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# The Harnack inequality for a class of degenerate elliptic operators

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## Abstract

We prove a Harnack inequality for distributional solutions to a type of degenerate elliptic PDEs in  $N$  dimensions. The differential operators in question are related to the Kolmogorov operator, made up of the Laplacian in the last  $N - 1$  variables, a first-order term corresponding to a shear flow in the direction of the first variable, and a bounded measurable potential term. The first-order coefficient is a smooth function of the last  $N - 1$  variables and its derivatives up to certain order do not vanish simultaneously at any point, making the operators in question hypoelliptic.

## 1 Introduction

We prove a Harnack inequality for distributional solutions to the degenerate elliptic PDE

$$\Delta_y u + \beta(y)u_x + \gamma(x, y)u = 0 \tag{1.1}$$

in cylindrical domains in  $\mathbb{R}^N$  with axes in the direction of the first variable  $x$ . Here  $\gamma$  is bounded measurable and  $\beta$  is a smooth function such that the operator

$$L = \sum_{n=1}^{N-1} X_n^2 + X_0 := \sum_{n=1}^{N-1} (\partial_{y_n})^2 + \beta(y)\partial_x = \Delta_y + \beta(y)\partial_x \tag{1.2}$$

satisfies Hörmander's hypoellipticity condition. That is, vector fields  $\{X_n\}_{n=0}^{N-1}$  and their commutators up to certain order span the whole tangent space  $\mathbb{R}^N$  at each  $(x, y)$ . Moreover,  $\beta$  changes sign so that  $L$  is not parabolic, since then the “elliptic” Harnack inequality (1.4) below would not hold in general. These conditions on  $\beta$  are equivalent to hypothesis (1.3) below and our result is then as follows:

**Theorem 1.1** *Let  $D \subseteq \mathbb{R}^{N-1}$  be open connected and  $u : (a, b) \times D \rightarrow [0, \infty)$  a bounded distributional solution of (1.1) with  $\gamma$  bounded measurable and  $\beta$  satisfying for some  $r \in \mathbb{N}$ ,*

$$\beta \in C^\infty(D), \quad \inf_D \beta < 0 < \sup_D \beta, \quad \text{and} \quad \sum_{0 \leq |\zeta| \leq r} |D^\zeta \beta(y)| > 0 \text{ for all } y \in D.^1 \quad (1.3)$$

*Then for each  $[a', b'] \subseteq (a, b)$  and bounded open  $D'$  with  $\overline{D'} \subseteq D$ , there is  $C > 0$ , depending only on  $D$ ,  $D'$ ,  $\beta$  and an upper bound on  $(a' - a)^{-1}$ ,  $(b - b')^{-1}$ ,  $b' - a'$ , and  $\|\gamma\|_\infty$ , such that*

$$\sup_{(a', b') \times D'} u \leq C \inf_{(a', b') \times D'} u. \quad (1.4)$$

*Remark.* We note that  $\Delta_y$  could be replaced by any  $x$ -independent, uniformly elliptic in  $y$  operator on  $D$ , but for the sake of simplicity we state the theorem with  $\Delta_y$  instead.

This result is motivated by its application in our work [6] on large amplitude  $A \rightarrow \infty$  asymptotics of traveling fronts in the  $x$ -direction, and their speeds, for the reaction-advection-diffusion equation

$$v_t + A \alpha(y) v_x = \Delta_{x,y} v + f(v) \quad (1.5)$$

on  $\mathbb{R}^{N+1}$ , with the first order term representing a shear flow in the  $x$ -direction and  $f$  a non-negative reaction function vanishing at 0 and 1. The front speeds in question are proved to satisfy  $\lim_{A \rightarrow \infty} c^*(A\alpha, f)/A = \kappa(\alpha, f)$  for some constant  $\kappa(\alpha, f) \geq 0$ , so after substituting the front ansatz  $v(t, x, y) = u(x - c^*(A\alpha, f)t, y)$  into (1.5) and scaling by  $A$  in the  $x$  variable, one formally recovers (1.1) in the limit  $A \rightarrow \infty$ , with  $\beta(y) := \kappa(\alpha, f) - \alpha(y)$  and  $\gamma(x, y) := -f(u(x, y))/u(x, y)$ .

