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Regularity and Singularities of Optimal Convex Shapes in the Plane

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Abstract

We focus here on the analysis of the regularity or singularity of solutions Ω_0 to shape optimization problems among convex planar sets, namely:

$$J(\Omega_0) = \min\{J(\Omega),\ \Omega \text{ convex},\ \Omega \in \mathcal{S}_{ad}\},$$

where $\mathcal{S}_{ad}$ is a set of 2-dimensional admissible shapes and $J : \mathcal{S}_{ad} \to \mathbb{R}$ is a shape functional.

Our main goal is to obtain qualitative properties of these optimal shapes by using first and second order optimality conditions, including the infinite dimensional Lagrange multiplier due to the convexity constraint. We prove two types of results:

i) under a suitable convexity property of the functional J, we prove that Ω_0 is a $W^{2,p}$-set, $p \in [1, \infty]$. This result applies, for instance, with $p = \infty$ when the shape functional can be written as $J(\Omega) = R(\Omega) + P(\Omega)$, where $R(\Omega) = F(|\Omega|, E_f(\Omega), \lambda_1(\Omega))$ involves the area $|\Omega|$, the Dirichlet energy $E_f(\Omega)$ or the first eigenvalue of the Laplace-Dirichlet operator $\lambda_1(\Omega)$, and $P(\Omega)$ is the perimeter of Ω,

ii) under a suitable concavity assumption on the functional J, we prove that Ω_0 is a polygon. This result applies, for instance, when the functional is now written as $J(\Omega) = R(\Omega) - P(\Omega)$, with the same notations as above.

Keywords: Shape optimization, convexity constraint, optimality conditions, regularity of free boundary.

1 Introduction

The goal of this paper is to develop general and systematic tools to prove the regularity or the singularity of optimal shapes in shape optimization problems *among convex planar sets*, namely problems like:

$$\min\{J(\Omega),\ \Omega \text{ convex},\ \Omega \in \mathcal{S}_{ad}\}, \tag{1}$$

where $\mathcal{S}_{ad}$ is a set of admissible shapes among subsets of $\mathbb{R}^2$ and $J : \mathcal{S}_{ad} \to \mathbb{R}$ is a shape functional. Our main objective is to obtain qualitative properties of optimal shapes by exploiting first and second order optimality conditions on (1) where *the convexity constraint* is included through appropriate infinite dimensional Lagrange multipliers.

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Our approach is analytic in the sense that convex sets are represented through adequate parametrizations and we work with the corresponding "shape functionals" defined on *spaces of functions*. In particular, we will use the classical polar coordinates representation of convex sets as follows:

$$\Omega_u := \left\{ (r,\theta) \in [0,\infty) \times \mathbb{R} \; ; \; r < \frac{1}{u(\theta)} \right\}, \tag{2}$$

where u is a positive and 2π-periodic function, often called "gauge function of Ω_u". It is well-known that

$$\Omega_u \text{ is convex } \iff u'' + u \geq 0. \tag{3}$$

Thus, Problem (1) may be transformed into the following:

$$\min \left\{ j(u) := J(\Omega_u),\ u'' + u \geq 0,\ u \in \mathcal{F}_{ad} \right\}, \tag{4}$$

where $\mathcal{F}_{ad}$ is a space of 2π-periodic functions which will be chosen appropriately to represent $\mathcal{S}_{ad}$ in (1).

We obtain two families of results depending on whether j is "of convex type" or "of concave type". In the first case, we prove regularity of the optimal shapes. In the second case, we prove that optimal shapes are polygons.

i) *"Optimal shapes are regular":* under a suitable convexity property on the "main part" of the functional j, we prove that any solution u_0 of (4) is $W^{2,p}$, which means that the curvature of $\partial\Omega_0 = \partial\Omega_{u_0}$ is an L^p function whereas it is a priori only a measure: see Theorems 2.4, 2.6 and Corollary 2.7. To that end, *we simply use the first optimality condition* for the problem (1).

   The functionals under consideration here are of the form $J(\Omega) = R(\Omega) + C(\Omega)$, where $r(u) := R(\Omega_u)$ has an L^p-derivative and C is like (6) below and satisfies a convexity condition. As a main example, we consider $R(\Omega) = F(|\Omega|, E_f(\Omega), \lambda_1(\Omega))$ which depends on the area $|\Omega|$, on the Dirichlet energy $E_f(\Omega)$ and/or on the first eigenvalue $\lambda_1(\Omega)$ of the Laplace operator on Ω (with Dirichlet boundary conditions), and $C(\Omega) = P(\Omega)$ is its perimeter: see Section 3.2. In this case, we actually prove that the optimal shape is $W^{2,\infty}$ which means that the curvature is bounded.

ii) *"Optimal shapes are polygons":* next, we prove that, under a suitable concavity assumption on the functional j, for any solution u_0 of (4), $u_0 + u_0''$ is (locally) a finite sum of Dirac masses, so that Ω_{u_0} is (locally) a polygon: see Theorems 2.9, 2.12 and Corollary 2.13. The proof of this result is *based on the second order optimality condition* for the problem (1). We apply this result to shape optimization problems where $J(\Omega) = R(\Omega) - P(\Omega)$ where $R(\Omega) = F(|\Omega|, E_f(\Omega), \lambda_1(\Omega))$ with the same notations as above, see Section 4.2. This application involves some sharp estimates on the second shape derivative of the energy which are interesting for themselves: see Section 4.3.2.

Our examples enlighten and exploit the fact that, in the context of shape optimization under convexity constraint, the perimeter is "stronger" than usual energies involving PDE, in terms of the influence on the qualitative properties of optimal shapes: if it appears in the energy as a positive term, it has a smoothing effect on optimal shapes, and on the opposite as a negative term, it leads to polygonal optimal shapes.

Dual parametrization: Since our results are stated for the analytic functionals (4), we may apply them to the dual parametrization of convex sets instead of the parametrization with the gauge function: each convex shape can also be associated to its support function $h_\Omega(\theta) = \max\{x \cdot e^{i\theta},\ x \in \Omega\}, \theta \in \mathbb{T}$ and (1) again leads to the problem:

$$\min \left\{ \widetilde{j}(h),\ h'' + h \geq 0,\ h \in \widetilde{\mathcal{F}_{ad}} \right\}, \tag{5}$$

where $\widetilde{j}(h) := J(\Omega^h)$, Ω^h being now the set whose support function is h, and $\widetilde{\mathcal{F}_{ad}}$ are all support functions of admissible shapes $\Omega \in \mathcal{S}_{ad}$. In this framework, if $\widetilde{j}$ satisfies the suitable convexity property, the regularity result (i) above holds for h_0 minimizer of (5). However, this regularity does not imply that the corresponding optimal shape $\Omega_0 := \Omega^{h_0}$ is regular, but it exactly means that Ω_0 is strictly convex: see Section 3.4.

The situation is more similar to the gauge representation when exploiting the results (ii). Indeed, when they apply, they imply that the optimal shape is polygonal as well: see Remark 2.14.

Situation with respect to previous results: The second family of results (ii) is an extension of previous results obtained in [14] by the two first authors for the specific following functionals of "local type":

$$J(\Omega_u) = \int_0^{2\pi} G\left(\theta, u(\theta), u'(\theta)\right) d\theta, \tag{6}$$

where $G = G(\theta, u, q) : \mathbb{T} \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is strictly concave in q. Among these functionals, we find for instance the area $|\Omega|$, the perimeter $P(\Omega)$ or also the famous Newton's problem of the body of minimal resistance as studied by T. Lachand-Robert and coauthors: see for example [5, 16] and see also [14, 7] for more examples arising in the operator theory. Actually, the techniques employed in [14], and here as well for (ii), are inspired from those introduced in [16]. The main novelty here in the results (ii) is that the functionals are not necessarily of the local form (6) and may include shape functionals defined through state functions which are solutions of partial differential equations (PDE). The "concavity condition" is then expressed in a functional way through the coercivity of the second derivative in an adequate functional space : see Theorem 2.9. In [2], a similar concavity phenomenon is used to get qualitative properties of minimizers in higher dimension, under assumptions about their regularity and convexity. We avoid here any assumption of this kind for the planar case.

The general optimality conditions including the infinite dimensional Lagrange multipliers were also provided (and exploited) in the same paper [14]. They are revisited here in an $W^{1,\infty}$-context which is better adapted to our more general functionals (see e.g. Proposition 3.1).

Similar arguments to those used here to obtain the first family of results (i) may also be found in [3] where optimality conditions with convexity constraints are developed in an N-dimensional setting. They are exploited for several examples in dimension 1 (or in radial situations) to obtain C^1-regularity of the optimal shapes. With our approach here, we are able to reach $W^{2,\infty}$-regularity and this is valid for a rather general family of functionals.

About a localization of the approach: Let us mention that our two families of results may be mixed in the same functional: indeed, as often the case, it may be that the required convexity property for (i) is valid on some part of the boundary of the optimal shape, while the concavity property for (ii) is valid on the other part. Then, the techniques developed here may be locally applied to each part and we can obtain at the same time smooth and polygonal pieces in the boundary. However, as one expects, it remains difficult to understand the portion of the boundary which remains at the intersection of these two parts. We refer to Section 5.1 for more details.

To end this introduction, let us say that many questions are of interest in shape optimization among convex sets. Here, we try to exploit as much as possible analytical tools to obtain precise qualitative results for optimal shapes among convex planar sets. But many questions are left open in higher dimensions. Among them, and besides the Newton's problem already mentioned, we can quote the famous Mahler conjecture about the minimization of the so-called Mahler-product $|K||K^\circ|$ among symmetric convex bodies in $\mathbb{R}^d$ (see [19]), which is of great interest in convex geometry and functional analysis, and the Pólya-Szegö conjecture about the minimization of the Newtonian capacity among convex bodies of $\mathbb{R}^3$ whose surface area is given (see for example [4] and reference therein).

This paper is structured as follows. In the following section we state our main results. In Section 3 we focus on the regularity result (i) and we apply it to some various examples. In Section 4, we deal with problems leading to

polygonal solutions (result (ii)), and we again consider in detail some classical examples. We conclude with some remarks and perspectives.

2 Main results

2.1 Notations and problems

We set $\mathbb{T} := [0, 2\pi)$. Throughout the paper, any function defined on $\mathbb{T}$ is considered as the restriction to $\mathbb{T}$ of a 2π-periodic function on $\mathbb{R}$. We define $W^{1,\infty}(\mathbb{T}) := \{u \in W^{1,\infty}_{loc}(\mathbb{R}), u \text{ is } 2\pi\text{-periodic}\}$, and similarly for any functional space. If $u \in W^{1,\infty}(\mathbb{T})$, we say that $u'' + u \geq 0$ if

$$\forall\, v \in W^{1,\infty}(\mathbb{T}) \text{ with } v \geq 0, \quad \int_{\mathbb{T}} \left(uv - u'v'\right) d\theta \geq 0.$$

In this case, $u'' + u$ is a nonnegative 2π-periodic measure on $\mathbb{R}$ and finite on $[0, 2\pi]$.

We denote by $\mathcal{S}_{ad}$ a class of open bounded sets in $\mathbb{R}^2$ (including constraints besides convexity). We will focus on two problems:

$$\min\{J(\Omega),\ \Omega \in \mathcal{S}_{ad},\ \Omega \text{ convex}\}, \tag{7}$$

$$\min\{J(\Omega),\ \Omega \in \mathcal{S}_{ad},\ \Omega \text{ convex}, M(\Omega) = M_0\}, \tag{8}$$

where $J : \mathcal{S}_{ad} \to \mathbb{R}$ is referred as the energy and $M : \mathcal{S}_{ad} \to \mathbb{R}^d$ is an extra constraint (M_0 given in $\in \mathbb{R}^d$).

