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VALUATION OF THE PREPAYMENT OPTION OF A PERPETUAL CORPORATE LOAN

TIMOTHEE PAPIN* AND GABRIEL TURINICI†

Abstract. We investigate in this paper a perpetual prepayment option related to a corporate loan. The default intensity of the firm is supposed to follow a CIR process. Two frameworks are discussed: first a constant interest rate and a secondly a multi-regime framework where the interest rate is augmented by a liquidity factor dependent on the regime. The prepayment option needs specific attention as the payoff itself is an implicit function of the parameters of the problem and of the dynamics. We establish in the unique regime case analytic formulas for the payoff of the option; in both cases we give a verification result that allows to compute the price of the option. Numerical results that implement the findings are also presented and are completely consistent with the theory; it is seen that when liquidity parameters are very different (i.e., when a liquidity crisis occur) in the high liquidity cost regime the exercise domain may entirely disappear meaning that it is not optimal for the borrower to prepay during such a liquidity crisis. The method allows to quantify and interpret these findings.

Key words. liquidity regime, loan prepayment, mortgage option, American option, perpetual option, option pricing, Snell envelope, prepayment option, CIR process, switching regimes, Markov modulated dynamics.

AMS subject classifications. 91G20, 91G30, 91G40, 91G50, 91G60, 91G80, 93E20

1. Introduction. When a firm needs money it can turn to its bank which lends it against e.g., periodic payments in a form of a loan. A loan contract issued by a bank for its corporate clients is a financial agreement that often comes with more flexibility than a retail loan contract. These options are designed to meet clients' expectations and can include e.g., a prepayment option (which entitles the client, if he desires so, to pay all or a fraction of its loan earlier than the maturity), a multi-currency option, a multi-index option, etc. On the other hand, there are also some mechanisms to protect the lender from the deterioration of the borrower's credit quality e.g., a pricing grid based on the borrower rating or protecting guaranties.

The main option remains however the prepayment option and it will be the subject of this entire paper. In almost every loan contract, the borrower has the free option to prepay a portion or all the nominal. Even if the technicalities are, as it will be seen in the following, different, the concept of this option is very close to the embedded option of a callable bond. When market interest rates have gone down the issuer of the bond can buy back his debt at a defined call price before the bond reaches its maturity date. It allows the issuer to refinance its debt at a cheaper rate.

In order to decide whether the exercise of the option is worthwhile the borrower compares the remaining payments (actualized by the interest rate he can obtain at that time) with the nominal value. If the remaining payments exceed the nominal value then it is optimal for the borrower to refinance his debt at a lower rate.

When the interest rates are not constant or borrower is subject to default the computation of the actualization is less straightforward. It starts with considering all possible scenarios of evolution for interest rate and default intensity in a risk-neutral framework and compute the average value of remaining payments (including the final payment of the principal if applicable); this quantity will be called "$PVRP$"(denoted ξ) and is the *present value of the remaining payments* i.e., the cash amount equivalent,

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both for borrower and lender in this model of the set of remaining payments. The $PVRP$ is compared with the nominal : if the $PVRP$ value is larger than the nominal then the borrower should prepay, otherwise not. Recall that at the initial time the payments correspond to a rate, the sum of the interest rate and a contractual margin $\overline{\rho_0}$, which is precisely making the two quantities equal. Note that in order to compute the price of the embedded prepayment option the lender also uses the $PVRP$ as it will be seen below.

For a bank, the prepayment option is essentially a reinvestment risk i.e., the risk that the borrower decides to repay earlier his/her loan and that the bank can not reinvest his/her excess of cash in a new loan. So the longest the maturity of the loan, the riskier the prepayment option. Therefore, it is interesting to study long-term loans that are set for more than three years and can run for more than twenty years. The valuation problem of the prepayment option can be modelled as an American embedded option on a risky debt owned by the borrower. As Monte-Carlo simulations are slow to converge to assess accurately the continuation value of the option during the life of the loan and that the binomial tree techniques are time-consuming for long-term loans (cf. works by D. Cossin et al. [8]), we decided to focus, in this paper, on the prepayment option for perpetual loan.

When valuing financial products with long maturity the robustness with respect to shocks and other exogenous variabilities is important. Among problems that have to be treated is the liquidity and its variability. Liquidity is the key of the stability of the entire financial system and can cause banks' failures if systemic liquidity squeezes appear in the financial industry. Historical events like the Asian crisis of 1997; the Russian financial crisis of 1998; the defaults of hedge funds and investment firms like LTCM, Enron, Worldcom and Lehman Brothers defaults, sovereign debts crisis of 2010-11 and so on prove that banks hold significant liquidity risk in their balance sheets. Even if liquidity problems have a very low probability to occur, a liquidity crisis can have a severe impact on a bank's funding costs, its market access (reputation risk) and short-term funding capabilities.

Following the state of the economic environment, the liquidity can be defined by distinct states. Between two crisis, investors are confident and banks find it easier to launch their long term refinancing programs through regular bonds issuances. Thus the liquidity market is stable. Unfortunately, during crisis, liquidity become scarce, pushing the liquidity curve to very high levels which can only decrease if confidence returns to the market. The transition is between these two distinct behaviors is rarely smooth but rather sudden.

In order to model the presence of distinct liquidity behaviors we will simulate the liquidity cost by a continuous time Markov chain that can have a discrete set of possible values, one for each regime that is encountered in the liquidity evolution.

From a technical point of view this paper faces several non-standard conditions: although the goal is to value a perpetual American option the payoff of the option is highly non-standard (is dependent on the $PVRP$). As a consequence the characterization of the exercice region is not standard and technical conditions have to be met. Furthermore our focus here is on a specific type of dynamics (of CIR type) with even more specific interest on the situation when several regimes are present.

The balance of the paper is as follows: in the remainder of this section (Sub-Section 1.1) we review the related existing literature; in Section 2, we consider that the liquidity cost is negligible and the borrower credit risk defined by his/her default intensity (called in the following simply "intensity") which follows a CIR stochastic

process. We are able to obtain in this situation quasi-analytic formulas for the prepayment option price. In Section 3 we explore the situation when the liquidity cost, defined as the cost of the lender to access the cash on the market, has several distinct regimes that we model by a Markov chain. We write the pricing formulas and theoretically support an algorithm to identify the boundary of the exercice region; final numerical examples close the paper.

1.1. Related literature. There exist few articles (e.g., works by D. Cossin et al. [8]) on the loan prepayment option but a close subject, the prepayment option in fixed-rate mortgage loan, has been widely covered in several papers by J.E. Hilliard and J.B. Kau [11] and more recent works by Chen et al. [6]. To approximate the PDE satisfied by the prepayment option, they define two state variables (interest rate and house price). Their approach is based on a bivariate binomial option pricing technique with a stochastic interest rate and a stochastic house value.

Another contribution by D. Cossin et al. [8] applies the binomial tree technique (but of course it is time-consuming for long-term loans due to the nature of binomial trees) to corporate loans. They consider a prepayment option with a 1 year loan with a quarterly step but it is difficult to have an accurate assessment of the option price for a 10 years loan.

There also exist mortgage prepayment decision models based on Poisson regression approach for mortgage loans. See, for example, E.S. Schwartz and W.N. Torous [21]. Unfortunately, the volume and history of data are very weak in the corporate loan market.

Due to the form of their approach, these papers did not have to consider the geometry of the exercice region because it is explicitly given by the numerical algorithm. This is not the case for us and requires that particular care be taken when stating the optimality of the solution. Furthermore, to the best of our knowledge, none of these approaches explored the circumstance when several regimes exist.

The analysis of Markov-modulated regimes has been investigated in the literature when the underlying(s) follow the Black& Scholes dynamics with drift and volatility having Markov jumps; several works are of interest in this area: Guo and Zhang [25] have derived the closed-form solutions for vanilla American put; Guo analyses in [10] Russian (i.e., perpetual look-back) options and is able to derive explicit solutions for the optimal stopping time; in [23] Y. Xu and Y. Wu analyse the situation of a two-asset perpetual American option where the pay-off function is a homogeneous function of degree one; Mamon and Rodrigo [17] find explicit solutions to vanilla European options. Buffington and Elliott [4] study European and American options and obtain equations for the price. A distinct approach (Hopf factorization) is used by Jobert and Rogers [14] to derive very good approximations of the option prices for, among others, American puts. Other contributions include [24, 22] etc.

Works involving Markov switched regimes and CIR dynamics appears in [9] where the bond valuation problem is considered (but not in the form of an American option; their approach will be relevant to the computation of the payoff of our American option although in their model only the mean reverting level is subject to Markov jumps) and in [26] where the term structure of the interest rates is analyzed.

On the other hand numerical methods are proposed in [12] where it is found that a fixed point policy iteration coupled with a direct control formulation seems to perform best.

