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Stability estimates for determination of potential from the impedance boundary map

M.I. Isaev and R.G. Novikov

Abstract

We study the impedance boundary map (or Robin-to-Robin map) for the Schrödinger equation in open bounded domain at fixed energy in multidimensions. We give global stability estimates for determining potential from these boundary data and, as corollary, from the Cauchy data set. Our results include also, in particular, an extension of the Alessandrini identity to the case of the impedance boundary map.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \quad E \in \mathbb{R}, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \quad (1.2)$$

$$\text{with } \partial D \in C^2,$$

$$v \in L^\infty(D). \quad (1.3)$$

We consider the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by

$$\hat{M}_\alpha[\psi]_\alpha = [\psi]_{\alpha-\pi/2} \quad (1.4)$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = [\psi(x)]_\alpha = \cos \alpha \psi(x) - \sin \alpha \frac{\partial \psi}{\partial \nu} |_{\partial D}(x), \quad x \in \partial D, \quad \alpha \in \mathbb{R} \quad (1.5)$$

and ν is the outward normal to ∂D . One can show (see Lemma 3.2) that there is not more than a countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition

$$\cos \alpha \psi |_{\partial D} - \sin \alpha \frac{\partial \psi}{\partial \nu} |_{\partial D} = 0. \quad (1.6)$$

Therefore, for any energy level E we can assume that for some fixed $\alpha \in \mathbb{R}$

$$E \text{ is not an eigenvalue for the operator } -\Delta + v \text{ in } D \quad (1.7)$$

$$\text{with boundary condition (1.6)}$$

and, as a corollary, \hat{M}_α can be defined correctly.

Note that the impedance boundary map \hat{M}_α is reduced to the Dirichlet-to-Neumann (DtN) map if $\alpha = 0$ and is reduced to the Neumann-to-Dirichlet (NtD) map if $\alpha = \pi/2$. The map \hat{M}_α can be called also as the Robin-to-Robin map.

We consider the following inverse boundary value problem for equation (1.1).

Problem 1.1. Given \hat{M}_α for some fixed E and α , find v .

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [8], [14]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [14]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems and global reconstruction methods for Problem 1.1 with $\alpha = 0$ were given for the first time in [14] in dimension $d \geq 3$ and in [4] in dimension $d = 2$.

Global stability estimates for Problem 1.1 with $\alpha = 0$ were given for the first time in [1] in dimension $d \geq 3$ and in [21] in dimension $d = 2$. A principal improvement of the result of [1] was given recently in [19] (for the zero energy case). Due to [13] these logarithmic stability results are optimal (up to the value of the exponent). An extension of the instability estimates of [13] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [10].

It should be noted that in most of previous works on inverse boundary value problems for equation (1.1) at fixed E it was assumed in one way or another that E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D , see [1], [13], [14], [19], [21], [22]. Nevertheless, the results of [4] can be considered as global uniqueness and reconstruction results for Problem 1.1 in dimension $d = 2$ with general α .

In the present work we give global stability estimates for Problem 1.1 in dimension $d \geq 3$ with general α . These results are presented in detail in Section 2. Global stability estimates for Problem 1.1 in dimension $d = 2$ for general α will be proved in a subsequent paper.

In addition, in the present work we establish some basic properties of the impedance boundary map with general α . In particular, we extend the Alessandrini identity to this general case. These results are presented in detail in Section 3.

In a subsequent paper we plan to give also global reconstruction method for Problem 1.1 in multidimensions with general α .

2. Stability estimates

In this section we always assume that D satisfies (1.2).

We will use the fact that if v_1, v_2 are potentials satisfying (1.3), (1.7) for some fixed E and α , then

$$\hat{M}_{\alpha, v_1}(E) - \hat{M}_{\alpha, v_2}(E) \text{ is a bounded operator in } \mathbb{L}^\infty(\partial D), \quad (2.1)$$

where $\hat{M}_{\alpha, v_1}(E), \hat{M}_{\alpha, v_2}(E)$ denote the impedance boundary maps for v_1, v_2 , respectively. Actually, under our assumptions, $\hat{M}_{\alpha, v_1}(E) - \hat{M}_{\alpha, v_2}(E)$ is a compact operator in $\mathbb{L}^\infty(\partial D)$ (see, Corollary 3.1).

Let

$$\begin{aligned} \|A\| \text{ denote the norm of an operator} \\ A : \mathbb{L}^\infty(\partial D) \rightarrow \mathbb{L}^\infty(\partial D). \end{aligned} \quad (2.2)$$

Let the Cauchy data set \mathcal{C}_v for equation (1.1) be defined by:

$$\mathcal{C}_v = \left\{ \left(\psi|_{\partial D}, \frac{\partial \psi}{\partial \nu} |_{\partial D} \right) : \begin{array}{l} \text{for all sufficiently regular solutions } \psi \text{ of} \\ \text{equation (1.1) in } \bar{D} = D \cup \partial D \end{array} \right\}. \quad (2.3)$$

In addition, the Cauchy data set \mathcal{C}_v can be represented as the graph of the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by (1.4) under assumptions (1.7).