The study of hypoelliptic operators of the form

$$L = \sum_{n=1}^M X_n^2 + X_0$$

(where  $X_n$  are first order differential operators with smooth coefficients), possibly with an additional potential term, has been systematically pursued since Hörmander's fundamental paper [7]. Although various regularity and maximum principle results have been obtained soon thereafter (see, e.g., [2, 3, 4, 14, 19, 20]), Harnack inequalities and related heat kernel estimates for such operators have initially been proved only in the case when the tangent space at each point is spanned by the fields  $\{X_n\}_{n=1}^M$  and their commutators, sometimes with  $X_0$  either zero or a linear combination of  $\{X_n\}_{n=1}^M$  [2, 9, 10, 12, 13].

More recently, Harnack inequalities have been obtained without this assumption for certain special classes of operators, not including (1.1) with general  $\beta, \gamma$ . Specifically, some operators with constant and linear coefficients, such as the Kolmogorov operator  $L = \partial_{yy}^2 + y\partial_x - \partial_t$ , were considered in [5, 16], and cases of more general coefficients satisfying somewhat rigid structural assumptions (see hypothesis [H.1] in [17]) were studied in [11, 17] and with a potential term in [18]. The domains involved in the obtained inequalities have to

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<sup>1</sup>For  $\zeta = (\zeta_1, \dots, \zeta_{N-1}) \in \mathbb{N}^{N-1}$ , we let  $|\zeta| = \zeta_1 + \dots + \zeta_{N-1}$  and  $D^\zeta \beta(y) = \frac{\partial^{|\zeta|} \beta}{\partial y_1^{\zeta_1} \dots \partial y_{N-1}^{\zeta_{N-1}}}(y)$ .

depend on the metrics associated to the operators rather than the Euclidian metric, as shows a counter-example to a Harnack inequality in [5]. This is related to the need for the sign-changing assumption on  $\beta$  here. We also note that the operators considered in these papers involve the term  $\partial_t$  and appropriate “parabolic-type” Harnack inequalities are obtained, but corresponding “elliptic” inequalities follow from these.

It was a mild surprise to us that we were not able to find in the literature a sufficiently general result which would include our case (1.1). It appears that Harnack inequalities and heat kernel estimates become much more involved when the field  $X_0$  is required for Hörmander’s condition to be satisfied. One hint in this direction is the fact that the sign-changing hypothesis on  $\beta$  is necessary for (1.4) to hold, so hypoellipticity of  $L$  is in itself not a sufficient condition.

We therefore believe that our method of proof of Theorem 1.1 in the next section is itself also a valuable contribution to the problem of quantitative estimates for hypoelliptic operators. The proof is based on the Feynman-Kac formula for the stochastic process associated with the operator  $L$ , and uses the independence of  $L$ , and thus also of the stochastic process, on  $x$ . It is not immediately obvious whether this requirement can be lifted and replaced, for instance, by some assumption on the relation of the stochastic processes associated to  $L$  and starting from two different points which can be connected by a path with tangent vector  $X_0$  at each point. We leave this as an open problem.

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## 2 Proof of Theorem 1.1

Without loss we can assume  $\inf_{D'} \beta < 0 < \sup_{D'} \beta$  and  $D'$  connected, after possibly enlarging  $D'$ . We will also assume  $a = -5$ ,  $a' = 0$ ,  $b' = 1$ ,  $b = 6$ ,  $D = B_3(0)$ ,  $D' = B_1(0)$ , and  $\|\gamma\|_\infty \leq 1$ , with  $C$  then only depending on  $\beta$ , because the general case is analogous. We also note that [19, Theorem 18(c)] and boundedness of  $u$  show that  $u$  is actually continuous.

We first claim that for each  $d > 0$  there is  $C_{d,\beta} \geq 1$  such that

$$\sup_{[0,1] \times A_d} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u, \quad (2.6)$$

with  $A_d := A_d^+ \cup A_d^-$  and  $A_d^\pm := \{y \in B_1(0) \mid \pm \beta(y) > d\}$ . Clearly it suffices to show this for all small enough  $d$  such that  $A_d^\pm \neq \emptyset$ , which we shall assume.

