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CONICAL STOCHASTIC MAXIMAL L^p-REGULARITY FOR $1 \leq p < \infty$

PASCAL AUSCHER, JAN VAN NEERVEN, AND PIERRE PORTAL

Abstract. Let $A = -\operatorname{div} a(\cdot)\nabla$ be a second order divergence form elliptic operator on $\mathbb{R}^n$ with bounded measurable real-valued coefficients and let W be a cylindrical Brownian motion in a Hilbert space H. Our main result implies that the stochastic convolution process

$$u(t) = \int_0^t e^{-(t-s)A} g(s)\, dW(s), \quad t \geq 0,$$

satisfies, for all $1 \leq p < \infty$, a conical maximal L^p-regularity estimate

$$\mathbb{E}\|\nabla u\|^p_{T^{p,2}_2(\mathbb{R}_+ \times \mathbb{R}^n)} \leq C_p^p \mathbb{E}\|g\|^p_{T^{p,2}_2(\mathbb{R}_+ \times \mathbb{R}^n; H)}.$$

Here, $T^{p,2}_2(\mathbb{R}_+ \times \mathbb{R}^n)$ and $T^{p,2}_2(\mathbb{R}_+ \times \mathbb{R}^n; H)$ are the parabolic tent spaces of real-valued and H-valued functions, respectively. This contrasts with Krylov's maximal L^p-regularity estimate

$$\mathbb{E}\|\nabla u\|^p_{L^p(\mathbb{R}_+; L^2(\mathbb{R}^n; \mathbb{R}^n))} \leq C^p \mathbb{E}\|g\|^p_{L^p(\mathbb{R}_+; L^2(\mathbb{R}^n; H))}$$

which is known to hold only for $2 \leq p < \infty$, even when $A = -\Delta$ and $H = \mathbb{R}$.

The proof is based on an L^2-estimate and extrapolation arguments which use the fact that A satisfies suitable off-diagonal bounds. Our results are applied to obtain conical stochastic maximal L^p-regularity for a class of nonlinear SPDEs with rough initial data.

1. Introduction

Let us consider the following stochastic heat equation in $\mathbb{R}^n$ driven by a cylindrical Brownian motion W with values in a (finite- or infinite-dimensional) Hilbert space H:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + g(t,x)\dot{W}(t), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Under suitable measurability and integrability conditions on the process $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$, the process $u : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ given formally by the stochastic convolution

$$u(t) = \int_0^t e^{(t-s)\Delta} g(s)\, dW(s), \quad t \geq 0,$$

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is well defined. This process is usually called the mild solution of (1.1), and it has stochastic maximal L^2-regularity in the sense that

$$\mathbb{E}\|\nabla u\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;\mathbb{R}^n))} \leq C^2\mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n;H))}$$

with a constant C independent of g and H. This follows from a classical result of Da Prato [15] (see [16] for further results along these lines). It was subsequently shown by Krylov [22, 24] that, for $p \geq 2$, u has stochastic maximal L^p-regularity in the sense that

$$\mathbb{E}\|\nabla u\|^p_{L^p(\mathbb{R}_+;L^2(\mathbb{R}^n;\mathbb{R}^n))} \leq C_p^p\mathbb{E}\|g\|^p_{L^p(\mathbb{R}_+;L^2(\mathbb{R}^n;H))}.$$

Krylov actually proves that $L^2(\mathbb{R}^n;H)$ may be replaced by $L^q(\mathbb{R}^n;H)$ for any $2 \leq q \leq p$ and that Δ may be replaced by any second-order uniformly elliptic operator under mild regularity assumptions on the coefficients. For $p > 2$, the condition $q \leq p$ was removed in [29] and the result was extended to arbitrary operators having a bounded H^∞-calculus on $L^q(X,\mu)$, where $q \geq 2$ and (X,μ) is an arbitrary σ-finite measure space.

The condition $p \geq 2$ in all these results is necessary, in the sense that the corresponding result for $1 \leq p < 2$ is false even for $H = \mathbb{R}$ [23]. The aim of this paper is to show that the stochastic heat equation (1.1) does have 'conical' stochastic maximal L^p-regularity in the full range of $1 \leq p < \infty$, provided the condition $g \in L^p(\mathbb{R}_+ \times \Omega; L^2(\mathbb{R}^n;H))$ is replaced by the condition $g \in L^p(\Omega; T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; H))$. Here $T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; H)$ is a weighted parabolic tent space of H-valued functions on $\mathbb{R}_+ \times \mathbb{R}^n$ (the definition is stated in Section 2; for $\beta = 1$ the classical parabolic tent space $T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n; H)$ is obtained). Our main result, stated somewhat informally (see Theorem 3.1 for the precise formulation), reads as follows.

Theorem 1.1. *Let $A = -\operatorname{div} a(\cdot)\nabla$ be a divergence form elliptic operator on $\mathbb{R}^n$ with bounded measurable real-valued coefficients. Then for all $1 \leq p < \infty$ and $\beta > 0$ the stochastic convolution process*

$$u(t) = \int_0^t e^{-(t-s)A} g(s)\, dW(s), \quad t \geq 0,$$

satisfies the conical stochastic maximal L^p-regularity estimate

$$\mathbb{E}\|\nabla u\|^p_{T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; \mathbb{R}^n)} \leq C^p_{p,\beta}\mathbb{E}\|g\|^p_{T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx; H)}.$$

The precise assumptions on A are stated in Example 2.2 below. The proof of Theorem 1.1 proceeds in two steps. First, a $T_2^{2,2}$-estimate is deduced from the Itô isometry (Section 4). Using off-diagonal bound techniques, this estimate is then extrapolated to a $T_2^{p,2}$-estimate (Section 5).

The results are applied to prove conical maximal L^p-regularity for a class of stochastic partial differential equations on $\mathbb{R}^n$ driven by space-time white noise (Section 6). We shall prove that if $b : \mathbb{R}^n \to \mathbb{R}$ satisfies appropriate Lipschitz and growth assumptions and A is as in Theorem 1.1, then the mild solution of the stochastic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + Au(t,x) = b(\nabla u(t,x))\dot{W}(t), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

has conical stochastic maximal L^p-regularity for all $1 < p < \infty$, in the sense that $\nabla u \in T_2^{p,2}(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dx)$ for all $0 < \beta < 1$ and all initial values $u_0 \in \mathsf{D}_p(A^{\frac{\beta}{2}})$, the domain of the L^p-realisation of $A^{\frac{\beta}{2}}$. Note that the weight $t^{-\beta}$ allows the handling of initial values in $\mathsf{D}_p(A^\theta)$ with $\theta > 0$ arbitrarily small. It is only the stochastic part that forces us to take $\beta > 0$, and it seems that our technique does not work when $\beta = 0$.

The present paper, as well as [7] which contains more elaborate developments not needed here, builds upon techniques developed in [10]. There, similar off-diagonal bound techniques are applied to obtain conical maximal L^p-regularity for a class of deterministic initial value problems. The key feature of both papers is that they depart from the traditional paradigm in the theory of evolution equations where a solution is a trajectory, indexed by time, in a suitably chosen state space. This could be called the 'Newtonian' paradigm, in which time and space are treated as separate entities. In the conical approach, space and time are inextricably mixed into one 'space-time'.

The idea of using tent space maximal regularity in PDEs goes back, as far as we know, to Koch and Tataru [21], who proved $T^{\infty,2}$-regularity of solutions of Navier-Stokes equations with rough initial data (see also [20]). The underlying ideas come from the theory of Hardy spaces and its application to boundary value problems (see, e.g. [17]). To the best of our knowledge, the present paper is the first to consider a tent space approach for stochastic PDEs.

The notations in this paper are standard. For unexplained terminology we refer to [28, 27, 30] (concerning cylindrical Brownian motions and vector-valued stochastic integration) and [32] (concerning tent spaces). We use the convention $\mathbb{R}_+ = (0, \infty)$. We work over the real scalar field.

2. Preliminaries

2.1. **Off-diagonal bounds.** Our results rely on off-diagonal bound techniques. A family $(T_t)_{t>0}$ of bounded linear operators on $L^2(\mathbb{R}^n)$ is said to satisfy L^q-L^2 *off-diagonal bounds* if there exist constants $c > 0$ and $C \geq 0$ such that for all Borel sets E, F in $\mathbb{R}^n$ and all $f \in L^2 \cap L^q(\mathbb{R}^n)$ we have

$$\|\mathbf{1}_E T_t \mathbf{1}_F f\|_{L^2(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})} \exp(-c(d(E,F))^2/t) \|\mathbf{1}_F f\|_{L^q(\mathbb{R}^n)},$$

with $d(E,F) := \inf\{|x-y| : \ x \in E, \ y \in F\}$.

Such bounds are substitutes for the classical pointwise kernel estimates of Calderón-Zygmund theory, which are not available when one deals with semigroups generated by elliptic operators with rough coefficients. Following the breakthrough paper [13], they have recently become a highly popular tool in harmonic analysis. Typical examples of their use are given in the memoir [1]. Note that L^2-L^2 off-diagonal bounds imply uniform boundedness in L^2 (taking $E = F = \mathbb{R}^n$). Observe that off-diagonal bounds form an ordered scale of conditions.