As in [19], we assume for simplicity that

$$v \in \mathbb{W}^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \text{ supp } v \subset D, \quad (2.4)$$

where

$$\mathbb{W}^{m,1}(\mathbb{R}^d) = \{v : \partial^J v \in \mathbb{L}^1(\mathbb{R}^d), |J| \leq m\}, \quad m \in \mathbb{N} \cup \{0\}, \quad (2.5)$$

where

$$J \in (\mathbb{N} \cup 0)^d, |J| = \sum_{i=1}^d J_i, \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \quad (2.6)$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{\mathbb{L}^1(\mathbb{R}^d)}. \quad (2.7)$$

Note also that (2.4) \Rightarrow (1.3).

Theorem 2.1. *Let D satisfy (1.2), where $d \geq 3$. Let v_1, v_2 satisfy (2.4) and (1.7) for some real fixed E and α . Let $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let $\hat{M}_{\alpha,v_1}(E)$ and $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1 and v_2 , respectively. Then*

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq C_\alpha (\ln(3 + \delta_\alpha^{-1}))^{-s}, \quad 0 < s \leq (m - d)/m, \quad (2.8)$$

where $C_\alpha = C_\alpha(N, D, E, m, s)$, $\delta_\alpha = \|\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)\|$ is defined according to (2.2).

Estimate (2.8) with $\alpha = 0$ is a variation of the result of [1] (see also [19]).

Proof of Theorem 2.1 is given in Section 5. This proof is based on results presented in Sections 3, 4.

Theorem 2.1 implies the following corollary.

Corollary 2.1. *Let D satisfy (1.2), where $d \geq 3$. Let potentials v_1, v_2 satisfy (2.4). Then*

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha (\ln(3 + \delta_\alpha^{-1}))^{-s}, \quad 0 < s \leq (m - d)/m, \quad (2.9)$$

where C_α and δ_α at fixed α are the same that in Theorem 2.1.

Actually, Corollary 2.1 can be considered as global stability estimate for determining potential v from its Cauchy data set \mathcal{C}_v for equation (1.1) at fixed energy E , where $d \geq 3$.

Let $\sigma_{\alpha,v}$ denote the spectrum of the operator $-\Delta + v$ in D with boundary condition (1.6).

Remark 2.1. In Theorem 2.1 we do not assume that $E \notin \sigma_{\alpha,v_1} \cup \sigma_{\alpha,v_2}$ namely for $\alpha = 0$ in contrast with [1], [19]. In addition, in fact, in Corollary 2.1 there are no special assumptions on E and α at all. Actually, the stability estimates of [1], [19] make no sense for $E \in \sigma_{0,v_1} \cup \sigma_{0,v_2}$ and are too weak if $\text{dist}(E, \sigma_{0,v_1} \cup \sigma_{0,v_2})$ is too small.

Remark 2.2. The stability estimates of Theorem 2.1 and Corollary 2.1 admit principal improvement in the sense described in [19], [20], [24]. In particular, Theorem 2.1 with $s = m - d$ (for $d = 3$ and $E = 0$) follows from results presented in Sections 3, 4 of the present work and results presented in Section 8 of [19].

Remark 2.3. Proceeding from results of [21], [23] and results given in Section 3 of the present work, one can give global stability estimates for Problem 1.1 in dimension $d = 2$ for general α .

In the present work we do not develop Remarks 2.2 and 2.3 in detail because of restrictions in time.

Note also that Theorem 2.1 remains valid with complex-valued potentials v_1, v_2 and complex E, α .

3. Some basic properties of the impedance boundary map

Lemma 3.1. *Let D satisfy (1.2). Let potential v satisfy (1.3) and (1.7) for some fixed E and α . Let $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ denote the impedance boundary map for v . Then*

$$\begin{aligned} \left(\sin \alpha \hat{M}_\alpha + \cos \alpha \hat{I} \right) [\psi]_\alpha &= \psi|_{\partial D}, \\ \left(\cos \alpha \hat{M}_\alpha - \sin \alpha \hat{I} \right) [\psi]_\alpha &= \frac{\partial \psi}{\partial \nu} \Big|_{\partial D}, \end{aligned} \quad (3.1)$$

$$\int_{\partial D} [\psi^{(1)}]_\alpha \hat{M}_\alpha [\psi^{(2)}]_\alpha dx = \int_{\partial D} [\psi^{(2)}]_\alpha \hat{M}_\alpha [\psi^{(1)}]_\alpha dx \quad (3.2)$$

for all sufficiently regular solutions $\psi, \psi^{(1)}, \psi^{(2)}$ of equation (1.1) in \bar{D} , where \hat{I} denotes the identity operator on ∂D and $[\psi]_\alpha$ is defined by (1.5).

