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The Harnack inequality for a class of degenerate elliptic operators

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Abstract

We prove a Harnack inequality for distributional solutions to a type of degenerate elliptic PDEs in N dimensions. The differential operators in question are related to the Kolmogorov operator, made up of the Laplacian in the last $N - 1$ variables, a first-order term corresponding to a shear flow in the direction of the first variable, and a bounded measurable potential term. The first-order coefficient is a smooth function of the last $N - 1$ variables and its derivatives up to certain order do not vanish simultaneously at any point, making the operators in question hypoelliptic.

1 Introduction

We prove a Harnack inequality for distributional solutions to the degenerate elliptic PDE

$$\Delta_y u + \beta(y)u_x + \gamma(x, y)u = 0 \tag{1.1}$$

in cylindrical domains in \mathbb{R}^N with axes in the direction of the first variable x . Here γ is bounded measurable and β is a smooth function such that the operator

$$L = \sum_{n=1}^{N-1} X_n^2 + X_0 := \sum_{n=1}^{N-1} (\partial_{y_n})^2 + \beta(y)\partial_x = \Delta_y + \beta(y)\partial_x \tag{1.2}$$

satisfies Hörmander's hypoellipticity condition. That is, vector fields $\{X_n\}_{n=0}^{N-1}$ and their commutators up to certain order span the whole tangent space \mathbb{R}^N at each (x, y) . Moreover, β changes sign so that L is not parabolic, since then the “elliptic” Harnack inequality (1.4) below would not hold in general. These conditions on β are equivalent to hypothesis (1.3) below and our result is then as follows:

Theorem 1.1 *Let $D \subseteq \mathbb{R}^{N-1}$ be open connected and $u : (a, b) \times D \rightarrow [0, \infty)$ a bounded distributional solution of (1.1) with γ bounded measurable and β satisfying for some $r \in \mathbb{N}$,*

$$\beta \in C^\infty(D), \quad \inf_D \beta < 0 < \sup_D \beta, \quad \text{and} \quad \sum_{0 \leq |\zeta| \leq r} |D^\zeta \beta(y)| > 0 \text{ for all } y \in D.^1 \quad (1.3)$$

Then for each $[a', b'] \subseteq (a, b)$ and bounded open D' with $\overline{D'} \subseteq D$, there is $C > 0$, depending only on D , D' , β and an upper bound on $(a' - a)^{-1}$, $(b - b')^{-1}$, $b' - a'$, and $\|\gamma\|_\infty$, such that

$$\sup_{(a', b') \times D'} u \leq C \inf_{(a', b') \times D'} u. \quad (1.4)$$

Remark. We note that Δ_y could be replaced by any x -independent, uniformly elliptic in y operator on D , but for the sake of simplicity we state the theorem with Δ_y instead.

This result is motivated by its application in our work [6] on large amplitude $A \rightarrow \infty$ asymptotics of traveling fronts in the x -direction, and their speeds, for the reaction-advection-diffusion equation

$$v_t + A \alpha(y) v_x = \Delta_{x,y} v + f(v) \quad (1.5)$$

on \mathbb{R}^{N+1} , with the first order term representing a shear flow in the x -direction and f a non-negative reaction function vanishing at 0 and 1. The front speeds in question are proved to satisfy $\lim_{A \rightarrow \infty} c^*(A\alpha, f)/A = \kappa(\alpha, f)$ for some constant $\kappa(\alpha, f) \geq 0$, so after substituting the front ansatz $v(t, x, y) = u(x - c^*(A\alpha, f)t, y)$ into (1.5) and scaling by A in the x variable, one formally recovers (1.1) in the limit $A \rightarrow \infty$, with $\beta(y) := \kappa(\alpha, f) - \alpha(y)$ and $\gamma(x, y) := -f(u(x, y))/u(x, y)$.

The study of hypoelliptic operators of the form

$$L = \sum_{n=1}^M X_n^2 + X_0$$

(where X_n are first order differential operators with smooth coefficients), possibly with an additional potential term, has been systematically pursued since Hörmander's fundamental paper [7]. Although various regularity and maximum principle results have been obtained soon thereafter (see, e.g., [2, 3, 4, 14, 19, 20]), Harnack inequalities and related heat kernel estimates for such operators have initially been proved only in the case when the tangent space at each point is spanned by the fields $\{X_n\}_{n=1}^M$ and their commutators, sometimes with X_0 either zero or a linear combination of $\{X_n\}_{n=1}^M$ [2, 9, 10, 12, 13].