Lemma 2.1. *Let $1 \leq q \leq r \leq 2$, and $(T_t)_{t>0}$ be a family of bounded linear operators on L^2, which satisfies L^q-L^2 off-diagonal bounds. Then $(T_t)_{t>0}$ satisfies L^r-L^2 off-diagonal bounds.*

Proof. This is a consequence of [8, Proposition 3.2], where it is proven that such off-diagonal bounds are equivalent, on $\mathbb{R}^n$, to off-diagonal bounds on balls (see [8, Definition 2.1]). The result for the latter follows from Hölder's inequality. □

Example 2.2 (Divergence form elliptic operators). We mostly consider second order operators in divergence form $A = -\operatorname{div} a\nabla$, with $a \in L^\infty(\mathbb{R}^n; M^n(\mathbb{R}))$ elliptic in the sense that there exist $C, C' > 0$ such that for all $x \in \mathbb{R}^n$ and $\xi, \xi' \in \mathbb{R}^n$ we have

$$a(x)\xi \cdot \xi \geq C|\xi|^2 \quad \text{and} \quad |a(x)\xi \cdot \xi'| \leq C'|\xi||\xi'|.$$

It is proven in [1, subsection 4.3] that $(t^{\frac{1}{2}}\nabla e^{-tA})_{t\geq 0}$ satisfies L^q-L^2 off-diagonal bounds for all $q \in (1, 2]$. In fact, $(t^{\frac{1}{2}}\nabla e^{-tA})_{t\geq 0}$ even satisfies L^1-L^2 off-diagonal bounds, as can be seen in [11, page 51] as a consequence of [11, Theorem 4 and Lemma 20]. We use these L^1-L^2 bounds in the results below. If we assume only L^q-L^2 bounds for some $q \in (1, 2]$, Theorem 5.2 still holds for all $p \in [1, 2] \cap (\frac{2n}{n+\beta q'}, 2]$ (where $\frac{1}{q} + \frac{1}{q'} = 1$), but the proof is technical (see [7]). This version suffices for proving Theorem 1.1.

Note that we assume that a has real-valued coefficients. In the stochastic setting, where the noise process W is also real-valued, this is a natural assumption.

2.2. **Conical maximal L^p-regularity.** The notion of maximal L^p-regularity has played an important role in much of the recent progress in the theory of nonlinear parabolic evolution equations. We refer to the lecture notes of Kunstmann and Weis [25] for an overview and references to the rapidly expanding literature on this topic.

Motivated by applications to boundary value problems with L^2-data, Auscher and Axelsson [4, 3] proved that for a bounded analytic C_0-semigroups $S = (S(t))_{t\geq 0}$ with generator $-A$ on a Hilbert space E, the classical maximal L^2-regularity estimate

$$\|AS * g\|_{L^2(\mathbb{R}_+;E)} \leq C\|g\|_{L^2(\mathbb{R}_+;E)}$$

implies, for any $\beta \in (-1, \infty)$, the weighted maximal L^2-regularity estimate

$$\|AS * g\|_{L^2(\mathbb{R}_+,t^{-\beta}dt;E)} \leq C_\beta\|g\|_{L^2(\mathbb{R}_+,t^{-\beta}dt;E)}. \tag{2.1}$$

Here,

$$S * g(t) = \int_0^t S(t-s)g(s)\,ds$$

denotes the convolution of g with the semigroup S and $AS * g := A(S * g)$. See also [31] for similar weighted maximal regularity estimates in L^p spaces.

With the aim of eventually extending the results of [4] to an L^p-setting, a 'conical' L^p-version of (2.1) was subsequently obtained in [10]. Observing that, for $E = L^2(\mathbb{R}^n)$, one has

$$\|g\|_{L^2(\mathbb{R}_+,t^{-\beta}dt;L^2(\mathbb{R}^n))} = \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty \fint_{B(x,t^{\frac{1}{2}})} |g(t,y)|^2\,dy\,\frac{dt}{t^\beta}\Big)\,dx\Big)^{\frac{1}{2}}, \tag{2.2}$$

where the dashed integral denotes the average over the ball $B(x, t^{\frac{1}{2}}) = \{y \in \mathbb{R}^n : |x - y| < t^{\frac{1}{2}}\}$. One defines, for $1 \leq p < \infty$,

$$\|g\|_{T_2^{p,2}(\mathbb{R}_+\times\mathbb{R}^n,t^{-\beta}dt\times dy)} := \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty \fint_{B(x,t^{\frac{1}{2}})} |g(t,y)|^2\,dy\,\frac{dt}{t^\beta}\Big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}}.$$

The Banach space

$$T^{p,2}_{2,\beta} := T^{p,2}_2(\mathbb{R}_+ \times \mathbb{R}^n; t^{-\beta}dt \times dy)$$

consisting of all measurable functions $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ for which this norm is finite is called the *tent space* of exponent p and weight β. The spaces $T^{p,2}_{2,\beta}$ are weighted,

parabolic versions of the spaces $T^{p,2}$ introduced by Coifman, Meyer and Stein [14], and have been studied by many authors. We refer to [32] for a thorough discussion and references to the literature. It is useful to observe that $T^{p,2}_{2,\beta}$ can be identified with a closed subspace of $L^p(\mathbb{R}^n; L^2(t^{-\frac{n}{2}-\beta}dt \times dy))$ for $1 < p < \infty$ and of the Hardy space $H^1(\mathbb{R}^n; L^2(t^{-\frac{n}{2}-\beta}dt \times dy))$ for $p = 1$ (see [18]).

Notation. From now on, whenever functions belong to a (vector-valued) Lebesgue space over $\mathbb{R}^n$, we shall suppress $\mathbb{R}^n$ from our notations. For instance, we shall write

$$L^2 := L^2(\mathbb{R}^n), \quad L^2(H) = L^2(\mathbb{R}^n; H)$$

and thus use the notation $L^2(\mathbb{R}^n)$ as an abbreviation for $L^2(\mathbb{R}^n; \mathbb{R}^n)$. Likewise we suppress $\mathbb{R}_+ \times \mathbb{R}^n$ from the notations for (vector-valued) tent spaces. In all other instances we shall be notationally more explicit.

The next estimate is the main result of [10].

Theorem 2.3 (Conical maximal L^p-regularity)**.** *Let $-A$ be the generator of a bounded analytic C_0-semigroup $S = (S(t))_{t\geq 0}$ on L^2, and suppose that the family $(tAS(t))_{t\geq 0}$ satisfies L^2-L^2 off-diagonal bounds. Then for all $\beta > -1$, $p > \sup\big(\frac{2n}{n+2(1+\beta)}, 1\big)$, and $g \in L^2(t^{-\beta}\,dt; \mathsf{D}(A)) \cap T^{p,2}_{2,\beta}$ one has*

$$\|AS * g\|_{T^{p,2}_{2,\beta}} \leq C_{p,\beta}\|g\|_{T^{p,2}_{2,\beta}},$$

with constant $C_{p,\beta}$ independent of g.

It is routine to see that the inclusions

$$L^2(t^{-\beta}\,dt; \mathsf{D}(A)) \cap T^{p,2}_{2,\beta} \hookrightarrow L^2(t^{-\beta}\,dt; \mathbb{R}^n) \cap T^{p,2}_{2,\beta} = T^{2,2}_{2,\beta} \cap T^{p,2}_{2,\beta} \hookrightarrow T^{p,2}_{2,\beta}$$

are dense, so the above result gives the unique extendability of $g \mapsto AS * g$ to a bounded operator on $T^{p,2}_{2,\beta}$.

The proof of this result, as well as that of Theorem 3.1 below, depends on a change of aperture result for tent spaces. Tent spaces with *aperture* $\alpha > 0$ are defined by the norms

$$\|g\|_{T^{p,2}_{2,\beta,\alpha}} := \Big(\int_{\mathbb{R}^n}\Big(\int_0^\infty \fint_{B(x,\alpha t^{\frac{1}{2}})} |g(y,t)|^2\,dy\,\frac{dt}{t^\beta}\Big)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{p}}.$$

For all $\alpha \geq 1$ one has

$$\|g\|_{T^{p,2}_{2,\beta,\alpha}} \leq C\alpha^{n/(p\wedge 2)}\|g\|_{T^{p,2}_{2,\beta}} \tag{2.3}$$

for some constant C independent of α and m. This was first proved in [19] in a vector-valued context, but with an additional logarithmic factor. A different proof in the scalar-valued case was obtained in [2]. The important point is that the right-hand side improves the classical bound from [14]. The weighted parabolic situation treated here follows from these results applied to the function $(t, y) \mapsto t^{\beta+1}f(t^2, y)$ (see [10]). For later use we mention that the bounds (2.3) extend to the Hilbert space-valued tent spaces $T^{p,2}_{2,\beta}(H)$ (which are defined in the obvious way).

3. The main result

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ and a real Hilbert space H, unless stated otherwise, $W = (W(s))_{s\geq 0}$ denotes a $\mathscr{F}$-cylindrical Brownian motion in H (see, e.g., [28] for the precise definition) which we consider to be fixed throughout the rest of the paper. In applications to stochastic partial differential equations, one typically takes H to be $L^2(D)$ for some domain $D \subseteq \mathbb{R}^n$; this provides the mathematically rigorous model for space-time white noise on D. Also note that for $H = \mathbb{R}^d$, W is just a standard $\mathscr{F}$-Brownian motion in $\mathbb{R}^d$.

