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# A NOETHER-DEURING THEOREM FOR DERIVED CATEGORIES

ALEXANDER ZIMMERMANN

ABSTRACT. We prove a Noether-Deuring theorem for the derived category of bounded complexes of modules over a Noetherian algebra.

## INTRODUCTION

The classical Noether-Deuring theorem states that given an algebra  $A$  over a field  $K$  and a finite extension field  $L$  of  $K$ , two  $A$ -modules  $M$  and  $N$  are isomorphic as  $A$ -modules, if  $L \otimes_K M$  is isomorphic to  $L \otimes_K N$  as an  $L \otimes_K A$ -module. In 1972 Roggenkamp gave a nice extension of this result to extensions  $S$  of local commutative Noetherian rings  $R$  and modules over Noetherian  $R$ -algebras.

For the derived category of  $A$ -modules no such generalisation was documented before. The purpose of this note is to give a version of the Noether-Deuring theorem, in the generalised version given by Roggenkamp, for right bounded derived categories of  $A$ -modules. If there is a morphism  $\alpha \in \text{Hom}_{D(\Lambda)}(X, Y)$ , then it is fairly easy to show that for a faithfully flat ring extension  $S$  over  $R$  the fact that  $id_S \otimes \alpha$  is an isomorphism implies that  $\alpha$  is an isomorphism. This is done in Proposition 1. More delicate is the question if only an isomorphism in  $\text{Hom}_{D(S \otimes_R \Lambda)}(S \otimes_R X, S \otimes_R Y)$  is given. Then we need further finiteness conditions on  $\Lambda$  and on  $R$  and proceed by completion of  $R$  and then a classical going-down argument. This is done in Theorem 4 and Corollary 9.

For the notation concerning derived categories we refer to Verdier [6]. In particular,  $D(A)$  (resp  $D^-(A)$ , resp  $D^b(A)$ ) denotes the derived category of complexes (resp. right bounded complexes, resp. bounded complexes) of finitely generated  $A$ -modules,  $K^-(A - \text{proj})$  (resp.  $K^b(A - \text{proj})$ , resp  $K^{-,b}(A - \text{proj})$ ) is the homotopy category of right bounded complexes (resp. bounded complexes, resp. right bounded complexes with bounded homology) of finitely generated projective  $A$ -modules. For a complex  $Z$  we denote by  $H_i(Z)$  the homology of  $Z$  in degree  $i$ , and by  $H(Z)$  the graded module given by the homology of  $Z$ .

## 1. THE RESULT

We start with an easy observation.

**Proposition 1.** *Let  $R$  be a commutative ring and let  $\Lambda$  be an  $R$ -algebra. Let  $S$  be a commutative faithfully flat  $R$ -algebra. Denote by  $D(\Lambda)$  the derived category of complexes of finitely generated  $\Lambda$ -modules. Then if there is  $\alpha \in \text{Hom}_{D(\Lambda)}(X, Y)$  so that  $id_S \otimes_R \alpha \in \text{Hom}_{D(S \otimes_R \Lambda)}(S \otimes_R^{\mathbb{L}} X, S \otimes_R^{\mathbb{L}} Y)$  is an isomorphism in  $D(S \otimes_R \Lambda)$ , then  $\alpha$  is an isomorphism in  $D(\Lambda)$ .*

Proof. Let  $Z$  be a complex in  $D(\Lambda)$ . Since  $S$  is flat over  $R$  the functor  $S \otimes_R - : R - \text{Mod} \rightarrow S - \text{Mod}$  is exact, and hence the left derived functor  $S \otimes_R^{\mathbb{L}} -$  coincides with the ordinary tensor product functor  $S \otimes_R -$ . We can therefore work with the usual tensor product and a complex  $Z$  of  $\Lambda$ -modules.

We claim that since  $S$  is flat,  $S \otimes_R -$  induces an isomorphism  $S \otimes_R H(Z) \simeq H(S \otimes_R^{\mathbb{L}} Z)$ . If  $\partial_Z$  is the differential of  $Z$ , then

$$0 \longrightarrow \ker(\partial_Z) \longrightarrow Z \xrightarrow{\partial_Z} \text{im}(\partial_Z) \longrightarrow 0$$

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is exact in the category of  $\Lambda$ -modules.