In order to analyze the regularity of an optimal shape, we transform these problems into minimization problems in a functional analytic setting as follows: choosing an origin O and using parameterization (2), we define

$$\mathcal{F}_{ad} := \{u \in W^{1,\infty}(\mathbb{T}),\ \ \Omega_u \in \mathcal{S}_{ad}\}, \tag{9}$$

the set of admissible gauge functions, endowed with the $\|\cdot\|_{W^{1,\infty}(\mathbb{T})}$-norm, and we assume that this set can be written

$$\mathcal{F}_{ad} = \{u \in W^{1,\infty}(\mathbb{T}) \ / \ k_1 \leq u \leq k_2 \text{ and } u > 0\}, \tag{10}$$

for some functions $k_1, k_2 : \mathbb{T} \to \overline{\mathbb{R}_+}$ respectively upper- and lower-semicontinuous (see Remark 2.1 below for this assumption).

A simple calculus of the curvature shows that Ω_u is convex if and only if $u'' + u \geq 0$. Moreover, the support of the measure $u'' + u$ gives a parametrization of the "strictly convex part" of the boundary, and a Dirac mass in this measure correspond to a corner of the associated shape; we have for instance that Ω_u is a convex polygon if and only if $u'' + u$ is a finite sum of positive Dirac masses.

If Ω_0 is a solution of problem (7) (resp. (8)), then its gauge function u_0 is respectively solution of:

$$j(u_0) = \min\left\{j(u),\quad u'' + u \geq 0,\ u \in \mathcal{F}_{ad}\right\}, \tag{11}$$

$$\text{resp. } j(u_0) = \min\left\{j(u) \ / \ u \in \mathcal{F}_{ad}, u'' + u \geq 0 \text{ and } m(u) = M_0\right\}, \tag{12}$$

where $j : \mathcal{F}_{ad} \mapsto \mathbb{R}$, $j(u) := J(\Omega_u)$, and $m : \mathcal{F}_{ad} \to \mathbb{R}$, $m(u) = M(\Omega_u)$.

Our main goal in this paper is the analysis of the convexity constraint. Thus, given an optimal shape Ω_0, we focus on the part of $\partial\Omega_0$ which does not saturate the other constraints defined by $\mathcal{S}_{ad}$. We therefore define, for $u_0 \in \mathcal{F}_{ad}$ and $\Omega_0 = \Omega_{u_0}$,

$$\mathbb{T}_{in} \ := \ \mathbb{T}_{in}(\mathcal{F}_{ad}, u_0) = \{\theta \in \mathbb{T} \ / \ k_1(\theta) < u_0(\theta) < k_2(\theta)\}, \tag{13}$$

$$(\partial\Omega_0)_{in} \ := \ \left\{x \in \partial\Omega_0 \ / \ \exists\theta \in \mathbb{T}_{in}, x = \frac{1}{u_0(\theta)}(\cos\theta, \sin\theta)\right\}. \tag{14}$$

See Example 2.2 and Figure 1 for examples.

Remark 2.1 If k_1 or k_2 happened not to be semicontinuous, we could replace them by

$$\overline{k_1} = \inf\{k : \mathbb{T} \to \mathbb{R} \text{ continuous }, k \geq k_1\}, \quad \underline{k_2} = \sup\{k : \mathbb{T} \to \mathbb{R} \text{ continuous }, k \leq k_2\}$$

and we have

$$\{u \in W^{1,\infty}(\mathbb{T}) \ / \ k_1 \leq u \leq k_2\} = \{u \in W^{1,\infty}(\mathbb{T}) \ / \ \overline{k_1} \leq u \leq \underline{k_2}\}.$$

Therefore, the assumptions on k_1 and k_2 are not restrictive. Note that, thanks to the regularity of u_0, k_1, k_2, the set $\mathbb{T}_{in}$ is open.

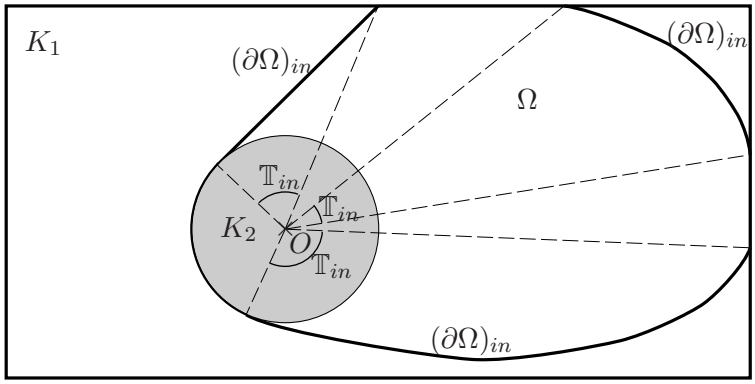

Figure 1: Inclusion constraints

Example 2.2 A frequent example for admissible shapes $\mathcal{S}_{ad}$ is:

$$\mathcal{S}_{ad} := \left\{\Omega \text{ bounded open set of } \mathbb{R}^2 \ / \ K_2 \subset \Omega \subset K_1\right\},$$

where K_2 and K_1 are two given bounded open sets. If for example K_1 and K_2 are starshaped with respect to a common point O, chosen as the origin, then

$$\mathcal{F}_{ad} = \{u \in W^{1,\infty}(\mathbb{T}) \ / \ k_1 \leq u \leq k_2\},$$

where k_1, k_2 are the gauge functions of K_2 and K_1 respectively. In that case, given a set $\Omega \in \mathcal{S}_{ad}$,

$$(\partial\Omega)_{in} = \partial\Omega \setminus (\partial K_1 \cup \partial K_2),$$

see Figure 1.

The analysis of the optimal shape around the set $\{\theta \ / \ u_0(\theta) \in \{k_1(\theta), k_2(\theta)\}\} = \mathbb{T} \setminus \mathbb{T}_{in}$, where the inclusion constraint is saturated, may require more efforts, see [14] for example. In this paper, we will not discuss this question.

Note that we can also consider the case $K_2 = \emptyset$ and/or $K_1 = \mathbb{R}^2$ with $k_1 = 0$ and $k_2 = +\infty$.

Example 2.3 With respect to the constraints m, M in (8), (12), a classical example is the area constraint:

$$m(u) := |\Omega_u| = A_0 \iff \int_{\mathbb{T}} \frac{1}{2u^2} d\theta = A_0,$$

where $|\Omega|$ denotes the area of Ω.

2.2 The main results

As explained in the introduction, Section 1, we will prove two types of results: they are described in the two following subsections.

2.2.1 "Optimal shapes are smooth"

First we consider the problem (7) and its associated analytical version (11). We assume that $J(\Omega) = R(\Omega) + C(\Omega)$, R satisfying some "regularity" assumption, and C being written like in (6), and satisfying a convexity like property. More precisely:

Theorem 2.4 *Let $u_0 > 0$ be an optimal solution of* (11) *with $\mathcal{F}_{ad}$ of the form* (10) *and*

$$j(u) := r(u) + \int_{\mathbb{T}} G\left(\theta, u(\theta), u'(\theta)\right) d\theta, \quad \forall u \in W^{1,\infty}(\Omega) \cap \{u > 0\}, \tag{15}$$

where r and G satisfy:

- *i)* *$r : W^{1,\infty}(\mathbb{T}) \to \mathbb{R}$ is C^1 around u_0 and $G : (\theta, u, q) \in \mathbb{T} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is C^2 around $\mathbb{T} \times u_0(\mathbb{T}) \times \mathrm{Conv}(u_0'(\mathbb{T}))$, where $\mathrm{Conv}(u_0'(\mathbb{T}))$ is the smallest (bounded) closed interval containing the values of the right- and left-derivatives $u_0'(\theta^+), u_0'(\theta^-), \theta \in \mathbb{T}$,*
- *ii)* *$r'(u_0) \in L^p(\mathbb{T})$ for some $p \in [1, \infty]$,*
- *iii)* *$G_{qq} > 0$ in $\mathbb{T} \times u_0(\mathbb{T}) \times \mathrm{Conv}(u_0'(\mathbb{T}))$.*

Then

$$u_0 \in W^{2,p}(\mathbb{T}_{in}), \text{ where } \mathbb{T}_{in} \text{ is defined in } (13).$$

See Section 3.1 for the proof, and Section 3.2 for explicit examples.

Remark 2.5 A C^1-regularity result has been proved for a similar problem with $r = 0$ in [3] with different boundary conditions, with a proof which is also based on first order optimality conditions. Here, for periodic boundary conditions (but this is not essential), we improve this result to the $C^{1,1}$-regularity, and generalize it to the case of non-trivial r, which is of great interest for our applications. Let us also refer to [6] for a higher dimensional result.

Let us remark that the same result is valid, with the same proof, if we only assume that $r'(u_0)$ is the sum of a function in $L^p(\mathbb{T})$ and of a nonpositive measure on $\mathbb{T}$. □

We can also get a similar result for the equality constrained problem (8) and the associated problem (12) as follows.

Theorem 2.6 *Let $u_0 > 0$ be an optimal solution of* (12) *with $j, \mathcal{F}_{ad}$ as in Theorem 2.4, and $m : W^{1,\infty} \to \mathbb{R}^d$ a C^1 function around u_0 with $m'(u_0) \in (L^p(\mathbb{T}))^d$ onto. Then*

$$u_0 \in W^{2,p}(\mathbb{T}_{in}).$$

See Section 3.1 for the proof, and Section 3.2 for explicit examples.

For a shape functional, using parametrization (2), Theorems 2.4 and 2.6 lead to the following.

Corollary 2.7 *Let $\mathcal{S}_{ad}$ be a class of open sets in $\mathbb{R}^2$ such that $\mathcal{F}_{ad} := \{u \ / \ \Omega_u \in \mathcal{S}_{ad}\}$ is of the form* (10) *(Ω_u is defined in* (2)*), and let $J : \mathcal{S}_{ad} \to \mathbb{R}$ be a shape functional:*
i) Let Ω_0 be an optimal shape for problem (7)*, and assume that $J = R + C$ with:*

$$\forall u \in \mathcal{F}_{ad}, \quad R(\Omega_u) = r(u) \text{ and } C(\Omega_u) = \int_{\mathbb{T}} G(\theta, u(\theta), u'(\theta))d\theta,$$

where r and G satisfy assumptions of Theorem 2.4 for some $p \in [1, \infty]$. Then $(\partial\Omega_0)_{in}$, as defined in (14)*, is $\mathcal{C}^1$ and its curvature is in $L^p((\partial\Omega_0)_{in})$.*
ii) A similar results holds for the problem (8)*, if $m(u) = M(\Omega_u)$ satisfies the hypotheses in Theorem 2.6.*

Remark 2.8 The results of this section are in an abstract analytical context, and do not depend on the characterization of the domain. Therefore, one could consider the classical characterization of a convex body with its support function instead of the gauge function. In Section 3.4, we give a geometrical interpretation of similar results associated to this parametrization.

2.2.2 "Optimal shapes are polygons"

Our second result is a generalization of Theorem 2.1 from [14]. We give a sufficient condition on the shape functional J so that any solution of (1) be a polygon. In [14], the first two authors only consider shape functionals of local type like (6). The following results deal with non-local functionals, which allow a much larger class of applications, including shape functionals depending on a PDE.