An *$\mathscr{F}$-adapted simple process* with values in H is a measurable mapping $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$ of the form

$$g(t,x,\omega) = \sum_{\ell=1}^{N} \mathbf{1}_{(t_\ell, t_{\ell+1}]}(t) \sum_{m=1}^{N} \mathbf{1}_{A_{m\ell}}(\omega)\phi_{m\ell}(x)$$

with $0 \leq t_1 < \cdots < t_N < t_{N+1} < \infty$, $A_{m\ell} \in \mathscr{F}_{t_\ell}$, and $\phi_{m\ell}$ simple functions on $\mathbb{R}^n$ with values in H. For such processes, the stochastic convolution process

$$S \diamond g(t) := \int_0^t S(t-s)g(s)\, dW(s)$$

is well-defined as an L^2-valued process whenever $S = (S(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on L^2 (see, e.g., [27]).

The main result of this paper reads as follows.

Theorem 3.1 (Conical stochastic maximal L^p-regularity)**.** *Let $A = -\operatorname{div} a(\cdot)\nabla$ be a divergence form elliptic operator on $\mathbb{R}^n$ with bounded measurable real-valued coefficients, and denote by $S = (S(t))_{t\geq 0}$ the analytic C_0-contraction semigroup generated by $-A$. Then for all $1 \leq p < \infty$ and $\beta > 0$, and all adapted simple processes $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$ one has*

$$\mathbb{E}\|\nabla S \diamond g\|^p_{T^{p,2}_{2,\beta}(\mathbb{R}^n)} \leq C^p_{p,\beta}\mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)},$$

with constant $C_{p,\beta}$ independent of g and H.

Remark 3.2. Compared to the results given in [22, 24, 29], Theorem 3.1 gives conical stochastic maximal L^p-regularity for $1 \leq p < \infty$, while stochastic maximal L^p-regularity can only hold for $2 \leq p < \infty$ even for $A = -\Delta$ (see [23] and the discussion in the Introduction).

The proof of Theorem 3.1 combines two ingredients: a $T^{2,2}_{2,\beta}$ estimate, and an extrapolation result based on off-diagonal bounds for L which gives the $T^{p,2}_{2,\beta}$ estimate. These steps are carried out in Sections 4 and 5, respectively.

4. Conical stochastic maximal L^2-regularity

A classical stochastic maximal L^2-regularity result due to Da Prato (see [16, Theorem 6.14]) asserts that if $-A$ generates an analytic C_0-contraction semigroup $(S(t))_{t\geq 0}$ on a Hilbert space E and g is an $\mathscr{F}$-adapted simple process with values in the vector space $H \otimes E$ of finite rank operators from H to E, then there exists a constant $C \geq 0$, independent of g and H, such that

$$\mathbb{E}\|A^{\frac{1}{2}} S \diamond g\|^2_{L^2(\mathbb{R}_+;E)} \leq C^2\mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+;\mathscr{L}_2(H,E))}. \tag{4.1}$$

Here, $\mathscr{L}_2(H,E)$ denotes the space of Hilbert-Schmidt operators from H to E.

This estimate has the following weighted analogue.

Proposition 4.1. *Suppose $-A$ generates an analytic C_0-contraction semigroup $S=(S(t))_{t\geq 0}$ on a Hilbert space E. Then for all $\beta\geq 0$ there exists a constant $C_\beta\geq 0$ such that for all $\mathscr{F}$-adapted simple processes $g:\mathbb{R}_+\times\Omega\to\mathscr{L}_2(H,E)$,*

$$\mathbb{E}\|A^{\frac{1}{2}}S\diamond g\|^2_{L^2(\mathbb{R}_+,t^{-\beta}dt;E)}\leq C_\beta^2\|g\|^2_{L^2(\mathbb{R}_+,t^{-\beta}dt;\mathscr{L}_2(H,E))}.$$

Proof. For $\beta=0$, this is Da Prato's result. We thus assume that $\beta>0$. The proof follows the lines of Theorem 2.3 in [3]. On the subinterval $(0,\frac{t}{2})$ we estimate, using the Itô isometry,

$$\begin{aligned}
&\mathbb{E}\Big\|t\mapsto\int_0^{\frac{t}{2}}A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+,t^{-\beta}dt;E)}\\
&\quad=\mathbb{E}\int_0^{\frac{t}{2}}\|A^{\frac{1}{2}}S(t-s)g(s)\|^2_{\mathscr{L}_2(H,E)}\,ds\,\frac{dt}{t^\beta}\\
&\quad\lesssim\mathbb{E}\int_0^\infty\int_0^{\frac{t}{2}}(t-s)^{-1}\|g(s)\|^2_{\mathscr{L}_2(H,E)}\,ds\,\frac{dt}{t^\beta}\\
&\quad\lesssim\mathbb{E}\int_0^\infty\|g(s)\|^2_{\mathscr{L}_2(H,E)}\,\frac{ds}{s^\beta}.
\end{aligned}$$

On the subinterval $(\frac{t}{2},t)$ we have, using (4.1),

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^tA^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+,t^{-\beta}dt;E)}\Big)^{\frac{1}{2}}\\
&\quad\lesssim\Big(\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^ts^{-\frac{\beta}{2}}A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+;E)}\Big)^{\frac{1}{2}}\\
&\qquad+\Big(\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^t(s^{-\frac{\beta}{2}}-t^{-\frac{\beta}{2}})A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+;E)}\Big)^{\frac{1}{2}}\\
&\quad\lesssim\Big(\mathbb{E}\Big\|t\mapsto\int_0^ts^{-\frac{\beta}{2}}A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\|^2_{L^2(\mathbb{R}_+;E)}\Big)^{\frac{1}{2}}\\
&\qquad+\Big(\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^t(s^{-\frac{\beta}{2}}-t^{-\frac{\beta}{2}})A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+;E)}\Big)^{\frac{1}{2}}\\
&\quad\lesssim\mathbb{E}\|g\|_{L^2(\mathbb{R}_+,\frac{ds}{s^\beta};\mathscr{L}_2(H;E))}\\
&\qquad+\Big(\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^t(s^{-\frac{\beta}{2}}-t^{-\frac{\beta}{2}})A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+;E)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Using once more the Itô isometry, the last part is estimated as follows:

$$\begin{aligned}
&\mathbb{E}\Big\|t\mapsto\int_{\frac{t}{2}}^t(s^{-\frac{\beta}{2}}-t^{-\frac{\beta}{2}})A^{\frac{1}{2}}S(t-s)g(s)\,dW(s)\Big\|^2_{L^2(\mathbb{R}_+;E)}\\
&\quad\lesssim\mathbb{E}\int_0^\infty\int_{\frac{t}{2}}^t\frac{|s^{-\frac{\beta}{2}}-t^{-\frac{\beta}{2}}|^2}{|s-t|}\|g(s)\|^2_{\mathscr{L}_2(H,E)}\,ds\,dt\\
&\quad\lesssim\mathbb{E}\int_0^\infty\|g(s)\|^2_{\mathscr{L}_2(H,E)}\Big(\int_s^{2s}\frac{|(\frac{s}{t})^{\frac{\beta}{2}}-1|^2}{|\frac{t}{s}-1|}\,dt\Big)\,\frac{ds}{s^{\beta+1}}
\end{aligned}$$

$$\lesssim \mathbb{E}\|g\|^2_{L^2(\mathbb{R}_+,t^{-\beta}dt;\mathscr{L}_2(H,E))}.$$

The proof is concluded by collecting the estimates. □

Following the principles described in Subsection 2.2, we shall specialise, in the next section, to the case $E = L^2(\mathbb{R}^n)$, and identify

$$L^2(\mathbb{R}_+, t^{-\beta}dt; \mathscr{L}_2(H, L^2(\mathbb{R}^n)) = L^2(\mathbb{R}_+, t^{-\beta}dt; L^2(\mathbb{R}^n; H)) = T^{2,2}_{2,\beta}(H)$$

(cf. (2.2)).

5. Extrapolating conical stochastic maximal L^2-regularity

In this section, we prove two abstract extrapolation results based on off-diagonal estimates. Proposition 5.1 is an extrapolation result for $p \in [1,\infty) \cap (\frac{2n}{n+2\beta}, \infty)$, assuming L^2-L^2 off-diagonal bounds, and Theorem 5.2 gives the result for $p \in [1,\infty)$, assuming L^1-L^2 off-diagonal bounds (the well-definedness of the stochastic integrals on the left-hand side of (5.1) and (5.3) being part of the assumptions).