Finally, we refer to [13] for theoretical results concerning the pricing of American options in general.

2. Perpetual prepayment option with a stochastic intensity CIR model. We assume throughout the paper that the interest rate r is constant. Therefore, the price of the prepayment option only depends on the intensity evolution over time. We model the intensity dynamics by a Cox-Ingersoll-Ross process (see [5, 2, 15] for theoretical and numerical aspects of CIR processes and the situations where the CIR process has been used in finance):

$$d\lambda_s = \gamma(\theta - \lambda_s)ds + \sigma\sqrt{\lambda_s}dW_s,\ \gamma, \theta, \sigma > 0,\ \lambda_0 = \overline{\lambda_0} \tag{2.1}$$

It is known that if $2\gamma\theta \geq \sigma^2$ then CIR process ensure an intensity strictly positive. Fortunately, as it will be seen in the following, the PVRP is given by an analytic formula.

2.1. Analytical formulas for the PVRP. Assuming a loan has a fixed coupon defined by the interest rate r and an initial contractual margin $\overline{\rho_0}$. Let $\xi(t, T, \lambda)$ be, the present value of the remaining payments at time t of a corporate loan with initial contractual margin $\overline{\rho_0}$ (depending on $\overline{\lambda_0}$), intensity at time t, λ_t, following the risk-neutral equation (2.1) with $\lambda_t = \lambda$, has nominal amount K and contractual maturity T.

Therefore the loan value $LV(t, T, \lambda)$ is equal to the present value of the remaining payments $\xi(t, T, \lambda)$ minus the prepayment option value $P(t, T, \lambda)$.

$$LV(t, T, \lambda) = \xi(t, T, \lambda) - P(t, T, \lambda) \tag{2.2}$$

The ξ is the present value of the cash flows discounted at the (instantaneous) risky rate $r + \lambda_t$,. The cash flows are $K(r + \overline{\rho_0})$ and the final payment of the principal K. Then:

$$\xi(t, T, \lambda) = \mathbb{E}\Big[K \cdot (r + \overline{\rho_0}) \int_t^T e^{-\int_t^{\tilde{t}}(r+\lambda_u)du} d\tilde{t} + Ke^{-\int_t^T r+\lambda_u du}\Big|\lambda_t = \lambda\Big] \tag{2.3}$$

For a perpetual loan the maturity $T = +\infty$. Since λ_t is always positive $r + \lambda_t \geq r > 0$ and thus the last term tend to zero when $T \to \infty$. A second remark is that since μ and σ independent of time, ξ is independent of the starting time t :

$$\xi(t, \lambda) = \mathbb{E}\left[K \cdot (r + \overline{\rho_0}) \int_t^{+\infty} e^{-\int_t^{\tilde{t}} r+\lambda_u du} d\tilde{t}\Big|\lambda_t = \lambda\right] \tag{2.4}$$

$$= \mathbb{E}\left[K \cdot (r + \overline{\rho_0}) \int_0^{+\infty} e^{-\int_0^{\tilde{t}} r+\lambda_u du} d\tilde{t}\Big|\lambda_0 = \lambda\right] =: \xi(\lambda), \tag{2.5}$$

where the last equality is a definition. For a CIR stochastic process, we obtain (see [5, 15]),

$$\xi(\lambda) = K \cdot (r + \overline{\rho_0}) \int_0^{+\infty} e^{-r\tilde{t}} B(0, \tilde{t}, \lambda) d\tilde{t} \tag{2.6}$$

where for general $t, \tilde{t}$ we use the notation:

$$B(t, \tilde{t}, \lambda) = \mathbb{E}\left[e^{-\int_t^{\tilde{t}} \lambda_u du}\Big|\lambda_t = \lambda\right]. \tag{2.7}$$

Note that $B(t, \tilde{t}, \lambda)$ is a familiar quantity: it is formally the same formula as the price of a zero-coupon where the interest rates follow a CIR dynamics. Of course here the

interest rate is constant and the intensity is following a CIR dynamics nevertheless the same formula applies for general $t, \tilde{t}$:

$$B(t,\tilde{t},\lambda) = \alpha(t,\tilde{t})e^{-\beta(t,\tilde{t})\lambda} \tag{2.8}$$

with,

$$\begin{aligned}\alpha(t,\tilde{t}) &= \left(\frac{2h\ \ e^{(\gamma+h)\frac{\tilde{t}-t}{2}}}{2h+(\gamma+h)(e^{(\tilde{t}-t)h}-1)}\right)^{\frac{2\gamma\theta}{\sigma^2}} \\ \beta(t,\tilde{t}) &= \frac{2(e^{(\tilde{t}-t)h}-1)}{2h+(\gamma+h)(e^{(\tilde{t}-t)h}-1)},\ \text{where } h=\sqrt{\gamma^2+2\sigma^2}.\end{aligned} \tag{2.9}$$

Obviously $B(0,t,\lambda)$ is monotonic with respect to λ, thus the same holds for ξ.

The margin $\overline{\rho_0}$ is the solution of the following equilibrium equation:

$$\xi(\overline{\lambda_0}) = K \tag{2.10}$$

which can be interpreted as the fact that the present value of the cash flows (according to the probability of survival) is equal to the nominal K:

$$\overline{\rho_0} = \frac{1}{\int_0^{+\infty} e^{-r\tilde{t}}B(0,\tilde{t},\overline{\lambda_0})d\tilde{t}} - r. \tag{2.11}$$

Note that we assume no additional commercial margin.

REMARK 1. *If an additional commercial margin μ_0 is considered then $\overline{\rho_0}$ is first computed as above and then replaced by $\overline{\overline{\rho_0}} = \overline{\rho_0} + \mu_0$ in Equation (2.6). Equations (2.10) and (2.11) will not be verified as such but will still hold with some $\overline{\overline{\lambda_0}}$ instead of $\overline{\lambda_0}$; for instance we will have*

$$\overline{\overline{\rho_0}} = \frac{1}{\int_0^{+\infty} e^{-r\tilde{t}}B(0,\tilde{t},\overline{\overline{\lambda_0}})d\tilde{t}} - r. \tag{2.12}$$

With these changes all results in the paper are valid, except that when computing for operational purposes once the price of the prepayment option is computed for all λ one will use $\lambda = \overline{\lambda_0}$ as price relevant to practice.

From definition (2.7) of $B(t,\tilde{t},\lambda)$ it follows that $B(t,\tilde{t},\lambda) < 1$ thus

$$e^{-r\tilde{t}}B(0,\tilde{t},\overline{\lambda_0}) < e^{-r\tilde{t}}$$

and as consequence

$$\int_0^{+\infty} e^{-r\tilde{t}}B(0,\tilde{t},\overline{\lambda_0})d\tilde{t} < \int_0^{+\infty} e^{-r\tilde{t}}d\tilde{t} = 1/r \tag{2.13}$$

which implies that $\overline{\rho_0} > 0$.

2.2. Valuation of the prepayment option. The valuation problem of the prepayment option can be modelled as an American call option on a risky debt owned by the borrower. Here the prepayment option allows borrower to buy back and refinance his/her debt according to the current contractual margin at any time during the life of the option. As the perpetual loan, the option value will be assumed independent of the time t.

As discussed above, the prepayment exercise results in a pay-off $(\xi(t,T,\lambda) - K)^+$ for the borrower. The option is therefore an American call option on the risky asset $\xi(t,T,\lambda_t)$ and the principal K (the amount to be reimbursed) being the strike. Otherwise we can see it as an American option on the risky λ_t with pay-off,

$$\chi(t,\lambda) := (\xi(t,\lambda) - K)^+ \tag{2.14}$$

or, for our perpetual option:

$$\chi(\lambda) := (\xi(\lambda) - K)^+. \tag{2.15}$$

We will denote by $\mathcal{A}$ the characteristic operator (cf. [27, Chapter 7.5]) of the CIR process i.e. the operator that acts on any C^2 class function v by

$$(\mathcal{A}v)(\lambda) = \gamma(\theta - \lambda)\partial_\lambda v(\lambda) + \frac{1}{2}\sigma^2\lambda\partial_{\lambda\lambda}v(\lambda). \tag{2.16}$$

Denote for $a, b \in \mathbb{R}$ and $x \geq 0$ by $U(a,b,x)$ the solution to the confluent hypergeometric differential (also known as the Kummer) equation [1]:

$$Xz'' + (b - X)z' - az = 0 \tag{2.17}$$

that increase at most polynomially at infinity and is finite (not null) at the origin. Recall also that this function is proportional to the the confluent hypergeometric function of the second kind $U(a,b,x)$ (also known as the Kummer's function of the second kind, Tricomi function, or Gordon function); for $a, x > 0$ the function $U(a,b,x)$ is given by the formula:

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^{+\infty} e^{-xt}t^{a-1}(1+t)^{b-a-1}dt. \tag{2.18}$$

When $a \leq 0$ one uses other representations (see the cited references; for instance one can use a direct computation or the recurrence formula $U(a,b,x) = (2a - b + z - 2)U(a+1,b,x) - (a+1)(a-b+2)U(a+2,b,x)$); it is known that $U(a,b,x)$ behaves as x^{-a} at infinity. Also introduce for $x \geq 0$:

$$W(x) = e^{x\frac{\gamma-\kappa}{\sigma^2}}x^{\frac{\sigma^2-2\gamma\theta}{\sigma^2}}U\left(-\frac{-r\sigma^2 - \sigma^2\kappa + \gamma^2\theta + \gamma\kappa\theta}{\sigma^2\kappa}, 2 - \frac{2\gamma\theta}{\sigma^2}, \frac{2\kappa}{\sigma^2}x\right), \tag{2.19}$$

where $\kappa = \sqrt{\gamma^2 + 2\sigma^2}$.