More recently, Harnack inequalities have been obtained without this assumption for certain special classes of operators, not including (1.1) with general β, γ . Specifically, some operators with constant and linear coefficients, such as the Kolmogorov operator $L = \partial_{yy}^2 + y\partial_x - \partial_t$, were considered in [5, 16], and cases of more general coefficients satisfying somewhat rigid structural assumptions (see hypothesis [H.1] in [17]) were studied in [11, 17] and with a potential term in [18]. The domains involved in the obtained inequalities have to

¹For $\zeta = (\zeta_1, \dots, \zeta_{N-1}) \in \mathbb{N}^{N-1}$, we let $|\zeta| = \zeta_1 + \dots + \zeta_{N-1}$ and $D^\zeta \beta(y) = \frac{\partial^{|\zeta|} \beta}{\partial y_1^{\zeta_1} \dots \partial y_{N-1}^{\zeta_{N-1}}}(y)$.

depend on the metrics associated to the operators rather than the Euclidian metric, as shows a counter-example to a Harnack inequality in [5]. This is related to the need for the sign-changing assumption on β here. We also note that the operators considered in these papers involve the term ∂_t and appropriate “parabolic-type” Harnack inequalities are obtained, but corresponding “elliptic” inequalities follow from these.

It was a mild surprise to us that we were not able to find in the literature a sufficiently general result which would include our case (1.1). It appears that Harnack inequalities and heat kernel estimates become much more involved when the field X_0 is required for Hörmander’s condition to be satisfied. One hint in this direction is the fact that the sign-changing hypothesis on β is necessary for (1.4) to hold, so hypoellipticity of L is in itself not a sufficient condition.

We therefore believe that our method of proof of Theorem 1.1 in the next section is itself also a valuable contribution to the problem of quantitative estimates for hypoelliptic operators. The proof is based on the Feynman-Kac formula for the stochastic process associated with the operator L , and uses the independence of L , and thus also of the stochastic process, on x . It is not immediately obvious whether this requirement can be lifted and replaced, for instance, by some assumption on the relation of the stochastic processes associated to L and starting from two different points which can be connected by a path with tangent vector X_0 at each point. We leave this as an open problem.

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2 Proof of Theorem 1.1

Without loss we can assume $\inf_{D'} \beta < 0 < \sup_{D'} \beta$ and D' connected, after possibly enlarging D' . We will also assume $a = -5$, $a' = 0$, $b' = 1$, $b = 6$, $D = B_3(0)$, $D' = B_1(0)$, and $\|\gamma\|_\infty \leq 1$, with C then only depending on β , because the general case is analogous. We also note that [19, Theorem 18(c)] and boundedness of u show that u is actually continuous.

We first claim that for each $d > 0$ there is $C_{d,\beta} \geq 1$ such that

$$\sup_{[0,1] \times A_d} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u, \quad (2.6)$$

with $A_d := A_d^+ \cup A_d^-$ and $A_d^\pm := \{y \in B_1(0) \mid \pm \beta(y) > d\}$. Clearly it suffices to show this for all small enough d such that $A_d^\pm \neq \emptyset$, which we shall assume.

To this end, note that parabolic regularity theory with x as the time variable, applied on $[-1, 5] \times \{y \in B_2(0) \mid -\beta(y) > d/2\}$, yields

$$\sup_{[0,1] \times A_d^-} u \leq C'_{d,\beta} \inf_{[2,5] \times A_d^-} u, \quad (2.7)$$

where $C'_{d,\beta} > 0$ depends only on d and β . Similarly, we obtain

$$\sup_{[3,4] \times A_d^+} u \leq C'_{d,\beta} \inf_{[-1,2] \times A_d^+} u, \quad (2.8)$$

Next, consider the stochastic process $(X_t^{x,y}, Y_t^{x,y})$ starting at $(x, y) \in \mathbb{R} \times B_2(0)$ and satisfying the stochastic differential equation

$$(dX_t^{x,y}, dY_t^{x,y}) = (\beta(Y_t^{x,y})dt, \sqrt{2}dB_t), \quad (X_0^{x,y}, Y_0^{x,y}) = (x, y).$$