Proposition 5.1 (Extrapolation via L^2-L^2 off-diagonal bounds)**.** *Let $(T_t)_{t>0}$ be a family of bounded linear operators on L^2, let $\beta > 0$, and suppose there exists a constant $C_\beta \geq 0$, independent of g and H, such that*

$$\mathbb{E}\Big\| \int_0^t T_{t-s} g(s,\cdot)\, dW(s)\Big\|^2_{T^{2,2}_{2,\beta}} \leq C_\beta^2 \mathbb{E}\|g\|^2_{T^{2,2}_{2,\beta}(H)} \tag{5.1}$$

for all $\mathscr{F}$-adapted simple $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$. If $(t^{\frac12}T_t)_{t>0}$ satisfies L^2-L^2-off-diagonal bounds, then, for $p \in [1,\infty) \cap (\frac{2n}{n+2\beta}, \infty)$, there exists a constant $C_{p,\beta} \geq 0$, independent of g and H, such that

$$\mathbb{E}\Big\| \int_0^t T_{t-s} g(s,\cdot)\, dW(s)\Big\|^p_{T^{p,2}_{2,\beta}} \leq C^p_{p,\beta} \mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)}.$$

Proof. We introduce the sets

$$C_j(x,t) = \begin{cases} B(x,t) & j = 0 \\ B(x,2^j t) \setminus B(x, 2^{j-1}t) & j = 1,2,\dots \end{cases}$$

Fix an $\mathscr{F}$-adapted simple process $g : \mathbb{R}_+ \times \mathbb{R}^n \times \Omega \to H$. Using the Itô isomorphism for stochastic integrals [27] in combination with a square function estimate [30, Corollary 2.10], we obtain

$$\begin{aligned}
&\mathbb{E}\Big\| \int_0^t T_{t-s} g(s,\cdot)\, dW(s)\Big\|^p_{T^{p,2}_{2,\beta}} \\
&\quad \eqsim \mathbb{E}\|\mathbf{1}_{\{t\geq s\}}(t,s) T_{t-s} g(s,\cdot)\|^p_{T^{p,2}_{2,\beta}(L^2(\mathbb{R}_+;H))} \\
&\quad = \mathbb{E}\Big\| \, \|\mathbf{1}_{\{t\geq s\}} T_{t-s} g(s,\cdot)\|_{L^2(\mathbb{R}_+;H)} \, \Big\|^p_{T^{p,2}_{2,\beta}} \\
&\quad = \mathbb{E}\int_{\mathbb{R}^n} \Big(\int_0^\infty \fint_{B(x,t^{\frac12})} \int_0^t \|T_{t-s}[g(s,\cdot)](y)\|^2_H \, ds\, dy\, \frac{dt}{t^\beta}\Big)^{\frac p2} dx \\
&\quad \leq \mathbb{E}\sum_{j=0}^\infty \sum_{k=1}^\infty I_{j,k} + \mathbb{E}\sum_{j=0}^\infty J_j,
\end{aligned} \tag{5.2}$$

where

$$I_{j,k} = \int_{\mathbb{R}^n} \Big(\int_0^\infty \fint_{B(x,t^{\frac12})} \int_{2^{-k-1}t}^{2^{-k}t} \|T_{t-s}[\mathbf{1}_{C_j(x,4t^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \frac{dt}{t^\beta} \Big)^{\frac p2} dx$$

and

$$J_j = \int_{\mathbb{R}^n} \Big(\int_0^\infty \fint_{B(x,t^{\frac12})} \int_{\frac t2}^{t} \|T_{t-s}[\mathbf{1}_{C_j(x,4s^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \frac{dt}{t^\beta} \Big)^{\frac p2} dx.$$

Following closely the proof given in [10] we shall estimate each of these contributions separately.

We begin with an estimate for $I_{j,k}$ for $j \geq 0$ and $k \geq 1$. Using the off-diagonal bounds, we find

$$\begin{aligned}
&\int_0^\infty \fint_{B(x,t^{\frac12})} \int_{2^{-k-1}t}^{2^{-k}t} \|T_{t-s}[\mathbf{1}_{C_j(x,4t^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \frac{dt}{t^\beta} \\
&= \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} \frac{1}{t-s} \big\| \mathbf{1}_{B(x,t^{\frac12})} (t-s)^{\frac12} T_{t-s}[\mathbf{1}_{C_j(x,4t^{\frac12})} g(s,\cdot)] \big\|_{L^2(H)}^2 \, ds \frac{dt}{t^{\frac n2+\beta}} \\
&\lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} \frac{1}{t} \exp(-\frac{c4^j t}{t-s}) \big\| \mathbf{1}_{C_j(x,4t^{\frac12})} g(s,\cdot) \big\|_{L^2(H)}^2 \, ds \frac{dt}{t^{\frac n2+\beta}} \\
&\lesssim \exp(-c4^j) \int_0^\infty \Big(\int_{2^k s}^{2^{k+1}s} \frac{dt}{t^{\frac n2+1+\beta}} \Big) \big\| \mathbf{1}_{B(x,2^{j+\frac k2+3}s^{\frac12})} g(s,\cdot) \big\|_{L^2(H)}^2 \, ds \\
&\lesssim \exp(-c4^j) 2^{-k(\frac n2+\beta)} \int_0^\infty \big\| \mathbf{1}_{B(x,2^{j+\frac k2+3}s^{\frac12})} g(s,\cdot) \big\|_{L^2(H)}^2 \frac{ds}{s^{\frac n2+\beta}}.
\end{aligned}$$

By (2.3) it follows that

$$\begin{aligned}
\mathbb{E} I_{j,k} &\lesssim \exp(-\tfrac{cp}{2}4^j) 2^{-\frac12 k(\frac n2+\beta)p} \mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta,2^{j+k/2+3}}(H)} \\
&\lesssim \exp(-\tfrac{cp}{2}4^j) 2^{-\frac12 k(\frac n2+\beta)p} 2^{(j+\frac k2+3)\frac{np}{p\wedge 2}} \mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)}.
\end{aligned}$$

The sum $\mathbb{E}\sum_{j,k} I_{j,k}$ thus converges since we assumed that $p > \frac{2n}{n+2\beta}$.

Next we estimate J_0. We have

$$\begin{aligned}
&\mathbb{E} \int_0^\infty \fint_{B(x,t^{\frac12})} \int_{\frac t2}^{t} \|T_{t-s}[\mathbf{1}_{B(x,4s^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \, \frac{dt}{t^\beta} \\
&\leq \mathbb{E} \int_0^\infty \fint_{B(x,t^{\frac12})} \int_0^{t} \|T_{t-s}[\mathbf{1}_{B(x,4s^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \, \frac{dt}{t^\beta} \\
&\leq \mathbb{E} \int_0^\infty \int_{\mathbb{R}^n} \int_0^{t} \|T_{t-s}[\mathbf{1}_{B(x,4s^{\frac12})} g(s,\cdot)](y)\|_H^2 \, ds\, dy \, \frac{dt}{t^{\frac n2+\beta}} \\
&= \mathbb{E} \int_0^\infty \int_{\mathbb{R}^n} \mathbb{E} \Big| \int_0^t T_{t-s}[\mathbf{1}_{B(x,4s^{\frac12})} g(s,\cdot)](y) \, dW(s) \Big|^2 \, dy \, \frac{dt}{t^{\frac n2+\beta}} \\
&= \mathbb{E} \Big\| (t,y) \mapsto \int_0^t T_{t-s}[\mathbf{1}_{B(x,4s^{\frac12})} g(s,\cdot)](y) \, dW(s) \Big\|^2_{L^2(t^{-\frac n2-\beta}dt\times dy;H)} \\
&\lesssim \mathbb{E}\| t \mapsto \mathbf{1}_{B(x,4t^{\frac12})} g(t,\cdot) \|^2_{L^2(t^{-\frac n2-\beta}dt\times dy;H)},
\end{aligned}$$

where the last inequality follows from the $T^{2,2}_{2,\beta}$-boundedness assumption on the stochastic convolution operator. It follows that

$$\begin{aligned}\mathbb{E}J_0 &\lesssim \mathbb{E}\int_{\mathbb{R}^n} \|t\mapsto \mathbf{1}_{B(x,4t^{\frac12})}g(t,\cdot)\|^p_{L^2(t^{-\frac n2-\beta}dt\times dy;H)}\,dx\\ &= \mathbb{E}\int_{\mathbb{R}^n}\Big(\int_0^\infty\int_{\mathbb{R}^n}\mathbf{1}_{B(x,4t^{\frac12})}(y)\|g(t,y)\|_H^2\,dy\,\frac{dt}{t^{\frac n2+\beta}}\Big)^{\frac p2}\,dx\\ &\lesssim \mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)},\end{aligned}$$

the last of these estimates being a consequence of (2.3).

Finally we estimate J_j for $j\geq 1$. We have

$$\begin{aligned}&\int_0^\infty \fint_{B(x,t^{\frac12})}\int_{\frac t2}^t \|T_{t-s}[\mathbf{1}_{C_j(x,4s^{\frac12})}g(s,\cdot)](y)\|_H^2\,ds\,dy\frac{dt}{t^\beta}\\ &\lesssim \int_0^\infty\int_{\frac t2}^t \frac{1}{t-s}\exp(-\frac{c4^js}{t-s})\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,ds\,\frac{dt}{t^{\frac n2+\beta}}\\ &\leq \int_0^\infty\int_{\frac t2}^t \frac{1}{t-s}\exp(-c\frac{4^js}{t-s})\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^{\frac n2+\beta}}\,dt\\ &= \int_0^\infty\Big(\int_s^{2s}\frac{1}{t-s}\exp(-c\frac{4^js}{t-s})\,dt\Big)\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^{\frac n2+\beta}}\\ &\leq \exp(-\frac c24^j)\int_0^\infty\Big(\int_s^{2s}\frac{1}{t-s}\exp(-\frac c2\frac{4^js}{t-s})\,dt\Big)\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^{\frac n2+\beta}}\\ &= \exp(-\frac c24^j)\int_0^\infty\Big(\int_1^\infty\exp(-\frac c24^ju)\,\frac{du}{u}\Big)\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^{\frac n2+\beta}}\\ &\lesssim \exp(-\frac c24^j)\int_0^\infty\|\mathbf{1}_{B(x,2^{j+2}s^{\frac12})}g(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^{\frac n2+\beta}}.\end{aligned}$$