Note that identities (3.1) imply that

$$\left(\sin(\alpha_1 - \alpha_2) \hat{M}_{\alpha_1} + \cos(\alpha_1 - \alpha_2) \hat{I} \right) \left(\sin(\alpha_2 - \alpha_1) \hat{M}_{\alpha_2} + \cos(\alpha_2 - \alpha_1) \hat{I} \right) = \hat{I}, \quad (3.3)$$

under the assumptions of Lemma 3.1 fulfilled simultaneously for $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

Proof of Lemma 3.1. Identities (3.1) follow from definition (1.4) of the map \hat{M}_α .

To prove (3.2) we use, in particular, the Green formula

$$\int_{\partial D} \left(\phi^{(1)} \frac{\partial \phi^{(2)}}{\partial \nu} - \phi^{(2)} \frac{\partial \phi^{(1)}}{\partial \nu} \right) dx = \int_D \left(\phi^{(1)} \Delta \phi^{(2)} - \phi^{(2)} \Delta \phi^{(1)} \right) dx, \quad (3.4)$$

where $\phi^{(1)}$ and $\phi^{(2)}$ are arbitrary sufficiently regular functions in \bar{D} . Using (3.4) and the identities

$$\psi^{(1)} \Delta \psi^{(2)} = (v - E) \psi^{(1)} \psi^{(2)} = \psi^{(2)} \Delta \psi^{(1)} \quad \text{in } D, \quad (3.5)$$

we obtain that

$$\int_{\partial D} \left(\psi^{(1)} \frac{\partial \psi^{(2)}}{\partial \nu} - \psi^{(2)} \frac{\partial \psi^{(1)}}{\partial \nu} \right) dx = 0. \quad (3.6)$$

Using (3.6), we get that

$$\begin{aligned} \int_{\partial D} \left(\cos \alpha \psi^{(1)} - \sin \alpha \frac{\partial \psi^{(1)}}{\partial \nu} \right) \left(\sin \alpha \psi^{(2)} + \cos \alpha \frac{\partial \psi^{(2)}}{\partial \nu} \right) dx = \\ = \int_{\partial D} \left(\cos \alpha \psi^{(2)} - \sin \alpha \frac{\partial \psi^{(2)}}{\partial \nu} \right) \left(\sin \alpha \psi^{(1)} + \cos \alpha \frac{\partial \psi^{(1)}}{\partial \nu} \right) dx. \end{aligned} \quad (3.7)$$

Identity (3.2) follows from (3.7) and definition (1.4) of the map \hat{M}_α . ■

Theorem 3.1. *Let D satisfy (1.2). Let two potentials v_1, v_2 satisfy (1.3), (1.7) for some fixed E and α . Let $\hat{M}_{\alpha,v_1} = \hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2} = \hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1, v_2 , respectively. Then*

$$\int_D (v_1 - v_2) \psi_1 \psi_2 dx = \int_{\partial D} [\psi_1]_\alpha \left(\hat{M}_{\alpha,v_1} - \hat{M}_{\alpha,v_2} \right) [\psi_2]_\alpha dx \quad (3.8)$$

for all sufficiently regular solutions ψ_1 and ψ_2 of equation (1.1) in \bar{D} with $v = v_1$ and $v = v_2$, respectively, where $[\psi]_\alpha$ is defined by (1.5).

Proof of Theorem 3.1. As in (3.5) we have that

$$\begin{aligned}\psi_1 \Delta \psi_2 &= (v_2 - E) \psi_1 \psi_2, \\ \psi_2 \Delta \psi_1 &= (v_1 - E) \psi_1 \psi_2.\end{aligned}\tag{3.9}$$

Combining (3.9) with (3.4), (3.1) and (3.2), we obtain that

$$\begin{aligned}\int_D (v_1(x) - v_2(x)) \psi_1(x) \psi_2(x) dx &= \int_{\partial D} \left(\psi_2 \frac{\partial \psi_1}{\partial \nu} - \psi_1 \frac{\partial \psi_2}{\partial \nu} \right) dx = \\ &= \int_{\partial D} \left(\sin \alpha \hat{M}_{\alpha, v_2} + \cos \alpha \hat{I} \right) [\psi_2]_\alpha \left(\cos \alpha \hat{M}_{\alpha, v_1} - \sin \alpha \hat{I} \right) [\psi_1]_\alpha dx - \\ &- \int_{\partial D} \left(\sin \alpha \hat{M}_{\alpha, v_1} + \cos \alpha \hat{I} \right) [\psi_1]_\alpha \left(\cos \alpha \hat{M}_{\alpha, v_2} - \sin \alpha \hat{I} \right) [\psi_2]_\alpha dx = \\ &= \int_{\partial D} [\psi_1]_\alpha \left(\hat{M}_{\alpha, v_1} - \hat{M}_{\alpha, v_2} \right) [\psi_2]_\alpha dx.\end{aligned}\tag{3.10}$$

■

Remark 3.1. Identity (3.8) for $\alpha = 0$ is reduced to Alessandrini's identity (Lemma 1 of [1]).