Here t is a new time variable and B_t is a normalized Brownian motion on \mathbb{R}^{N-1} with $B_0 = 0$ (defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$). We then have

$$(X_t^{x,y}, Y_t^{x,y}) = (X_t^{0,y} + x, \sqrt{2}B_t + y). \quad (2.9)$$

for any $(x, y) \in \mathbb{R} \times B_2(0)$, in particular, $Y_t^{x,y}$ is independent from x . For any $y \in B_2(0)$ we also define the stopping time

$$\tau = \tau_y := \inf \{t > 0 \mid Y_t^{x,y} \notin B_2(0)\}.$$

If $t \wedge \tau := \min\{t, \tau\}$, then by the Feynman-Kac formula, $\|\gamma\|_\infty \leq 1$, and the parabolic comparison principle, we have for each $t \geq 0$ and $(x, y) \in \mathbb{R} \times B_2(0)$,

$$e^{-t}\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})) \leq u(x, y) \leq e^t\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})). \quad (2.10)$$

(The Feynman-Kac formula is usually stated for C^2 functions so we provide a proof of (2.10) in Lemma 2.1 below.) Here

$$\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})) = \int_{\Omega} u(X_{t \wedge \tau}^{x,y}(\omega), Y_{t \wedge \tau}^{x,y}(\omega))d\mathbb{P}(\omega) = \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y')d\mu_t^{x,y}(x', y'), \quad (2.11)$$

with the probability measure $\mu_t^{x,y}$ on $\mathbb{R} \times \overline{B_2(0)}$ such that $\mu_t^{x,y}(A) = \mathbb{P}((X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}) \in A)$ for Borel sets $A \subseteq \mathbb{R} \times \overline{B_2(0)}$. Notice that $\mu_t^{x,y}$ is supported on $[x - \|\beta\|_\infty t, x + \|\beta\|_\infty t] \times \overline{B_2(0)}$ and $\mu_t^{x,y}(\mathbb{R} \times \partial B_2(0)) = \mathbb{P}(\tau_y \leq t)$.

By (2.9), translation in x equally translates the $\mu_t^{x,y}$, and the $(x$ -independent) measure on $\overline{B_2(0)}$ given by $\nu_t^y(A) = \mu_t^{x,y}(\mathbb{R} \times A)$ is just the law of $\sqrt{2}B_{t \wedge \tau_y} + y$, the Brownian motion on $B_2(0)$ starting at y , with stopping time τ_y , and with time scaled by a factor of two. In particular for each $t > 0$ there is $h_t > 0$ such that for any $y_1, y_2 \in B_1(0)$ and any Borel sets $A_1 \subseteq B_1(0)$ and $A_2 \subseteq \overline{B_2(0)}$,

$$h_t \nu_t^{y_1}(A_2) \leq \nu_t^{y_2}(A_2) \leq h_t^{-1} \nu_t^{y_1}(A_2) \quad \text{and} \quad h_t |A_1| \leq \nu_t^{y_1}(A_1) \leq h_t^{-1} |A_1|. \quad (2.12)$$

From this it follows for $t := \|\beta\|_\infty^{-1}$ that

$$\inf_{[0,1] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[-1,2] \times A_d^+} u \quad (2.13)$$

with $C''_{d,\beta} := e^{-t} h_t \min\{|A_d^+|, |A_d^-|\}$ and, similarly, we obtain

$$\inf_{[3,4] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[2,5] \times A_d^-} u. \quad (2.14)$$

Using (2.7), (2.14), (2.8), and (2.13) (in that order) yields

$$\sup_{[0,1] \times A_d^-} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,$$

with $C_{d,\beta} > 0$ depending only on d and β . An analogous argument gives

$$\sup_{[0,1] \times A_d^+} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,$$

and (2.6) follows.

Next we let $v(x, y) := \int_{-z}^z u(x + s, y) ds$ for some $z \in (0, 1/3]$

$$\sup_{[0,1] \times B_1(0)} v \leq \tilde{C}_{z,\beta} \inf_{[0,1] \times B_1(0)} u \quad (2.15)$$

holds for some $\tilde{C}_{z,\beta} \geq 1$. Indeed, it follows from (2.9), (2.10), (2.11) that for each $(x, y) \in \mathbb{R} \times B_2(0)$,

$$e^{-t} \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y') d\mu_t^{x,y;z}(x', y') \leq v(x, y) \leq e^t \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y') d\mu_t^{x,y;z}(x', y'),$$

where $\mu_t^{x,y;z}(x', y') = \mu_t^{x,y}(x', y') * (\chi_{[-z,z]}(x') dx' \delta_0(y'))$. The above claims about $\mu_t^{x,y}$ and the definition of ν_t^y imply that