With (2.3) it follows that

$$\mathbb{E}J_j\lesssim \exp(-c4^{j-1}p)\mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta,2^{j+2}}(H)}\lesssim \exp(-c4^{j-1}p)2^{(j+2)\frac{np}{p\wedge 2}}\mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)},$$

and the sum $\mathbb{E}\sum_j J_j$ thus converges. □

Theorem 5.2 (Extrapolation via L^1-L^2 off-diagonal bounds)**.** *Let $(T_t)_{t>0}$ be a family of bounded linear operators on L^2, let $\beta>0$, and suppose there exists a constant $C_\beta\geq 0$,independent of g and H, such that*

$$\mathbb{E}\Big\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\Big\|^2_{T^{2,2}_{2,\beta}}\leq C_\beta^2\mathbb{E}\|g\|^2_{T^{2,2}_{2,\beta}(H)} \tag{5.3}$$

for all $\mathscr{F}$-adapted simple process $g:\mathbb{R}_+\times\mathbb{R}^n\times\Omega\to H$. If $(t^{\frac12}T_t)_{t>0}$ is a family of bounded linear operators on L^2 which satisfies L^1-L^2 off-diagonal bounds, then, for all $p\in[1,\infty)$, there exists a constant $C_{p,\beta}\geq 0$, independent of g and H, such that

$$\mathbb{E}\Big\|\int_0^t T_{t-s}g(s,\cdot)\,dW(s)\Big\|^p_{T^{p,2}_{2,\beta}}\leq C^p_{p,\beta}\mathbb{E}\|g\|^p_{T^{p,2}_{2,\beta}(H)}.$$

Recall that L^1-L^2 off-diagonal bounds are stronger than L^2-L^2 off-diagonal bounds by Lemma 2.1, so the previous proposition applies, and gives the result for $p\in[2,\infty)$.

The proof of Theorem 5.2 will be based on two lemmas. The first gives a simple sufficient condition for membership of $T^{p,2}_{2,\beta}(H)$.

Lemma 5.3. *If $a \in L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dy; H)$ is supported in a set of the form $(0, r^2) \times B(x_0, r)$ with $r > 0$ and $x_0 \in \mathbb{R}^n$, then, for all $1 \leq p \leq 2$, we have $a \in T^{p,2}_{2,\beta}(H)$ and*

$$\|a\|_{T^{p,2}_{2,\beta}(H)} \lesssim r^{n(\frac{1}{p}-\frac{1}{2})} \|a\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dy; H)}$$

with implied constant depending on n and p, but not on β, r, and x_0.

Proof. Noting that, for $t \in (0, r^2)$, $B(x_0, r) \cap B(x, t^{\frac{1}{2}}) \neq \emptyset$ only if $|x - x_0| < t^{\frac{1}{2}} + r \leq 2r$, from Hölder's inequality we obtain

$$\begin{aligned}
\|a\|^p_{T^{p,2}_{2,\beta}(H)} &= \int_{\mathbb{R}^n} \Big(\int_0^\infty \fint_{B(x,t^{\frac{1}{2}})} \|a(t,y)\|^2_H \, dy \frac{dt}{t^\beta} \Big)^{\frac{p}{2}} dx \\
&= \int_{B(x_0,2r)} \Big(\int_0^{r^2} \fint_{B(x,t^{\frac{1}{2}})} \|a(t,y)\|^2_H \, dy \frac{dt}{t^\beta} \Big)^{\frac{p}{2}} dx \\
&\leq \Big(\int_{B(x_0,2r)} dx \Big)^{1-\frac{p}{2}} \Big(\int_{B(x_0,2r)} \int_0^{r^2} \fint_{B(x,t^{\frac{1}{2}})} \|a(t,y)\|^2_H \, dy \frac{dt}{t^\beta} \, dx \Big)^{\frac{p}{2}} \\
&\lesssim r^{n(1-\frac{p}{2})} \Big(\int_{B(x_0,2r)} \int_0^{r^2} \int_{\mathbb{R}^n} \frac{\mathbf{1}_{B(x,t^{\frac{1}{2}})}(y)}{|B(x,t^{\frac{1}{2}})|} \|a(t,y)\|^2_H \, dy \, \frac{dt}{t^\beta} \, dx \Big)^{\frac{p}{2}} \\
&= r^{n(1-\frac{p}{2})} \Big(\int_0^{r^2} \int_{\mathbb{R}^n} \int_{B(x_0,2r)} \frac{\mathbf{1}_{B(y,t^{\frac{1}{2}})}(x)}{|B(y,t^{\frac{1}{2}})|} \|a(t,y)\|^2_H \, dx \, dy \, \frac{dt}{t^\beta} \Big)^{\frac{p}{2}} \\
&= r^{n(1-\frac{p}{2})} \Big(\int_0^{r^2} \int_{\mathbb{R}^n} \frac{|B(x_0,2r) \cap B(y,t^{\frac{1}{2}})|}{|B(y,t^{\frac{1}{2}})|} \|a(t,y)\|^2_H \, dy \, \frac{dt}{t^\beta} \Big)^{\frac{p}{2}} \\
&\leq r^{n(1-\frac{p}{2})} \Big(\int_0^\infty \int_{\mathbb{R}^n} \|a(t,y)\|^2_H \, dy \, \frac{dt}{t^\beta} \Big)^{\frac{p}{2}}.
\end{aligned}$$

□

For the second lemma we need to introduce some terminology. An *atom* with values in H is a function $a : \mathbb{R}_+ \times \mathbb{R}^n \to H$ supported in a set of the form $(0, r^2) \times B(x_0, r)$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$ and satisfying the estimate

$$\|a\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dy; H)} \leq r^{-\frac{n}{2}}.$$

By the previous lemma, any atom belongs to $T^{1,2}_{2,\beta}(H)$ with norm $\|a\|_{T^{1,2}_{2,\beta}(H)} \lesssim 1$.

The next lemma is a consequence of the well-known fact that $T^{1,2}_{2,\beta}(H)$ admits an atomic decomposition, and interpolation.

Lemma 5.4. *Let $\beta \in \mathbb{R}$ and let $\mathscr{H}$ be a Hilbert space. A bounded linear operator from $T^{2,2}_{2,\beta}(H)$ to $T^{2,2}_{2,\beta}(\mathscr{H})$, which is uniformly bounded on atoms, extends to a bounded operator from $T^{1,2}_{2,\beta}(H)$ to $T^{1,2}_{2,\beta}(\mathscr{H})$.*

A subtle point here is that an operator that is uniformly bounded on atoms is not necessarily defined on $T^{1,2}_{2,\beta}(H)$. However, if the operator is also bounded on $T^{2,2}_{2,\beta}(H)$, then a simple modification of [9, Theorem 4.9, Step 3] takes care of this issue.

Proof of Theorem 5.2. Given a simple function $f : \mathbb{R}_+ \to L^2 \otimes H$, let

$$Mf(t,x) := \mathbf{1}_{\{\frac{t}{2} \geq s\}}[T_{t-s}f(s,\cdot)](x).$$

As in (5.2), given an adapted simple process $g : \mathbb{R}_+ \times \Omega \to L^2 \otimes H$, for all $1 \leq p < \infty$, we have

$$\mathbb{E}\Big\| \int_0^{\frac{t}{2}} T_{t-s}g(s,\cdot)\, dW(s)\Big\|^p_{T^{p,2}_{2,\beta}} \lesssim \mathbb{E}\|Mg\|^p_{T^{p,2}_{2,\beta}(L^2(\mathbb{R}_+;H))}.$$

Hence the theorem is proved once we show that the linear mapping M is bounded from $T^{p,2}_{2,\beta}(H)$ to $T^{p,2}_{2,\beta}(L^2(\mathbb{R}_+;H))$ for $p \in [1,2]$. Indeed, the stochastic integral over the interval $(\frac{t}{2}, t)$ has already been estimated in the proof of Proposition 5.1.

By interpolation, it suffices to consider the exponents $p = 1$ and $p = 2$.

Step 1 – We start with the case $p = 2$. Proceeding as in (5.2), using the isometry $T^{2,2}_{2,\beta}(H) = L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^{-\beta}dt \times dy; H)$ (first with H replaced by $\mathbb{R}$ and at the end of the computation with H), Fubini's theorem, the uniform boundedness of the operators $t^{\frac{1}{2}}T_t$, we obtain

$$\begin{aligned}
\|Mf\|^2_{T^{p,2}_{2,\beta}(L^2(\mathbb{R}_+;H))} &= \Big\| \big\|\mathbf{1}_{\{\frac{t}{2}\geq s\}}T_{t-s}f(s,\cdot)\big\|_{L^2(\mathbb{R}_+;H)} \Big\|^2_{T^{p,2}_{2,\beta}} \\
&= \int_0^\infty \int_0^{\frac{t}{2}} \|T_{t-s}f(s,\cdot)\|^2_{L^2(H)}\, ds \frac{dt}{t^\beta} \\
&\lesssim \int_0^\infty \int_0^{\frac{t}{2}} \frac{s}{t}\|f(s,\cdot)\|^2_{L^2(H)}\, \frac{ds}{s}\frac{dt}{t^\beta} \\
&= \int_0^{\frac{1}{2}} \int_0^\infty \|f(tu,\cdot)\|^2_{L^2(H)}\, \frac{dt}{t^\beta}\, du \\
&= \int_0^{\frac{1}{2}} \int_0^\infty u^{\beta-1}\|f(t,\cdot)\|^2_{L^2(H)}\, \frac{dt}{t^\beta}\, du \\
&\lesssim \int_0^\infty \|f(t,\cdot)\|^2_{L^2(H)}\, \frac{dt}{t^\beta} \\
&= \|f\|^2_{T^{p,2}_{2,\beta}(H)}.
\end{aligned}$$

Step 2 – Next we consider the case $p = 1$. We will prove that there exists a constant $C_\beta \geq 0$ such that for every atom a we have

$$\|Ma\|_{T^{1,2}_{2,\beta}(L^2(\mathbb{R}_+;H))} \leq C_\beta. \tag{5.4}$$

An appeal to Lemma 5.4 will then finish the proof.