Theorem 2.9 *Let $u_0 > 0$ be a solution for* (11) *with $\mathcal{F}_{ad}$ of the form* (10)*, and assume that $j : W^{1,\infty}(\mathbb{T}) \to \mathbb{R}$ is $\mathcal{C}^2$ around u_0 and satisfies (see Section 4.1 for definitions of H^s-(semi-)norms):*

$$\begin{aligned} &\exists s \in [0,1), \alpha > 0, \beta, \gamma \in [0, \infty), \text{ such that} \\ &\forall v \in W^{1,\infty}(\mathbb{T}), \quad j''(u_0)(v,v) \leq -\alpha |v|^2_{H^1(\mathbb{T})} + \gamma |v|_{H^1(\mathbb{T})} \|v\|_{H^s(\mathbb{T})} + \beta \|v\|^2_{H^s(\mathbb{T})}. \end{aligned} \tag{16}$$

If I is a connected component of $\mathbb{T}_{in}$ (defined in (13)*), then*

$$u_0'' + u_0 \text{ is a finite sum of Dirac masses in } I.$$

See Section 4.1 for a proof and Section 4.2 for explicit examples.

Remark 2.10 We can even get an estimate of the number of Dirac masses in terms of α, β, γ, see Remark 4.2.

Remark 2.11 Theorem 2.9 remains true if (16) holds only for any v such that (denoting $\mu = u_0'' + u_0$):

$$\exists\, \varphi \in L^\infty(\mathbb{T}, \mu) \text{ with } v'' + v = \varphi\, \mu.$$

Indeed, the proof of Theorem 2.9 uses only this kind of perturbations v which preserve the convexity of the shape. □

As in Section 2.2.1, we can also handle the problem with an equality constraint as follows.

Theorem 2.12 *Let $u_0 > 0$ be any optimal solution of* (12) *with $j, \mathcal{F}_{ad}$ as in Theorem 2.9, and the new assumptions:*

$$\begin{aligned} &j'(u_0) \in \left(C^0(\mathbb{T})\right)', \quad \text{and} \quad m : W^{1,\infty} \to \mathbb{R}^d \text{ is } C^2 \text{ around } u_0, \\ &\qquad\qquad m'(u_0) \in \left(C^0(\mathbb{T})'\right)^d \quad \text{is onto,} \quad \|m''(u_0)(v,v)\| \leq \beta' \|v\|^2_{H^s(\mathbb{T})}, \quad \text{for some } \beta' \in \mathbb{R}. \end{aligned}$$

Then, if I is a connected component of $\mathbb{T}_{in}$ (defined in (13)*),*

$$u_0'' + u_0 \text{ is a finite sum of Dirac masses in } I.$$

See Section 4.1 for the proof.

Again, using the parametrization (2), we get the following result.

Corollary 2.13 *Let $\mathcal{S}_{ad}$ be a class of open sets in $\mathbb{R}^2$ such that $\mathcal{F}_{ad} := \{u \ / \ \Omega_u \in \mathcal{S}_{ad}\}$ is of the form* (10)*, Ω_0 be an optimal shape for the problem* (7) *(or* (8) *for the constrained problem), and assume that $j : u \in \mathcal{F}_{ad} \mapsto J(\Omega_u)$ satisfies assumptions of Theorem 2.9 (and $m : u \in \mathcal{F}_{ad} \mapsto M(\Omega_u)$ satisfies assumption in Theorem 2.12 in the case of the constrained problem). Then:*

$$\textit{each connected component of } (\partial\Omega_0)_{in} \textit{ is polygonal}.$$

Remark 2.14 When one uses the parametrization of convex sets by the gauge function u, Ω_u is a polygon if and only if $u'' + u$ is a sum of Dirac masses. When parametrizing Ω with the support function as in Section 3.4, one has the same characterization. Therefore, the results of this section hold if we work with the optimization problems as in Section 3.4.

3 Shape functionals containing a local-convex term

In this section, we give the proof of the results in Section 2.2.1, that is to say regularity results for solutions of (11) or (12). Using the parametrization (2), since the regularity of a shape and of its gauge functions are the same, we consider several applications of regularity for optimal shapes to classical examples of energies. We conclude with a few remarks about the application of our results when we use another parametrization of convex bodies, namely the support function. In that case, we get the regularity of the support function, which does not imply the regularity of the corresponding shape, but only the fact that this one is strictly convex.

3.1 Proof of Theorem 2.4 and 2.6

First order optimality condition:

A first optimality condition for the problem (11) is stated in [14, Proposition 3.1, 3.2] when j is defined and differentiable in the Sobolev Hilbert space $H^1(\mathbb{T})$. We give here an adaptation to state this result in $W^{1,\infty}$ instead (which is important for our applications involving a PDE, since the shape functionals are known to be differentiable for Lipschitz deformations only).

Proposition 3.1 *Let $u_0 > 0$ be a solution of (11) with $j : W^{1,\infty}(\mathbb{T}) \to \mathbb{R}$ of class C^1 and such that $j'(u_0) \in C^0(\mathbb{T})'$. Then there exists $\zeta_0 \in W^{1,\infty}(\mathbb{T})$, such that*

$$\begin{cases} \zeta_0 \geq 0 \text{ on } \mathbb{T}, \quad \zeta_0 = 0 \text{ on } \mathrm{Supp}(u_0'' + u_0), \quad \text{and} \\ \forall\, v \in W^{1,\infty}(\mathbb{T}_{in}), \quad j'(u_0)v = \langle \zeta_0 + \zeta_0'', v\rangle_{(W^{1,\infty})' \times W^{1,\infty}} := \displaystyle\int_{\mathbb{T}} \zeta_0 v - \zeta_0' v'. \end{cases} \tag{17}$$

Remark 3.2 Without any assumption on $j'(u_0)$, we would a priori get a Lagrange multiplier $\zeta_0 \in L^\infty(\mathbb{T})$ (see the proof below). The non-continuity of ζ_0 may lead to some difficulties, especially to state that $\zeta_0 = 0$ on $\mathrm{Supp}(u_0'' + u_0)$. Though a restriction, the assumption $j'(u_0) \in C^0(\mathbb{T})'$ will be satisfied in all of our applications.

Proof. We set

$$g : v \in W^{1,\infty} \mapsto v'' + v \in (W^{1,\infty})' \text{ in the sense that } \langle v'' + v, \varphi\rangle_{W^{1,\infty\prime} \times W^{1,\infty}} = \int_{\mathbb{T}} v\varphi - v'\varphi',$$

and we consider $Y := \mathrm{Im}(g) = \{f \in W^{1,\infty}(\mathbb{T})',\ \langle f, \cos\rangle_{(W^{1,\infty})'\times W^{1,\infty}} = \langle f, \sin\rangle_{(W^{1,\infty})'\times W^{1,\infty}} = 0\}$, which is a closed subspace of $(W^{1,\infty}(\mathbb{T}))'$.

Applying the same strategy as in [14], one gets $l_0 \in Y'$ such that $l_0(g(u_0)) = 0$ and

$$\forall f \in Y,\ f \geq 0 \Rightarrow l_0(f) \geq 0, \quad \text{and} \quad \forall v \in W^{1,\infty},\ \langle j'(u_0), v\rangle_{(W^{1,\infty})'\times W^{1,\infty}} = \langle l_0, v'' + v\rangle_{Y'\times Y}.$$

We restrict ourselves to $v \in \mathcal{D}(\mathbb{T}) := C^\infty(\mathbb{T})$, and consider

$$\zeta_0 : f \in \mathcal{D}(\mathbb{T}) \cap Y \mapsto \langle \zeta_0, f\rangle_{\mathcal{D}'\times\mathcal{D}} := \langle l_0, f\rangle_{Y'\times Y}.$$

Our aim is to prove that ζ_0 can be extended to a continuous linear form on $L^1(\mathbb{T})$. First, for $f \in \mathcal{D}(\mathbb{T}) \cap Y = \{f \in \mathcal{D}(\mathbb{T}),\ \int_{\mathbb{T}} f\sin = \int_T f\cos = 0\}$ we choose the unique $v \in W^{2,1}(\mathbb{T})$ such that $\{\int_{\mathbb{T}} v\sin = \int_{\mathbb{T}} v\cos = 0\}$ and $v'' + v = f$ in $\mathbb{T}$. Then there exists $C < \infty$ independant of v or f such that

$$\|v\|_{W^{1,\infty}(\mathbb{T})} \leq C\|f\|_{L^1(\mathbb{T})}. \tag{18}$$

Indeed, we first get an L^∞-estimate using Fourier series: if $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n$ with $e_n(\theta) = e^{in\theta}$ and $\widehat{f}(n) = \int_{\mathbb{T}} f(\theta)e^{-in\theta}\frac{d\theta}{2\pi}$, then $v = \sum_{|n|\neq 1} \frac{1}{1-n^2}\widehat{f}(n)e_n$, and therefore

$$\|v\|_{L^\infty} \leq \left(\sum_{|n|\neq 1} \frac{1}{|1-n^2|}\right) \max_n |\widehat{f}(n)| \leq C\|f\|_{L^1},$$

with $C < \infty$. Then we get a $W^{1,\infty}$-estimate by choosing θ_0 such that $v'(\theta_0) = 0$ (which is always possible, thanks to regularity and periodicity of v), and getting from $v'' + v = f$ that

$$|v'(\theta)| = \left|-\int_{\theta_0}^{\theta} (f(s) - v(s))ds\right| \leq 2\pi\left(\|v\|_{L^\infty} + \|f\|_{L^1}\right),$$

which concludes the proof of the estimate (18).

Therefore, we can write (C may define different universal constants)

$$\forall f \in Y \cap \mathcal{D}(\mathbb{T}), \quad |\langle \zeta_0, f\rangle_{\mathcal{D}'\times\mathcal{D}}| = |\langle l, v'' + v\rangle_{Y'\times Y}| = |\langle j'(u_0), v\rangle_{(W^{1,\infty})'\times W^{1,\infty}}| \leq C\|v\|_{W^{1,\infty}} \leq C\|f\|_{L^1}. \tag{19}$$

We now extend ζ_0 on $\mathcal{D}(\mathbb{T})$ by

$$\forall f \in \mathcal{D}(\mathbb{T}), \quad \langle \zeta_0, f\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle \zeta_0, f - \widehat{f}(1)e_1 - \widehat{f}(-1)e_{-1}\rangle_{\mathcal{D}'\times\mathcal{D}}.$$

Then, applying (19) to $f - \widehat{f}(1)e_1 - \widehat{f}(-1)e_{-1}$, we get

$$\forall f \in \mathcal{D}(\mathbb{T}), \quad |\langle \zeta_0, f\rangle_{\mathcal{D}'\times\mathcal{D}}| \leq C\|f - \widehat{f}(1)e_1 - \widehat{f}(-1)e_{-1}\|_{L^1} \leq C\|f\|_{L^1},$$

and therefore by density, we extend ζ_0 to a continuous linear form in L^1, which can be identified with $\zeta_0 \in L^\infty$. Moreover, in the sense of distributions:

$$\langle \zeta_0, v'' + v\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle j'(u_0), v\rangle_{\mathcal{D}'\times\mathcal{D}}, \text{ that is to say } \zeta_0'' + \zeta_0 = j'(u_0).$$

From the hypothesis for $j'(u_0)$ it follows $\zeta_0'' + \zeta_0 \in (C^0(\mathbb{T}))'$ which implies $\zeta_0 \in W^{1,\infty}(\mathbb{T})$. Using the continuity of ζ_0 and the fact $j'(u_0)(u_0) = 0$ we get $\int_{\mathbb{T}} \zeta_0 d(u_0'' + u_0) = 0$ by a density argument. Therefore, the rest of the proof stays as in [14], namely we prove that we can add a combination of $\cos$ and $\sin$ to ζ_0 so that $\zeta_0 \geq 0$. □

Proof of Theorem 2.4.