THEOREM 2.

1. *Introduce for $\Lambda > 0$ the family of functions: $P_\Lambda(\lambda)$ such that:*

$$P_\Lambda(\lambda) = \chi(\lambda)\ \forall\lambda \in [0,\Lambda] \tag{2.20}$$
$$(\mathcal{A}P_\Lambda)(\lambda) - (r+\lambda)P_\Lambda(\lambda) = 0,\ \forall\lambda > \Lambda \tag{2.21}$$
$$\lim_{\lambda\to\Lambda} P_\Lambda(\lambda) = \chi(\Lambda), \tag{2.22}$$
$$\lim_{\lambda\to\infty} P_\Lambda(\lambda) = 0. \tag{2.23}$$

Then

$$P_\Lambda(\lambda) = \begin{cases} \chi(\lambda) & \forall\lambda \in [0,\Lambda] \\ \frac{\chi(\Lambda)}{W(\Lambda)}W(\lambda) & \forall\lambda \geq \Lambda. \end{cases} \tag{2.24}$$

2. Suppose now a $\Lambda^* \in]0, \overline{\rho_0} \wedge \overline{\lambda_0}[$ exists such that:

$$\frac{dP_{\Lambda^*}(\lambda)}{d\lambda}\Big|_{\lambda=(\Lambda^*)^+} = \frac{d\chi(\lambda)}{d\lambda}\Big|_{\lambda=(\Lambda^*)^-}. \tag{2.25}$$

Then the price of the prepayment option is $P(\lambda) = P_{\Lambda^*}(\lambda)$.

Proof. We start with the first item: it is possible to obtain a general solution of (2.21) in an analytic form. We recall that $z(X) = U(a, b, X)$ is the solution of the Kummer equation (2.17). A cumbersome but straightforward computation shows that the general solution vanishing at infinity of the PDE (2.21) is $W(\lambda)$ thus

$$P_\Lambda(\lambda) = C_\Lambda W(\lambda) \ \forall \lambda > \Lambda \tag{2.26}$$

with some $C_\Lambda > 0$ to be determined. Now use the boundary conditions. If $\lambda = \Lambda$ by continuity $\chi(\Lambda) = P_\Lambda(\Lambda) = C_\Lambda W(\Lambda)$. Thus, $C_\Lambda = \frac{\chi(\Lambda)}{W(\Lambda)}$. Division by W is legitimate because by definition, $W(x) > 0$ for all $x > 0$.

We now continue with the second part of the theorem. The valuation problem of an American option goes through several steps: first one introduces the admissible trading and consumptions strategies cf. [18, Chapter 5]; then one realizes using results in cited reference (also see [19, 15]) that the price $P(\lambda)$ of the prepayment option involves computing a stopping time associated to the pay-off. Denote by $\mathcal{T}$ the ensemble of (positive) stopping times; we conclude that:

$$P(\lambda) = \sup_{\tau \in \mathcal{T}} \mathbb{E}(e^{-\int_0^\tau r + \lambda_u du} \chi(\lambda_\tau) | \lambda_0 = \lambda). \tag{2.27}$$

Further results derived for the situation of a perpetual (standard) American put options [13, 3] show that the stopping time has a simple structure: a critical level exists that split the positive axis into two regions: to the left the exercice region where it is optimal to exercice and where the price equals the payoff and a continuation region (to the right) where the price satisfies a partial differential equation similar to Black-Scholes equation. We refer to [7] for how to adapt the theoretical arguments for the situation when the dynamics is not Black-Scholes like but a CIR process.

The result builds heavily on the fact that the discounted payoff of the standard situation of an American put $e^{-rt}(S-K)^-$, is a submartingale. For us the discounted payoff is

$$e^{-\int_0^t r + \lambda_u du} \chi(\lambda_t) = e^{-\int_0^t r + \lambda_u du} (\xi(\lambda_t) - K)^+ \tag{2.28}$$

and checking this condition requires here more careful examination which is the object of Lemma 2.1. It is now possible to apply Thm. 10.4.1 [27, Section 10.4 page 227] (see also [7] for specific treatment of the CIR process) which will show that $P(\lambda)$ is the true option price if the following conditions are satisfied:

1. on $]0, \Lambda^*[$ we have $P(\lambda) = \chi(\lambda) = (\xi(\lambda) - K)^+$ and the relation (2.34) holds;
2. on $]\Lambda^*, \infty[$ the solution candidate $P(\lambda)$ satisfies the relation

$$(\mathcal{A}P)(\lambda) - (r + \lambda)P(\lambda) = 0, \ \forall \lambda > \Lambda^*. \tag{2.29}$$

3. the function $P(\lambda)$ is C^1 everywhere, continuous at the origin and C^2 on each sub-interval $]0, \Lambda^*[$ and $]\Lambda^*, \infty[$.

The theorem also says that the borrower exercises his option on *the exercise region* $[0, \Lambda^*]$ while on the *continuation region* $]\Lambda^*, \infty[$ the borrower keeps the option because it is worth more non-exercised.

We now show that P_{Λ^*} verifies all conditions above which will allow to conclude that $P = P_{\Lambda^*}$. The requirement 1 is treated in Lemma 2.1; the requirement 3 amounts at asking that the optimal frontier value Λ^* be chosen such that:

$$\frac{dP_{\Lambda^*}(\lambda)}{d\lambda}\Big|_{\lambda=(\Lambda^*)^+} = \frac{d\chi(\lambda)}{d\lambda}\Big|_{\lambda=(\Lambda^*)^-}. \tag{2.30}$$

The requirement 2 implies that in the continuation region the price is the solution of the following PDE:

$$(\mathcal{A}P)(\lambda) - (r+\lambda)P(\lambda) = 0, \ \forall \lambda > \Lambda^*. \tag{2.31}$$

For this PDE we need boundary conditions. The condition at $\lambda = \Lambda^*$ is

$$P(\lambda)\Big|_{\lambda=\Lambda^*} = \chi(\lambda)\Big|_{\lambda=\Lambda^*}. \tag{2.32}$$

When $\lambda = +\infty$ the default intensity is infinite thus the time to failure is zero thus the borrower has failed ; in this case the option is worthless i.e.

$$\lim_{\lambda\to\infty} P(\lambda) = 0. \tag{2.33}$$

These conditions give exactly the definition of P_{Λ^*}, q.e.d. □

LEMMA 2.1. *The following inequality holds:*

$$(\mathcal{A}\chi)(\lambda) - (r+\lambda)\chi(\lambda) < 0, \ \forall \lambda < \overline{\rho_0} \wedge \overline{\lambda_0}. \tag{2.34}$$

Proof. Recall that $\chi(\lambda) = (\xi(\lambda) - K)^+$; the definition (2.5) of ξ implies (cf. [27, Section 8.2 and exercice 9.12 p 203]) that ξ is solution of the following PDE:

$$(\mathcal{A}\xi)(\lambda) - (r+\lambda)\xi(\lambda) + (r+\overline{\rho_0})K = 0, \ \forall \lambda > 0. \tag{2.35}$$

For $\lambda < \overline{\lambda_0}$ we have $\xi(\lambda) > K = \xi(\overline{\lambda_0})$ thus

$$\begin{aligned}
&\Big(\mathcal{A}(\xi(\cdot) - K)^+\Big)(\lambda) - (r+\lambda)(\xi(\lambda) - K)^+ && (2.36)\\
&= \Big(\mathcal{A}(\xi(\cdot) - K)\Big)(\lambda) - (r+\lambda)\Big(\xi(\lambda) - K\Big) && (2.37)\\
&= (\mathcal{A}\xi)(\lambda) - (r+\lambda)\xi(\lambda) + (r+\lambda)K && (2.38)\\
&= -(r+\overline{\rho_0})K + (r+\lambda)K = (\lambda - \overline{\rho_0})K < 0 \ \forall \lambda < \overline{\rho_0} \wedge \overline{\lambda_0}. && (2.39)
\end{aligned}$$

□

Note that the Theorem 2 is only a sufficient result (a so-called "verification" result) ; under the assumption that a Λ^* fulfilling the hypotheses of the Theorem exist the question is how to find it.