To this end, note that parabolic regularity theory with  $x$  as the time variable, applied on  $[-1, 5] \times \{y \in B_2(0) \mid -\beta(y) > d/2\}$ , yields

$$\sup_{[0,1] \times A_d^-} u \leq C'_{d,\beta} \inf_{[2,5] \times A_d^-} u, \quad (2.7)$$

where  $C'_{d,\beta} > 0$  depends only on  $d$  and  $\beta$ . Similarly, we obtain

$$\sup_{[3,4] \times A_d^+} u \leq C'_{d,\beta} \inf_{[-1,2] \times A_d^+} u, \quad (2.8)$$

Next, consider the stochastic process  $(X_t^{x,y}, Y_t^{x,y})$  starting at  $(x, y) \in \mathbb{R} \times B_2(0)$  and satisfying the stochastic differential equation

$$(dX_t^{x,y}, dY_t^{x,y}) = (\beta(Y_t^{x,y})dt, \sqrt{2}dB_t), \quad (X_0^{x,y}, Y_0^{x,y}) = (x, y).$$

Here  $t$  is a new time variable and  $B_t$  is a normalized Brownian motion on  $\mathbb{R}^{N-1}$  with  $B_0 = 0$  (defined on a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ ). We then have

$$(X_t^{x,y}, Y_t^{x,y}) = (X_t^{0,y} + x, \sqrt{2}B_t + y). \quad (2.9)$$

for any  $(x, y) \in \mathbb{R} \times B_2(0)$ , in particular,  $Y_t^{x,y}$  is independent from  $x$ . For any  $y \in B_2(0)$  we also define the stopping time

$$\tau = \tau_y := \inf \{t > 0 \mid Y_t^{x,y} \notin B_2(0)\}.$$

If  $t \wedge \tau := \min\{t, \tau\}$ , then by the Feynman-Kac formula,  $\|\gamma\|_\infty \leq 1$ , and the parabolic comparison principle, we have for each  $t \geq 0$  and  $(x, y) \in \mathbb{R} \times B_2(0)$ ,

$$e^{-t}\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})) \leq u(x, y) \leq e^t\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})). \quad (2.10)$$

(The Feynman-Kac formula is usually stated for  $C^2$  functions so we provide a proof of (2.10) in Lemma 2.1 below.) Here

$$\mathbb{E}(u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})) = \int_{\Omega} u(X_{t \wedge \tau}^{x,y}(\omega), Y_{t \wedge \tau}^{x,y}(\omega))d\mathbb{P}(\omega) = \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y')d\mu_t^{x,y}(x', y'), \quad (2.11)$$

with the probability measure  $\mu_t^{x,y}$  on  $\mathbb{R} \times \overline{B_2(0)}$  such that  $\mu_t^{x,y}(A) = \mathbb{P}((X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}) \in A)$  for Borel sets  $A \subseteq \mathbb{R} \times \overline{B_2(0)}$ . Notice that  $\mu_t^{x,y}$  is supported on  $[x - \|\beta\|_\infty t, x + \|\beta\|_\infty t] \times \overline{B_2(0)}$  and  $\mu_t^{x,y}(\mathbb{R} \times \partial B_2(0)) = \mathbb{P}(\tau_y \leq t)$ .

By (2.9), translation in  $x$  equally translates the  $\mu_t^{x,y}$ , and the  $(x$ -independent) measure on  $\overline{B_2(0)}$  given by  $\nu_t^y(A) = \mu_t^{x,y}(\mathbb{R} \times A)$  is just the law of  $\sqrt{2}B_{t \wedge \tau_y} + y$ , the Brownian motion on  $B_2(0)$  starting at  $y$ , with stopping time  $\tau_y$ , and with time scaled by a factor of two. In particular for each  $t > 0$  there is  $h_t > 0$  such that for any  $y_1, y_2 \in B_1(0)$  and any Borel sets  $A_1 \subseteq B_1(0)$  and  $A_2 \subseteq \overline{B_2(0)}$ ,

$$h_t \nu_t^{y_1}(A_2) \leq \nu_t^{y_2}(A_2) \leq h_t^{-1} \nu_t^{y_1}(A_2) \quad \text{and} \quad h_t |A_1| \leq \nu_t^{y_1}(A_1) \leq h_t^{-1} |A_1|. \quad (2.12)$$