Let $G_\alpha(x, y, E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) under assumptions (1.2), (1.3) and (1.7). Note that

$$G_\alpha(x, y, E) = G_\alpha(y, x, E), \quad x, y \in \bar{D}.\tag{3.11}$$

The symmetry (3.11) is proved in Section 6.

Theorem 3.2. *Let D satisfy (1.2). Let potential v satisfy (1.3) and (1.7) for some fixed E and α such that $\sin \alpha \neq 0$. Let $G_\alpha(x, y, E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6). Then for $x, y \in \partial D$*

$$M_\alpha(x, y, E) = \frac{1}{\sin^2 \alpha} G_\alpha(x, y, E) - \text{ctg } \alpha \delta_{\partial D}(x - y),\tag{3.12}$$

where $M_\alpha(x, y, E)$ and $\delta_{\partial D}(x - y)$ denote the Schwartz kernels of the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha, v}(E)$ and the identity operator \hat{I} on ∂D , respectively, where \hat{M}_α and \hat{I} are considered as linear integral operators.

Proof of Lemma 3.2. Note that

$$[\phi]_{\alpha - \pi/2} = \frac{1}{\sin^2 \alpha} \sin \alpha \phi|_{\partial D} - \text{ctg } \alpha [\phi]_\alpha.\tag{3.13}$$

for all sufficiently regular functions ϕ in some neighbourhood of ∂D in D . Since G_α is the Green function for equation (1.1) we have that

$$\psi(y) = \int_{\partial D} \left(\psi(x) \frac{\partial G_\alpha}{\partial \nu_x}(x, y, E) - G_\alpha(x, y, E) \frac{\partial \psi}{\partial \nu}(x) \right) dx, \quad y \in D,\tag{3.14}$$

for all sufficiently regular solutions ψ of equation (1.1). Using (3.14) and impedance boundary condition (1.6) for G_α , we get that

$$\begin{aligned} \sin \alpha \psi(y) &= \sin \alpha \int_{\partial D} \left(\psi(x) \frac{\partial G_\alpha}{\partial \nu_x}(x, y, E) - G_\alpha(x, y, E) \frac{\partial \psi}{\partial \nu}(x) \right) dx = \\ &= \int_{\partial D} [\psi(x)]_\alpha G_\alpha(x, y, E) dx, \quad y \in D. \end{aligned} \quad (3.15)$$

Due to (3.2) we have that

$$M_\alpha(x, y, E) = M_\alpha(y, x, E), \quad x, y \in \partial D. \quad (3.16)$$

Combining (1.4), (3.13), (3.15) and (3.16), we obtain (3.12). \blacksquare

Corollary 3.1. *Let assumptions of Theorem 3.1 hold. Then*

$$\hat{M}_{\alpha, v_1}(E) - \hat{M}_{\alpha, v_2}(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D). \quad (3.17)$$

Scheme of the proof of Corollary 3.1. Let $G_{\alpha, v_1}(x, y, E)$ and $G_{\alpha, v_2}(x, y, E)$ be the Green functions for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $v = v_1$ and $v = v_2$, respectively. Using (3.11), we find that

$$\begin{aligned} G_{\alpha, v_1}(x, y, E) &= \int_D G_{\alpha, v_1}(x, \xi, E) (\Delta_\xi - v_2(\xi) + E) G_{\alpha, v_2}(\xi, y, E) d\xi, \\ G_{\alpha, v_2}(x, y, E) &= \int_D (\Delta_\xi - v_1(\xi) + E) G_{\alpha, v_1}(x, \xi, E) G_{\alpha, v_2}(\xi, y, E) d\xi, \\ \int_{\partial D} \left(G_{\alpha, v_1}(x, \xi, E) \frac{\partial G_{\alpha, v_2}}{\partial \nu_\xi}(\xi, y, E) - G_{\alpha, v_2}(\xi, y, E) \frac{\partial G_{\alpha, v_1}}{\partial \nu_\xi}(x, \xi, E) \right) d\xi &= 0, \\ & \quad x, y \in D. \end{aligned} \quad (3.18)$$

Combining (3.18) with (3.4), we get that

$$\begin{aligned} G_{\alpha, v_1}(x, y, E) - G_{\alpha, v_2}(x, y, E) &= \int_D (v_1(\xi) - v_2(\xi)) G_{\alpha, v_1}(x, \xi, E) G_{\alpha, v_2}(\xi, y, E) d\xi, \\ & \quad x, y \in D. \end{aligned} \quad (3.19)$$

The proof of (3.17) for the case of $\sin \alpha \neq 0$ can be completed proceeding from (3.3), (3.12), (3.19) and estimates of [11] and [3] on $G_\alpha(x, y, E)$ for $v \equiv 0$.