Fix an atom a supported in $(0, r^2) \times B(x_0, r)$, and define the following sets:

$$\begin{aligned}
C_0 :=& \{(t,x) \in (0,\infty) \times \mathbb{R}^n \,;\, |x - x_0| < 2r \text{ and } t < (2r)^2\}, \\
C_j :=& \{(t,x) \in (0,\infty) \times \mathbb{R}^n \,;\, 2^j r \leq |x - x_0| < 2^{j+1}r \text{ and } t < (2^j r)^2\},\ j \geq 1, \\
C'_j :=& \{(t,x) \in (0,\infty) \times \mathbb{R}^n \,;\, |x - x_0| < 2^{j+1}r \text{ and } (2^j r)^2 \leq t < (2^{(j+1)}r)^2\},\ j \geq 1.
\end{aligned}$$

We write

$$\big\|\mathbf{1}_{\{\frac{t}{2}\geq s\}}T_{t-s}f(s,\cdot)\big\|_{L^2(\mathbb{R}_+;H)}$$

$$= \Big(\mathbf{1}_{C_0}(t,x) + \sum_{j\geq 1}(\mathbf{1}_{C_j}(t,x) + \mathbf{1}_{C_j'}(t,x))\Big)\Big(\int_0^{\frac{t}{2}} \|[T_{t-s}a(s,\cdot)](x)\|_H^2\,ds\Big)^{\frac{1}{2}}$$

and, using Lemma 5.3, show that each term is in $T^{1,2}_{2,\beta}$ with suitable bounds.
1. Estimate on C_0: Estimating as before, using the uniform boundedness of $(t^{\frac{1}{2}}T_t)_{t>0}$ we have

$$\begin{aligned}
&\int_{B(x_0,2r)}\int_0^{4r^2}\int_0^{\frac{t}{2}} \|[T_{t-s}a(s,\cdot)](x)\|_H^2\,ds\,\frac{dt}{t^\beta}\,dx\\
&\quad\lesssim \int_0^{4r^2}\int_0^{\frac{t}{2}} \frac{s}{t}\|a(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s}\,\frac{dt}{t^\beta}\\
&\quad= \int_0^{2r^2}\int_{2s}^{4r^2} \frac{s}{t}\|a(s,\cdot)\|^2_{L^2(H)}\,\frac{dt}{t^\beta}\,\frac{ds}{s}\\
&\quad\lesssim \int_0^{\infty} s^{1-\beta}\|a(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s}\\
&\quad\lesssim r^{-n}.
\end{aligned}$$

Therefore, by Lemma 5.3,

$$\Big\|(t,x)\mapsto \mathbf{1}_{C_0}(t,x)\Big(\int_0^{\frac{t}{2}} \|T_{t-s}[a(s,\cdot)](x)\|_H^2\,ds\Big)^{\frac{1}{2}}\Big\|_{T^{1,2}_{2,\beta}} \lesssim 1.$$

2. Estimate on C_j: Let us write $\tilde{B}_j = B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)$. Using the L^1-L^2 off-diagonal estimates, and the fact that a is supported on $(0,r^2)\times B(x_0,r)$, we have

$$\begin{aligned}
&\int_{\tilde{B}_j}\int_0^{4^jr^2}\int_0^{\frac{t}{2}} \|T_{t-s}[a(s,\cdot)](x)\|_H^2\,ds\,\frac{dt}{t^\beta}\,dx\\
&= \int_0^{4^jr^2}\int_0^{\frac{t}{2}}\int_{\tilde{B}_j} \|t^{\frac{1}{2}}T_{t-s}[a(s,\cdot)\mathbf{1}_{B(x_0,r)}(\cdot)](x)\|_H^2\,dx\,ds\,\frac{dt}{t^{1+\beta}}\\
&\lesssim \int_0^{4^jr^2}\int_0^{\frac{t}{2}} t^{-\frac{n}{2}}\exp(-c4^jr^2/t)\|a(s,\cdot)\|^2_{L^1(\mathbb{R}^n;H)}\,ds\,\frac{dt}{t^{1+\beta}}\\
&= \int_0^{4^jr^2}\Big(\int_0^{\frac{t}{2}} s^\beta\|a(s,\cdot)\|^2_{L^1(\mathbb{R}^n;H)}\,\frac{ds}{s^\beta}\Big)t^{-\beta-\frac{n}{2}}\exp(-c4^jr^2/t)\,\frac{dt}{t}\\
&\lesssim \Big(r^n\int_0^{r^2} r^{2\beta}\|a(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^\beta}\Big)\int_0^{4^jr^2} t^{-\beta-\frac{n}{2}}\exp(-c4^jr^2/t)\,\frac{dt}{t}\\
&\lesssim r^{2\beta}(4^jr^2)^{-\beta-\frac{n}{2}}\\
&= r^{-n}4^{-j(\beta+\frac{n}{2})}.
\end{aligned}$$

Therefore, by Lemma 5.3,

$$\begin{aligned}
&\Big\|(t,x)\mapsto \mathbf{1}_{C_j}(t,x)\Big(\int_0^{\frac{t}{2}} \|T_{t-s}[a(s,\cdot)](x)\|_H^2\,ds\Big)^{\frac{1}{2}}\Big\|_{T^{1,2}_{2,\beta}}\\
&\qquad\lesssim (2^{j+1}r)^{\frac{n}{2}}r^{-\frac{n}{2}}2^{-j(\beta+\frac{n}{2})} \lesssim 2^{-j\beta}.
\end{aligned}$$

3. Estimate on C_j': Using the L^1-L^2 off-diagonal bounds, we have

$$\begin{aligned}
&\int_{B(x_0,2^{j+1}r)}\int_{(2^jr)^2}^{(2^{j+1}r)^2}\int_0^{\frac{t}{2}} \|T_{t-s}[a(s,\cdot)](x)\|_H^2\,ds\,\frac{dt}{t^\beta}\,dx\\
&\quad=\int_{(2^jr)^2}^{(2^{j+1}r)^2}\int_0^{\frac{t}{2}}\int_{B(x_0,2^{j+1}r)} \|t^{\frac12}T_{t-s}[a(s,\cdot)\mathbf{1}_{B(x_0,r)}(\cdot)](x)\|_H^2\,dx\,ds\,\frac{dt}{t^{1+\beta}}\\
&\quad\lesssim\int_{(2^jr)^2}^{(2^{j+1}r)^2}\int_0^{\frac{t}{2}} t^{-\frac{n}{2}}\|a(s,\cdot)\|^2_{L^1(\mathbb{R}^n;H)}\,ds\,\frac{dt}{t^{1+\beta}}\\
&\quad\lesssim\Big(r^n\int_0^{r^2} r^{2\beta}\|a(s,\cdot)\|^2_{L^2(H)}\,\frac{ds}{s^\beta}\Big)\int_{(2^jr)^2}^{(2^{j+1}r)^2} t^{-\beta-\frac{n}{2}}\,\frac{dt}{t}\\
&\quad\lesssim r^{2\beta}(4^jr^2)^{(-\beta-\frac{n}{2})}\\
&\quad= r^{-n}4^{-j(\beta+\frac{n}{2})}.
\end{aligned}$$

Therefore, by Lemma 5.3,

$$\|(t,x)\mapsto \mathbf{1}_{C_j'}(t,x)\Big(\int_0^{\frac{t}{2}}\|T_{t-s}[a(s,\cdot)](x)\|_H^2\,ds\Big)^{\frac12}\|_{T^{1,2}_{2,\beta}}\lesssim 2^{-j\beta}.$$

4. Collecting the estimates: Summing the above three estimates over j gives (5.4). □

Proof of Theorem 3.1. For A as in Example 2.2, $(t^{\frac12}\nabla e^{-tA})_{t\geq 0}$ has L^1-L^2 off-diagonal bounds. By the solution of Kato's square root problem [5],

$$\|A^{\frac12}u\|_{L^2}\eqsim\|\nabla u\|_{L^2(\mathbb{R}^n)}. \tag{5.5}$$

Moreover, A is maximal accretive on L^2 and therefore the bounded analytic semigroup generated by $-A$ is contractive on L^2. By Proposition 4.1, the mapping $g\mapsto A^{\frac12}S\diamond g$ (and hence the mapping $g\mapsto\nabla S\diamond g$) extends to a bounded operator from $L^2_{\mathscr{F}}(\Omega;T^{2,2}_{2,\beta}(H))$ to $L^2_{\mathscr{F}}(\Omega;T^{2,2}_{2,\beta})$ (respectively, from $L^2_{\mathscr{F}}(\Omega;T^{2,2}_{2,\beta}(H))$ to $L^2_{\mathscr{F}}(\Omega;T^{2,2}_{2,\beta}(\mathbb{R}^n))$). The result thus follows from Theorem 5.2. □

Remark 5.5. The results in this section are stated in way that is suitable for applications to the divergence form elliptic operators from Example 2.2. Introducing an homogeneity parameter m as in [7, 10], one can prove analogue results suitable for the study of differential operators of order m. We have chosen not to do so here to make the paper more readable.