Two approaches can be considered; first, it is enough to find a zero of the following function $\Lambda \mapsto \Upsilon(\Lambda) := \left(\frac{\partial P_\Lambda(\lambda)}{\partial\lambda}\Big|_{\lambda=\Lambda^+} - \frac{\partial\chi(\lambda)}{\partial\lambda}\Big|_{\lambda=\Lambda^-}\right)$ (the last equality is a definition). Of course $\frac{\partial\chi(\lambda)}{\partial\lambda}\Big|_{\lambda=\overline{\lambda_0}+\epsilon} = 0$ and $\frac{\partial P_{\overline{\lambda_0}+\epsilon}(\lambda)}{\partial\lambda}\Big|_{\lambda=\overline{\lambda_0}+\epsilon} < 0$ thus $\Upsilon(\overline{\lambda_0}+\epsilon) < 0$ for any $\epsilon > 0$

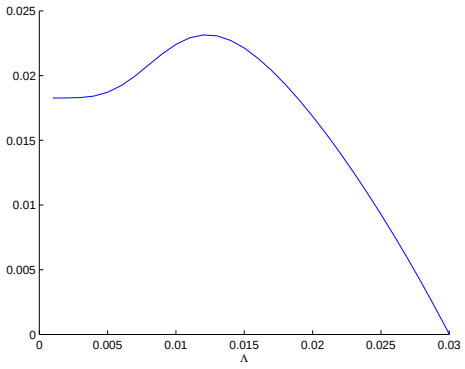

FIG. 2.1. *We illustrate here the dependence of $P_\Lambda(\overline{\lambda_0})$ as a function of Λ; this allows to find the optimal value Λ^* that maximize the option price. For the numerical example described here we obtain $\Lambda^* = 123$ bps.*

hence $\Upsilon(\overline{\lambda_0}) \leq 0$. Thus is it natural not to look for Λ^* outside the interval $[0, \overline{\lambda_0}]$. The theorem asks furthermore to restrict the search to the interval $[0, \overline{\lambda_0} \wedge \overline{\rho_0}]$.

A different convenient procedure to find the critical Λ^* is to consider the dependence $\Lambda \mapsto P_\Lambda(\overline{\lambda_0})$. Let us consider the stopping time τ_Λ that stops upon entering the domain $[0, \Lambda]$. We remark that by a Feynman-Kac formula (cf. [27, p 203])

$$P_\Lambda(\lambda) = \mathbb{E}(e^{-\int_0^{\tau_\Lambda} r+\lambda_u du} \chi(\lambda_{\tau_\Lambda}) | \lambda_0 = \lambda). \tag{2.40}$$

From (2.27) $P(\lambda) \geq P_\Lambda(\lambda)$ for any λ thus Λ^* is the value that maximizes (with respect to Λ) the function $\Lambda \mapsto P_\Lambda(\overline{\lambda_0})$. To comply with the theorem the maximization is performed in the interval $[0, \overline{\lambda_0} \wedge \overline{\rho_0}]$.

2.3. Numerical Application. We consider a perpetual loan ($T = +\infty$) with a nominal amount $K = 1$ and the borrower default intensity λ_t follows a CIR dynamics with parameters: initial intensity $\overline{\lambda_0} = 300$ bps, volatility $\sigma = 0.05$, average intensity $\theta = 200$ bps, reversion coefficient $\gamma = 0.5$. We assume a constant interest rate $r = 300bps$ i.e., $r = 3\%$. Recall that a basis point, denoted "1 bps" equals 10^{-4}.

In order to find the initial contractual margin we use equation (2.11) and find $\overline{\rho_0} = 208$ bps.

At inception, the present value of cash flows is at par, so $\xi(\overline{\lambda_0}) = 1$. The prepayment option price is $P(+\infty, \overline{\lambda_0}) = 0.0232$ i.e., $P(\overline{\lambda_0}) = 2.32\% \cdot K$. Therefore the loan value equals $\xi(\overline{\lambda_0}) - P(\overline{\lambda_0}) = 0,9768$.

The value $\Lambda^* = 123$ bps is obtained by maximizing $P_\Lambda(\overline{\lambda_0})$ as indicated in the Remarks above; the dependence of $P_\Lambda(\overline{\lambda_0})$ with respect to Λ is illustrated in Figure 2.1. The loan value will equal to par if the intensity decreases until the exercise region ($\lambda < \Lambda^*$) see Figures 2.2. The continuation and exercise regions are depicted in Figure 2.3. We postpone to Section 3.5 the description of the numerical method to solve (2.21).

3. Perpetual prepayment option with a switching regime. In this second part, the perpetual prepayment option is still an option on the credit risk, intensity, but now also the liquidity cost. The liquidity cost is defined as the specific cost of a bank to access the cash on the market. This cost will be modelled with a switching regime with a Markov chain of finite states of the economy. We assume an *interbank*

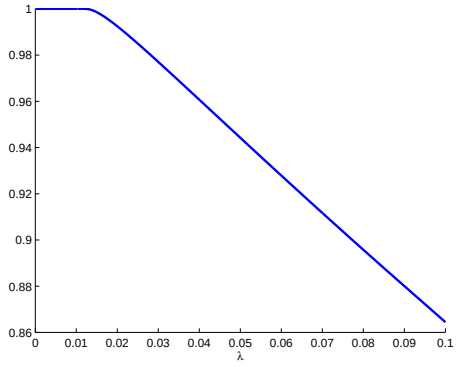

FIG. 2.2. *Loan value as a function of the intensity. The loan value is decreasing when there is a degradation of the credit quality (i.e., λ increases) and converges to* 0.

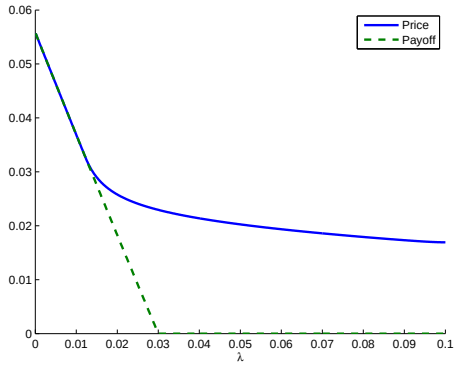

FIG. 2.3. *Prepayment option price $P(\lambda)$ (solid line) and payoff $\chi(\lambda)$ (dashed line) as a function of the intensity λ. Two regions appear : the continuation region $\lambda > \Lambda^*$ and the exercise region $\lambda \leq \Lambda^*$.*

offered rate IBOR r to be constant. Therefore, the assessment of the loan value and its prepayment option is a N-dimensional problem. The intensity is still defined by a Cox-Ingersoll-Ross process with $2k\theta \geq \sigma^2$:

$$d\lambda_t = \gamma(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \ \lambda_0 = \overline{\lambda_0}. \tag{3.1}$$

3.1. Theoretical regime switching framework. We assume the economic state of the market is described by a finite state Markov chain $\mathbf{X} = \{X_t, t \geq 0\}$. The state space $\mathbf{X}$ can be taken to be, without loss of generality, the set of unit vectors $E = \{e_1, e_2, ..., e_N\}$, $e_i = (0, ..., 0, 1, 0, ..., 0)^T \in \mathbb{R}^N$. Here T is the transposition operator.

Assuming the process X_t is homogeneous in time and has a rate matrix A, then if $p_t = \mathbb{E}[X_t] \in \mathbb{R}^N$,

$$\frac{dp_t}{dt} = Ap_t \tag{3.2}$$

and,

$$X_t = X_0 + \int_0^t AX_u du + M_t, \tag{3.3}$$

where $\mathbf{M} = \{M_t, t \geq 0\}$ is a martingale with respect to the filtration generated by $\mathbf{X}$. In differential form

$$dX_t = AX_t dt + dM_t,\ X_0 = \overline{X_0}. \tag{3.4}$$

We assume the instantaneous liquidity cost of the bank depends on the state $\mathbf{X}$ of the economy, so that

$$l_t = \langle \boldsymbol{l}, X_t \rangle \tag{3.5}$$

Denote by $a_{k,j}$ the entry on the line k and the column j of the $N \times N$ matrix A with $a_{k,j} \geq 0$ for $j \neq k$ and $\sum_{j=1}^{N} a_{k,j} = 0$ for any k.

3.2. Analytical formulas for the PVRP. Assume a loan has a fixed coupon defined by the interest rate r and an initial contractual margin $\overline{\rho_0}$ calculated at the inception for a par value of the loan. Let $\xi(t, T, \lambda_t, X_t)$ be, the present value of the remaining payments at time t of a corporate loan where: λ_t is the intensity at time t; T is the contractual maturity; K is the nominal amount and X_t is the state of the economy at time t.