Corollary 3.1 for the Dirichlet-to-Neumann case ($\sin \alpha = 0$) was given in [14]. \blacksquare

Lemma 3.2. *Let D satisfy (1.2). Let v be a real-valued potential satisfying (1.3). Then for any fixed $E \in \mathbb{R}$ there is not more than countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with boundary condition (1.6).*

Proof of Lemma 3.2. Let $\psi^{(1)}, \psi^{(2)}$ be eigenfunctions for the operator $-\Delta + v$ in D with boundary condition (1.6) for $\alpha = \alpha^{(1)}$ and $\alpha = \alpha^{(2)}$, respectively. Then

$$\sin(\alpha^{(1)} - \alpha^{(2)}) \int_{\partial D} \psi^{(1)} \psi^{(2)} dx = \sin \alpha^{(1)} \sin \alpha^{(2)} \int_{\partial D} \left(\psi^{(1)} \frac{\partial \psi^{(2)}}{\partial \nu} - \psi^{(2)} \frac{\partial \psi^{(1)}}{\partial \nu} \right) dx = 0. \quad (3.20)$$

Since in the separable space $\mathbb{L}^2(\partial D)$ there is not more than countable orthogonal system of functions, we obtain the assertion of Lemma 3.2. \blacksquare

Remark 3.1 The assertion of Lemma 3.2 remains valid for the case of $\alpha \in \mathbb{C}$.

4. Faddeev functions

We consider the Faddeev functions G, ψ, h (see [6], [7], [9], [14]):

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \quad (4.1)$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (4.2)$$

where $x \in \mathbb{R}^d, k \in \mathbb{C}^d, \text{Im } k \neq 0, d \geq 3$,

$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (4.3)$$

where

$$k, l \in \mathbb{C}^d, k^2 = l^2, \text{Im } k = \text{Im } l \neq 0. \quad (4.4)$$

One can consider (4.1), (4.3) assuming that

$$v \text{ is a sufficiently regular function on } \mathbb{R}^d \text{ with sufficient decay at infinity.} \quad (4.5)$$

For example, in connection with Problem 1.1, one can consider (4.1), (4.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \quad (4.6)$$

We recall that (see [6], [7], [9], [14]):

- The function G satisfies the equation

$$(\Delta + k^2)G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \quad (4.7)$$

- Formula (4.1) at fixed k is considered as an equation for

$$\psi = e^{ikx} \mu(x, k), \quad (4.8)$$

where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$;

- As a corollary of (4.1), (4.2), (4.7), ψ satisfies (1.1) for $E = k^2$;
- The Faddeev functions G , ψ , h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized „scattering“ amplitude).

In addition, G , ψ , h in their zero energy restriction, that is for $E = 0$, were considered for the first time in [2]. The Faddeev functions G , ψ , h were, actually, rediscovered in [2].

Let

$$\begin{aligned}\Sigma_E &= \{k \in \mathbb{C}^d : k^2 = k_1^2 + \dots + k_d^2 = E\}, \\ \Theta_E &= \{k \in \Sigma_E, l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\}.\end{aligned}\quad (4.9)$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x, k) \rightarrow 1 \quad \text{as} \quad |\operatorname{Im} k| \rightarrow \infty \quad (4.10)$$

and, for any $\sigma > 1$,

$$|\mu(x, k)| + |\nabla \mu(x, k)| \leq \sigma \quad \text{for} \quad |\operatorname{Im} k| \geq r_1(N, D, E, m, \sigma), \quad (4.11)$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k, l) \in \Theta_E, k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \rightarrow \infty}} h(k, l) \quad \text{for any } p \in \mathbb{R}^d, \quad (4.12)$$

$$\begin{aligned}|\hat{v}(p) - h(k, l)| &\leq \frac{c_1(D, E, m)N^2}{\rho} \quad \text{for } (k, l) \in \Theta_E, \quad p = k - l, \\ &|\operatorname{Im} k| = |\operatorname{Im} l| = \rho \geq r_2(N, D, E, m), \\ &p^2 \leq 4(E + \rho^2),\end{aligned}\quad (4.13)$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (4.14)$$

Results of the type (4.10) go back to [2]. Results of the type (4.12), (4.13) (with less precise right-hand side in (4.13)) go back to [9]. In the present work estimate (4.11) is given according to [16], [18]. Estimate (4.13) follows, for example, from the estimate

$$\begin{aligned}\|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{L}^2(\mathbb{R}^d)} &= O(|k|^{-1}) \quad \text{as } |k| \rightarrow \infty, \\ k &\in \mathbb{C}^d \setminus \mathbb{R}^d, \quad |k| = (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2},\end{aligned}\quad (4.15)$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x - y, k)$ and Λ denotes the multiplication operator by the function $(1 + |x|^2)^{1/2}$. Estimate (4.15) was formulated, first, in [12] for $d \geq 3$. Concerning proof of (4.15), see [25].