Since  $S$  is flat,

$$0 \longrightarrow S \otimes_R \ker(\partial_Z) \longrightarrow S \otimes_R Z \xrightarrow{id_S \otimes_R \partial_Z} S \otimes_R im(\partial_Z) \longrightarrow 0$$

is exact. Hence

$$\ker(id_S \otimes_R \partial_Z) = S \otimes_R \ker(\partial_Z) \text{ and } im(id_S \otimes_R \partial_Z) = S \otimes_R im(\partial_Z).$$

This shows the claim.

Since  $id_S \otimes_R \alpha$  is an isomorphism, its cone  $C(id_S \otimes_R \alpha)$  is acyclic. Moreover,  $C(id_S \otimes_R \alpha) = S \otimes_R C(\alpha)$  by the very construction of the mapping cone. But now,

$$0 = H(C(id_S \otimes_R \alpha)) = H(S \otimes_R C(\alpha)) = S \otimes_R H(C(\alpha)).$$

Since  $S$  is faithfully flat, this implies  $H(C(\alpha)) = 0$  and therefore  $C(\alpha)$  is acyclic. We conclude that  $\alpha$  is an isomorphism in  $D(\Lambda)$  which shows the statement. ■

**Remark 2.** Observe that we assumed that  $X \xrightarrow{\alpha} Y$  is assumed to be a morphism in  $D(\Lambda)$ . The question if the existence of an isomorphism  $S \otimes_R X \xrightarrow{\hat{\alpha}} S \otimes_R Y$  in  $D(S \otimes_R \Lambda)$  implies the existence of a morphism  $\alpha : X \longrightarrow Y$  in  $D(\Lambda)$  so that  $id_S \otimes_R^{\mathbb{L}} \alpha$  is an isomorphism is left open. Under stronger hypotheses this is the purpose of Theorem 4 below. The proof follows [5] which deals with the module case.

**Lemma 3.** *If  $S$  is a faithfully flat  $R$ -module and  $\Lambda$  is a Noetherian  $R$ -algebra, then for all objects  $X$  and  $Y$  of  $D^-(\Lambda)$  we get*

$$Hom_{D^-(S \otimes_R \Lambda)}(S \otimes_R X, S \otimes_R Y) \simeq S \otimes_R Hom_{D^-(\Lambda)}(X, Y).$$

Proof. We use the equivalence of categories  $K^-(\Lambda\text{-proj}) \simeq D^-(\Lambda)$  and suppose therefore that  $X$  and  $Y$  are right bounded complexes of finitely generated projective  $\Lambda$ -modules. We even may assume that  $X$  and  $Y$  are right bounded complexes of finitely generated free  $\Lambda$ -modules. But

$$S \otimes_R Hom_{\Lambda}(\Lambda^n, U) = S \otimes_R U^n = (S \otimes_R U)^n = Hom_{S \otimes_R \Lambda}((S \otimes_R \Lambda)^n, S \otimes_R U)$$

which proves the statement. ■

**Theorem 4.** *Let  $R$  be a commutative Noetherian ring, let  $S$  be a commutative Noetherian  $R$ -algebra and suppose that  $S$  is faithfully flat as  $R$ -module. Suppose  $S \otimes_R rad(R) = rad(S)$ . Let  $\Lambda$  be a Noetherian  $R$ -algebra, let  $X$  and  $Y$  be two objects of  $D^-(\Lambda)$  and suppose that  $End_{D^-(X)}(X)$  is a finitely generated  $R$ -module. Then*

$$S \otimes_R^{\mathbb{L}} X \simeq S \otimes_R^{\mathbb{L}} Y \Leftrightarrow X \simeq Y.$$

**Remark 5.** We observe that if  $R$  is local and  $S = \hat{R}$  is the  $rad(R)$ -adic completion, then  $S$  is faithfully flat as  $R$ -module and  $S \otimes_R rad(R) = rad(S)$ .