The loan value $LV(t, T, \lambda)$ is still equal to the present value of the remaining payments $\xi(t, T, \lambda)$ minus the prepayment option value $P(t, T, \lambda)$.

$$LV(t, T, \lambda) = \xi(t, T, \lambda) - P(t, T, \lambda) \tag{3.6}$$

The PVRP ξ is the present value of the cash flows discounted at the risky rate, where the risky rate at time t is the constant risk-free rate r plus the liquidity cost l_t and the intensity λ_t. Similar to the discussion in the Subsection 2.1, ξ is not depending on time when $T = +\infty$ (perpetual loan). So we denote,

$$\xi(\lambda, X) := K\left(r + \overline{\rho_0}\right) \mathbb{E}\left[\int_0^{+\infty} e^{-\int_0^{\tilde{t}} r + l_u + \lambda_u du} d\tilde{t} \,\middle|\, \lambda_0 = \lambda, X_0 = X\right] \tag{3.7}$$

We consider that there is no correlation between the credit risk, i.e., the intensity λ_t, of the borrower and the cost to access the cash on the market, i.e. the liquidity cost l_t, of the lender. Therefore, we have,

$$\xi(\lambda, X) = K\left(r + \overline{\rho_0}\right) \int_0^{+\infty} e^{-r\tilde{t}} \mathbb{E}\left[e^{-\int_0^{\tilde{t}} \lambda_u du} \middle| \lambda_0 = \lambda\right] \times \mathbb{E}\left[e^{-\int_0^{\tilde{t}} l_u du} \middle| X_0 = X\right] d\tilde{t} \tag{3.8}$$

Remark 3. *The crucial information here is that the coefficients γ, θ, σ of the CIR process are* **not** *depending on the regime X thus we can separate the CIR dynamics and the Markov dynamics at this level. A different approach can extend this result by using the properties of the PVRP as explained in the next section.*

Note that (cf. Subsection 2.1 equation (2.7))

$$\mathbb{E}\left[e^{-\int_0^t \lambda_u du}\middle|\lambda_0 = \lambda\right] = B(0,t,\lambda) \tag{3.9}$$

and $B(0,t,\lambda)$ is evaluated using equations (2.8) - (2.11). In order to compute

$$\mathbb{E}\left[e^{-\int_0^t l_u du}\middle|X_0 = X\right]$$

let $f_k(t)$ be defined by:

$$f_k(t) = \mathbb{E}\left[e^{-\int_0^t l_u du}\middle|X_0 =< \mathbf{X}, e_k >\right]. \tag{3.10}$$

Let τ, the time of the first jump from $X_0 =< \mathbf{X}, e_k >$ to some other state. We know (cf. Lando [16] paragraph 7.7 p 211) that τ is a random variable following an exponential distribution of parameter α_k with,

$$\alpha_k = \sum_{j \neq k} a_{k,j} \tag{3.11}$$

We also know that conditional to the fact that a jump has occurred at time τ the probability that the jump is from state e_k to state e_j is $p_{k,j}$, where

$$p_{k,j} = \frac{a_{k,j}}{\alpha_k} \tag{3.12}$$

Thus,

$$\begin{aligned} f_k(t) &= \mathbb{P}(\tau > t)e^{-l_k t} + \mathbb{P}(\tau \leq t)e^{-l_k \tau}\textstyle\sum_{j\neq k}\mathbb{P}(l_\tau = l_j)\mathbb{E}\left[e^{-\int_\tau^t l_u du}\middle|X_\tau =< \mathbf{X}, e_j >\right] \\ &= e^{-(l_k+\alpha_k)t} + \alpha_k \textstyle\int_0^t e^{-(l_k+\alpha_k)\tau}\sum_{j\neq k} p_{k,j} f_j(t-\tau)d\tau \end{aligned}$$

Then,

$$\begin{aligned} e^{(l_k+\alpha_k)t} f_k(t) &= 1 + \alpha_k \textstyle\int_0^t e^{-(l_k+\alpha_k)(t-\tau)}\sum_{j\neq k} p_{k,j} f_j(t-\tau)d\tau \\ &= 1 + \alpha_k \textstyle\int_0^t e^{-(l_k+\alpha_k)s}\sum_{j\neq k} p_{k,j} f_j(s)ds \end{aligned}$$

By differentiation with respect to t:

$$\frac{d}{dt}\left[e^{(l_k+\alpha_k)t} f_k(t)\right] = \alpha_k e^{-(l_k+\alpha_k)t}\sum_{j\neq k} p_{k,j} f_j(t)$$

Then

$$\frac{df_k(t)}{dt} + (l_k+\alpha_k)f_k(t) = \alpha_k \sum_{j\neq k} p_{k,j} f_j(t)$$

Thus,

$$\frac{df_k(t)}{dt} = \left[\sum_{j\neq k} \alpha_k p_{k,j} f_j(t)\right] - (l_k+\alpha_k)f_k(t) \tag{3.13}$$

Denote $F(t) = (f_1(t), f_2(t), ..., f_N(t))^T$ and introduce the $N \times N$ matrix B,

$$B_{i,j} = \begin{cases} \alpha_i p_{i,j} & if \quad i \neq j \\ -(\alpha_i + l_i) & if \quad i = j \end{cases} \tag{3.14}$$

From equation (3.13) we obtain,

$$\frac{dF(t)}{dt} = BF(t) \; thus \; F(t) = e^{Bt}F(0) \tag{3.15}$$

with the initial condition,

$$F(0) = \Big(f_k(0)\Big)_{k=1}^N = (1, 1, ..., 1)^T \in \mathbb{R}^N. \tag{3.16}$$

We have therefore analytical formulas for the PVRP $\xi(\lambda, X)$. We refer the reader to [9] for similar considerations on a related CIR switched dynamics.

REMARK 4. *When all liquidity parameters l_k are equal (to some quantity l) then $B = A - l \cdot Id$ and then we obtain (after some computations) that $f_k(t) = e^{-lt}$ thus the payoff is equal to that of a one-regime dynamics with interest rate $r + l$, which is consistent with intuitive image we may have. Another limiting case is when the switching is very fast, see also Remark 7 item 6 for further details.*

The margin $\overline{\rho_0}$ is set to satisfy the equilibrium equation

$$\xi(\overline{\lambda_0}, \overline{X_0}) = K. \tag{3.17}$$

Similar arguments to that in previous section show that $\overline{\rho_0} > \min_k l_k > 0$. See Remark 1 for the situation when a additional commercial margin is to be considered.

We will also need to introduce for any $k = 1, ..., N$ the value $\overline{\Lambda}_k^0$ such that

$$\xi(\overline{\Lambda}_k^0, e_k) = K. \tag{3.18}$$

Of course, $\overline{\Lambda}_{\overline{X_0}}^0 = \overline{\lambda_0}$. Recall that $\xi(\lambda, e_k)$ is decreasing with respect to λ; when $\xi(0, e_k) < K$ there is no solution to eqn. (3.17) and we will chose by convention $\overline{\Lambda}_k^0 = 0$.

3.3. Further properties of the PVRP ξ. It is useful for the following to introduce a PDE formulation for ξ. To ease the notations we introduce the operator $\mathcal{A}^{\mathcal{R}}$ that acts on functions $v(\lambda, X)$ as follows:

$$(\mathcal{A}^{\mathcal{R}}v)(\lambda, e_k) = (\mathcal{A}v)(\lambda, e_k) - (r + l_k + \lambda)v(\lambda, e_k) + \sum_{j=1}^N a_{k,j}\Big(v(\lambda, e_j) - v(\lambda, e_k)\Big). \tag{3.19}$$

Having defined the dynamics (3.1) and (3.4) one can use an adapted version of the Feynman-Kac formula in order to conclude that PVRP defined by (3.7) satisfies the equation:

$$(\mathcal{A}^{\mathcal{R}}\xi)(\lambda, e_k) + (r + \overline{\rho_0})K = 0. \tag{3.20}$$

REMARK 5. *When the dynamics involves different coefficients of the CIR process for different regimes (cf. also Remark 3) the Equation* (3.20) *changes in that it will involve, for $\xi(\cdot, e_k)$, the operator*

$$\mathcal{A}_k(v)(\lambda) = \gamma_k(\theta_k - \lambda)\partial_\lambda v(\lambda) + \frac{1}{2}\sigma_k^2 \lambda \partial_{\lambda\lambda} v(\lambda). \tag{3.21}$$

instead of $\mathcal{A}$.

3.4. Valuation of the prepayment option. The valuation problem of the prepayment option can be modelled as an American call option on a risky debt owned by the borrower with payoff:

$$\chi(\lambda, X) = (\xi(\lambda, X) - K)^+. \tag{3.22}$$

Here the prepayment option allows borrower to buy back and refinance his/her debt according to the current contractual margin at any time during the life of the option.