In addition, we have that:

$$\begin{aligned}h_2(k, l) - h_1(k, l) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \psi_1(x, -l)(v_2(x) - v_1(x))\psi_2(x, k) dx \\ &\quad \text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0, \\ &\quad \text{and } v_1, v_2 \text{ satisfying (4.5),}\end{aligned}\quad (4.16)$$

$$\begin{aligned}
 h_2(k, l) - h_1(k, l) &= (2\pi)^{-d} \int_{\partial D} [\psi_1(\cdot, -l)]_\alpha \left(\hat{M}_{\alpha, v_2} - \hat{M}_{\alpha, v_1} \right) [\psi_2(\cdot, k)]_\alpha dx \\
 &\quad \text{for } (k, l) \in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0, \\
 &\quad \text{and } v_1, v_2 \text{ satisfying (1.7), (4.6),}
 \end{aligned} \tag{4.17}$$

where h_j, ψ_j denote h and ψ of (4.3) and (4.1) for $v = v_j$, and \hat{M}_{α, v_j} denotes the impedance boundary map of (1.4) for $v = v_j$, where $j = 1, 2$.

Formula (4.16) was given in [15]. Formula (4.17) follows from Theorem 3.1 and (4.16). Formula (4.17) for $\alpha = 0$ was given in [17].

5. Proof of Theorem 2.1

Let

$$\begin{aligned}
 \mathbb{L}_\mu^\infty(\mathbb{R}^d) &= \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\
 \|u\|_\mu &= \operatorname{ess\,sup}_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0.
 \end{aligned} \tag{5.1}$$

Note that

$$\begin{aligned}
 w \in \mathbb{W}^{m,1}(\mathbb{R}^d) &\implies \hat{w} \in \mathbb{L}_\mu^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\
 \|\hat{w}\|_\mu &\leq c_3(m, d) \|w\|_{m,1} \quad \text{for } \mu = m,
 \end{aligned} \tag{5.2}$$

where $\mathbb{W}^{m,1}, \mathbb{L}_\mu^\infty$ are the spaces of (2.5), (5.1),

$$\hat{w}(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \tag{5.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \tag{5.4}$$

we have that

$$\begin{aligned}
 \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq \sup_{x \in \bar{D}} \left| \int_{\mathbb{R}^d} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) dp \right| \leq \\
 &\leq I_1(r) + I_2(r) \quad \text{for any } r > 0,
 \end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
 I_1(r) &= \int_{|p| \leq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\
 I_2(r) &= \int_{|p| \geq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp.
 \end{aligned} \tag{5.6}$$

Using (5.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_3(m, d) N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{5.7}$$

Due to (4.13), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq |h_2(k, l) - h_1(k, l)| + \frac{2c_1(D, E, m)N^2}{\rho}, \\ p &\in \mathbb{R}^d, \quad p = k - l, \quad (k, l) \in \Theta_E, \\ |\operatorname{Im} k| = |\operatorname{Im} l| &= \rho \geq r_2(N, D, E, m), \\ p^2 &\leq 4(E + \rho^2). \end{aligned} \quad (5.8)$$

Let

$$\begin{aligned} c_4 &= (2\pi)^{-d} \int_{\partial D} dx, \quad L = \max_{x \in \partial D} |x|, \\ \delta_\alpha &= \|\hat{M}_{\alpha, v_2}(E) - \hat{M}_{\alpha, v_1}(E)\|, \end{aligned} \quad (5.9)$$

where $\|\hat{M}_{\alpha, v_2}(E) - \hat{M}_{\alpha, v_1}(E)\|$ is defined according to (2.2).

Due to (4.16), (4.17), we have that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 \|\psi_1(\cdot, -l)\|_{\mathbb{L}^\infty(\partial D)} \delta_\alpha \|\psi_2(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)}, \\ (k, l) &\in \Theta_E, \quad |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0. \end{aligned} \quad (5.10)$$

Using (1.5), (4.11), we find that

$$\begin{aligned} \|\psi(\cdot, k)\|_{\mathbb{L}^\infty(\partial D)} &\leq c_5(E) \sigma \exp\left(|\operatorname{Im} k|(L + 1)\right), \\ k &\in \Sigma_E, \quad |\operatorname{Im} k| \geq r_1(N, D, E, m, \sigma). \end{aligned} \quad (5.11)$$

Here and bellow in this section the constant σ is the same that in (4.11).