Proof of Theorem 4. According to the hypotheses we now suppose that  $End_{D^-(\Lambda)}(X)$  is a finitely generated  $R$ -module and that  $S \otimes_R rad(R) = rad(S)$ . We only need to show " $\Rightarrow$ " and assume therefore that  $X$  and  $Y$  are in  $K^-(\Lambda\text{-proj})$ , and that  $S \otimes_R X$  and  $S \otimes_R Y$  are isomorphic.

Let  $X_S := S \otimes_R X$  and  $S \otimes_R Y =: Y_S$  in  $D^-(S \otimes_R \Lambda)$  to shorten the notation and denote by  $\varphi_S$  the isomorphism  $X_S \longrightarrow Y_S$ . Since then  $X_S$  is a direct factor of  $Y_S$  by means of  $\varphi_S$ , the mapping

$$\varphi_S = \sum_{i=1}^n s_i \otimes \varphi_i : X_S \longrightarrow Y_S$$

for  $s_i \in S$  and  $\varphi_i \in Hom_{D^-(\Lambda)}(X, Y)$  has a left inverse  $\psi : Y_S \longrightarrow X_S$  so that

$$\psi \circ \varphi_S = id_{X_S}.$$

Then,

$$0 \longrightarrow \text{rad}(R) \longrightarrow R \longrightarrow R/\text{rad}(R) \longrightarrow 0$$

is exact and since  $S$  is flat over  $R$  we get that

$$0 \longrightarrow S \otimes_R \text{rad}(R) \longrightarrow S \longrightarrow S \otimes_R (R/\text{rad}(R)) \longrightarrow 0$$

is exact. This shows

$$S \otimes_R (R/\text{rad}(R)) \simeq S/(S \otimes_R \text{rad}(R)).$$

By hypothesis we have  $S \otimes_R \text{rad}(R) = \text{rad}(S)$ , identifying canonically  $S \otimes_R R \simeq S$ . Then there are  $r_i \in R$  so that  $1_S \otimes r_i - s_i \in \text{rad}(S)$  for all  $i \in \{1, \dots, n\}$ .

Put

$$\varphi := \sum_{i=1}^n r_i \varphi_i \in \text{Hom}_{D^-(\Lambda)}(X, Y).$$

Then

$$\begin{aligned} \sum_{i=1}^n \psi \circ (1_S \otimes (r_i \varphi_i)) - 1_S \otimes \text{id}_X &= \sum_{i=1}^n (\psi \circ (1_S \otimes r_i \varphi_i) - \psi \circ (s_i \otimes \varphi_i)) \\ &= \sum_{i=1}^n (1_S \otimes r_i - s_i) \cdot (\psi \circ (\text{id}_S \otimes \varphi_i)) \\ &\in (\text{rad}(S) \otimes_R \text{End}_{D^-(\Lambda)}(X)) \end{aligned}$$

and since  $\text{End}_{D^-(\Lambda)}(X)$  is a Noetherian  $R$ -module, using Nakayama's lemma we obtain that  $\psi \circ (\sum_{i=1}^n 1_S \otimes r_i \varphi_i)$  is invertible in  $S \otimes_R \text{End}_{D^-(\Lambda)}(X)$ . Hence  $\text{id}_S \otimes_R \varphi$  is left split and therefore

$$X_S \xrightarrow{\text{id}_S \otimes_R \varphi} Y_S \longrightarrow C(\text{id}_S \otimes_R \varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle, with  $C(\text{id}_S \otimes_R \varphi)$  being the cone of  $\text{id}_S \otimes_R \varphi$ . However,

$$C(\text{id}_S \otimes_R \varphi) = S \otimes_R C(\varphi)$$

and hence

$$X_S \xrightarrow{\text{id}_S \otimes_R \varphi} Y_S \longrightarrow S \otimes_R C(\varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle.