From this it follows for  $t := \|\beta\|_\infty^{-1}$  that

$$\inf_{[0,1] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[-1,2] \times A_d^+} u \quad (2.13)$$

with  $C''_{d,\beta} := e^{-t} h_t \min\{|A_d^+|, |A_d^-|\}$  and, similarly, we obtain

$$\inf_{[3,4] \times B_1(0)} u \geq C''_{d,\beta} \inf_{[2,5] \times A_d^-} u. \quad (2.14)$$

Using (2.7), (2.14), (2.8), and (2.13) (in that order) yields

$$\sup_{[0,1] \times A_d^-} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,$$

with  $C_{d,\beta} > 0$  depending only on  $d$  and  $\beta$ . An analogous argument gives

$$\sup_{[0,1] \times A_d^+} u \leq C_{d,\beta} \inf_{[0,1] \times B_1(0)} u,$$

and (2.6) follows.

Next we let  $v(x, y) := \int_{-z}^z u(x + s, y) ds$  for some  $z \in (0, 1/3]$

$$\sup_{[0,1] \times B_1(0)} v \leq \tilde{C}_{z,\beta} \inf_{[0,1] \times B_1(0)} u \quad (2.15)$$

holds for some  $\tilde{C}_{z,\beta} \geq 1$ . Indeed, it follows from (2.9), (2.10), (2.11) that for each  $(x, y) \in \mathbb{R} \times B_2(0)$ ,

$$e^{-t} \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y') d\mu_t^{x,y;z}(x', y') \leq v(x, y) \leq e^t \int_{\mathbb{R} \times \overline{B_2(0)}} u(x', y') d\mu_t^{x,y;z}(x', y'),$$

where  $\mu_t^{x,y;z}(x', y') = \mu_t^{x,y}(x', y') * (\chi_{[-z,z]}(x') dx' \delta_0(y'))$ . The above claims about  $\mu_t^{x,y}$  and the definition of  $\nu_t^y$  imply that

$$\mu_t^{x,y;z}(x', y') \leq \kappa_t^{x;z}(x') \times \nu_t^y(y') \leq \sum_{m=-M}^M \mu_t^{x+2mz,y;z}(x', y'),$$

where  $\kappa_t^{x;z}$  is the measure on  $\mathbb{R}$  with  $\kappa_t^{x;z}(B) = |B \cap [x-z-\|\beta\|_\infty t, x+z+\|\beta\|_\infty t]|$  for any Borel set  $B \subseteq \mathbb{R}$ , and  $M$  is such that  $(2M+1)z \geq 2(z+\|\beta\|_\infty t)$ , for instance,  $M := \lceil 1/2 + \|\beta\|_\infty t/z \rceil$ . This and the first claim in (2.12) means that

$$v(x, y_1) \leq e^{2t} h_t^{-2} \sum_{m=-M}^M v(x + 2mz, y_2) \quad (2.16)$$

for any  $x \in \mathbb{R}$ ,  $y_1, y_2 \in B_1(0)$  and  $t > 0$ .

Now we take any  $x \in [0, 1]$ ,  $y_1 \in B_1(0)$ , and  $y_2 \in A_d$  for some fixed  $d > 0$  such that  $A_d^\pm \neq \emptyset$ . Pick  $t := (2\|\beta\|_\infty)^{-1}z$  and  $M = 1$  to obtain using (2.16),

$$v(x, y_1) \leq e^{2t} h_t^{-2} \int_{-3z}^{3z} u(x + s, y_2) ds \leq e^{2t} h_t^{-2} \int_{-1}^2 u(x', y_2) dx'.$$