Applying the previous proposition, and using the hypotheses on the functional j, we get:

$$\forall v \in C_c^\infty(\mathbb{T}_{in}),\ \ j'(u_0)v = r'(u)v + \int_{\mathbb{T}} G_u(\theta,u_0,u_0')v + G_q(\theta,u_0,u_0')v' \ \ = \ \ \langle \zeta_0 + \zeta_0'', v\rangle_{(W^{1,\infty}(\mathbb{T}))'\times W^{1,\infty}(\mathbb{T})}.$$

To integrate by part in this formula, since u_0' is only in $BV(\mathbb{T})$, we may look in [20] (see also [1]) to get:

$$r'(u_0) + G_u(\theta,u_0,u_0') - G_{\theta q}(\theta,u_0,u_0') - G_{uq}(\theta,u_0,u_0')u_0' - u_0''\widetilde{G_{qq}}(\theta,u_0,u_0') = \zeta_0 + \zeta_0'' \text{ in } \mathcal{D}'(\mathbb{T}_{in}). \quad (20)$$

where $\widetilde{G_{qq}}(\theta,u_0,u_0') = \int_0^1 G_{qq}(\theta,u_0(\theta),(1-t)u_0'(\theta^+)+tu_0'(\theta^-))dt$. For simplicity, we will drop the indication of the dependence in (θ,u_0,u_0') and write more simply

$$r'(u_0) + G_u - G_{\theta q} - G_{uq}u_0' - u_0''\widetilde{G_{qq}} = \zeta_0 + \zeta_0'' \text{ in } \mathcal{D}'(\mathbb{T}_{in}). \quad (21)$$

Equality (21) implies that ζ_0'' is a Radon measure, and also that the singular parts of the measures in the two sides of (21) are equal. To study the sign of these measures, we will use the following lemma.

Lemma 3.3 *The measure ζ_0'' satisfies: $\zeta_0'' \geq 0$ on $[\zeta_0 = 0]$.*

Proof of Lemma 3.3. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}), \varphi \geq 0$ and let $p_n : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$\forall r \in [0,1/n],\ p_n(r) = 1 - nr;\ \forall r \in [1/n,+\infty),\ p_n(r) = 0.$$

Recall that $\zeta_0 \in W^{1,\infty}(\mathbb{T})$ and $\zeta_0 \geq 0$. Then

$$\int \varphi p_n(\zeta_0) d(\zeta_0'') = -\left(\int \zeta_0'\varphi' p_n(\zeta_0) + \varphi p_n'(\zeta_0){\zeta_0'}^2\right) \geq -\int \zeta_0'\varphi' p_n(\zeta_0).$$

Letting n tend to $+\infty$ leads to

$$\int_{[\zeta_0=0]} \varphi d(\zeta_0'') \geq -\int_{[\zeta_0=0]} \zeta_0'\varphi' = 0,$$

the last integral being equal to 0 thanks to the known property $\zeta_0' = 0$ a.e. on $[\zeta_0 = 0]$. □

End of the proof of Theorem 2.4:

Denote $K := \mathrm{Supp}(u_0'' + u_0)$. Recall that $\zeta_0 = 0$ on K by Proposition 3.1. By Lemma 3.3, $\zeta_0'' \geq 0$ on K. Let $u_0'' = \mu_{ac} + \mu_s$ and $\zeta_0'' = n_{ac} + n_s$ be the Radon-Nikodym decompositions of the measures u_0'', ζ_0'' in their absolutely continuous and singular parts. Note that: $[u_0'' + u_0 \geq 0 \Rightarrow \mu_s \geq 0]$ and $n_s \geq 0$ on K.

Identifying the singular parts in the identity (21), and using that $r'(u_0), G_u, G_{\theta q}, G_{uq}u_0', u_0\widetilde{G_{qq}}$ are at least L^p-functions, we are led to $-\mu_s\widetilde{G_{qq}} = n_s$ in $\mathbb{T}_{in}$. Since $\widetilde{G_{qq}} > 0, \mu_s \geq 0, n_s \geq 0$ on $K \supset \mathrm{Supp}(\mu_s)$, we deduce $\mu_s = 0 = n_s$ in $\mathbb{T}_{in}$. Thus, $u_0 \in W^{2,1}(\mathbb{T}_{in})$ and u_0' is absolutely continuous on $\mathbb{T}_{in}$. In particular, $\widetilde{G_{qq}} = G_{qq}$ on $\mathbb{T}_{in}$.

We can now obtain higher regularity, using again the multiplier ζ_0''. Indeed, on one hand, we deduce from Lemma 3.3, from (21) and from the inequality $-u_0''G_{qq} \leq u_0G_{qq}$, that, on the set $\mathbb{T}_{in} \cap K$

$$0 \leq \zeta_0'' \leq r'(u_0) + G_u - G_{\theta q} - G_{uq}u_0' + u_0G_{qq} \in L^p(\mathbb{T}).$$

Thus, $\zeta_0'' \in L^p(\mathbb{T}_{in} \cap K)$. Going back to (21) and using that $\widetilde{G_{qq}} = G_{qq}$ is bounded from below on the compact set $\mathbb{T} \times u_0(\mathbb{T}) \times Conv(u_0'(\mathbb{T}))$, we deduce $u_0'' \in L^p(\mathbb{T}_{in} \cap K)$.

On the other hand, in the open set $\mathbb{T}_{in} \setminus K$, we have $u_0'' + u_0 = 0$ so that $u_0'' \in L^\infty(\mathbb{T}_{in} \setminus K)$. As a conclusion $u_0'' \in L^p(\mathbb{T}_{in})$. □

Proof of Theorem 2.6.

Optimality conditions are written with the Lagrangian (since $m'(u_0)$ is onto, see also [14, Proposition 2.3.3]):

$$\forall v \in C_c^\infty(\mathbb{T}_{in}), \quad j'(u_0)v + \mu \cdot (m'(u_0)v) = \langle \zeta_0 + \zeta_0'', v\rangle_{(W^{1,\infty})' \times W^{1,\infty}},$$

for some $\mu \in \mathbb{R}^d$. The regularity of $m'(u_0)$ implies that the strategy used in the proof of Theorem 2.4 remains valid. □

3.2 Examples

In this section, we apply Corollary 2.7 to a number of classical energy functionals. For the proof of the differentiability of the shape functionals see Section 3.3. We start by reminding some classical PDE functionals that we use in our examples.

Dirichlet energy - Torsional rigidity

For Ω an open bounded set in $\mathbb{R}^2$, we consider the solution of the following PDE, in a variational sense:

$$U_\Omega \in H_0^1(\Omega), \quad -\Delta U_\Omega = f \ \text{ in } \Omega, \tag{22}$$

and we define the Dirichlet energy of Ω by

$$\begin{aligned} E_f(\Omega) &:= \int_\Omega \left(\frac{1}{2}|\nabla U_\Omega|^2 - f\, U_\Omega\right) = \min\left\{\int_\Omega \left(\frac{1}{2}|\nabla U|^2 - fU\right),\ U \in H_0^1(\Omega)\right\} \\ &= -\frac{1}{2}\int_\Omega |\nabla U_\Omega|^2 = -\frac{1}{2}\int_\Omega U_\Omega f. \end{aligned}$$

About the regularity of the state function, we are going to use the following classical result (see [13], [9]).

Lemma 3.4 *Let Ω be convex, $f \in L^p_{loc}(\mathbb{R}^2)$ with $p > 2$, and U_Ω be the solution of (22). Then $U_\Omega \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$.*

Remark 3.5 When $f \equiv 1$, the Dirichlet energy is linked to the so-called torsional rigidity $T(\Omega)$, with the formula $T(\Omega) = -2E_1(\Omega)$.

First Dirichlet-eigenvalue of the Laplace operator

We define $\lambda_1(\Omega)$ as the first eigenvalue for the Laplacian with Dirichlet's boundary conditions on $\partial\Omega$. It is well-known that, if we define U_Ω as a solution of the following minimization problem,

$$\lambda_1(\Omega) := \int_\Omega |\nabla U_\Omega|^2 = \min\left\{\int_\Omega |\nabla U|^2,\ U \in H_0^1(\Omega),\ \int_\Omega U^2 = 1\right\},$$

then U_Ω is (up to the sign) the positive first eigenfunction of $-\Delta$ in Ω:

$$U_\Omega \in H_0^1(\Omega), \quad -\Delta U_\Omega = \lambda_1(\Omega) U_\Omega, \quad \int_\Omega U_\Omega^2 = 1.$$

Again, like in Lemma 3.4, if Ω is convex then $U_\Omega \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and $U_\Omega > 0$ in Ω.

We are now in position to state some applications of Corollary 2.7:

Example 3.6 (Penalization by perimeter) One can study

$$\min\{J(\Omega) := F\left(|\Omega|, E_f(\Omega), \lambda_1(\Omega)\right) + P(\Omega) \ / \ \Omega \text{ convex}, D_1 \subset \Omega \subset D_2\} \tag{23}$$

where $F : (0, +\infty) \times (-\infty, 0) \times (0, \infty) \to \mathbb{R}$ is C^1, $f \in H^1_{loc}(\mathbb{R}^2)$, D_1, D_2 are bounded open sets, $E_f(\Omega)$ is the Dirichlet energy and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ defined as above.

Proposition 3.7 *If Ω_0 is an optimal set for the problem* (23)*, then the free boundary $\partial\Omega_0 \cap (D_2 \setminus \overline{D_1})$ is $C^{1,1}$ (or equivalently $W^{2,\infty}$), that is to say $\partial\Omega_0 \cap (D_2 \setminus \overline{D_1})$ has a bounded curvature.*

The proof is a simple consequence of Section 3.3, which asserts that $R(\Omega) = F(|\Omega|, E_f(\Omega), \lambda_1(\Omega))$ and $C(\Omega) = P(\Omega)$ satisfy the assumptions in Corollary 2.7 with $p = \infty$.

Note that in Proposition 3.7 we could also add a dependence of F in the capacity of Ω or in any shape functional which is shape differentiable and whose shape derivative can be represented as a function of $L^\infty(\partial\Omega)$ when Ω is convex.

Remark 3.8 The constraints $D_1 \subset \Omega \subset D_2$ helps existence for the problem (23). Of course, if one can prove existence of an optimal shape without these constraints (mainly, one need to prove that a minimizing sequence remains bounded and does not converge to a segment), the result of Proposition 3.7 remains a fortiori true for the whole boundary of the optimal shape, i.e. $\partial\Omega_0$ is $C^{1,1}$.

Example 3.9 (Volume constraint and Perimeter penalization) We can also consider a similar problem with a volume constraint:

$$\min\{J(\Omega) := F(E_f(\Omega), \lambda_1(\Omega)) + P(\Omega) \ / \ \Omega \text{ convex, and } |\Omega| = V_0\}, \quad V_0 \in (0, +\infty).$$

In this case, the first optimality condition will be similar to the one for the problem (23) with $F(E_f(\Omega), \lambda_1(\Omega)) + \mu|\Omega| + P(\Omega)$ where μ is a Lagrange multiplier for the constraint $|\Omega| = V_0$. Theorem 2.6 applies and one gets globally the same regularity result (but global) as in Proposition 3.7 on any optimal shape.

Example 3.10 (Perimeter constraint) If one considers again a problem with a perimeter constraint,

$$\min\{J(\Omega) := F(|\Omega|, E_f(\Omega), \lambda_1(\Omega)) \ / \ \Omega \text{ convex, and } P(\Omega) = P_0\} \tag{24}$$

where $P_0 \in (0, +\infty)$, one needs to be more careful. In this case, the first optimality condition will be similar to the one for the problem (23), with $F(|\Omega|, E_f(\Omega), \lambda_1(\Omega)) + \mu P(\Omega)$, where μ is a Lagrange multiplier for the constraint $P(\Omega) = P_0$. Therefore if we are able to prove $\mu > 0$ then we can apply the same strategy as in Theorem 2.4, and we therefore get the same regularity result as in Proposition 3.7. However, if $\mu < 0$, we refer to Example 4.9.