Since  $\varphi_S$  is an isomorphism,  $\varphi_S$  has a right inverse  $\chi : Y_S \longrightarrow X_S$  as well. Now, since  $X_S \simeq Y_S$ , since  $S$  is faithfully flat over  $R$ , and since  $\text{End}_{D^-(\Lambda)}(X)$  is finitely generated as  $R$ -module, using Lemma 3 we obtain that  $\text{End}_{D^-(\Lambda)}(Y)$  is finitely generated as  $R$ -module as well. The same argument as for the left inverse  $\psi$  shows that  $(\text{id}_S \otimes \varphi) \circ \chi$  is invertible in  $S \otimes_R \text{End}_{D^-(\Lambda)}(Y)$ . Hence

$$X_S \xrightarrow{\text{id}_S \otimes_R \varphi} Y_S \xrightarrow{0} S \otimes_R C(\varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle. This shows that  $S \otimes_R C(\varphi)$  is acyclic, and hence

$$0 = H(S \otimes_R C(\varphi)) = S \otimes_R H(C(\varphi)).$$

Since  $S$  is faithfully flat over  $R$  also  $H(C(\varphi)) = 0$ , which implies that  $C(\varphi)$  is acyclic and therefore  $\varphi$  is an isomorphism.

This proves the theorem. ■

Let  $A$  be an algebra over a complete discrete valuation ring  $R$  which is finitely generated as modules over  $R$ . We shall need a Krull-Schmidt theorem for the derived category of bounded complexes over  $A$ . This fact seems to be well-known, but for the convenience of the reader we give a proof.

**Proposition 6.** *Let  $R$  be a complete discrete valuation ring and let  $A$  be an  $R$ -algebra, finitely generated as  $R$ -module. Then the Krull-Schmidt theorem holds for  $K^b(A\text{-proj})$ .*

Proof. We first show a Fitting lemma for  $K^b(A - proj)$ .

Let  $X$  be a complex in  $K^b(A - proj)$  and let  $u$  be an endomorphism of the complex  $X$ . Then  $X = X' \oplus X''$  as graded modules, by Fitting's lemma in the version for algebras over complete discrete valuation rings [1, Lemma 1.9.2]. The restriction of  $u$  on  $X'$  is an automorphism in each degree and the restriction of  $u$  on  $X''$  is nilpotent modulo  $rad(R)^m$  for each  $m$ . Therefore  $u$  is a diagonal matrix  $\begin{pmatrix} \iota & 0 \\ 0 & \nu \end{pmatrix}$  in each degree where  $\iota : X' \rightarrow X'$  is invertible, and  $\nu : X'' \rightarrow X''$  is nilpotent modulo  $rad(R)^m$  for each  $m$  in each degree. The differential  $\partial$  on  $X$  is given by  $\begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}$  and the fact that  $u$  commutes with  $\partial$  shows that  $\partial_3 \iota = \nu \partial_3$  and  $\partial_2 \nu = \iota \partial_2$ . Therefore,  $\partial_3 \iota^s = \nu^s \partial_3$  and  $\partial_2 \nu^s = \iota^s \partial_2$  for all  $s$ . Since  $\nu$  is nilpotent modulo  $rad(R)^m$  for each  $m$  in each degree, and  $\iota$  is invertible,  $\partial_2 = \partial_3 = 0$ . Hence the differential of  $X$  restricts to a differential on  $X'$  and a differential on  $X''$ . Moreover,  $X'$  and  $X''$  are both projective modules, since  $X$  is projective. Since  $X$  is actually in  $K^b(A - proj)$ , there are only finitely many degrees with non zero components, the restriction of  $u$  to  $X''$  is nilpotent modulo  $rad(R)^m$  for each  $m$ .

Hence, the endomorphism ring of an indecomposable object is local and the Krull-Schmidt theorem is an easy consequence by the classical proof as in [4] or in [1].

This shows the proposition. ■

**Remark 7.** If  $R$  is a field and  $A$  is a finite dimensional  $R$ -algebra, then we would be able to argue more directly. Indeed,  $X' = \text{im}(u^N)$  and  $X'' = \text{ker}(u^N)$  for large enough  $N$ . Then it is obvious that  $X'$  and  $X''$  are both subcomplexes of  $X$ . Observe that  $R$  may be a field in Proposition 6.