Combining (5.10) and (5.11), we obtain that

$$\begin{aligned} |h_2(k, l) - h_1(k, l)| &\leq c_4 (c_5(E)\sigma)^2 \exp\left(2\rho(L + 1)\right) \delta_\alpha, \\ (k, l) &\in \Theta_E, \quad \rho = |\operatorname{Im} k| = |\operatorname{Im} l| \geq r_1(N, D, E, m, \sigma). \end{aligned} \quad (5.12)$$

Using (5.8), (5.12), we get that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| &\leq c_4 (c_5(E)\sigma)^2 \exp\left(2\rho(L + 1)\right) \delta_\alpha + \frac{2c_1(D, E, m)N^2}{\rho}, \\ p &\in \mathbb{R}^d, \quad p^2 \leq 4(E + \rho^2), \quad \rho \geq r_3(N, D, E, m, \sigma), \end{aligned} \quad (5.13)$$

where $r_3(N, D, E, m, \sigma)$ is such that

$$\rho \geq r_3(N, D, E, m, \sigma) \implies \begin{cases} \rho \geq r_1(N, D, E, m, \sigma), \\ \rho \geq r_2(N, D, E, m), \\ \rho^{2/m} \leq 4(E + \rho^2). \end{cases} \quad (5.14)$$

Let

$$c_6 = \int_{p \in \mathbb{R}^d, |p| \leq 1} dp, \quad c_7 = \int_{p \in \mathbb{R}^d, |p|=1} dp. \quad (5.15)$$

Using (5.6), (5.13), we get that

$$I_1(r) \leq c_6 r^d \left(c_4 (c_5(E)\sigma)^2 \exp\left(2\rho(L+1)\right) \delta_\alpha + \frac{2c_1(D, E, m)N^2}{\rho} \right), \quad (5.16)$$

$$r > 0, \quad r^2 \leq 4(\rho^2 + E), \quad \rho \geq r_3(N, D, E, m, \sigma).$$

Using (5.6), (5.7), we find that for any $r > 0$

$$I_2(r) \leq 2c_3(m, d)Nc_7 \int_r^{+\infty} \frac{dt}{t^{m-d+1}} \leq \frac{2c_3(m, D)Nc_7}{m-d} \frac{1}{r^{m-d}}. \quad (5.17)$$

Combining (5.5), (5.16), (5.17) for $r = \rho^{1/m}$ and (5.14), we get that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq c_8(D, \sigma)\rho^{d/m} e^{2\rho(L+1)} \delta_\alpha + c_9 d(N, D, E, m) \rho^{-\frac{m-d}{m}}, \quad (5.18)$$

$$\rho \geq r_3(N, D, E, m, \sigma).$$

We fix some $\tau \in (0, 1)$ and let

$$\beta = \frac{1-\tau}{2(L+1)}, \quad \rho = \beta \ln(3 + \delta_\alpha^{-1}), \quad (5.19)$$

where δ_α is so small that $\rho \geq r_3(N, D, E, m, \sigma)$. Then due to (5.18), we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} &\leq c_8(D, \sigma) (\beta \ln(3 + \delta_\alpha^{-1}))^{d/m} (3 + \delta_\alpha^{-1})^{2\beta(L+1)} \delta_\alpha + \\ &\quad + c_9(N, D, E, m) (\beta \ln(3 + \delta_\alpha^{-1}))^{-\frac{m-d}{m}} = \\ &= c_8(D, \sigma) \beta^{d/m} (1 + 3\delta_\alpha)^{1-\tau} \delta_\alpha^\tau (\ln(3 + \delta_\alpha^{-1}))^{d/m} + \\ &\quad + c_9(N, D, E, m) \beta^{-\frac{m-d}{m}} (\ln(3 + \delta_\alpha^{-1}))^{-\frac{m-d}{m}}, \end{aligned} \quad (5.20)$$

where τ, β and δ_α are the same as in (5.19).

Using (5.20), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq c_{10}(N, D, E, m, \sigma) (\ln(3 + \delta_\alpha^{-1}))^{-\frac{m-d}{m}} \quad (5.21)$$

for $\delta_\alpha = \|\hat{M}_{\alpha, v_2} - \hat{M}_{\alpha, v_1}\| \leq \delta^{(0)}(N, D, E, m, \sigma)$, where $\delta^{(0)}$ is a sufficiently small positive constant. Estimate (5.21) in the general case (with modified c_{10}) follows from (5.21) for $\delta_\alpha \leq \delta^{(0)}(N, D, E, m, \sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(D)} \leq c_{11}(D, m)N$.

Thus, Theorem 2.1 is proved for $s = \frac{m-d}{m}$ and, since $\ln(3 + \delta_\alpha^{-1}) > 1$, for any $0 < s \leq \frac{m-d}{m}$.