Since (2.6) and its shifts in  $x$  give for  $c = -1, 0, 1$ ,

$$\begin{aligned} \sup_{[c-1,c] \times A_d} u &\leq C_{d,\beta} \inf_{\{c\} \times A_d} u \leq C_{d,\beta} \sup_{[c,c+1] \times A_d} u, \\ \sup_{[c-1,c] \times A_d} u &\leq C_{d,\beta} \inf_{\{c-1\} \times A_d} u \leq C_{d,\beta} \sup_{[c-2,c-1] \times A_d} u, \end{aligned}$$

we obtain (2.6) with  $[0, 1]$  and  $C_{d,\beta}$  replaced by  $[-1, 2]$  and  $C_{d,\beta}^3$ . This proves (2.15). Similarly, (2.15) with  $[-1, 0]$  and  $[1, 2]$  in place of  $[0, 1]$ , together with (2.6), yield

$$\sup_{[-1,2] \times B_1(0)} v \leq \tilde{C}_{z,\beta} C_{d,\beta} \inf_{[0,1] \times B_1(0)} u.$$

In a similar way one can also obtain

$$\sup_{[-1,2] \times B_2(0)} v \leq C_{z,d,\beta} \inf_{[0,1] \times B_2(0)} u. \quad (2.17)$$

for some  $C_{z,d,\beta} > 0$  (recall that  $B_2(0) \subset\subset D = B_3(0)$ ).

We will now need to use (1.3) to finish the proof. This assumption makes the differential operator on the left-hand side of (1.1) hypoelliptic in the sense of Hörmander. It follows that for  $t > 0$ , the measure  $\mu_t^{x,y}$  is absolutely continuous when restricted to  $\mathbb{R} \times B_2(0)$  and also to  $\mathbb{R} \times \partial B_2(0)$  (as an  $(N-1)$ -dimensional measure in the latter case), with densities  $p_t(x, y, \cdot, \cdot), q_t(x, y, \cdot, \cdot) \geq 0$  such that

$$\begin{aligned} p_t(x, y, x', y') &= p_t(0, y, x' - x, y'), \\ q_t(x, y, x', y') &= q_t(0, y, x' - x, y'), \end{aligned}$$

and  $p_t, q_t$  are bounded functions when restricted to  $y \in B_1(0)$  (with  $y' \in B_2(0)$  for  $p_t$  and  $y' \in \partial B_2(0)$  for  $q_t$ ). For  $p_t$  this follows from the same claim for the corresponding measure  $\tilde{\mu}_t^{x,y}$  on  $\mathbb{R}^N$  given by (2.11) with  $t$  in place of  $t \wedge \tau$  and  $\beta$  smoothly extended to a periodic function on  $\mathbb{R}^{N-1}$  (whose density is smooth in all arguments, [8, Theorem 3]). This is because  $\tilde{\mu}_t^{x,y}(A) \geq \mu_t^{x,y}(A)$  for any Borel set  $A \subseteq \mathbb{R} \times B_2(0)$ .

For  $q_t$  this would follow from the same claim for the corresponding escape measure  $\tilde{\mu}_\tau^{x,y}$  on  $\mathbb{R} \times \partial B_2(0)$  given by (2.11) with  $\tau = \tau_y$  in place of  $t \wedge \tau$ . We know of such a result for bounded domains only [1, Corollary 2.11] but since  $\mu_t^{x,y}$  is supported on a bounded cylinder, it applies in our case as well. Specifically, take any  $a_- < -\|\beta\|_\infty t$  and  $a_+ > \|\beta\|_\infty t$ . There is a convex open domain  $G$  with a smooth boundary whose intersection with  $[a_-, a_+] \times \mathbb{R}^{N-1}$  is  $[a_-, a_+] \times B_2(0)$ , and the intersection with  $((-\infty, a_-) \cup (a_+, \infty)) \times \mathbb{R}^{N-1}$  are two smooth “slanted” conical caps  $G_\pm \subseteq \mathbb{R} \times B_2(0)$  over the  $(N-1)$ -dimensional balls  $\{a_\pm\} \times B_2(0)$  with the two (rounded) tips at points with  $y'$  coordinates  $y'_\pm$  such that  $\pm\beta(y'_\pm) > 0$  and sufficiently long so that for any  $(x', y') \in \partial G_\pm \cap \partial G$ , the unit outer normal vector  $n(x', y')$  to  $\partial G_\pm$  at  $(x', y')$  satisfies

$$|n(x', y') \cdot (1, 0, \dots, 0)| \leq \frac{1}{2}(\|\beta\|_\infty^{-1} + 1) \text{ whenever } \pm\beta(y') \leq 0.$$