We obtain the following consequence of Proposition 6.

**Proposition 8.** *Let  $R$  be a complete discrete valuation ring and let  $A$  be a Noetherian  $R$ -algebra, finitely generated as  $R$ -module. Then any complex  $X$  of  $D^b(A)$  admits a unique decomposition  $X \simeq \bigoplus_{i \in I} X_i$  into indecomposable complexes  $X_i$  and the decomposition is unique up to isomorphism and permutation of factors.*

Proof. Since  $A$  is Noetherian,  $D^b(A) \simeq K^{-,b}(A - proj)$  and given a bounded complex  $X$ , it is isomorphic to a complex  $Y := P_X$  in  $K^{-,b}(A - proj)$ . Since the homology of  $Y$  is bounded, we may find  $N_0$  so that  $H_n(Y) = 0$  if  $|n| \geq N_0$ . We cut the complex  $Y$  in degree  $N_0$ , in the sense that there is a complex  $\sigma_{N_0} Y$  given by  $(\sigma_{N_0} Y)_m = Y_m$  if  $m \leq N_0$  and  $(\sigma_{N_0} Y)_m = 0$  if  $m > N_0$ . For  $m \leq N_0$  the degree  $m$  differential of  $\sigma_{N_0} Y$  is the same as the differential of  $Y$ , and is 0 in larger degrees.

$$\begin{array}{cccccccccccc} \ker(\partial_{N_0})[N_0 + 1] & : & \dots & \xrightarrow{\partial_{N_0+2}} & P_{N_0+1} & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ Y & : & \dots & \xrightarrow{\partial_{N_0+2}} & P_{N_0+1} & \xrightarrow{\partial_{N_0+1}} & P_{N_0} & \xrightarrow{\partial_{N_0}} & \dots & \xrightarrow{\partial_{m+1}} & P_m & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \sigma_{N_0} Y & : & \dots & \longrightarrow & 0 & \longrightarrow & P_{N_0} & \xrightarrow{\partial_{N_0}} & \dots & \xrightarrow{\partial_{m+1}} & P_m & \longrightarrow & 0 \end{array}$$

which gives the distinguished triangle

$$\sigma_{N_0} Y \longrightarrow Y \longrightarrow \ker(\partial_{N_0})[N_0 + 1] \xrightarrow{\partial_{N_0+1}} \sigma_{N_0} Y[1].$$

The choice of  $N_0$  implies that  $H_{N_0}(\sigma_{N_0} Y) = \ker(\partial_{N_0})$ . By Proposition 6 we see that  $\sigma_{N_0} Y$  decomposes

$$\sigma_{N_0} Y \simeq Q_1 \oplus \dots \oplus Q_s$$

uniquely into a direct sum of indecomposable objects  $Q_i$  for  $i \in \{1, \dots, s\}$ . The decomposition is induced by idempotent endomorphisms  $e_1, \dots, e_s$  of  $\sigma_{N_0} Y$  and hence the idempotent

endomorphisms  $e_1, \dots, e_s$  of  $\sigma_{N_0}Y$  induce idempotent endomorphisms  $H_{N_0}(e_1), \dots, H_{N_0}(e_s)$  of  $H_{N_0}(\sigma_{N_0}Y)$ . Therefore

$$\ker(\partial_{N_0}) = \bigoplus_{i=1}^s (H_{N_0}(e_i))(\ker(\partial_{N_0}))$$

so that the cone of  $\sigma_{N_0}Y \rightarrow Y$  is a projective resolution of  $\ker(\partial_{N_0})$ . Since this module decomposes into a direct sum, also the resolution decomposes into a direct sum, and hence the idempotent endomorphisms  $e_i$  of  $\sigma_{N_0}Y$  extend to idempotent endomorphisms  $\hat{e}_i$  of the initial object  $Y$ .