THEOREM 6. *For any N-tuple* $\Lambda = (\Lambda_k)_{k=1}^N \in (\mathbb{R}_+)^N$ *introduce the function* $P_\Lambda(\lambda, X)$ *such that:*

$$P_\Lambda(\lambda, e_k) = \chi(\lambda, e_k) \ \forall \lambda \in [0, \Lambda_k] \tag{3.23}$$

$$(\mathcal{A}^{\mathcal{R}} P_\Lambda)(\lambda, e_k) = 0, \ \forall \lambda > \Lambda_k, \ k = 1, ..., N \tag{3.24}$$

$$\lim_{\lambda \to \Lambda_k} P_\Lambda(\lambda, e_k) = \chi(\Lambda_k, e_k), \ k = 1, ..., N \tag{3.25}$$

$$\lim_{\lambda \to \infty} P_\Lambda(\lambda, e_k) = 0, \ k = 1, ..., N. \tag{3.26}$$

Suppose a $\Lambda^* \in \prod_{k=1}^N [0, (\overline{\rho_0} - l_k)^+ \wedge \overline{\Lambda}_k^0]$ *exists such that for all* $k = 1, ..., N$*:*

$$P_{\Lambda^*}(\lambda, X) \geq \chi(\lambda, X) \ \forall \lambda, X \tag{3.27}$$

$$\frac{\partial P_{\Lambda^*}(\lambda, e_k)}{\partial \lambda}\Big|_{\lambda = (\Lambda_k^*)^+} = \frac{\partial \chi(\lambda, e_k)}{\partial \lambda}\Big|_{\lambda = (\Lambda_k^*)^-} \ if \ \Lambda_k^* > 0 \tag{3.28}$$

$$\sum_{j=1}^N a_{k,j} \Big(P_{\Lambda^*}(\lambda, e_j) - \chi(\lambda, e_j) \Big) + K(\lambda + l_k - \overline{\rho_0}) \leq 0 \ \forall \lambda \in] \min_j \Lambda_j^*, \Lambda_k^* [. \tag{3.29}$$

Then $P = P_{\Lambda^*}$.

Proof. Similar arguments as in the proof of Thm. 2 lead to consider the American option price in the form

$$P(\lambda, X) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\int_0^\tau r + l_u + \lambda_u du} \chi(\lambda_\tau, X_\tau) \Big| \lambda_0 = \lambda, X_0 = X \right].$$

We note that for $\Lambda \in (\mathbb{R}_+^*)^N$ if τ_Λ is the stopping time that stops upon exiting the domain $\lambda > \Lambda_k$ when $X = e_k$ then

$$P_\Lambda(\lambda, X) = \mathbb{E}\left[e^{-\int_0^{\tau_\Lambda} r + l_u + \lambda_u du} \chi(\lambda_{\tau_\Lambda}, X_{\tau_\Lambda}) \Big| \lambda_0 = \lambda, X_0 = X \right].$$

Remark that for $\Lambda \in (\mathbb{R}_+^*)^N$ the stopping time τ_Λ is finite a.e. Thus for any $\Lambda \in (\mathbb{R}_+^*)^N$ we have $P \geq P_\Lambda$; when Λ has some null coordinates the continuity (ensured among others by the boundary condition (3.23)) shows that we still have $P \geq P_\Lambda$. In particular for Λ^* we obtain $P \geq P_{\Lambda^*}$; all that remains to be proved is the reverse inequality i.e. $P \leq P_{\Lambda^*}$.

To this end we use a similar technique as in Thm. 10.4.1 [27, Section 10.4 page 227] (see also [25] for similar considerations). First one can invoke the same arguments as in cited reference (cf. Appendix D for technicalities) and work as if P_{Λ^*} is C^2 (not only C^1 as the hypothesis ensures).

Denote $D_{\Lambda^*} = \{(\lambda, e_k) | \lambda \in [0, \Lambda_k^*], k = 1, ..., N\}$ (which will be the exercice region) and C_{Λ^*} its complementary with respect to $\mathbb{R}_+ \times E$ (which will be the continuation region).

The Lemma 3.1 shows that $\mathcal{A}^{\mathcal{R}}P_{\Lambda^*}$ is non-positive everywhere (and is null on C_{Λ^*}). The Îto formula shows that

$$d\left(e^{-\int_0^t r+l_s+\lambda_s ds}P_{\Lambda^*}(\lambda_t,X_t))\right)=e^{-\int_0^t r+l_s+\lambda_s ds}(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(\lambda_t,X_t))dt+d(martingale) \tag{3.30}$$

Taking averages and integrating from 0 to some stopping time τ it follows from $\mathcal{A}^{\mathcal{R}}P_{\Lambda^*}\leq 0$ that

$$\begin{aligned}P_{\Lambda^*}(\lambda,X)&\geq\mathbb{E}\left[e^{-\int_0^\tau r+l_u+\lambda_u du}P_{\Lambda^*}(\lambda_\tau,X_\tau)\Big|\lambda_0=\lambda,X_0=X\right]\\&\geq\mathbb{E}\left[e^{-\int_0^\tau r+l_u+\lambda_u du}\chi(\lambda_\tau,X_\tau)\Big|\lambda_0=\lambda,X_0=X\right].\end{aligned}$$

Since this is true for any stopping time τ the conclusion follows. □

LEMMA 3.1. *Under the hypothesis of the Thm. 6 the following inequality holds (strongly except for the values $(\lambda,X)=(\Lambda_j^*,e_k)$ and everywhere in a weak sense):*

$$(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(\lambda,X)\leq 0,\ \forall\lambda>0,\forall X. \tag{3.31}$$

Proof. The non-trivial part of this lemma comes from the fact that if for fixed k we have for λ in a neighborhood of some λ_1: $P_{\Lambda^*}(\lambda,e_k)=\chi(\lambda,e_k)$ this does not necessarily imply $(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(\lambda_1,e_k)=(\mathcal{A}^{\mathcal{R}}\chi)(\lambda_1,e_k)$ because $\mathcal{A}^{\mathcal{R}}$ depends on other values $P_{\Lambda^*}(\lambda,e_j)$ with $j\neq k$.

From (3.24) the conclusion is trivially verified for $X=e_k$ for any $\lambda\in]\Lambda_k^*,\infty[$.

We now analyze the situation when $\lambda<\min_j\Lambda_j^*$; this means in particular that $0\leq\lambda<\min_j\Lambda_j^*\leq\overline{\Lambda}_\ell^0$ for any ℓ thus $\overline{\Lambda}_\ell^0>0$. Note that $\Lambda_k^*<\overline{\Lambda}_k^0$ implies $\xi(\Lambda_k^*,e_k)\geq\xi(\overline{\Lambda}_k^0,e_k)=K$ for any $k=1,...,N$ thus $\chi(\lambda,e_k)=\xi(\lambda,e_k)-K$ for any $\lambda\in[0,\Lambda_k^*]$ and any k. Furthermore since $\lambda<\min_j\Lambda_j^*$ we have $P_{\Lambda^*}(\lambda,e_k)=\chi(\lambda,e_k)=\xi(\lambda,e_k)-K$ for any k. Fix $X=e_k$; then

$$\begin{aligned}&(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(\lambda,e_k)=(\mathcal{A}^{\mathcal{R}}\chi)(\lambda,e_k)=(\mathcal{A}^{\mathcal{R}}(\xi-K))(\lambda,e_k)=(\mathcal{A}^{\mathcal{R}}\xi)(\lambda,e_k)-\mathcal{A}^{\mathcal{R}}(K)\\&=-(r+\overline{\rho_0})K-(r+l_k+\lambda)K=K(l_k+\lambda-\overline{\rho_0})\leq K(l_k+\Lambda_k^*-\overline{\rho_0})\leq 0\end{aligned} \tag{3.32}$$

the last inequality being true by hypothesis.

A last situation is when $\lambda\in]\min_j\Lambda_j^*,\Lambda_k^*[$; there $P_{\Lambda^*}(\lambda,e_k)=\chi(\lambda,e_k)$ but some terms $P_{\Lambda^*}(\lambda,e_j)$ for $j\neq k$ may differ from $\chi(\lambda,e_j)$. The computation is more subtle is this case. This point is specific to the fact that the payoff χ itself has a complex structure and as such was not emphasized in previous works (e.g., [25], etc.).