Then  $G$  satisfies the hypotheses of [1, Corollary 2.11] (it satisfies the escape condition and all points of  $\partial G$  are  $\tau'$ -regular). It follows that the escape measure  $\tilde{\mu}_\tau^{x,y}$  has a density  $\tilde{q}_\tau(x, y, \cdot, \cdot)$  which is a continuous function of  $(x, y, x', y') \in G \times \partial^* G$ , where  $\partial^* G$  is the set of “good” points of  $\partial G$ , that is, all  $(x', y') \in \partial G$  except of the two cone tips, where  $n(x', y') = (\pm 1, 0, \dots, 0)$ . Thus  $\tilde{q}_\tau$  is bounded on  $S := \{0\} \times B_1(0) \times (a_-, a_+) \times \partial B_2(0)$ . Since  $\{X_s^{0,y}\}_{s \leq t \wedge \tau}$  almost surely stays in  $(a_-, a_+)$ , we obtain  $q_t \leq \tilde{q}_\tau$  on  $S$  and  $q_t = 0$  on  $(\{0\} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)) \setminus S$ . Finally,  $q_t(x, y, x', y') = q_t(0, y, x' - x, y')$  shows that  $q_t$  is bounded on  $\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)$ .

Let  $d > 0$  be such that  $A_d^\pm \neq \emptyset$ , let  $z := 1/3$ ,  $t := \|\beta\|_\infty^{-1}$ , and

$$C_t := \max\left\{ \sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times B_2(0)} p_t, \sup_{\mathbb{R} \times B_1(0) \times \mathbb{R} \times \partial B_2(0)} q_t \right\} < \infty.$$

Then  $p_t(x, y, x', y'), q_t(x, y, x', y') \leq C_t \chi_{[x-1, x+1]}(x')$  because the measure  $\mu_t^{x, y}$  is supported on  $[x-1, x+1] \times \overline{B_2(0)}$ , so we obtain from (2.10) and (2.11)

$$\begin{aligned} \sup_{[0,1] \times B_1(0)} u &\leq C_t e^t \int_{[-1,2] \times B_2(0)} u(x', y') dx' dy' + C_t e^t \int_{[-1,2] \times \partial B_2(0)} u(x', y') dx' dy' \\ &\leq 10 C_t e^t \sup_{[-1,2] \times B_2(0)} v \\ &\leq 10 C_t C_{z,d,\beta} e^t \inf_{[0,1] \times B_1(0)} u \end{aligned}$$

by using  $[-1, 2] = [-1, -1/3] \cup [-1/3, 1/3] \cup [1/3, 1] \cup [1, 5/3] \cup [4/3, 2]$  and (2.17). This is (1.4), so the theorem will be proved once we establish (2.10).

**Lemma 2.1** *If  $u, \beta, \gamma, X_t^{x,y}, Y_t^{x,y}, \tau_y$  are as in the proof of Theorem 1.1 (in particular,  $\|\gamma\|_\infty \leq 1$ ), then (2.10) holds for  $(x, y) \in \mathbb{R} \times B_2(0)$ .*

**Proof.** Let  $Z_t^{x,y} = t$  so that  $dZ_t^{x,y} = dt$  and  $K := \Delta_y + \beta(y)\partial_x + \partial_z$  is the generator of the process  $(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})$ . If we let  $v(x, y, z) := e^z u(x, y)$ , then  $Kv \geq 0$  on  $\mathbb{R} \times B_3(0) \times \mathbb{R}$  in the sense of distributions, that is,

$$\int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v K^* \phi dx dy dz \geq 0$$

for any  $\phi \in C_0^\infty(\mathbb{R} \times B_3(0) \times \mathbb{R})$ , with  $K^* := \Delta_y - \beta(y)\partial_x - \partial_z$  the adjoint of  $K$ .