6. Appendix

Lemma 6.1. *Let D satisfy (1.2). Let potential v satisfy (1.3) and (1.7) for some fixed E and α . Let $G_\alpha(x, y, E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6). Then*

$$G_\alpha(x, y, E) = G_\alpha(y, x, E), \quad x, y \in D. \quad (6.1)$$

Proof of Lemma 6.1. Let D' be an open bounded domain in \mathbb{R}^d such that

- $D \subset D'$,
- D' satisfies (1.2),
- E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D' .

Here and bellow in this section we assume that $v \equiv 0$ on $D' \setminus D$. Let $R(x, y, E)$ denote the Green function for the operator $-\Delta + v - E$ in D' with the Dirichlet boundary condition. We recall that

$$R(x, y, E) = R(y, x, E), \quad x, y \in D'. \quad (6.2)$$

Using (3.4), (6.2), we find that for $x, y \in D$

$$\begin{aligned} & \int_{\partial D} \left(R(x, \xi, E) \frac{\partial R}{\partial \nu_\xi}(y, \xi, E) - R(y, \xi, E) \frac{\partial R}{\partial \nu_\xi}(x, \xi, E) \right) d\xi = \\ & = \int_D \left(R(x, \xi, E) (\Delta_\xi - v + E) R(y, \xi, E) - R(y, \xi, E) (\Delta_\xi - v + E) R(x, \xi, E) \right) d\xi = \\ & = -R(x, y, E) + R(y, x, E) = 0. \end{aligned} \quad (6.3)$$

Note that $W = G_\alpha + R(E)$ is the solution of the equation

$$(-\Delta_x + v - E)W(x, y) = 0, \quad x, y \in D \quad (6.4)$$

with the boundary condition

$$\begin{aligned} & \left(\cos \alpha W(x, y) - \sin \alpha \frac{\partial W}{\partial \nu_x}(x, y) \right) \Big|_{x \in \partial D} = \\ & = \left(\cos \alpha R(x, y, E) - \sin \alpha \frac{\partial R}{\partial \nu_x}(x, y, E) \right) \Big|_{x \in \partial D}, \quad y \in D. \end{aligned} \quad (6.5)$$

Using (3.4) and (6.4), we find that for $x, y \in D$

$$\begin{aligned} & \int_{\partial D} \left(W(\xi, x) \frac{\partial W}{\partial \nu_\xi}(\xi, y) - W(\xi, y) \frac{\partial W}{\partial \nu_\xi}(\xi, x) \right) d\xi = \\ & = \int_D \left(W(\xi, x) (\Delta_\xi - v + E) W(\xi, y) - W(\xi, y) (\Delta_\xi - v + E) W(\xi, x) \right) d\xi = 0 \end{aligned} \quad (6.6)$$

Note that

$$W(x, y) = - \int_D W(\xi, y) (\Delta_\xi - v + E) R(\xi, x, E) d\xi, \quad x, y \in D. \quad (6.7)$$

Combining (3.4), (6.4) and (6.7), we obtain that

$$\begin{aligned} W(x, y) = - \int_{\partial D} \left(W(\xi, y) \frac{\partial R}{\partial \nu_\xi}(\xi, x, E) - R(\xi, x, E) \frac{\partial W}{\partial \nu_\xi}(\xi, y) \right) d\xi, \\ x, y \in D. \end{aligned} \quad (6.8)$$

Using (6.5) and (6.8), we get that

$$\begin{aligned} \sin \alpha W(x, y) &= \\ &= \int_{\partial D} W(\xi, y) \left(\cos \alpha W(\xi, x) - \sin \alpha \frac{\partial W}{\partial \nu_\xi}(\xi, x) - \cos \alpha R(\xi, x, E) \right) d\xi - \\ &\quad - \int_{\partial D} R(\xi, x, E) \left(\cos \alpha R(\xi, y, E) - \sin \alpha \frac{\partial R}{\partial \nu_\xi}(\xi, x, E) - \cos \alpha W(\xi, y) \right) d\xi, \end{aligned} \tag{6.9}$$

$x, y \in D.$

Combining similar to (6.9) formula for $\sin \alpha W(y, x)$, (6.3) and (6.6), we obtain that

$$\sin \alpha W(x, y) - \sin \alpha W(y, x) = 0, \quad x, y \in D. \tag{6.10}$$

In the case of $\sin \alpha = 0$, combining (6.5) and (6.8), we get that

$$W(x, y) = \int_{\partial D} \left(-R(\xi, y, E) \frac{\partial R}{\partial \nu_\xi}(\xi, x, E) + W(\xi, x) \frac{\partial W}{\partial \nu_\xi}(\xi, y) \right) d\xi, \tag{6.11}$$

$x, y \in D.$

Hence, one can get that for any α

$$W(x, y) = W(y, x), \quad x, y \in D. \tag{6.12}$$

Combining (6.2) and (6.12), we obtain (6.1). ■

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