Recalling the properties of ξ one obtains (and since $P_{\Lambda^*}(\lambda,e_k)=\chi(\lambda,e_k)$):

$$\begin{aligned}&(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(\lambda,e_k)=(\mathcal{A}\chi)(\lambda,e_k)-(r+l_k+\lambda)\chi(\lambda,e_k)+\sum_{j=1}^N a_{k,j}\Big(P_{\Lambda^*}(\lambda,e_j)-\chi(\lambda,e_k)\Big)\\&=(\mathcal{A}^{\mathcal{R}}\chi)(\lambda,e_k)+\sum_{j=1}^N a_{k,j}\Big(P_{\Lambda^*}(\lambda,e_j)-\chi(\lambda,e_j)\Big)\\&=(\mathcal{A}^{\mathcal{R}}\xi)(\lambda,e_k)-\mathcal{A}^{\mathcal{R}}(K)+\sum_{j=1}^N a_{k,j}\Big(P_{\Lambda^*}(\lambda,e_j)-\chi(\lambda,e_j)\Big)\\&=-K(r+\overline{\rho_0})+(r+l_k+\lambda)K+\sum_{j=1}^N a_{k,j}\Big(P_{\Lambda^*}(\lambda,e_j)-\chi(\lambda,e_j)\Big)\leq 0,\end{aligned} \tag{3.33}$$

where for the last inequality we use hypothesis (3.29). Finally, since we proved that $(\mathcal{A}^{\mathcal{R}} P_{\Lambda^*})(\lambda, X) \leq 0$ strongly except for the values $(\lambda, X) = (\Lambda_j^*, e_k)$ and since P_{Λ^*} is of C^1 class we obtain the conclusion (the weak formulation only uses the first derivative of P_{Λ^*}). □

REMARK 7. *Several remarks are in order at this point:*

1. *when only one regime is present i.e., $N = 1$ the hypothesis of the Theorem are identical to that of Thm. 2 since (3.29) is automatically satisfied.*
2. *when $N > 1$ checking (3.29) does not involve any computation of derivatives and is straightforward.*
3. *as mentioned in the previous section, the Theorem is a verification result i.e., only gives sufficient conditions for a candidate to be the option price. Two possible partial converse results are possible: a first one to prove that the optimal price is indeed an element of the family P_Λ. A second converse result is to prove that supposing $P = P_{\Lambda^*}$ then $\Lambda^* \in \prod_{k=1}^N [0, (\overline{\rho_0} - l_k)^+ \wedge \overline{\Lambda}_k^0]$ and (3.27)-(3.29) are satisfied.*
4. *a more general verification result for different payoff function χ can be proven, cf [20] for details.*
5. *the search for the candidate Λ^* can be done either by looking for a zero of the function $\Lambda \mapsto \Upsilon(\Lambda) := \left(\frac{\partial P_{\Lambda^*}(\lambda, e_k)}{\partial \lambda}\Big|_{\lambda=(\Lambda_k^*)^+} - \frac{\partial \chi(\lambda, e_k)}{\partial \lambda}\Big|_{\lambda=(\Lambda_k^*)^-} \right)_{k=1}^N$ or by maximizing on $\prod_{k=1}^N]0, (\overline{\rho_0} - l_k) \wedge \overline{\Lambda}_k^0[$ the function $\Lambda \mapsto P_\Lambda(\overline{\lambda_0}, \overline{X_0})$.*
6. *if the optimization of $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$ is difficult to perform, one can use a continuation argument with respect to the coupling matrix A. Denote by $\Lambda^*(A)$ the optimal value of Λ^* as function of A. When $A = 0$ each Λ_k^* is found as in Section 2 (the problem separates into N independent i.e., no coupled, valuation problems, each of which requiring to solve a one dimensional optimization) and we construct thus $\Lambda^*(0)$. When considering μA with $\mu \to \infty$ at the limit the optimal $\Lambda_*(\infty A)$ has all entries equal to Λ_{mean}^* where Λ_{mean}^* is the optimal value for a one-regime ($N = 1$) dynamics with riskless interest rate r being replaced by $r + \frac{\sum_{k=1}^N l_k/\alpha_k}{\sum_{k=1}^N 1/\alpha_k}$. Having established the two extremal points the candidate $\Lambda^*(A)$ is searched within the N-dimensional segment $[\Lambda^*(0), \Lambda^*(\infty A)]$.*
7. *note that this continuation procedure above works even when the CIR parameters depend on k (cf. [20] for details).*

3.5. Numerical Application. The numerical solution of the partial differential equation (3.24) is required. We use a finite difference method. The first derivative is approximated by the finite difference formula:

$$\frac{\partial}{\partial \lambda} P_\Lambda(\lambda, X) = \frac{P_\Lambda(\lambda + \delta\lambda, X) - P_\Lambda(\lambda - \delta\lambda, X)}{2\delta\lambda} + O(\delta\lambda^2) \tag{3.34}$$

while the second derivative is approximated by:

$$\frac{\partial^2}{\partial \lambda^2} P_\Lambda(\lambda, X) = \frac{P_\Lambda(\lambda + \delta\lambda, X) - 2P_\Lambda(\lambda + \delta\lambda, X) + P_\Lambda(\lambda - \delta\lambda, X)}{\delta\lambda^2} + O(\delta\lambda^2) \tag{3.35}$$

To avoid working with an infinite domain a well-known approach is to define an artificial boundary λ_{max}. Then a boundary condition is imposed on λ_{max} which leads to a numerical problem in the finite domain $\cup_{k=1}^N [\Lambda_k^*, \lambda_{max}]$. In this numerical application, $\lambda_{max} = 400$ bps. We discretize $[\Lambda^*, \lambda_{max}]$ with a grid such that $\delta\lambda = 1bps$.

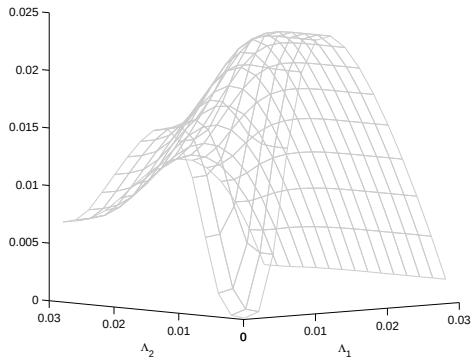

FIG. 3.1. *We illustrate here the dependence of* $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$ *as a function of* Λ*; this allows to find the optimal* $(\Lambda_1^* = 122bps, \Lambda_2^* = 64bps)$ *that maximizes the option price.*

Two approaches have been considered for imposing a boundary value at λ_{max}: either consider that $P_\Lambda(\lambda_{max}, e_k) = 0$, $\forall k = 1, ..., N$ (homogenous Dirichlet boundary condition) or that $\frac{\partial}{\partial\lambda} P_\Lambda(\lambda_{max}, e_k) = 0$, $\forall k = 1, ..., N$ (homogeneous Neuman boundary condition). Both are correct in the limit $\lambda_{max} \to \infty$. We tested the precision of the results by comparing with numerical results obtained on a much larger grid (10 times larger) while using same $\delta\lambda$. The Neumann boundary condition gives much better results for the situations we considered and as such was always chosen (see also Figure 3.3).

We consider a perpetual loan with a nominal amount $K = 1$ and the borrower default intensity λ_t follows a CIR dynamics with parameters: initial intensity $\overline{\lambda_0} =$ 300bps, volatility $\sigma = 0.05$, average intensity $\theta = 200$bps, reversion coefficient $\gamma = 0.5$. We assume a constant interest rate $r = 1\%$ and a liquidity cost defined by a Markov chain of two states $l_1 = 150$bps and $l_2 = 200$bps. For $N = 2$ the rate A matrix is completely defined by $\alpha_1 = 1/3$, $\alpha_2 = 1$.

In order to find the initial contractual margin we use equation (2.11) and find $\overline{\rho_0} = 331$ bps in the state 1. The contractual margin takes into account the credit risk (default intensity) and the liquidity cost. We have thus $\overline{\Lambda}_1^0 = \overline{\lambda_0}$; we obtain then $\overline{\Lambda}_2^0 = 260bps$.

The optimal value Λ^* is obtained by maximizing $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$ and turns out to be $(\Lambda_1^*, \Lambda_2^*) = (122bps, 64bps)$, see Figure 3.1. To be accepted, this numerical solution has to verify all conditions of the Theorem 6. The hypothesis (3.27) and (3.29) are satisfied (see Figure 3.3) and the hypothesis (3.29) is accepted after calculation. Moreover $\Lambda_1^* \leq (\overline{\rho_0} - l_1) \wedge \overline{\Lambda}_1^0$ and the analogous holds for Λ_2^*.

In the state $\overline{X_0} = 1$, the present value of cash flows is at par, so $\xi(\overline{\lambda_0}, \overline{X_0}) = 1$. The prepayment option price is $P(\overline{\lambda_0}, \overline{X_0}) = 0.0240$. Therefore the loan value equals $\xi(\overline{\lambda_0}, \overline{X_0}) - P(\overline{\lambda_0}, \overline{X_0}) = 0.9760$.

The loan value will equal to the nominal if the intensity decreases until the exercise region $\lambda \leq \Lambda^*$ see Figure 3.2. The continuation and exercise regions are depicted in Figure 3.3.