Example 3.11 In a more abstract context, one can consider

$$\min\{J(\Omega) - \alpha|\Omega| + P(\Omega) \ / \ \Omega \text{ convex } \subset D\}, \tag{25}$$

where J is a shape differentiable functional, increasing with respect to the domain inclusion, D is an open set, and $\alpha > 0$ (if $\alpha = 0$, the empty set is clearly solution of the problem). Again, we get that $\partial\Omega_0 \cap D$ has a locally bounded curvature. Indeed, the derivative of $j(u) := J(\Omega_u)$ is a nonpositive measure, thanks to the monotonicity of J (see [15]), and we apply Theorem 2.4 combined with the end of Remark 2.5.

3.3 Computation and estimate of first order shape derivatives

In this section we will prove the differentiability of the shape functionals involved in the examples of Section 3.2, which are needed in Proposition 3.7.

3.3.1 Volume and perimeter

About geometrical functionals, it is easy to write the area and the perimeter as functional of u, namely

$$a(u) := |\Omega_u| = \int_{\mathbb{T}} \frac{1}{2u^2} d\theta, \qquad p(u) := P(\Omega_u) = \int_{\mathbb{T}} \frac{\sqrt{u^2+u'^2}}{u^2} d\theta, \quad u \in W^{1,\infty}(\mathbb{T}) \cap \{u > 0\}. \tag{26}$$

Note that $p(u) = \int_{\mathbb{T}} G(\theta, u(\theta), u'(\theta)) d\theta$ with $G(\theta, u, q) = \frac{\sqrt{u^2+q^2}}{u^2}$ and one can easily check that $G_{qq} = \frac{1}{(u^2+q^2)^{3/2}} > 0$.

3.3.2 Dirichlet Energy - Torsional rigidity

We focus our analysis around a convex open set Ω_0 with parametrization $u_0 > 0$. For $\|u - u_0\|_{W^{1,\infty}(\mathbb{T})}$ small, consider

$$\begin{array}{rccc} e_f: & W^{1,\infty}(\mathbb{T}) \cap \{u > 0\} & \to & \mathbb{R}, \\ & u & \mapsto & E_f(\Omega_u). \end{array}$$

In order to study the differentiability of e_f near u_0, we use the classical framework of shape derivatives. As usual, we need to work with an extension operator: the deformation $\partial\Omega_0$ to $\partial\Omega_u$ allows to define the vector field $\xi(u) : \partial\Omega_0 \to \mathbb{R}^2$ such that $\partial\Omega_u = (Id + \xi(u))(\partial\Omega_0)$. We will consider an extension to $\mathbb{R}^2$ of this transformation, since we need to study the differentiability of $u \to \hat{U}_u := U_{\Omega_u} \circ (Id + \xi(u)) \in H_0^1(\Omega_0)$, where $U_u := U_{\Omega_u}$ (see [10] for example).

If we consider a smooth extension operator $\xi : W^{1,\infty}(\mathbb{T}) \to W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$, we have $(Id + \xi(u))(\partial\Omega_0) = \partial\Omega_u$ if

$$\xi(u)\left(\frac{1}{u_0(\theta)}, \theta\right) = \left(\frac{1}{u(\theta)} - \frac{1}{u_0(\theta)}\right) e^{i\theta}, \forall \theta \in \mathbb{T}, \tag{27}$$

where $(\frac{1}{u_0}(\theta), \theta)$ are polar coordinates (for simplicity, we will often write u_0, u or ξ instead of $u_0(\theta)$, $u(\theta)$ or $\xi(u)(r, \theta)$).

Remark 3.12 The transformation $\xi(u)$ can be extended to $\mathbb{R}^2$ in different ways. The easiest way is to take

$$\xi(u)(r, \theta) = \left(\frac{1}{u(\theta)} - \frac{1}{u_0(\theta)}\right) e^{i\theta} \eta(r, \theta) \quad \text{in} \quad \mathbb{R}^2, \tag{28}$$

where $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta = 0$ in a neighborhood of the origin and $\eta = 1$ in a neighborhood of $\partial\Omega_0$.

This (polar) extension of $\xi(u)$ is such that $\xi \in C^\infty(W^{1,\infty}(\mathbb{T}); W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2))$ near u_0, and is sufficient for the results of this section. More work will be needed for the second order shape derivatives, see Section 4.3.2.

Let us point out that if ξ is C^2 in a neighborhood of u_0 and satisfies (27), then

$$\forall v \in W^{1,\infty}(\mathbb{T}): \quad \xi'(u_0)(v) = -\frac{v}{u_0^2} e^{i\theta}, \quad \xi''(u_0)(v, v) = 2\frac{v^2}{u_0^3} e^{i\theta} \quad \text{on } \partial\Omega_0. \tag{29}$$

Note also that the method used in the proof of Lemma 3.14, which is needed in the proof of Proposition 3.13, allows to say that the method a priori fails if we consider an extension operator $\xi : H^1(\mathbb{T}) \to H^1(\mathbb{R}^2; \mathbb{R}^2)$. This explains our choice to work with $v \in W^{1,\infty}(\mathbb{T})$ rather than $v \in H^1(\mathbb{T})$, even though it introduces extra difficulties (like in Proposition 3.1 and in the proof of Proposition 4.11). □

The main result of this section is the following.

Proposition 3.13 *Let $\Omega_0 = \Omega_{u_0}$ convex, $f \in H^k_{loc}(\mathbb{R}^2)$, $k \in \mathbb{N}^*$ and $\xi \in C^k(W^{1,\infty}(\mathbb{T}); W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2))$ near u_0. We have:*
i) e_f is C^k near u_0.
ii) If ξ satisfies (27)*, then for any $v \in W^{1,\infty}(\mathbb{T})$ we have*

$$e'_f(u_0)(v) \;=\; -\int_{\partial\Omega_0} \frac{1}{2}|\nabla U_0|^2(\xi'(u_0)(v)\cdot\nu_0)ds_0 = \int_{\mathbb{T}} \frac{1}{2}|\nabla U_0(x_\theta)|^2 \frac{v(\theta)}{u_0^3(\theta)}d\theta, \tag{30}$$

where $U_0 \in H^2(\Omega_0)$ is the solution of (22) *in Ω_0, ν_0 is the exterior unit normal vector on $\partial\Omega_0$, $x_\theta = \frac{1}{u_0(\theta)}(\cos\theta, \sin\theta) \in \partial\Omega_0$.*
iii) Furthermore, $e'_f(u_0) \in L^\infty(\mathbb{T})$.

The proof of this proposition is classical and uses the following lemma, which will be needed in the following section.

Lemma 3.14 *Let $u_0 \in W^{1,\infty}(\mathbb{T})$, $u_0 > 0$, $f \in H^k_{loc}(\mathbb{R}^2)$, $k \in \mathbb{N}^*$. We have:*

i) The map $u \in W^{1,\infty}(\mathbb{T}) \mapsto \hat{U}_u \in H^1(\Omega_0)$ is C^k near u_0.

ii) For $v \in W^{1,\infty}(\mathbb{T})$, set

$$\hat{U}'_0 := \hat{U}'_u(u_0)(v), \quad U'_0 \;:=\; \hat{U}'_0 - \nabla U_0 \cdot \xi'(u_0)(v). \tag{31}$$

Then

$$U'_0 \in L^2(\Omega_0), \quad \Delta U'_0 \;=\; 0 \text{ in } \mathcal{D}'(\Omega_0), \tag{32}$$
$$U'_0 + \nabla U_0 \cdot \xi'(u_0)(v) \;\in\; H^1_0(\Omega_0). \tag{33}$$

iii) Furthermore, if $u''_0 + u_0 \geq 0$, then $U'_0 \in H^1(\Omega_0)$.

Remark 3.15 Here we are not interested in the differentiability of $u \mapsto U_u$ and the function U'_0 is directly defined by (31). In fact, the map $u \mapsto U_u$ (with U_u extended by zero in $\mathbb{R}^2$) is differentiable in $L^2(\mathbb{R}^2)$ and its derivative equals U'_0 in Ω_0, see Théorème 5.3.1, [10] for example.

Proof of Lemma 3.14:

i) The map $\theta \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \mapsto U_{(Id+\theta)(\Omega_0)} \circ (Id+\theta) \in H^1_0(\Omega_0)$ is C^k in a neighborhood of 0, see for example [10, Proposition 5.3.7]. We conclude by using the composition of this map with ξ.

ii) It is clear that $U'_0 \in L^2(\Omega_0)$ and that $U'_0 + \nabla U_0 \cdot \xi'(u_0)(v) = \hat{U}'_0 \in H^1_0(\Omega_0)$. To prove $\Delta U'_0 = 0$ we consider the map $S : W^{1,\infty}(\mathbb{T}) \mapsto W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$, $S(u) = (Id + \xi(u))^{-1}$, which is well defined and C^k in a neighborhood of u_0. From $S(u) \circ (Id + \xi(u)) = Id$, it is easy to check that for $v \in W^{1,\infty}(\mathbb{T})$ we have

$$S'(u_0)((v) = -\xi'(u_0)(v), \qquad S''(u_0)(v,v) = 2\nabla\xi'(u_0)(v)\cdot\xi'(u_0)(v) - \xi''(u_0)(v,v). \tag{34}$$

Let $\varphi \in \mathcal{D}(\Omega_0)$. From (22), for all u near u_0 we have $\int_{\Omega_0} \hat{U}_u \circ S(u)\Delta\varphi - f\varphi = 0$. Differentiating this equality on the direction v gives

$$\int_{\Omega_0} \left(\hat{U}'_u \circ S(u) + \nabla\hat{U}_u \circ S(u)\cdot S'(u_0)(v)\right)\Delta\varphi = 0. \tag{35}$$

Replacing $u = u_0$ in (35) and using (34) gives

$$\int_{\Omega_0} \left(\hat{U}'_0 - \nabla U_0 \cdot \xi'(u_0)(v)\right)\Delta\varphi = 0,$$

which proves ii).

iii) If $u_0'' + u_0 \geq 0$ then Ω_0 is convex. From Lemma 3.4 we obtain $U_0 \in H^2(\Omega_0)$, which implies $U_0' \in H^1(\Omega_0)$. □

Proof of Proposition 3.13:
i) The functional $u \mapsto e_f(u)$ can be seen as $e_f(u) = \mathcal{E}_{f,\Omega_0} \circ \xi(u)$, where $\mathcal{E}_{f,\Omega_0}$ is a classical functional, introduced to compute shape derivatives:

$$\begin{array}{rccc} \mathcal{E}_{f,\Omega_0}(\theta): & W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) & \to & \mathbb{R} \\ & \theta & \mapsto & E_f((Id+\theta)(\Omega_0)). \end{array} \tag{36}$$

As ξ is C^k near u_0 and $\mathcal{E}_{f,\Omega_0}$ is C^k near $\theta = 0$ in $W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$, see [10, Corollaire 5.3.8]), the differentiability of $e_f(u)$ follows.
ii) As we have $e_f(u) = -\frac{1}{2}\int_{\Omega_u} \hat{U}_u \circ S(u) f$ and $\hat{U}_u = 0$ on $\partial\Omega_0$, from Corollaire 5.2.5, [10], we obtain

$$e_f'(u)(v) = -\frac{1}{2}\int_{\Omega_u} \left(\hat{U}_u' \circ S(u) + \nabla \hat{U}_u \circ S(u) \cdot S'(u)(v)\right) f. \tag{37}$$

