

# On the bursting of gene products

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## Abstract

In this article we demonstrate that the so-called bursting production of molecular species during gene expression may be an artifact caused by low time resolution in experimental data collection and not an actual burst in production. We reach this conclusion through an analysis of a two-stage and binary model for gene expression, and demonstrate that in the limit when mRNA degradation is much faster than protein degradation they are equivalent. The negative binomial distribution is shown to be a limiting case of the binary model for fast “on to off” state transitions and high values of the ratio between protein synthesis and degradation rates. The gene products population increases by unity but multiple times in a time interval orders of magnitude smaller than protein half-life or the precision of the experimental apparatus employed in its detection. This rare-and-fast one-by-one protein synthesis has been interpreted as bursting.

Understanding the origin of fluctuations at the single cell level and how organisms deal them to guarantee both developmental viability and evolutionary adaptation to a constantly changing environment conditions is a challenge of the post-genomic era [3, 4]. Often, stochasticity at the single cell level is due to the presence of biochemical reactants in low copy number inside the cell [5] and heterogeneous spatial distribution [6]. Experimental techniques to investigate these phenomena have been greatly enhanced by the use of fluorescent molecules and technology to track the spatial and temporal behavior of individual molecules [7, 8]. Despite the striking nature of the data these techniques provide, these advances do not necessarily give the full picture of the dynamics of events at the single cell level such as transcription and translation. One example is the measurement of the bursting production of molecules, defined as an incremental increase in mRNA or protein number greater than one at a given time. Bursting is often held to be the *usual* mechanism for the synthesis of gene products [7, 8]. As we show here, the inference of bursting molecular production may be an artifact due to a lack of sufficiently fine temporal resolution in experimental data. Also, from a modeling perspective the inference of bursting may be flawed due to a reliance on the shape of stationary probability distributions rather than an analysis of the underlying dynamical processes giving rise to them.

The experimental observation of these jumps in molecular numbers has motivated several models for the prediction and fitting of observed data, *e.g.* by employing a Langevin approach (continuous) or the master equation (discrete). In the continuous case, stationary gamma distributions for molecular concentrations are predicted along with discontinuous trajectories for the corresponding stochastic process [9, 10]. In the discrete framework, bursts appear in models for gene expression with two stochastic variables (so called two-stage models with mRNA and protein), in the limit where the mRNA degradation rate is much larger than the degradation rate for protein (a common experimental finding). In these cases, the model predicted probability distributions are well described by a negative binomial probability distribution [11] and simulations exhibit temporal bursting in protein numbers [8].

In this article we use a discrete modeling framework to show that the bursting limit actually corresponds to a particular regime of a model of a switching gene between “on” and “off” states with a *one step variation* in the stochastic variable corresponding to protein numbers [12–14]. The model we develop here is an approximation to a model which is inte-

grable in both the stationary [14, 15] and time-dependent regimes [16, 17] with symmetries underlying the existence of analytical solution, and whose biological implications have been explored elsewhere [18, 19].

The steady state solution for the binary model, in the limit of rapid transitions from the “on” to the “off” state, approaches a negative binomial distribution that also describes the bursting model. Simulation of the binary model shows apparent bursting, but examined on a finer time scale reveals the protein numbers actually increase in a unitary fashion. This clearly suggests that the experimental detection of bursting in gene product numbers may be due to lack of temporal resolution in the data. Our results give clear guidelines for the conditions on the transcription and translation processes for artifactual bursts to appear. As such, the modeling establishes necessary conditions on the experimental temporal resolution necessary to establish the existence of true bursting.

Gene expression is a cascade of first transcription to produce mRNA followed by translation of that mRNA to protein. Thus it makes sense to take the system state variables to be the numbers of mRNA and protein molecules in a given cell. mRNA molecules are produced at a rate that is dependent on the interaction between the RNA polymerase and the promoter site. The number of mRNA molecules in the cell and their interactions with ribosomes controls the protein synthesis. In this framework, *self regulation* is introduced in the model by considering the transcription rate to be dependent on the number of protein molecules produced. It is worth noting that specific regulation of the state of the promoter site (active or repressed) is not taken into account on this model.

This picture for non-regulated genes has been treated in the literature previously [20].