3.6. Regimes when is never optimal to exercise. When the liquidity parameters corresponding to given regimes are very different it may happen that the optimization of $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$ over Λ gives an optimum value Λ^* with some null coordi-

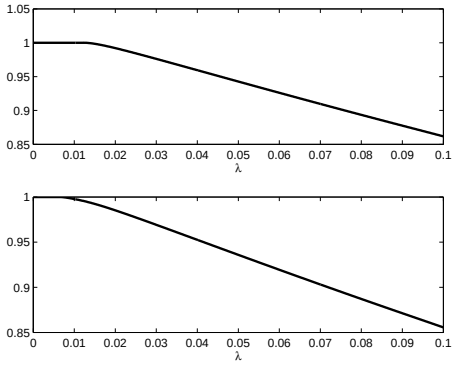

FIG. 3.2. *Loan value as a function of the intensity. Top: regime $X = 1$; bottom: regime $X = 2$. The loan value is decreasing when there is a degradation of the credit quality (i.e. when λ increases) and converges to 0.*

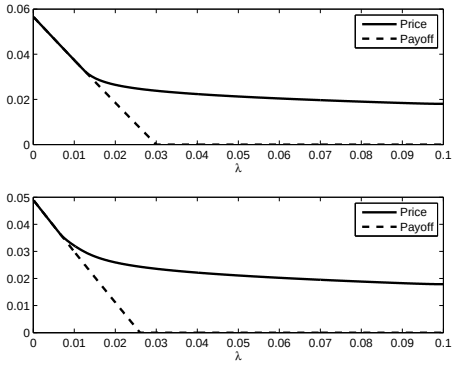

FIG. 3.3. *The price of the prepayment option $P_{\Lambda^*}(\lambda)$ (solid line) and the payoff $\chi(\lambda)$ (dashed line) as function of the intensity λ. Top: regime $X = 1$; bottom: regime $X = 2$. For each regime two regions appear : the continuation region $\lambda > \Lambda_i^*$ and the exercise region $\lambda \leq \Lambda_i^*$.*

nates Λ_{k_i}, $i = 1, ...$ This may hint to the fact that in this situation it is never optimal to exercise during the regimes e_{k_i}, $i = 1, ...$ This is not surprising in itself (remember that this is the case of an American call option) but needs more care when dealing with. Of course when in addition $\overline{\Lambda}_{k_i}^0 = 0$ the payoff being null it is intuitive that the option should not be exercised.

REMARK 8. *Further examination of the Theorem 2 calls for the following remarks:*

1. *the boundary value set in eqn.* (3.23) *for some regime e_k with $\Lambda_k^* = 0$ deserves an interpretation. The boundary value does not serve to enforce continuity of $\lambda \mapsto P_\Lambda(\lambda)$ because there is no exercise region in this regime thus any value will do. Moreover when $2\gamma\theta \geq \sigma^2$ the intensity λ_u does not touch 0 thus the stopping time τ_{Λ^*} is infinite in the regime e_k (thus the boundary value in 0 can be set to any arbitrary number since it is never used). The real meaning of the value $P_{\Lambda^*}(0, e_k)$ comes from arbitrage considerations: when one proves in the demonstration of the Theorem that $P \geq P_{\Lambda^*}$ one uses continuity of P_Λ with respect to the parameter Λ; in order to still have this*

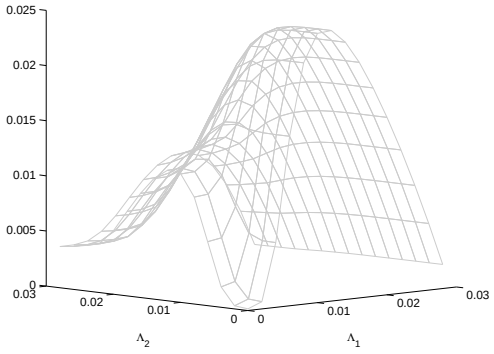

FIG. 3.4. *We illustrate here the dependence of* $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$ *as a function of the exercise boundary* Λ*; this allows to find the optimal* $(\Lambda_1^* = 121bps, \Lambda_2^* = 0)$ *that maximizes the option price.*

conclusion one has to set $P_{\Lambda^*}(0, e_k) \leq \lim_{\Lambda \in (\mathbb{R}_+^*)^N \to \Lambda^*} P_\Lambda(0, e_k) = \chi(0, e_k)$*. On the contrary, in order to have* $P \leq P_{\Lambda^*}$*, since* $P \geq \chi$ *is it required that* $P_{\Lambda^*}(0, e_k) \geq P(0, e_k) \geq \chi(0, e_k)$*. Thus only* $P_{\Lambda^*}(0, e_k) = \chi(0, e_k)$ *can prevent arbitrage.*

2. *it is interesting to know when such a situation can occur and how can one interpret it. Let us take a two-regime case* $(N = 2)$*:* l_1 *a "normal" regime and* l_2 *the "crisis" regime* $(l_2 \geq l_1)$*; when the agent contemplates prepayment the more severe the crisis (i.e. larger* $l_2 - l_1$ *) less he/she is likely to prepay during the crisis the cash is expensive (high liquidity cost). We will most likely see that for* $l_1 = l_2$ *some exercise region exists while starting from some large* l_2 *the exercise region will disappear in regime* e_2*. This is completely consistent with the numerical results reported in this paper.*

3.7. Numerical Application. We consider the same situation as in Section 3.7 except that $l_1 = 50$bps and $l_2 = 250$ bps. In order to find the initial contractual margin we use equation (2.11) and find $\overline{\rho_0} = 305$ bps in the state 1. The contractual margin takes into account the credit risk (default intensity) and the liquidity cost. As before $\overline{\Lambda}_1^0 = \overline{\lambda_0}$ but here we obtain $\overline{\Lambda}_2^0 = 221bps$.

The couple $(\Lambda_1^* = 121bps, \Lambda_2^* = 0)$ (see Figure 3.4) maximizes $P_\Lambda(\overline{\lambda_0}, \overline{X_0})$. There does not exist a exercise boundary in the state 2. The loan value will equal the par if the intensity decreases until the exercise region $\lambda \leq \Lambda^*$ see Figure 3.5. The continuation and exercise regions are depicted in Figure 3.6.

To be accepted as true price the numerical solution P_{Λ^*} has to verify all hypothesis and conditions of the Theorem 6. In the regime $X = 1$, the hypothesis (3.27) and (3.28) are verified numerically (see also Figure 3.6) and the hypothesis (3.29) is accepted after calculation. Moreover $\Lambda_k^* \leq (\overline{\rho_0} - l_k) \wedge \overline{\Lambda}_k^0$ for $k = 1, 2$.

In the state $X = 1$, the present value of cash flows is at par, so $\xi(\overline{\lambda_0}, \overline{X_0}) = K = 1$. The prepayment option price is $P(\overline{\lambda_0}) = 0.0245$. Therefore the loan value LV equals $\xi(\overline{\lambda_0}) - P(\overline{\lambda_0}) = 0.9755$.

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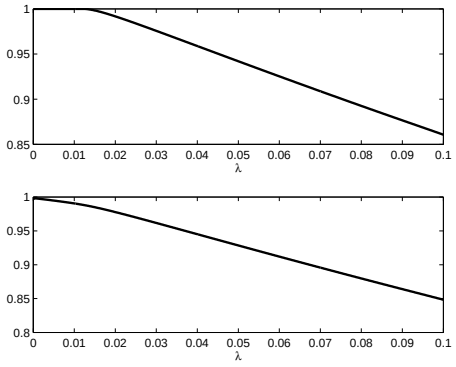

FIG. 3.5. *Loan value as a function of the intensity. Top: regime $X = 1$; bottom: regime $X = 2$. The loan value is decreasing when there is a degradation of the credit quality (i.e. when λ increases) and converges to 0.*

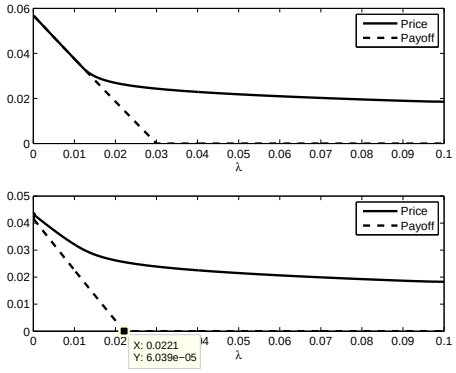

FIG. 3.6. *The price of the prepayment option $P_{\Lambda^*}(\lambda)$ (solid line) and the payoff $\chi(\lambda)$ (dashed line) as function of the intensity λ. Top: regime $X = 1$; bottom: regime $X = 2$. Two regions appear : the continuation region $\lambda > \Lambda_1^*$ and the exercise region $\lambda \leq \Lambda_1^*$. For the second regime there is no exercise region.*

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