$$

Now, witting $v_\varepsilon = u + u_1 + u_\varepsilon$, we get that $u_\varepsilon \in \mathbf{H}_{div,0}$ satisfies

$$\begin{aligned} \varepsilon u_\varepsilon - \nabla \operatorname{div} u_\varepsilon &= -\varepsilon u - \varepsilon u_1 + \varepsilon g, \\ \operatorname{curl} u_\varepsilon &= 0, \end{aligned}$$

and we have the estimate

$$\varepsilon \|u_\varepsilon\|_{L^2}^2 + \|\operatorname{div} u_\varepsilon\|_{L^2}^2 \leq \varepsilon \|u + u_1 - g\|_{L^2} \|u_\varepsilon\|_{L^2}.$$

We observe that, according to [8], $\mathbf{H}_\nu^1(\Omega)$ endowed with the norm

$$\left(\|\operatorname{div} v\|^2 + \|\operatorname{curl} v\|^2\right)^{\frac{1}{2}}$$

is equal to $\{w \in (H^1(\Omega))^d, w \cdot \nu_\gamma\}$. So the previous estimate gives

$$\begin{aligned} \varepsilon \|u_\varepsilon\|_{L^2}^2 + \|\operatorname{div} u_\varepsilon\|_{L^2}^2 + \|\operatorname{curl} u_\varepsilon\|_{L^2}^2 &\leq \varepsilon \|u + u_1 - g\|_{L^2} \|u_\varepsilon\|_{H_\nu^1} \\ &\leq C\varepsilon \|u + u_1 - g\|_{L^2} \left(\|\operatorname{div} u_\varepsilon\|_{L^2}^2 + \|\operatorname{curl} u_\varepsilon\|_{L^2}^2\right)^{\frac{1}{2}}, \end{aligned}$$

which implies using Young inequality

$$2\varepsilon \|u_\varepsilon\|_{L^2}^2 + \|\operatorname{div} u_\varepsilon\|_{L^2}^2 \leq \varepsilon^2 \|u + u_1 - g\|_{L^2}^2,$$

and the proof of proposition 4 follows. \square

Remark 4.

1. The function u belongs to $\mathbf{H}_\nu^1(\Omega)$, which is not the case for v_ε or u_1 , without some additional regularity hypotheses on the function g . Nevertheless, the function $u_\varepsilon = v_\varepsilon - u - u_1$ belongs to $\mathbf{H}_\nu^1(\Omega)$.
2. In the case where $g = 0$, the function v_ε belongs to $\mathbf{H}_\nu^1(\Omega)$.

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Received 23/07 2011; revised 17/12 2011.

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