6. An application to SPDE

In this section we apply our results to prove conical stochastic maximal L^p-regularity for a class of nonlinear stochastic evolution equations. We consider the problem

$$\begin{cases} du(t,x)=\operatorname{div}a(x)\nabla u(t,x)\,dt+b(\nabla u(t,x))\,dW(t), & t\geq 0,\ x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n.\end{cases} \tag{6.1}$$

Here, W is an $\mathscr{F}$-Brownian motion relative to some given filtration $\mathscr{F}$, the function $a:\mathbb{R}^n\to M^n(\mathbb{R})$ is bounded and measurable, the operator $A=-\operatorname{div}a\nabla$ satisfies

the ellipticity conditions of Example 2.2, the function $b : \mathbb{R}^n \to \mathbb{R}$ is globally Lipschitz continuous, with Lipschitz constant L_b, and satisfies

$$|b(x)| \leq C_b|x|, \quad x \in \mathbb{R}^n. \tag{6.2}$$

The initial value $u_0 : \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathsf{D}_p(A^{\frac{\beta}{2}})$, the domain of the L^p-realisation of $A^{\frac{\beta}{2}}$, for some $0 < \beta < 1$. At the expense of making the arguments more involved, we could also add an additional semilinear term and consider cylindrical Brownian motions, but in order to bring out the principles more clearly we have chosen to consider a simple model problem.

In order to arrive at a notion of solution we proceed as follows. At least formally, we reformulate (6.1) as an abstract initial value problem as follows:

$$\begin{cases} dU(t) + AU(t)\,dt = B(\nabla U(t))\,dW(t), & t \geq 0, \\ U(0) = u_0. \end{cases}$$

Here $A = -\operatorname{div} a(\cdot)\nabla$ and

$$(B(u))(t,x) := b(u(t,x))$$

is the Nemytskii operator associated with b. Denoting by $L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta})$ the closed subspace of all $\mathscr{F}$-adapted processes belonging to $L^p(\Omega; T^{p,2}_{2,\beta})$, it is immediate from (6.2) that B maps $L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta})$ into itself, and the Lipschitz continuity of b implies that of B, with the same constant.

In order to be consistent with the terminology used in the Introduction, at least formally, a "mild solution" should be an adapted "process" U that "satisfies" the variation of constants equation

$$U(t) = S(t)u_0 + \int_0^t S(t-s)B(\nabla U(s))\,dW(s), \tag{6.3}$$

where S is the bounded analytic C_0-semigroup generated by $-A$. By *conical stochastic maximal L^p-regularity* we then understand that the "gradient" of U is in $L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$. In order to make this rigorous, we formally apply ∇ to both sides of the identity (6.3) and, again formally, substitute $V = \nabla U$ to arrive at the equation

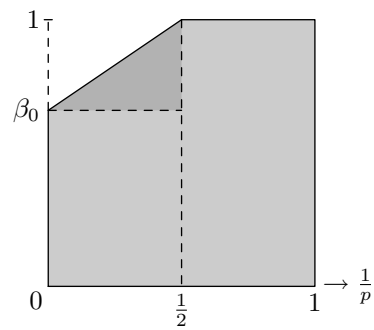
$$V = \nabla S(\cdot)u_0 + \nabla S \diamond B(V). \tag{6.4}$$

Definition 6.1. The problem (6.1) is said to have *conical stochastic maximal L^p-regularity with weight β* if for every initial value $u_0 \in \mathsf{D}_p(A^{\frac{\beta}{2}})$ there exists a unique element V in $L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ such that (6.4) holds.

Thus we solve for V, rather than for U. The above heuristic discussion shows that we may think of V as the "gradient of the mild solution of (6.3)".

Remark 6.2. If V solves (6.4), then, at least formally, we have $V = \nabla U$ with $U := S(\cdot)u_0 + S \diamond B(V)$. This definition makes sense provided stochastic convolution on the right-hand side is well defined in one way or the other. We are not asserting, however, that this process is a "mild solution" to (6.3) in any rigorous sense.

In the next lemma, which is of interest in its own right, we denote by S the semigroup generated by $-A$ on L^p.



The polytope of Lemma 6.3. Shaded in gray are the regions of steps 3 and 2; shaded dark is the region of step 4.

Lemma 6.3. *There exists $\beta_0 \in (0,1]$ with the following property. If $p \in (1,\infty)$ and $0 < \beta < 1$ are such that the pair $(\frac{1}{p}, \beta)$ belongs to the interior of the planar polytope with vertices $(0,0), (0,\beta_0), (\frac{1}{2}, 1), (1,1), (1,0)$, then for all $u_0 \in \mathsf{D}_p(A^{\frac{\beta}{2}})$ the function $(t,x) \mapsto \nabla S(t)u_0(x)$ belongs to $T^{p,2}_{2,\beta}(\mathbb{R}^n)$.*

Remark 6.4. The constant β_0 is related to elliptic regularity theory, and can be arbitrary small. For certain specific classes of operators A, however, it is known that one can take $\beta_0 = 1$, viz. in the case of constant coefficients (in this case, the arguments can be simplified using standard Littlewood-Paley estimates), and in the case of continuous periodic coefficients with common periods (see [1, Section 5.4], [11, page 139], and references therein).

Remark 6.5. For $\beta = 0$, the lemma holds for $1 < p < \infty$ using [6, Theorem 3.1] and $p_-(A) = 1$ (the number $p_-(A)$ being defined in [6]). The argument given here for $1 < p < 2$ and $0 < \beta < 1$ applies to $\beta = 0$ as well, and gives a different proof of this case.

Proof. For this proof, we use complexified spaces. Let $v_0 = A^{\frac{\beta}{2}} u_0$. The proof proceeds in four steps.

Step 1 – In the case $p = 2$ and $0 < \beta < 1$, one has

$$\begin{aligned}
\|(t,y) \mapsto \nabla S(t)u_0(y)\|^2_{T^{p,2}_{2,\beta}(\mathbb{C}^n)} &\eqsim \int_0^\infty \int_{\mathbb{R}^n} |\nabla e^{-tA} u_0|^2 \, dy \, \frac{dt}{t^\beta} \\
&\eqsim \int_0^\infty \int_{\mathbb{R}^n} |A^{\frac{1}{2}} e^{-tA} u_0|^2 \, dy \, \frac{dt}{t^\beta} \\
&= \int_0^\infty \int_{\mathbb{R}^n} |(tA)^{\frac{1-\beta}{2}} e^{-tA} v_0|^2 \, dy \, \frac{dt}{t} \\
&\eqsim \|v_0\|_2^2
\end{aligned}$$

where the second equivalence uses the equivalence $\|A^{\frac{1}{2}} u\|_{L^2} \eqsim \|\nabla u\|_{L^2(\mathbb{R}^n)}$ (cf. (5.5)), and the last inequality uses the boundedness of the H^∞-functional calculus of A in L^2 (a result going back to [26], see also [25, Theorem 11.9]).

Step 2 – In the case $1 < p < 2$ and $0 < \beta < 1$, we use the first-order approach of [19]. Consider the operator

$$D = \begin{pmatrix} 0 & -\operatorname{div} a(\cdot) \\ \nabla & 0 \end{pmatrix}$$

acting on $L^2 \oplus L^2(\mathbb{C}^n)$. This operator is bisectorial with angle $\eta \in (0, \frac{\pi}{2})$, its resolvent satisfies L^2-L^2 off-diagonal estimates with arbitrary large decay and has a bounded H^∞ functional calculus. The operator A is sectorial of angle $\omega \in (0, \frac{\pi}{2})$ and has the same properties. In particular, $-A$ generates a bounded analytic C_0-semigroup on $L^2 \oplus L^2(\mathbb{C}^n)$ (see [12]).

Note that

$$t^{\frac{1-\beta}{2}} \nabla S(t) u_0 = t^{\frac{1}{2}} \nabla e^{-tA} (tA)^{-\frac{\beta}{2}} v_0$$

and

$$(0, t^{\frac{1}{2}} \nabla e^{-tA} (tA)^{-\frac{\beta}{2}} v_0) = \psi(\sqrt{t} D)(v_0, 0)$$

for the function $\psi \in H^\infty(S_\theta)$ defined by

$$\psi(z) = z (z^2)^{-\frac{\beta}{2}} e^{-z^2}$$

(where $\theta \in (\omega, \frac{\pi}{2})$, $S_\theta = \Sigma_\theta \cup (-\Sigma_\theta) \cup \{0\}$, and $\Sigma_\theta = \{z \in \mathbb{C} \backslash \{0\} \mid |\arg(z)| < \theta\}$).