Taking $u = u_0$ in the last equality and using (34) gives

$$e_f'(u_0)(v) = -\frac{1}{2}\int_{\Omega_0} (\hat{U}_0' - \nabla U_0 \cdot \xi'(u_0)(v)) f = -\frac{1}{2}\int_{\Omega_0} U_0' f = -\frac{1}{2}\int_{\partial\Omega_0} |\nabla U_0|^2 (\xi'(u_0)(v)\cdot\nu_0) ds_0.$$

Finally, by changing the variable $s_0 = \frac{\sqrt{u_0^2+(u_0')^2}}{u_0^2} d\theta$, taking into account that $\nu_0 = \left(\frac{1}{u_0}e^{i\theta} + \frac{u_0'}{u_0^2}(ie^{i\theta})\right)\frac{u_0^2}{\sqrt{u_0^2+u_0'^2}}$, and after using (29). we obtain (30).
iii) As $k \in N^*$ it follows $f \in L^p(\Omega_0)$, for all $p \in [1,\infty)$. Then Lemma 3.4 gives $U_0 \in W^{1,\infty}(\Omega_0)$, so $e_f'(u_0) \in L^\infty(\mathbb{T})$. □

3.3.3 First eigenvalue of the Laplace operator with Dirichlet boundary conditions

We consider

$$\begin{array}{rccc} l_1: & \{u \in W^{1,\infty}(\mathbb{T}),\ u > 0\} & \to & \mathbb{R} \\ & u & \mapsto & l_1(u) := \lambda_1(\Omega_u) \end{array}$$

and we have the same result as in Proposition 3.13, see for example Théorème 5.7.1, [10], and (29), with

$$l_1'(u_0)(v) = \int_{\mathbb{T}} |\nabla U_0|^2(x_\theta)\frac{v(\theta)}{u_0^3(\theta)} d\theta, \quad \forall v \in W^{1,\infty}(\mathbb{T}).$$

3.4 Application with the dual parametrization

Instead of using parametrization by the gauge function, one can also use the well-known parametrization by the support function of a body, namely

$$\forall \theta \in \mathbb{T},\ \ h_\Omega(\theta) := \max\{x \cdot e^{i\theta},\ x \in \Omega\}.$$

We get a characterization of the convexity in a similar way to (3):

$$\Omega \text{ is convex } \Rightarrow h_\Omega'' + h_\Omega \geq 0.$$

Conversely, if $h \in W^{1,\infty}(\mathbb{T})$ satisfies $h'' + h \geq 0$, then one can find a unique (after a choice of an origin) open convex set, denoted Ω^h, whose support function is h (see [18] for example). This parametrization is the dual of the one

with the gauge function. Indeed, the gauge function of Ω is the support function of the dual body of Ω and vice versa.

Therefore the optimization problem

$$\min\{J(\Omega) \ / \ \Omega \in \mathcal{S}_{ad},\ \Omega \text{ convex}\}, \tag{38}$$

where $\mathcal{S}_{ad}$ is a class of open planar sets, becomes

$$\begin{cases} \text{find } h_0 \in \widetilde{\mathcal{F}_{ad}} \text{ such that } \widetilde{j}(h_0) = \min\{\widetilde{j}(h),\ h \in \widetilde{\mathcal{F}_{ad}},\ h'' + h \geq 0\}, \quad \text{where} \\ \widetilde{j}(h) = J(\Omega^h), \text{ and } \widetilde{\mathcal{F}_{ad}} = \{h \in W^{1,\infty}(\mathbb{T}) \ / \ \Omega^h \in \mathcal{S}_{ad}\}, \end{cases} \tag{39}$$

which is the same as (11).

Again, if the set of admissible functions can be written

$$\widetilde{\mathcal{F}_{ad}} = \{h \in W^{1,\infty}(\mathbb{T}) \ / \ k_1 \leq h \leq k_2\}, \tag{40}$$

we can define $\widetilde{\mathbb{T}_{in}} = \{\theta \in \mathbb{T} \ / \ k_1(\theta) < h(\theta) < k_2(\theta)\}$, and then $\widetilde{(\partial\Omega)_{in}} = \{x \in \partial\Omega \text{ s.t. } \exists \theta \in \widetilde{\mathbb{T}_{in}}, x \cdot e^{i\theta} = h(\theta)\}$, i.e. the set of points of $\partial\Omega$ whose supporting plane is orthogonal to $(\cos(\theta), \sin(\theta))$ with $\theta \in \widetilde{\mathbb{T}_{in}}$.

As in Example 2.2, if $\mathcal{S}_{ad} = \{\Omega \ / \ K_1 \subset \Omega \subset K_2\}$, where K_1 and K_2 are two convex open sets, then (40) is satisfied with k_1, k_2 the supports functions of K_1, K_2, and in that case $\widetilde{(\partial\Omega)_{in}} = \partial\Omega \setminus (\partial K_1 \cup \partial K_2)$.

Therefore one gets a dual version of Corollary 2.7 as follows.

Corollary 3.16 *Let $\Omega_0 = \Omega^{h_0}$ be an optimal shape for the problem* (38) *with $J = R + C$, and assume that,*

$$\forall h \in \widetilde{\mathcal{F}_{ad}}, \quad R(\Omega^h) = r(h) \text{ and } C(\Omega^h) = \int_{\mathbb{T}} G(\theta, h(\theta), h'(\theta)) d\theta$$

where r and G satisfy the assumptions of Theorem 2.4 for some $p \in [1, \infty]$. Then

$$h_0 \in W^{2,p}(\widetilde{\mathbb{T}_{in}}).$$

This implies in particular that $\widetilde{(\partial\Omega_0)_{in}}$ is strictly convex.

Remark 3.17 This parametrization is especially interesting when one has to deal with the perimeter because in this case $P(\Omega^h) = \int_{\mathbb{T}} h d\theta$. An example of a function $C(\Omega^h)$ satisfying the hypotheses of Corollary 3.16 is now the opposite of the area, since

$$|\Omega^h| = \frac{1}{2} \int_{\mathbb{T}} (h^2 - h'^2) d\theta.$$

However, it is not easy to work now with functionals coming from PDE. Indeed, it is well-known for example, that the derivative of λ_1 in terms of h is not more regular than a measure on $\mathbb{T}$, see [11, 12]. We think that this can be explained by the fact that some solutions of problems like (23) may not be strictly convex.

4 Optimization of concave non-local shape functionals

In this section, we prove the results of Section 2.2.2. The main proof relies on the analysis of the second order shape derivatives. Next we apply these results to various energy functionals involving the Dirichlet energy or the first eigenvalue of the Laplace-Dirichlet operator. Since the optimal shapes come with no a priori regularity except the convexity condition, one needs some delicate computations to check the required assumptions. This leads to rather sharp estimates on second derivatives which are interesting for themselves.

4.1 Proof of Theorems 2.9 and 2.12

We first introduce the classical Sobolev semi-norms on $\mathbb{T}$. For $s \in \mathbb{R}_+$, we set:

$$|u|^2_{H^s(\mathbb{T})} := \sum_{n\in\mathbb{Z}} |n|^{2s}|\widehat{u}(n)|^2 \text{ where } \widehat{u}(n) := \int_{\mathbb{T}} u(\theta)e^{-in\theta}\frac{d\theta}{2\pi}.$$

We also define $H^s(\mathbb{T}) := \{u \in L^2(\mathbb{T}) \text{ such that } |u|_{H^s(\mathbb{T})} < +\infty\}$ and $\|u\|^2_{H^s(\mathbb{T})} := \|u\|^2_{L^2(\mathbb{T})} + |u|^2_{H^s(\mathbb{T})}$.

Proof of Theorem 2.9.

The main idea is to prove that for a deformation supported by a small set, the estimate (16) is a concavity estimate, and so it violates the second order optimality condition. This relies of the following Poincaré-type inequality:

Lemma 4.1 *Let $s \in [0,1)$ and $\varepsilon \in (0,\pi)$. Then there exists a constant $C = C(s)$ independant on ε such that,*

$$\forall u \in H^1(\mathbb{T}) \text{ such that } \mathrm{Supp}(u) \subset [0,\varepsilon],\ \|u\|_{H^s(\mathbb{T})} \le C\varepsilon^{1-s}|u|_{H^1(\mathbb{T})}.$$

Proof of Lemma 4.1. Let $u \in C^\infty(\mathbb{T})$ with $\mathrm{Supp}(u) \subset [0,\varepsilon]$. If we first assume that $s = 0$, then we have the classical Poincaré inequality (with the optimal constant), proved using the fact that $|u|^2_{H^1(\mathbb{T})} = \int_{\mathbb{T}} u'^2$, so

$$\|u\|_{L^2(\mathbb{T})} \le \frac{\varepsilon}{\pi}|u|_{H^1(\mathbb{T})}.$$

If one has now $s \in (0,1)$, one can proceed with an interpolation inequality, easily obtained by Hölder inequality:

$$|u|^2_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |n|^{2s}|\widehat{u}(n)|^{2s}|\widehat{u}(n)|^{2(1-s)} \le \left(\sum_{n\in\mathbb{Z}} |n|^2|\widehat{u}(n)|^2\right)^s \left(\sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2\right)^{1-s},$$

and so

$$|u|_{H^s(\mathbb{T})} \le |u|^s_{H^1(\mathbb{T})}\|u\|^{1-s}_{L^2(\mathbb{T})} \le \frac{\varepsilon^{1-s}}{\pi^{1-s}}|u|_{H^1(\mathbb{T})}.$$

□

Let $K := \mathrm{Supp}(u_0'' + u_0)$. Assume that, for a connected component I of $\mathbb{T}_{in}$, $K \cap I$ is infinite. Then, there exists $\theta_0 \in \overline{I}$ an accumulation point of $K \cap I$. Without loss of generality we can assume $\theta_0 = 0$ and also that there exists a decreasing sequence (ε_n) tending to 0 such that $K \cap (0,\varepsilon_n) \subset I$ is infinite. Then, we follow an idea of T. Lachand-Robert and M.A. Peletier as in [14] (see also [16]). We can always find $0 < \varepsilon_n^i < \varepsilon_n$, $i = 1,\dots,4$, increasing with respect to i, such that $\mathrm{Supp}(u_0'' + u_0) \cap (\varepsilon_n^i, \varepsilon_n^{i+1}) \neq \emptyset$, $i = 1,3$. We consider $v_{n,i} \in W^{1,\infty}(\mathbb{T})$ solving

$$v''_{n,i} + v_{n,i} = \mathbb{1}_{(\varepsilon_n^i,\varepsilon_n^{i+1})}(u_0'' + u_0), \quad v_{n,i} = 0 \text{ in } (0,\varepsilon_n)^c,\ i = 1,\dots,3.$$

Such $v_{n,i}$ exist since we avoid the spectrum of the Laplace operator with Dirichlet boundary conditions. Next, we look for $\lambda_{n,i},\ i = 1,3$ such that $v_n = \sum_{i=1,3} \lambda_{n,i}v_{n,i}$ satisfies

$$v_n'(0^+) = v_n'(\varepsilon_n^-) = 0.$$

The above derivatives exist since $v_{n,i}$ are regular near 0 and ε_n in $(0,\varepsilon_n)$. We can always find such $\lambda_{n,i}$ so as they satisfy two linear equations. It implies that v_n'' does not have any Dirac mass at 0 and ε_n. It even implies that the

support of v_n is included in $[\varepsilon_n^1, \varepsilon_n^4]$. In particular, $v_n'' + v_n = \varphi(u_0'' + u_0)$ where φ is bounded and with support in $[\varepsilon_n^1, \varepsilon_n^4]$. As $\mathrm{Supp}(u_0) \cap (\varepsilon_n^i, \varepsilon_n^{i+1}) \neq \emptyset$, we also have $v_n \neq 0$.