$$\mu_t^{x,y;z}(x', y') \leq \kappa_t^{x;z}(x') \times \nu_t^y(y') \leq \sum_{m=-M}^M \mu_t^{x+2mz,y;z}(x', y'),$$

where $\kappa_t^{x;z}$ is the measure on \mathbb{R} with $\kappa_t^{x;z}(B) = |B \cap [x-z-\|\beta\|_\infty t, x+z+\|\beta\|_\infty t]|$ for any Borel set $B \subseteq \mathbb{R}$, and M is such that $(2M+1)z \geq 2(z+\|\beta\|_\infty t)$, for instance, $M := \lceil 1/2 + \|\beta\|_\infty t/z \rceil$. This and the first claim in (2.12) means that

$$v(x, y_1) \leq e^{2t} h_t^{-2} \sum_{m=-M}^M v(x + 2mz, y_2) \quad (2.16)$$

for any $x \in \mathbb{R}$, $y_1, y_2 \in B_1(0)$ and $t > 0$.

Now we take any $x \in [0, 1]$, $y_1 \in B_1(0)$, and $y_2 \in A_d$ for some fixed $d > 0$ such that $A_d^\pm \neq \emptyset$. Pick $t := (2\|\beta\|_\infty)^{-1}z$ and $M = 1$ to obtain using (2.16),

$$v(x, y_1) \leq e^{2t} h_t^{-2} \int_{-3z}^{3z} u(x + s, y_2) ds \leq e^{2t} h_t^{-2} \int_{-1}^2 u(x', y_2) dx'.$$

Since (2.6) and its shifts in x give for $c = -1, 0, 1$,

$$\begin{aligned} \sup_{[c-1,c] \times A_d} u &\leq C_{d,\beta} \inf_{\{c\} \times A_d} u \leq C_{d,\beta} \sup_{[c,c+1] \times A_d} u, \\ \sup_{[c-1,c] \times A_d} u &\leq C_{d,\beta} \inf_{\{c-1\} \times A_d} u \leq C_{d,\beta} \sup_{[c-2,c-1] \times A_d} u, \end{aligned}$$

we obtain (2.6) with $[0, 1]$ and $C_{d,\beta}$ replaced by $[-1, 2]$ and $C_{d,\beta}^3$. This proves (2.15). Similarly, (2.15) with $[-1, 0]$ and $[1, 2]$ in place of $[0, 1]$, together with (2.6), yield

$$\sup_{[-1,2] \times B_1(0)} v \leq \tilde{C}_{z,\beta} C_{d,\beta} \inf_{[0,1] \times B_1(0)} u.$$

In a similar way one can also obtain

$$\sup_{[-1,2] \times B_2(0)} v \leq C_{z,d,\beta} \inf_{[0,1] \times B_2(0)} u. \quad (2.17)$$

for some $C_{z,d,\beta} > 0$ (recall that $B_2(0) \subset\subset D = B_3(0)$).

We will now need to use (1.3) to finish the proof. This assumption makes the differential operator on the left-hand side of (1.1) hypoelliptic in the sense of Hörmander. It follows that for $t > 0$, the measure $\mu_t^{x,y}$ is absolutely continuous when restricted to $\mathbb{R} \times B_2(0)$ and also to $\mathbb{R} \times \partial B_2(0)$ (as an $(N-1)$ -dimensional measure in the latter case), with densities $p_t(x, y, \cdot, \cdot), q_t(x, y, \cdot, \cdot) \geq 0$ such that

$$\begin{aligned} p_t(x, y, x', y') &= p_t(0, y, x' - x, y'), \\ q_t(x, y, x', y') &= q_t(0, y, x' - x, y'), \end{aligned}$$

and p_t, q_t are bounded functions when restricted to $y \in B_1(0)$ (with $y' \in B_2(0)$ for p_t and $y' \in \partial B_2(0)$ for q_t). For p_t this follows from the same claim for the corresponding measure $\tilde{\mu}_t^{x,y}$ on \mathbb{R}^N given by (2.11) with t in place of $t \wedge \tau$ and β smoothly extended to a periodic function on \mathbb{R}^{N-1} (whose density is smooth in all arguments, [8, Theorem 3]). This is because $\tilde{\mu}_t^{x,y}(A) \geq \mu_t^{x,y}(A)$ for any Borel set $A \subseteq \mathbb{R} \times B_2(0)$.