Let  $m$  and  $n$  denote, respectively, the number of mRNA and protein molecules. The probability of finding the system in a state  $(m, n)$ ,  $m, n \geq 0$ , at time  $t$  is denoted by  $P_{m,n}(t)$ , while the synthesis rates for mRNA and protein are denoted by  $\mu_M^0$ ,  $\mu_M^1$  and  $\nu_P$ , and the corresponding degradation rates are  $\rho_M$  and  $\rho_P$ . Then the evolution of the probability is governed by a master equation for two coupled birth-death processes:

$$\begin{aligned} \frac{dP_{m,n}}{dt} = & (\mu_M^0 + \mu_M^1 n)(P_{m-1,n} - P_{m,n}) \\ & + \nu_P m(P_{m,n-1} - P_{m,n}) \\ & + \rho_M [(m+1)P_{m+1,n} - P_{m,n}] \\ & + \rho_P [(n+1)P_{m,n+1} - nP_{m,n}], \end{aligned} \tag{1}$$

where we have assumed that the transcription rate is a function of the protein number ( $\mu_M^1 n$ ), indicating positive self regulation, with the requirement that  $\mu_M^0 \neq 0$ . We have assumed a linear dependence between the protein translation rate ( $\nu_P$ ) and the number of available mRNA molecules in the cytoplasm. We have been unable to construct an analytic solution to the complete system of Eq. (1). However, exact quadrature is achieved in the limiting case when the mRNA degradation rate is much greater than the protein degradation rate ( $\rho_M/\rho_P \gg 1$ ) [11] so the mRNA lifetime is quite short relative to the protein lifetime.

That suggests scaling the model parameters by the protein lifetime ( $\sim \rho_P^{-1}$ ), which results in the dimensionless quantities

$$\mu^0 = \frac{\mu_M^0}{\rho_P}, \quad \mu^1 = \frac{\mu_M^1}{\rho_P}, \quad \gamma = \frac{\rho_M}{\rho_P}, \quad \nu = \frac{\nu_P}{\rho_P}. \quad (2)$$

The approximate Eq. (1) becomes

$$\begin{aligned} \frac{dP_{0,n}}{d\tau} &= [(n+1)P_{0,n+1} - nP_{0,n}] \\ &\quad - (\mu^0 + \mu^1 n)P_{0,n} + \gamma P_{1,n}, \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{dP_{1,n}}{d\tau} &= \nu(P_{1,n-1} - P_{1,n}) + [(n+1)P_{1,n+1} - nP_{1,n}] \\ &\quad + (\mu^0 + \mu^1 n)P_{0,n} - \gamma P_{1,n}, \end{aligned} \quad (4)$$

where we have introduced the dimensionless time  $\tau = \rho_P t$  scale and the approximations  $P_{m,n} \sim 0, m \geq 2$  and  $(\mu^0 + \mu^1 n)P_{1,n}/(\gamma P_{2,n}) \sim 1$ , for all  $n$ . Since our simplifying assumption implies that the mRNA lifetime is short relative to that of the protein, we would expect that mRNA probabilities will be peaked around zero for  $\mu^0, \mu^1$  of the same order as  $\rho_M$ . This offers some justification for assuming that Eqs. (3) and (4) are valid for describing gene expression (Supplementary information).

Eqs. (3) and (4) have the same form as the master equation for a binary gene with the state  $(1, n)$  (or  $(0, n)$ ) as the active (or repressed) state of protein synthesis with rate  $\nu$  (or zero). The ‘‘on-off’’ switching rate is given in terms of the mRNA degradation rate  $\gamma$  and the ‘‘off-on’’ transition depends on the mRNA synthesis rates,  $\mu^0$  (for external regulation) or  $\mu^0 + \mu^1 n$  (self regulation). To write the solutions of the model presented at the Eqs. (3) and (4), we define constants  $(a, b, \theta)$  as follows:

$$a = \frac{\mu^0}{1 + \mu^1}, \quad b = \frac{\mu^0 + \gamma}{1 + \mu^1}, \quad \theta = \frac{1}{1 + \mu^1}, \quad (5)$$

where external regulation is recovered by setting  $\theta = 1$ . For simplicity we consider only the steady state solutions for Eqs. (3) and (4),  $P_{0,n}, P_{1,n}$ , and the probabilities for finding  $n$  proteins inside the cell,  $P_n = P_{0,n} + P_{1,n}$ , namely:

$$P_{0,n} = \frac{b-a}{Cb} \frac{\nu^n}{n!} \frac{(a)_n}{(1+b)_n} M(a+n, 1+b+n, -\nu\theta), \quad (6)$$

$$P_{1,n} = \frac{a}{Cb} \frac{\nu^n}{n!} \frac{(1+a)_n}{(1+b)_n} M(1+a+n, 1+b+n, -\nu\theta), \quad (7)$$

$$P_n = \frac{\nu^n}{Cn!} \frac{(a)_n}{(b)_n} M(a+n, b+n, -\nu\theta), \quad (8)$$

where  $M(a, b, z)$  denotes the Kummer M function [21] and the normalization constant

$$C = M(a, b, \nu(1-\theta)),$$

assures conservation of probability  $\sum_{n=0}^{\infty} (P_{0,n} + P_{1,n}) = 1$ . Note that for external regulation,  $C = M(a, b, 0) = 1$ .

The denominator in Eq. (5),  $1+\mu^1$ , should be interpreted as the total protein removal rate from the cytoplasm by degradation plus transcription stimulus. The constant  $a$  characterizes the rate of spontaneous (basal) mRNA synthesis relative to the protein removal rate and states a relation with the probability for finding one mRNA ( $p_1$ ), defined as  $\sum_{n=0}^{\infty} P_{1,n}$ , namely

$$p_1 = \frac{a}{Cb} M(a+1, b+1, \nu(1-\theta)), \quad (9)$$

and, for the external regulating gene,  $\theta = 1$ , it implies

$$p_1 = \frac{a}{b}.$$

Phenomenologically,  $b$  is a compound relation between the rate for a cycle of mRNA synthesis-degradation and the protein removal rate. Its role in determining the statistics of protein numbers is seen from the average number of protein molecules,  $\langle n \rangle = p_1 \nu$ , and the variance relative to the average (Fano factor),  $\sigma^2 / \langle n \rangle = (\langle n^2 \rangle - \langle n \rangle^2) / \langle n \rangle$ , given by

$$\begin{aligned} \frac{\sigma^2}{\langle n \rangle} &= 1 + \nu \frac{a+1}{b+1} \frac{M(a+2, b+2, \nu(1-\theta))}{M(a+1, b+1, \nu(1-\theta))} \\ &\quad - \nu \frac{a}{b} \frac{M(a+1, b+1, \nu(1-\theta))}{M(a, b, \nu(1-\theta))}. \end{aligned} \quad (10)$$

For the case where  $\theta = 1$ , we have

$$\frac{\sigma^2}{\langle n \rangle} = 1 + \frac{\nu}{b} \frac{1-a/b}{1+1/b}. \quad (11)$$

In the limit  $b \rightarrow +\infty$  (fast switching) and the parameters  $(a, \theta, \nu)$  are finite, Eq. (10) is equal to one and the distribution of protein is Poissonian.

To get some intuition into this system, consider the steady state (or equilibrium) of the Eqs. (3) and (4), when probabilities are fixed with time, but the variables  $(m, n)$  change in time with the probabilities given by Eqs. (6), (7) and (8).  $b$  gives the average time for the system to complete one switching cycle – *e.g.* from *off* to *on* and back to *off* again. Then the probabilities  $p_1$  and  $1 - p_1$  are the fractions of the total switching time that the system spends in the active and inactive states respectively.

The transition from the dynamic to the stationary regime, as noted previously [15, 17], has an approach to equilibrium characterized by two of the time scales of the model,  $\rho_P^{-1}$  and  $(\rho_P + \mu_M^1)^{-1}b^{-1}$ . The former is the typical lifetime of the protein, whereas the second one is related to the switching. For rapid protein degradation, compared to the switching, the steady state is achieved after the equilibrium of *on-off* transitions that are slow and can result in super Poisson stationary distributions (eventually, bi-modality occurs with each peak related to one state of the system). On the other hand, when protein degradation dominates, and there is fast switching, the distributions are uni-modal. In that case, the gene regulatory mechanism (*e.g.* if binary or constitutive) is indistinguishable by simple protein counting.

This reasoning suggests that bursting would occur for systems with a large value of  $\nu$  and  $p_1 \sim 0$ . Biologically, this would mean that the mRNA number is mostly zero during an entire switching cycle. For a  $p_1$  fraction of that cycle, there will be one mRNA that is rapidly translated (at rate  $\nu$ ) and thus several unitary increments in  $n$  take place during a very short time. This will appear to be a single near-instantaneous increase in protein number by more than one. A mechanism for a rapid increase in  $n$  from one mRNA is the binding of several ribosomes to the mRNA.