Since $\mathrm{Supp}(v_n) \subset \mathbb{T}_{in}$ and $v_n'' + v_n = \varphi(u_0'' + u_0)$, it follows that $u_0 + tv_n$ is admissible for $|t|$ small enough (and n fixed). Consequently, since $j(u_0 + tv_n) \geq j(u_0)$ for $|t|$ small, we have $j'(u_0)(v) = 0$ and then by using the assumption (16) and Lemma 4.1, we get

$$\begin{aligned} 0 \leq j''(u_0)(v_n, v_n) &\leq -\alpha |v_n|^2_{H^1(\mathbb{T})} + \gamma |v_n|_{H^1(\mathbb{T})} \|v_n\|_{H^s(\mathbb{T})} + \beta \|v_n\|^2_{H^s(\mathbb{T})} \quad (41) \\ &\leq (-\alpha + C\gamma \varepsilon_n^{1-s} + C^2 \beta (\varepsilon_n)^{2(1-s)}) |v_n|^2_{H^1(\mathbb{T})}. \quad (42) \end{aligned}$$

As ε_n tends to 0, inequality (41) becomes impossible and proves that $\mathrm{Supp}(u_0'' + u_0)$ has no accumulation points in $\mathbb{T}_{in}$. It follows that $u_0'' + u_0$ is a finite sum of positive Dirac masses. □

Remark 4.2 More precisely, we can get an estimate of the number of corners in each connected component I of $\mathbb{T}_{in}$:

$$\#\{\mathrm{Supp}(u_0'' + u_0) \cap I\} \leq \frac{2|I|}{A} + 2 \text{ where } A^{1-s} := \frac{-\gamma + \sqrt{\gamma^2 + 4\alpha\beta}}{2\beta C}$$

($C = \frac{1}{\pi^{1-s}}$ appears in Lemma 4.1). Indeed, let us consider three consecutive Dirac masses $\theta_1, \theta_2, \theta_3$ in I. Then

- if $\beta > 0, \gamma \geq 0$, we have
$$(\theta_3 - \theta_1)^{1-s} \geq \frac{-\gamma + \sqrt{\gamma^2 + 4\alpha\beta}}{2\beta C}. \quad (43)$$
- if $\beta = \gamma = 0$, then we have a contradiction, that is to say $u_0'' + u_0$ is the some of at most two Dirac masses I.

To prove this estimate, we define $v \in H_0^1(\theta_1, \theta_3)$ satisfying $v'' + v = \delta_{\theta_2}$ in (θ_1, θ_3), $v = 0$ in $\mathbb{T} \setminus (\theta_1, \theta_3)$. In $\mathbb{T}$, the measure $v'' + v$ is supported in $\{\theta_1, \theta_2, \theta_3\}$, and since these points are in $\mathrm{Supp}(u_0'' + u_0)$, and $[\theta_1, \theta_3] \subset \mathbb{T}_{in}$, $u_0 + tv$ is admissible for small $|t|$. The second order optimality condition and then the assumption (16) together with Lemma 4.1 lead to

$$\begin{aligned} 0 \leq j''(u_0)(v, v) &\leq -\alpha |v|^2_{H^1(\mathbb{T})} + \gamma |v|_{H^1(\mathbb{T})} \|v\|_{H^s(\mathbb{T})} + \beta \|v\|^2_{H^s(\mathbb{T})} \\ &\leq (-\alpha + C\gamma X + C^2 \beta X^2) |v|^2_{H^1(\mathbb{T})}, \end{aligned}$$

where $X = (\theta_3 - \theta_1)^{1-s}$, which implies (43) when β is positive, and gives a contradiction if $\beta = \gamma = 0$.

Remark 4.3 When one uses the parametrization of convex sets by the gauge function u, Ω_u is a polygon if and only if $u'' + u$ is a sum of Dirac masses. With the support function (see Section 3.4), one has the same characterization. Therefore, the conclusion is the same if we work with the optimization problem (39). Estimate (43) remains valid. However, θ_i is no longer the polar angle of a corner of the shape, but is the angle of the normal vectors to the successive segments of the polygonal boundary of the shape. □

As in Section 2.2.1, one can also handle problem with the equality constraint.

Proof of Theorem 2.12.

We now need an abstract result for second order optimality conditions. Adapting [14, Proposition 3.3] similarly to the first order condition given in Proposition 3.1 (this explains the assumption $j'(u_0) \in (C^0(\mathbb{T}))'$), we get that there exist $\zeta_0 \in W^{1,\infty}(\mathbb{T})$ nonnegative, $\mu \in \mathbb{R}^d$ such that

$$\begin{cases} \zeta_0 = 0 \ \text{ on } \ \mathrm{Supp}(u_0'' + u_0) \quad \text{and} \\ \forall\, v \in W^{1,\infty}(\mathbb{T}_{in}), \quad j'(u_0)v + \mu \cdot m'(u_0)v = \langle \zeta_0 + \zeta_0'', v\rangle_{(W^{1,\infty})' \times W^{1,\infty}}. \end{cases} \tag{44}$$

Furthermore, for all $v \in H^1(\mathbb{T}_{in})$ such that $\exists \lambda \in \mathbb{R}$, with $v'' + v \geq \lambda(u_0'' + u_0)$, and $\langle \zeta_0 + \zeta_0'', v\rangle - \mu \cdot m'(u_0)(v) = 0$,

$$j''(u_0)(v,v) + \mu \cdot m''(u_0)(v,v) \geq 0. \tag{45}$$

Then we proceed as in the proof of [14, Theorem 2.1]. Compared to the first step of the proof of Theorem 2.9, we add one degree of freedom introducing 4 functions $v_{n,i}$ on a partition of $(0, \varepsilon_n)$, and we look for $\lambda_{n,i}, i = 1 \dots 4$ such that $v_n = \sum_{i=1,4} \lambda_{n,i} v_{n,i}$ satisfies

$$v_n'(0^+) = v_n'(\varepsilon_n^-) = \mu \cdot m'(u_0) v_n = 0.$$

Such a choice of $\lambda_{n,i}$ is always possible as $\lambda_{n,i}$ satisfy three linear equations. Moreover, v_n is not zero and using (44), we get $\int_{\mathbb{T}} v_n(\zeta_0 + \zeta_0'') = 0$, which implies

$$0 = j'(u_0)(v_n) = \int_{\mathbb{T}} v_n(\zeta_0 + \zeta_0'') = \mu \cdot m'(u_0)(v_n).$$

As $v_n'' + v_n \geq \lambda(u_0'' + u_0)$ for $\lambda \ll 0$, it follows that v_n is eligible for the second order necessary condition (45). Then, it follows

$$\begin{aligned} 0 \ &\leq \ j''(u_0)(v_n, v_n) + \mu \cdot m''(u_0)(v_n, v_n) \leq -\alpha |v_n|^2_{H^1(\mathbb{T})} + \gamma |v_n|_{H^1(\mathbb{T})} \|v_n\|_{H^s(\mathbb{T})} + (\beta + \|\beta' \mu\|) \|v_n\|^2_{H^s(\mathbb{T})} \\ &\leq \ (-\alpha + C\gamma \varepsilon_n^{1-s} + C^2(\beta + \|\beta'\mu\|)(\varepsilon_n)^{2(1-s)}) |v_n|^2_{H^1(\mathbb{T})} \end{aligned}$$

As n tends to ∞, the inequality $0 \leq j''(u_0)(v_n, v_n) + \mu \cdot m''(u_0)(v_n, v_n)$ becomes impossible and this concludes the proof. □

Remark 4.4 An estimate similar to the one in Remark 4.2 is not straightforward anymore, since the Lagrange multiplier μ is unknown.

4.2 Examples

We analyze the same examples as in Section 3.2, with $-P$ instead of P:

Example 4.5 (Negative perimeter penalization) One can study

$$\min\{J(\Omega) := F(|\Omega|, E_f(\Omega), \lambda_1(\Omega)) - P(\Omega) \ / \ \Omega \text{ convex}, D_1 \subset \Omega \subset D_2\} \tag{46}$$

where $F : (0, +\infty) \times (-\infty, 0) \times (0, +\infty) \to \mathbb{R}$ is C^2, $f \in H^2(\mathbb{R}^2)$, and D_1, D_2 are bounded open sets. We can prove the following.

Proposition 4.6 *If Ω_0 is an optimal set for the problem* (46)*, then each connected component of the free boundary $\partial\Omega_0 \setminus (\partial D_1 \cup \partial D_2)$ is polygonal.*

Proof. The proof is a direct consequence of Corollary 2.13 and of the estimates given in Section 4.3.2. Indeed, Proposition 4.11 for $E_f(\Omega)$, the similar result for λ_1 (See Section 4.3.3) and Proposition 4.10 for the volume, imply

$$|r''(0)| \leq C\|v\|^2_{H^{1/2+\varepsilon}(\mathbb{T})},$$

where $r(t) = F(|\Omega_t|, E_f(\Omega_t), \lambda_1(\Omega_t))$, $\Omega_t = \Omega_{u_0+tv}$ and $\varepsilon \in (0, \frac{1}{2})$. Next, the estimate for the perimeter in Proposition 4.10 provides the concavity condition. □

Remark 4.7 As in Remark 3.8, if we consider problems of type (46) where the constraint $D_1 \subset \Omega \subset D_2$ can be dropped, then the solution is a polygon.

Example 4.8 (Volume constraint and negative perimeter penalization) We can also consider a similar problem with a volume constraint:

$$\min\{J(\Omega) := F(E_f(\Omega), \lambda_1(\Omega)) - P(\Omega) \ / \ \Omega \text{ convex, and } |\Omega| = V_0\} \tag{47}$$

where $V_0 \in (0, +\infty)$. Again, Corollary 2.13 applies and leads to the fact that any optimal shape of (47) is a polygon.

Example 4.9 [Perimeter constraint] We consider again a problem with a perimeter constraint, as in Example 3.10

$$\min\{J(\Omega) := F(|\Omega|, E_f(\Omega), \lambda_1(\Omega)) \ / \ \Omega \text{ convex, and } P(\Omega) = P_0\} \tag{48}$$

where $P_0 \in (0, +\infty)$. The optimality conditions are written for $F(|\Omega|, E_f(\Omega), \lambda_1(\Omega)) + \mu P(\Omega)$, where μ is a Lagrange multiplier for the constraint $P(\Omega) = P_0$, so if we prove that $\mu < 0$, then the strategy of this section applies, and we get that any optimal shape is polygonal.

4.3 Computations and estimates of second order shape derivatives

4.3.1 Volume and perimeter

Let $a(u)$, $p(u)$ be the area and perimeter functionals, see (26).

Proposition 4.10 *Let $0 < u \in W^{1,\infty}(\mathbb{T})$. Then a and p are twice differentiable around u in $W^{1,\infty}(\mathbb{T})$ and there exists some real numbers $\beta_1, \beta_2, \beta_3, \gamma$ and $\alpha > 0$ (depending on u) such that, $\forall v \in W^{1,\infty}(\mathbb{T})$*

$$\begin{cases} |a''(u)(v,v)| \leq \beta_1 \|v\|^2_{L^2(\mathbb{T})} \\ \alpha |v|^2_{H^1(\mathbb{T})} - \gamma |v|_{H^1(\mathbb{T})} \|v\|_{L^2(\mathbb{T})} - \beta_2 \|v\|^2_{L^2(\mathbb{T})} \leq p''(u)(v,v) \leq \beta_3 \|v\|^2_{H^1(\mathbb{T})} \end{cases} \tag{49}$$

Proof. This is done by easy computations, using formulas of Section 3.3.1. □

4.3.2 The Dirichlet energy - Torsional rigidity

We now analyze the second order derivative of $e_f(u) = E_f(\Omega_u)$ introduced in Section 3.3. The main result is the following.