For q_t this would follow from the same claim for the corresponding escape measure $\tilde{\mu}_\tau^{x,y}$ on $\mathbb{R} \times \partial B_2(0)$ given by (2.11) with $\tau = \tau_y$ in place of $t \wedge \tau$. We know of such a result for bounded domains only [1, Corollary 2.11] but since $\mu_t^{x,y}$ is supported on a bounded cylinder, it applies in our case as well. Specifically, take any $a_- < -\|\beta\|_\infty t$ and $a_+ > \|\beta\|_\infty t$. There is a convex open domain G with a smooth boundary whose intersection with $[a_-, a_+] \times \mathbb{R}^{N-1}$ is $[a_-, a_+] \times B_2(0)$, and the intersection with $((-\infty, a_-) \cup (a_+, \infty)) \times \mathbb{R}^{N-1}$ are two smooth “slanted” conical caps $G_\pm \subseteq \mathbb{R} \times B_2(0)$ over the $(N-1)$ -dimensional balls $\{a_\pm\} \times B_2(0)$ with the two (rounded) tips at points with y' coordinates y'_\pm such that $\pm\beta(y'_\pm) > 0$ and sufficiently long so that for any $(x', y') \in \partial G_\pm \cap \partial G$, the unit outer normal vector $n(x', y')$ to ∂G_\pm at (x', y') satisfies

$$|n(x', y') \cdot (1, 0, \dots, 0)| \leq \frac{1}{2}(\|\beta\|_\infty^{-1} + 1) \text{ whenever } \pm\beta(y') \leq 0.$$

Then G satisfies the hypotheses of [1, Corollary 2.11] (it satisfies the escape condition and all points of ∂G are τ' -regular). It follows that the escape measure $\tilde{\mu}_\tau^{x,y}$ has a density $\tilde{q}_\tau(x, y, \cdot, \cdot)$ which is a continuous function of $(x, y, x', y') \in G \times \partial^* G$, where $\partial^* G$ is the set of “good” points of ∂G , that is, all $(x', y') \in \partial G$ except of the two cone tips, where $n(x', y') = (\pm 1, 0, \dots, 0)$. Thus \tilde{q}_τ is bounded on $S := \{0\} \times B_1(0) \times (a_-, a_+) \times \partial B_2(0)$. Since $\{X_s^{0,y}\}_{s \leq t \wedge \tau}$ almost surely stays in (a_-, a_+) , we obtain $q_t \leq \tilde{q}_\tau$ on S and $q_t = 0$ on $(\{0\} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)) \setminus S$. Finally, $q_t(x, y, x', y') = q_t(0, y, x' - x, y')$ shows that q_t is bounded on $\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)$.

Let $d > 0$ be such that $A_d^\pm \neq \emptyset$, let $z := 1/3$, $t := \|\beta\|_\infty^{-1}$, and

$$C_t := \max\left\{ \sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times B_2(0)} p_t, \sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)} q_t \right\} < \infty.$$

Then $p_t(x, y, x', y'), q_t(x, y, x', y') \leq C_t \chi_{[x-1, x+1]}(x')$ because the measure $\mu_t^{x, y}$ is supported on $[x-1, x+1] \times \overline{B_2(0)}$, so we obtain from (2.10) and (2.11)

$$\begin{aligned} \sup_{[0,1] \times B_1(0)} u &\leq C_t e^t \int_{[-1,2] \times B_2(0)} u(x', y') dx' dy' + C_t e^t \int_{[-1,2] \times \partial B_2(0)} u(x', y') dx' dy' \\ &\leq 10 C_t e^t \sup_{[-1,2] \times \overline{B_2(0)}} v \\ &\leq 10 C_t C_{z,d,\beta} e^t \inf_{[0,1] \times B_1(0)} u \end{aligned}$$

by using $[-1, 2] = [-1, -1/3] \cup [-1/3, 1/3] \cup [1/3, 1] \cup [1, 5/3] \cup [4/3, 2]$ and (2.17). This is (1.4), so the theorem will be proved once we establish (2.10).