Mathematically, the negative binomial distribution is assumed to describe a random variable characterizing a bursting process. We can show (Supplementary information) that the negative binomial distribution is a particular case of the probabilities of the Eq. (8) at the limit of  $b, \nu \rightarrow \infty$  with their ratio

$$\delta = \frac{\nu}{b}, \quad (12)$$

kept finite, namely:

$$P_n \rightarrow \frac{(a)_n}{n!} \left( \frac{\delta}{1 + \delta\theta} \right)^n \left( \frac{1 + \delta(\theta - 1)}{1 + \delta\theta} \right)^a, \quad (13)$$

where  $(a)_n = a(a + 1) \dots (a + n - 1)$ ,  $(a)_0 = 1$ . For the self-regulating case, an approximate negative binomial distribution occurs for  $\theta \sim 1$ , which implies weak induction of mRNA synthesis by proteins, ( $\mu^1 \ll 1$ ). For the externally regulated gene,  $\theta = 1$ , and the probabilities at equation above become:

$$P_n \rightarrow \frac{(a)_n}{n!} \left( \frac{\delta}{1 + \delta} \right)^n \left( \frac{1}{1 + \delta\theta} \right)^a, \quad (14)$$

that is the negative binomial distribution [11].

We illustrate our results in FIG. 1 where the left hand column is for the external regulated gene and the right hand column is for the self regulating case. FIG. 1.A and 1.D are the steady state probability distributions as obtained from the expression of the binary (Eq. 8) and negative binomial (Eq. 14) models. For the parameter values we choose, inspection shows high agreement with the externally regulated while a slight difference appears for the self-regulating gene. The corresponding trajectories are obtained from the binary model, with the protein numbers shown in FIG. 1.B and 1.E. The *apparent* bursts appear explicitly at protein half-life time scale. However, an expansion of the time scale reveals that the protein synthesis is occurring one by one. Finally, the corresponding mRNA number dynamics is shown in FIG. 1.C and 1.F. As expected, it is switching between very short time intervals with one mRNA and long intervals with no mRNA.

Experimentally, the temporal resolution necessary for avoiding anomalous bursting detection should be of the order of the average time for translation of one protein,  $\sim 1/\nu_P$ . In what follows, we shall consider the system approached in Ref. [22] to show an example of a measurement of apparent bursting. In their work, the authors monitored the expression of the  $\beta$ -gal protein under the control of the *lac* promoter. They have detected the occurrence of burstings in protein numbers and measured the average bursting size to be of  $\sim 8$  proteins synthesized per burst. Our aim is to calculate the protein synthesis rate of the system reported in Ref. [22] using the average bursting size calculated by the authors. We shall employ our approximation at Eqs. (3) and (4) and estimate the necessary time resolution for avoiding the measurement of apparent bursting.

We start by setting the average bursting size in terms of the rates of the Eq. (1). In literature [11], the average bursting size is usually given by the parameter  $\delta$ , of the negative

binomial probability distribution at the Eqs. (13) and (14). Therefore, the protein synthesis rate,  $\nu_P$  at the Eq. (1), can be written as a function of  $\delta$  as follows:

$$\nu_P = \delta \rho_P \frac{\mu_M^0 + \rho_M}{\rho_P + \mu_M^1}, \quad (15)$$

that is deduced by combining the Eqs. (12), (5), and (2).

To proceed calculating  $\nu_P$ , we set  $\mu_M^1 = 0$  – and assume external regulation – since the *lac* promoter interacts with the Lac repressor protein that *is not* encoded in the *lac operon* genes. Then, the expression for calculating the protein synthesis rate at the Eq. (12) is reduced to  $\nu_P = \delta(\mu_M^0 + \rho_M)$ . The values of the remaining unknown constants, mRNA synthesis and degradation, are determined in terms of experimental measures.

The mRNA degradation rate of the  $\beta$ -gal mRNA –  $\rho_M$  – is taken to be  $\sim 0.1 \text{ min}^{-1}$  based on data reported in Ref. [23].