Therefore, by [19, Theorem 7.10], one has the following equivalence of norms in the tent spaces upon changing ψ in the appropriate class (more precisely, as $1 < p < 2$, it is enough that ψ has slight decay at 0 and sufficiently large decay at ∞: see the condition (d) in Proposition 7.5 there and note that we have cotype 2),

$$\|(t,y) \mapsto \psi(\sqrt{t} D)(v_0, 0)(y)\|_{T^{p,2}_{2,1}(\mathbb{C}^{n+1})} \eqsim \|(t,y) \mapsto \widetilde{\psi}(\sqrt{t} D)(v_0, 0)(y)\|_{T^{p,2}_{2,1}(\mathbb{C}^{n+1})}$$

for $\widetilde{\psi}(z) = z e^{-z^2}$. As

$$\widetilde{\psi}(\sqrt{t} D)(v_0, 0) = (0, t^{\frac{1}{2}} \nabla S(t) v_0)$$

we have shown that

$$\begin{aligned} \|(t,y) \mapsto \nabla S(t) u_0(y)\|_{T^{p,2}_{2,\beta}(\mathbb{C}^n)} &= \|(t,y) \mapsto t^{\frac{1-\beta}{2}} \nabla S(t) A^{-\frac{\beta}{2}} v_0(y)\|_{T^{p,2}_{2,1}(\mathbb{C}^n)} \\ &\eqsim \|(t,y) \mapsto t^{\frac{1}{2}} \nabla S(t) v_0(y)\|_{T^{p,2}_{2,1}(\mathbb{C}^n)} \\ &= \|(t,y) \mapsto \nabla S(t) v_0(y)\|_{T^{p,2}_{2,0}(\mathbb{C}^n)}. \end{aligned}$$

Using [6, Theorem 3.1], this gives

$$\|(t,y) \mapsto \nabla S(t) u_0(y)\|_{T^{p,2}_{2,\beta}(\mathbb{C}^n)} \lesssim \|v_0\|_{L^p} = \|A^{\frac{\beta}{2}} u_0\|_{L^p}.$$

Step 3 – We turn to the case $2 < p < \infty$ and $0 < \beta < \beta_0$, where $\beta_0 \in (0, 1]$ will be determined in a moment. We use the fact that the spaces $T^{p,2}_{2,\beta}$ interpolate (by either the complex or the real method, see e.g. [14]) between $p = 2$ and $p = \infty$. For $p = \infty$, $T^{p,2}_{2,\beta}$ is defined as the space of all locally square integrable functions such that the Carleson measure condition

$$\int_0^{r^2} \int_B |g(t,y)|^2 \, dy \, \frac{dt}{t^\beta} \leq C r^n$$

holds whenever B is a ball of radius $r > 0$, with C independent of B.

We claim that there exist $\beta_0 \in (0, 1]$ such that for all $0 < \beta < \beta_0$ and $f \in L^\infty$,

$$\|(t,y) \mapsto t^{\frac{1-\beta}{2}} \nabla A^{-\frac{\beta}{2}} e^{-tA} f(y)\|_{T^{\infty,2}_{2,1}(\mathbb{C}^n)} \lesssim \|f\|_{L^\infty}.$$

Assuming the claim, and using the $p=2$ result for all $0<\beta<1$, one concludes, by interpolation, that for $0<\beta<\beta_0$ and $p\in[2,\infty)$,

$$\|(t,y)\mapsto t^{\frac{1-\beta}{2}}\nabla A^{-\frac{\beta}{2}}e^{-tA}v_0(y)\|_{T^{p,2}_{2,1}(\mathbb{C}^n)}\lesssim\|v_0\|_{L^p},$$

i.e.,

$$\|(t,y)\mapsto\nabla S(t)u_0(y)\|_{T^{p,2}_{2,\beta}(\mathbb{C}^n)}\lesssim\|A^{\frac{\beta}{2}}u_0\|_{L^p}.$$

We now prove the claim. The argument is scale and translation invariant (up to changing the matrix $a(x)$ to $a(rx+x_0)$ which does not change the ellipticity constants), so we assume that B is the unit ball. The same proof works in the general case, and produces the required factor r^n. Let f be a bounded measurable function with compact support. Let $f_0=f1_{2B}$ and $f_1=f-f_0$. By the $p=2$ result, one has

$$\int_0^1\int_B|t^{\frac{1-\beta}{2}}\nabla A^{-\frac{\beta}{2}}e^{-tA}f_0(y)|^2\,dy\,\frac{dt}{t}\lesssim\|f_0\|_{L^2}^2\lesssim\|f\|_{L^\infty}^2.$$

To control the other term, we use the representation formula

$$\nabla A^{-\frac{\beta}{2}}e^{-tA}f_1=C\int_0^\infty\nabla s^{\beta/2}(sA)e^{-(s+t)A}f_1\frac{ds}{s},$$

for some constant $C>0$ independent of f_1, which holds in $L^2(B;\mathbb{C}^n)$ thanks to the following estimate on the kernel $\tilde{K}_s(x,y)$ of Ae^{-sA}. There are constants $c,C>0$ and $\gamma_0>0$ in $(n-2,n]$ (for $n=1$, one has $\gamma_0=1$) such that

$$\forall y\notin 2B\quad\Big(\int_B|\nabla_x\tilde{K}_{s+t}(x,y)|^2\,dx\Big)^{\frac12}\le\frac{C}{(s+t)^{\frac{n}{4}+\frac32+\frac{\gamma_0}{4}}}\cdot\exp\Big(-\frac{c|y|^2}{s+t}\Big).$$

This estimate is proven in [11, Lemma 33, p. 139] for e^{-tA} but the same proof (based on estimates for the kernel of the resolvent) gives the estimate for Ae^{-sA}. Define $\beta_0=\frac12(\gamma_0-n+2)\in(0,1]$. By the kernel estimate and the fact that f_1 is supported away from $2B$, we have

$$\begin{aligned}
&\|\nabla A^{-\frac{\beta}{2}}e^{-tA}f_1\|_{L^2(B;\mathbb{C}^n)}\\
&\quad\eqsim\Big\|\int_0^\infty\nabla s^{\frac{\beta}{2}}(sA)e^{-(s+t)A}f_1\frac{ds}{s}\Big\|_{L^2(B;\mathbb{C}^n)}\\
&\quad\le\int_0^\infty s^{1+\frac{\beta}{2}}\int_{|y|\ge2}\|\nabla\tilde{K}_{s+t}(\cdot,y)f_1(y)\Big\|_{L^2(B;\mathbb{C}^n)}\,dy\,\frac{ds}{s}\\
&\quad\lesssim\int_0^\infty\int_{|y|\ge2}\frac{s^{1+\frac{\beta}{2}}}{(s+t)^{\frac{n}{4}+\frac32+\frac{\gamma_0}{4}}}\cdot\exp\Big(-\frac{c|y|^2}{s+t}\Big)|f_1(y)|\,dy\,\frac{ds}{s}\\
&\quad\lesssim\|f_1\|_{L^\infty}\int_{|y|\ge2}\frac{1}{|y|^{n+\beta_0-\beta}}\,dy\\
&\quad\lesssim\|f_1\|_{L^\infty},
\end{aligned}$$

for $0<\beta<\beta_0$. Therefore $\int_0^1\|t^{\frac{1-\beta}{2}}\nabla A^{-\frac{\beta}{2}}e^{-tA}f_1\|^2_{L^2(B;\mathbb{C}^n)}\frac{dt}{t}\lesssim\|f\|^2_{L^\infty}$.

Step 4 – The arguments so far show that the $T^{p,2}_{2,\beta}(\mathbb{C}^n)$ estimate holds for $0<\operatorname{Re}\beta<1$ and $p=2$, with controlled growth in terms of $\operatorname{Im}\beta$. It also shows that the $T^{p,2}_{2,\beta}(\mathbb{C}^n)$ estimate holds for $0<\operatorname{Re}\beta<\beta_0$ and $p=\infty$ again with controlled growth in $\operatorname{Im}\beta$. Using Stein's complex interpolation, one also gets the conclusion of the lemma for all pairs (p,β) such that the point $(\frac1p,\operatorname{Re}\beta)$ is inside the planar

polytope with vertices $(0,0), (0,\beta_0), (\frac{1}{2},1), (\frac{1}{2},1), (0,1)$. We leave the details to the reader. □

Theorem 6.6. *Let $p \in (1,\infty)$ and $0 < \beta < 1$ be such that the pair $(\frac{1}{p},\beta)$ belongs to the interior of the planar polytope with vertices $(0,0), (0,\beta_0), (\frac{1}{2},1), (1,1), (1,0)$, where β_0 is defined in Lemma 6.3. Suppose that $K_{p,\beta}L_b < 1$, where $K_{p,\beta}$ is the norm of the mapping $g \mapsto \nabla S \diamond g$ from $L^p(\Omega; T^{p,2}_{2,\beta})$ to $L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{C}^n))$ and L_b is the Lipschitz constant of b. Then the problem (6.1) has conical stochastic maximal L^p-regularity with weight β, i.e., (6.4) holds for all initial values $u_0 \in \mathsf{D}_p(A^{\frac{\beta}{2}})$.*

Proof. Consider the fixed point mapping F on $L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ defined by

$$F(v) := \nabla S(\cdot)u_0 + \nabla S \diamond B(v).$$

By Theorem 5.2, we have $\nabla S \diamond B(v) \in L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ for all $v \in T^{p,2}_{2,\beta}(\mathbb{R}^n)$. This, in combination with the previous lemma, shows that F maps $L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ into itself.

For $v_1, v_2 \in L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ we may estimate

$$\begin{aligned}\|F(v_1) - F(v_2)\|_{L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))} &= \|\nabla S \diamond (B(v_1) - B(v_2))\|_{L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))} \\ &\leq K_{p,\beta}\|B(v_1) - B(v_2)\|_{L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))} \\ &\leq K_{p,\beta}L_b\|v_1 - v_2\|_{L^p(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))}.\end{aligned}$$

Since by assumption $K_{p,\beta}L_b < 1$, F has a unique fixed point V in $L^p_{\mathscr{F}}(\Omega; T^{p,2}_{2,\beta}(\mathbb{R}^n))$ and the theorem is proved. □

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