For any  $\varepsilon > 0$  let  $\delta_\varepsilon \in (0, 1/2\sqrt{N-1})$  be such that  $|\beta(y) - \beta(y')| \leq \varepsilon^2$  whenever  $y, y' \in B_{5/2}(0)$  and  $|y - y'| \leq \sqrt{N-1}\delta_\varepsilon$ . Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be a smooth non-negative function supported in  $[-1, 1]$ , with  $\int_{-1}^1 \eta(x') dx' = 1$  and  $\|\eta'\|_\infty \leq 2$ . For  $\varepsilon > 0$  define the mollifier

$$\eta^\varepsilon(x, y, z) := \varepsilon^{-2} \delta_\varepsilon^{1-N} \eta\left(\frac{x}{\varepsilon}\right) \eta\left(\frac{z}{\varepsilon}\right) \prod_{n=1}^{N-1} \eta\left(\frac{y_n}{\delta_\varepsilon}\right),$$

and let  $v^\varepsilon := v * \eta^\varepsilon$  and  $\phi^{\varepsilon; x, y, z}(x', y', z') := \eta^\varepsilon(x - x', y - y', z - z')$ . For  $\varepsilon \in (0, 1)$  the smooth function  $v^\varepsilon$  then satisfies on  $\mathbb{R} \times B_2(0) \times \mathbb{R}$

$$\begin{aligned} (Kv^\varepsilon)(x, y, z) &= \int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v K^* \phi^{\varepsilon; x, y, z} dx' dy' dz' \\ &\quad + \int_{\mathbb{R} \times B_3(0) \times \mathbb{R}} v(x', y', z') [\beta(y') - \beta(y)] \phi_{x'}^{\varepsilon; x, y, z}(x', y', z') dx' dy' dz'. \end{aligned}$$

The first integral is non-negative. The integrand in the second vanishes when  $|x' - x| > \varepsilon$  or  $|y'_n - y_n| > \delta_\varepsilon$  for some  $n$  or  $|z' - z| > \varepsilon$ , and  $|\phi_{x'}^{\varepsilon; x, y, z}(x', y', z')| \leq 2\varepsilon^{-3} \delta_\varepsilon^{1-N}$ , so we have

$$(Kv^\varepsilon)(x, y, z) \geq -2^{N+2} \varepsilon e^{z+\varepsilon} \|u\|_\infty.$$



We next apply Dynkin's formula [15, Theorem 7.4.1] to the smooth function  $v^\varepsilon$ , the process  $(X_t^{x,y}, Y_t^{x,y}, Z_t^{x,y})$ , and stopping time  $t \wedge \tau$  (with  $\tau = \tau_y$ ), to obtain

$$\begin{aligned} \mathbb{E}[v^\varepsilon(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}, Z_{t \wedge \tau}^{x,y})] &= v^\varepsilon(x, y, 0) + \mathbb{E}\left[\int_0^{t \wedge \tau} (Kv^\varepsilon)(X_s^{x,y}, Y_s^{x,y}, Z_s^{x,y}) ds\right] \\ &\geq v^\varepsilon(x, y, 0) - 2^{N+2}\varepsilon e^{t+\varepsilon}\|u\|_\infty t. \end{aligned}$$

Since  $v^\varepsilon \rightarrow v$  uniformly on  $[x - \|\beta\|_\infty t, x + \|\beta\|_\infty t] \times \overline{B_2(0)} \times [0, t]$  as  $\varepsilon \rightarrow 0$  (by continuity of  $v$ ) and  $Z_{t \wedge \tau}^{x,y} \leq t$ , it follows that

$$e^t \mathbb{E}[u(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y})] \geq \mathbb{E}[v(X_{t \wedge \tau}^{x,y}, Y_{t \wedge \tau}^{x,y}, Z_{t \wedge \tau}^{x,y})] \geq u(x, y).$$

This is the second inequality in (2.10). The first inequality is obtained in the same way, this time with  $v(x, y, z) := e^{-z}u(x, y)$ , so that  $Kv \leq 0$  on  $\mathbb{R} \times B_3(0) \times \mathbb{R}$  and

$$(Kv^\varepsilon)(x, y, z) \leq 2^{N+2}\varepsilon e^{-z+\varepsilon}\|u\|_\infty.$$

□

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