Lemma 2.1 *If $u, \beta, \gamma, X_t^{x,y}, Y_t^{x,y}, \tau_y$ are as in the proof of Theorem 1.1 (in particular, $\|\gamma\|_\infty \leq 1$), then (2.10) holds for $(x, y) \in \mathbb{R} \times B_2(0)$.*

Proof. Let $Z_t^{x,y} = t$ so that $dZ_t^{x,y} = dt$ and $K := \Delta_y + \beta(y)\partial_x + \partial_z$ is the generator of the process $(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})$. If we let $v(x, y, z) := e^z u(x, y)$, then $Kv \geq 0$ on $\mathbb{R} \times B_3(0) \times \mathbb{R}$ in the sense of distributions, that is,

$$\int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v K^* \phi dx dy dz \geq 0$$

for any $\phi \in C_0^\infty(\mathbb{R} \times B_3(0) \times \mathbb{R})$, with $K^* := \Delta_y - \beta(y)\partial_x - \partial_z$ the adjoint of K .

For any $\varepsilon > 0$ let $\delta_\varepsilon \in (0, 1/2\sqrt{N-1})$ be such that $|\beta(y) - \beta(y')| \leq \varepsilon^2$ whenever $y, y' \in B_{5/2}(0)$ and $|y - y'| \leq \sqrt{N-1}\delta_\varepsilon$. Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth non-negative function supported in $[-1, 1]$, with $\int_{-1}^1 \eta(x') dx' = 1$ and $\|\eta'\|_\infty \leq 2$. For $\varepsilon > 0$ define the mollifier

$$\eta^\varepsilon(x, y, z) := \varepsilon^{-2} \delta_\varepsilon^{1-N} \eta\left(\frac{x}{\varepsilon}\right) \eta\left(\frac{z}{\varepsilon}\right) \prod_{n=1}^{N-1} \eta\left(\frac{y_n}{\delta_\varepsilon}\right),$$

and let $v^\varepsilon := v * \eta^\varepsilon$ and $\phi^{\varepsilon;x,y,z}(x', y', z') := \eta^\varepsilon(x - x', y - y', z - z')$. For $\varepsilon \in (0, 1)$ the smooth function v^ε then satisfies on $\mathbb{R} \times B_2(0) \times \mathbb{R}$

$$\begin{aligned} (Kv^\varepsilon)(x, y, z) &= \int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v K^* \phi^{\varepsilon;x,y,z} dx' dy' dz' \\ &\quad + \int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v(x', y', z') [\beta(y') - \beta(y)] \phi_{x'}^{\varepsilon;x,y,z}(x', y', z') dx' dy' dz'. \end{aligned}$$

The first integral is non-negative. The integrand in the second vanishes when $|x' - x| > \varepsilon$ or $|y'_n - y_n| > \delta_\varepsilon$ for some n or $|z' - z| > \varepsilon$, and $|\phi_{x'}^{\varepsilon;x,y,z}(x', y', z')| \leq 2\varepsilon^{-3} \delta_\varepsilon^{1-N}$, so we have

$$(Kv^\varepsilon)(x, y, z) \geq -2^{N+2} \varepsilon e^{z+\varepsilon} \|u\|_\infty.$$

We next apply Dynkin's formula [15, Theorem 7.4.1] to the smooth function v^ε , the process $(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})$, and stopping time $t \wedge \tau$ (with $\tau = \tau_y$), to obtain

$$\begin{aligned} \mathbb{E}[v^\varepsilon(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}, Z_{t \wedge \tau}^{x,y})] &= v^\varepsilon(x, y, 0) + \mathbb{E}\left[\int_0^{t \wedge \tau} (Kv^\varepsilon)(X_s^{x,y}, Y_s^{x,y}, Z_s^{x,y}) ds\right] \\ &\geq v^\varepsilon(x, y, 0) - 2^{N+2}\varepsilon e^{t+\varepsilon}\|u\|_\infty t. \end{aligned}$$

Since $v^\varepsilon \rightarrow v$ uniformly on $[x - \|\beta\|_\infty t, x + \|\beta\|_\infty t] \times \overline{B_2(0)} \times [0, t]$ as $\varepsilon \rightarrow 0$ (by continuity of v) and $Z_{t \wedge \tau}^{x,y} \leq t$, it follows that

$$e^t \mathbb{E}[u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})] \geq \mathbb{E}[v(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}, Z_{t \wedge \tau}^{x,y})] \geq u(x, y).$$

This is the second inequality in (2.10). The first inequality is obtained in the same way, this time with $v(x, y, z) := e^{-z}u(x, y)$, so that $Kv \leq 0$ on $\mathbb{R} \times B_3(0) \times \mathbb{R}$ and

$$(Kv^\varepsilon)(x, y, z) \leq 2^{N+2}\varepsilon e^{-z+\varepsilon}\|u\|_\infty.$$

□

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