A maybe more direct way of seeing this is to observe that the decomposition  $\sigma_{N_0}Y \simeq Q_1 \oplus \dots \oplus Q_s$  induces a decomposition  $\partial_{N_0} = \bigoplus_{i=1}^s \partial_{N_0}^i$  for the corresponding differential  $\partial^i$  of  $Q_i$ . Hence

$$\ker(\partial_{N_0}) = \ker\left(\bigoplus_{i=1}^s \partial_{N_0}^i\right) = \bigoplus_{i=1}^s \ker(\partial_{N_0}^i)$$

and  $Y$  in degrees bigger than  $N_0$  is a projective resolution of the module  $\bigoplus_{i=1}^s \ker(\partial_{N_0}^i)$ , whence a direct sum of projective resolutions of the modules  $\ker(\partial_{N_0}^i)$  for all  $i \in \{1, \dots, s\}$ .

Hence, the Krull-Schmidt theorem holds for  $D^b(A)$  as well. ■

For the next Corollary we follow closely [5].

**Corollary 9.** *Let  $R$  be a commutative semilocal Noetherian ring, let  $S$  be a commutative  $R$ -algebra so that  $\hat{S} := \hat{R} \otimes_R S$  is a faithful projective  $\hat{R}$ -module of finite type. Let  $\Lambda$  be a Noetherian  $R$ -algebra, finitely generated as  $R$ -module, and let  $X$  and  $Y$  be two objects of  $D^b(\Lambda)$  and suppose that  $\text{End}_{D^b(\Lambda)}(X)$  is a finitely generated  $R$ -module. Then*

$$S \otimes_R^{\mathbb{L}} X \simeq S \otimes_R^{\mathbb{L}} Y \Leftrightarrow X \simeq Y.$$

Proof. If  $S \otimes_R^{\mathbb{L}} X \simeq S \otimes_R^{\mathbb{L}} Y$  in  $D^b(S \otimes_R \Lambda)$ , we get  $\hat{S} \otimes_R^{\mathbb{L}} X \simeq \hat{S} \otimes_R^{\mathbb{L}} Y$  in  $D^b(\hat{S} \otimes_R \Lambda)$ . Since  $R$  is semilocal with maximal ideals  $m_1, \dots, m_s$  we get  $\hat{R} = \prod_{i=1}^s \hat{R}_{m_i}$  for the completion  $\hat{R}_{m_i}$  of  $R$  at  $m_i$ . Now,  $\hat{S}$  is projective faithful of finite type, and so there are  $n_1, \dots, n_s$  with

$$\hat{S} \simeq \prod_{i=1}^s (\hat{R}_{m_i})^{n_i}$$

and therefore  $\hat{S} \otimes_R^{\mathbb{L}} X \simeq \hat{S} \otimes_R^{\mathbb{L}} Y$  implies

$$\prod_{i=1}^s (\hat{R}_{m_i})^{n_i} \otimes_R^{\mathbb{L}} X \simeq \prod_{i=1}^s (\hat{R}_{m_i})^{n_i} \otimes_R^{\mathbb{L}} Y.$$

Hence

$$(\hat{R}_{m_i} \otimes_R^{\mathbb{L}} X)^{n_i} \simeq (\hat{R}_{m_i} \otimes_R^{\mathbb{L}} Y)^{n_i}$$

for each  $i$ , and therefore by Proposition 8

$$\hat{R}_{m_i} \otimes_R^{\mathbb{L}} X \simeq \hat{R}_{m_i} \otimes_R^{\mathbb{L}} Y$$

for each  $i$ . By Theorem 4 we obtain  $X \simeq Y$ . ■

We get cancellation of factors from this statement.

**Corollary 10.** *Under the hypothesis of Theorem 4 or of Corollary 9 we get  $X \oplus U \simeq Y \oplus U$  in  $D^b(\Lambda)$  implies  $X \simeq Y$ .*

Proof. This is clear by Corollary 9 in combination with Proposition 8. ■

**Remark 11.** In [3] we developed a theory to roughly speaking parameterise geometrically objects in  $D^b(A)$  by orbits of a group action on a variety. For this purpose we need to assume that  $A$  is a finite dimensional algebra over an algebraically closed field  $K$ , so that it is possible to use arguments and constructions from algebraic geometry. Using Theorem 4 we can extend the theory to non algebraically closed fields  $K$  as well.

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