We use the data provided in Ref. [22] to estimate the mRNA synthesis rate ( $\mu_M^0$ ) at  $\sim 10^{-3} \text{ min}^{-1}$ . This number is achieved dividing the average frequency of bursting of 0.16 per cell cycle by the average period of a cell cycle, 145 min (both data from Ref. [22]). The bursts of proteins occur whenever one mRNA arrives at the cytoplasm. Therefore, we can convert the average bursting frequency into the average mRNA synthesis rate.

Based on the values of  $\delta$ ,  $\mu_M^0$ ,  $\mu_M^1$ , and  $\rho_M$  above we compute the protein synthesis rate as  $\nu_P \sim 1 \text{ min}^{-1}$ . Thus, the time scale for the synthesis of one protein is  $\sim 1 \text{ min}$  or,  $\ln 2/\nu_P \sim 0.5 \text{ min}$  in case of exponential growth of the protein population. Such time scales are smaller than the time resolution of protein detection of 4 min reported in Ref. [22]. Intuitively, one might expect an average of at least 4 protein synthesis during the temporal range of the experimental resolution. The detection of an increase greater than one in protein population should be interpreted as a bursting. However, under the conditions we are considering in this manuscript, an experimental time resolution of  $\sim 1 \text{ min}$  would re-establish a one-by-one protein population increase.

We also stress another aspect of experimental measurements: the sampling time. Usually, this is the time interval between two gatherings of cells from the cell culture. In the experiments presented in Ref. [22], the sampling time is 20 s. Note that it appears to be enough to detect the one-by-one protein increment. However, the time resolution for protein detection is obtained indirectly, from fluorescence measurements, which results a 4 min precision. Therefore, this is not enough to detect individual proteins.



It is worth to discuss the phenomenology of the burst like behavior reported in Ref. [22]. Theoretically, it relates to the value of the ratio  $\nu = \nu_P/\rho_P$  that is a large number,  $\gg 1$ . Then, whenever an mRNA appears in the cytoplasm, a plethora of proteins is fastly synthesized while their degradation is very slow. In a plot of the protein number versus time, that condition appears as a fast increment in protein population, followed by a plateau, if the experiment stands shorter than the protein half-life time.

In that case, a description of protein numbers inside the cell in terms of the negative binomial distribution is appropriate. However, it must be emphasized that the fitting of the measured histogram by a known probability distribution does not imply the occurrence of the stochastic process from where the distribution is derived. The predictive power of a model based on such approach might be lowered. In that sense, our discussion is an increment in the capability of interpreting experimental data. That is done establishing the time resolution necessary for experimental demonstration of bursting.

We establish two possible sceneries for the occurrence of real burst of proteins. It is assumed experimental time resolution of the order of  $\sim 1/\nu_P$  and the measurement of a greater than one instantaneous increment in protein population. The first scenery requires the existence of only one mRNA in the cytoplasm. A possible mechanism to underlie this effect is: multiple polipeptide chains present in cytoplasm that were translated individually and start their functional activity simultaneously. Note that that implies a delay between the translation process and the protein folding. In our second bursting scenery, there is abundant fast degradating mRNA's in the cytoplasm that can be translated synchronously in an one-by-one fashion. Hence, multiple proteins could be synthesized simultaneously in a time scale of the order of  $\sim 1/\nu_P$ .

Our results also suggest a picture of gene expression where the bursting (or burst-like) dynamics corresponds to one among other possible behaviors. Different regimes of gene expression are possible depending on the specific relations among the effective rates of the reactions participating of a gene network. For example, the model at Eq. (1) has a precise biological interpretation. Its approximation in terms of the binary model, Eqs. (3) and (4), shows the utility of the “on” and “off” model for the analysis of gene products synthesis. In terms of probability distributions one expect that, besides the negative binomial, the gene products should also satisfy the probabilities based on Eqs. (6), (7) and (8).

These probabilities satisfy the Eq. (1) when the probability to find more than one mRNA

in the cytoplasm is negligible. In this sense, one might provide further approximations to the solution of the Eq. (1) for negligible probability of detecting three, four mRNA's in the cytoplasm, respectively, in terms of ternary, quaternary models. Different insights in the workings of gene expression could also be provided by this kind of approach.

In this manuscript we are proposing a theoretical framework, based on the binary model to gene expression, that generalizes the negative binomial distribution for the description of the stochasticity in gene products number. The burst like behavior occurs in a well defined regime, when the ratio between synthesis and degradation rates is of the order of  $10^3$  and the synthesis of gene products very rare. As we predict from our model, measurements aiming to detect the one-by-one increments in gene products number must have temporal resolution of the order of the synthesis rate ( $1/\nu_P$ ), *e.g.* in conditions reported in Ref. [22] that would imply a time resolution of  $\sim 10 - 60$  s.

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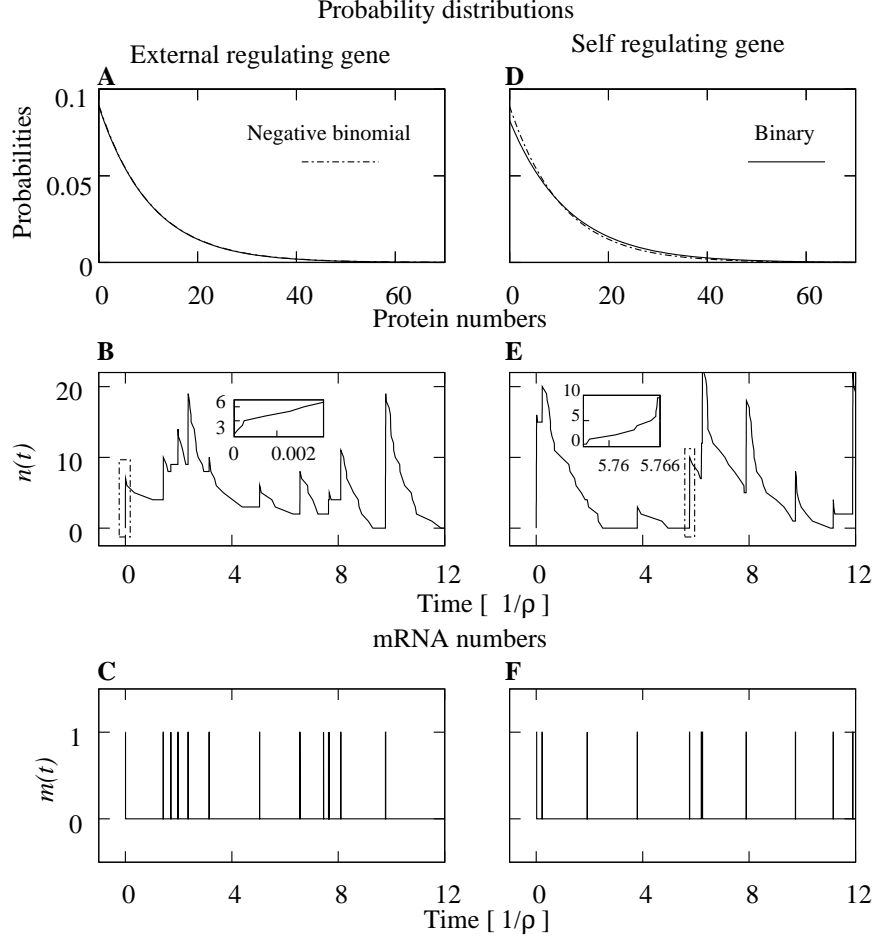


FIG. 1. **Steady state probability distributions and trajectories.** The parameter values for the externally regulated gene are  $\mu^0 = 1, \gamma = 99$  and for the self regulating gene  $\mu^0 = 100/99, \mu^1 = 1/99, \gamma = 100$ . The parameters  $(a, b, \theta, \nu)$ , for the binary models, are then  $(1, 100, 1, 1000)$  for the external and  $(1, 100, 0.99, 1000)$  for the self regulating probabilities. For the negative binomial distribution in FIGS. A and B we took  $a = 1$  and  $\delta = \nu/b = 10$ . In FIG. 1. B, E, C, and F, the trajectories for the binary model are shown in protein half-life time scale. An expansion of the time scale in the region bounded by the dashed lines in FIG. 1. B and E is shown in the insets where the time scales are magnified by  $10^3$ , the same as the ratio between protein synthesis and degradation rates. For the external and self regulating genes, the probabilities for finding one mRNA are, respectively, 0.01 and 0.011, the mean protein number  $\langle n \rangle$  are, 10 and 11.08, and finally the Fano factor's values are 10.8 and 11.08. The Fano factor value of the negative binomial distribution is 11, considering the same set of parameters, and is closely related to the one for the binary probabilities.

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