Proposition 4.11 *Assume $\Omega_0 := \Omega_{u_0}$, $u_0 > 0$, $u_0'' + u_0 \geq 0$, $f \in H^2_{loc}(\mathbb{R}^2)$. Then e_f is C^2 in a neighborhood of u_0 (in $W^{1,\infty}(\mathbb{T})$). Furthermore, there exist β_1, β_2 positive such that, for all $v \in W^{1,\infty}(\mathbb{T})$,*

$$|e_f'(u_0)v| \leq \beta_1 \|v\|_{L^2(\mathbb{T})}, \tag{50}$$

$$|e_f''(u_0)(v,v)| \leq \beta_2 (\|v\|^2_{H^{1/2}(\mathbb{T})} + \|v\|^2_{L^\infty(\mathbb{T})}). \tag{51}$$

The differentiability of e_f and the estimate (50) follow easily from Proposition 3.13. The estimate (51) is easy to prove when working with smooth sets and one can then even drop the L^∞ term. However, this result is more difficult for a general convex set and the rest of this section is devoted to its proof.

Let v be given as in Proposition 4.11. To prove the estimate (51), it is appropriate to consider a transformation ξ such that

$$\xi \in C^2((-\eta,\eta), W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)),\ \eta \in (0,1), \qquad \xi(t) = \left(\frac{1}{u_0+tv} - \frac{1}{u_0}\right) e^{i\theta} \quad \text{on } \partial\Omega_0. \tag{52}$$

Then, we will differentiate twice $t \in (-\eta,\eta) \to e(t) = E(\Omega_{u_0+tv})$. We will use the following notation and identities:

$$\Omega_t := \Omega_{u_0+tv}, \quad U_t := U_{\Omega_{u_0+tv}}, \quad \hat{U}_t := U_t \circ (I+\xi(t)), \quad e(t) := E(\Omega_{u_0+tv}). \tag{53}$$

Note that $e(t) = e_f(u_0+tv) = \mathcal{E}_{f,\Omega_0}(\xi(t))$ and we have

$$e'(0) = e_f'(u_0)(v) = \mathcal{E}'_{f,\Omega_0}(0)(\xi'(0)), \tag{54}$$

$$e''(0) = e_f''(u_0)(v,v) = \mathcal{E}''_{f,\Omega_0}(0)(\xi'(0),\xi'(0)) + \mathcal{E}'_{f,\Omega_0}(0)(\xi''(0)). \tag{55}$$

In the smooth case, $e''(0)$ can be written in terms of boundary integrals, which involve in particular the boundary trace of D^2U_0 and $\nabla U_0'$. These terms are not well defined in the non-smooth setting (even in the case Ω_0 convex). To overcome this difficulty, our strategy will be to write all non-smooth terms of $e''(0)$ as "interior" integrals in Ω_0.

Estimate of $e''(0)$: Note that we have proven in Section 3.3 that e_f is C^2 if $f \in H^2_{loc}(\mathbb{R}^2)$) (so, e is C^2). We remind the following classical formulation of $e''(0)$

Lemma 4.12 *Let $f \in H^2_{loc}(\mathbb{R}^2)$ and $\xi \in C^2(\mathbb{R}; W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2))$ near 0. Then we have*

$$e''(0) = -\frac{1}{2}\left(\int_{\Omega_0} fU_0'' + \int_{\partial\Omega_0} fU_0'(\xi'(0)\cdot\nu_0)\right), \tag{56}$$

where $\hat{U}_0'' := \hat{U}_u''(u_0)(v,v)$ and U_0'' is defined by

$$U_0'' := \hat{U}_0'' - \left(2\nabla U_0'\cdot\xi'(0) + \xi'(0)\cdot D^2U_0\cdot\xi'(0) + \nabla U_0\cdot\xi''(0)\right) \quad \text{in} \quad \Omega_0, \tag{57}$$

and satisfies

$$U_0'' \in L^2(\Omega_0), \quad \Delta U_0'' = 0 \quad \text{in } \mathcal{D}'(\Omega_0). \tag{58}$$

Proof. Differentiating (37) at $u = u_0$ (see Corollaire 5.2.5, [10]) and then using (34) gives

$$\begin{aligned} e''(0) &= -\frac{1}{2}\int_{\Omega_0}\left(\hat{U}_0'' - 2\nabla\hat{U}_0'\cdot\xi'(0) + \xi'(0)\cdot D^2U_0\cdot\xi'(0) + \nabla\hat{U}_0\cdot(2\nabla\xi'(0)\cdot\xi'(0) - \xi''(0))\right) f, \\ &\quad -\frac{1}{2}\int_{\partial\Omega_0} U_0' f(\xi'(0)\cdot\nu_0). \end{aligned} \tag{59}$$

After replacing $\hat{U}_0' = U_0' - \nabla U_0\cdot\xi'(0)$, (59) gives (56).

Clearly $U_0'' \in L^2(\Omega_0)$. To prove that $\Delta U_0'' = 0$ we differentiate (35) at $u = u_0$ and use (34). Then we obtain

$$\int_{\Omega_0}\left(\hat{U}_0'' - 2\nabla\hat{U}_0'\cdot\xi'(0) + \xi'(0)\cdot D^2U_0\cdot\xi'(0) + \nabla U_0\cdot(2\nabla\xi'(0)\cdot\xi'(0) - \xi''(0))\right)\Delta\varphi = 0.$$

Replacing $\hat{U}_0'$ as given by (31) gives $\int_{\Omega_0} U_0''\Delta\varphi = 0$, which proves (58). □

Proof of Proposition 4.11.

We will often write ξ, ξ', ξ'' for $\xi(0), \xi'(0), \xi''(0)$. Let us rewrite (56) in the form $e''(0) = \frac{1}{2}(I_1 + I_2)$. The second term I_2 is easy to estimate: from (31) we have

$$I_2 \;:=\; -\int_{\partial\Omega_0} f U_0'(\xi'\cdot\nu_0) = \int_{\partial\Omega_0} f\partial_{\nu_0}U_0(\xi'\cdot\nu_0)^2 \le C\|\xi'\|^2_{L^2(\partial\Omega)}, \quad C = C(\|f\|_{L^\infty(\Omega_0)}, \|U_0\|_{W^{1,\infty}(\Omega_0)}) \tag{60}$$

The first term $I_1 = \int_{\Omega_0} U_0''\Delta U_0$ requires more investigation. To go around the non regularity of Ω_0, we introduce

$$U_0 = U_1 - U_2, U_i \in H^1_0(\Omega_0), -\Delta U_1 = f^+, \;\; -\Delta U_2 = f^-, \;\; U_i > 0 \text{ on } \Omega_0.$$

Recall that $U_i \in W^{1,\infty}(\Omega_0)\cap H^2(\Omega_0)$. We will compute on the level sets $\Omega^i_\varepsilon := \{x\in\Omega_0, U_i(x) > \epsilon\}$ (only on one of them if $f^+\equiv 0$ or $f^-\equiv 0$). Indeed, by Sard's theorem, the Ω^i_ε are at least C^1 for a.e. ε. By strict positivity of U_i, $\lim_{\varepsilon\to 0}\mathbb{1}_{\Omega^i_\varepsilon} = \mathbb{1}_{\Omega_0}$, so that

$$I_1 = \lim_{\varepsilon\to 0}\int_{\Omega^1_\varepsilon} U_0''\Delta U_1 - \int_{\Omega^2_\varepsilon} U_0''\Delta U_2.$$

Note that $U_0', U_0'' \in C^\infty_{loc}(\Omega_0)$ and as $f\in H^2_{loc}(\mathbb{R}^2)$ we have $U_0\in H^4_{loc}(\Omega_0)$. We obtain

$$\begin{aligned}\int_{\Omega^i_\varepsilon} U_0''\Delta U_i = \int_{\partial\Omega^i_\varepsilon} U_0''\partial_{\nu_\varepsilon}U_i \;&=\; \int_{\partial\Omega_\varepsilon}\hat{U}_0''\partial_{\nu_\varepsilon}U_i - 2(\nabla U_0'\cdot\xi')\partial_{\nu_\varepsilon}U_i - (\xi'\cdot D^2U_0\cdot\xi')\partial_{\nu_\varepsilon}U_i - (\nabla U_0\cdot\xi'')\partial_{\nu_\varepsilon}U_i\\ &=:\; I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon.\end{aligned} \tag{61}$$

For the term I_1^ε, we have

$$I_1^\varepsilon \;=\; \int_{\Omega^i_\varepsilon}\hat{U}_0''\Delta U_i + \nabla\hat{U}_0''\cdot\nabla U_i \xrightarrow{\varepsilon\to 0} \int_{\Omega_0}\hat{U}_0''\Delta U_i + \nabla\hat{U}_0''\cdot\nabla U_i = \int_{\partial\Omega_0}\hat{U}_0''\partial_{\nu_0}U_i = 0. \tag{62}$$

To deal with I_2^ε and I_3^ε, we will need the following generalized formula of integration by parts.

Lemma 4.13 *Let Ω be a C^1 open set, $U\in W^{1,\infty}_0(\Omega)\cap H^2(\Omega)$, $V\in H^1(\Omega)\cap\{\Delta V\in L^2(\Omega_0\}$, $g\in W^{1,\infty}(\Omega;\mathbb{R}^2)$. Then*

$$J := \int_{\partial\Omega}\partial_\nu U(g\cdot\nabla V) = \int_\Omega \nabla(\nabla U\cdot g)\cdot\nabla V + (\nabla U\cdot g)\Delta V - \nabla(\nabla^\perp U\cdot g)\cdot\nabla^\perp V, \tag{63}$$

where the operator $\perp$ acts on a vector and is defined by ${}^\perp(a_1,a_2) = (-a_2, a_1)$. As a consequence

$$|J| \le \|\nabla U\|_{L^\infty(\Omega)}\|g\|_{L^2(\Omega)}\|\Delta V\|_{L^2(\Omega)} + 2\left\{\|V\|_{H^1(\Omega)}\left[\|U\|_{H^2(\Omega)}\|g\|_{L^\infty(\Omega)} + \|\nabla U\|_{L^\infty(\Omega)}\|\nabla g\|_{L^2(\Omega)}\right].\right\} \tag{64}$$

Proof. If ν is the exterior normal unit vector to $\partial\Omega$ and $\tau = {}^\perp\nu$ the unit tangent vector, then, for $\varphi\in H^1(\Omega)$ and $a = (a_1,a_2)$, using that $\nabla U\cdot\tau = 0$ on $\partial\Omega_0$, we have

$${}^\perp\tau = -\nu, \quad (a\cdot\nu)\partial_\nu U = a\cdot\nabla U, \quad (a\cdot\tau)\partial_\nu U = a\cdot\nabla^\perp U, \quad \tau\cdot\nabla\varphi = -\nu\cdot\nabla^\perp\varphi.$$

Then we obtain

$$\begin{aligned}\int_{\partial\Omega}\partial_\nu U(g\cdot\nabla V) \;&=\; \int_{\partial\Omega}(\nabla U\cdot\nu)(g\cdot\nu)(\nabla V\cdot\nu) + (\nabla U\cdot\nu)(g\cdot\tau)(\nabla V\cdot\tau)\\ &=\; \int_{\partial\Omega}\Big((\nabla U\cdot g)\nabla V - (\nabla^\perp U\cdot g)\nabla^\perp V\Big)\cdot\nu \qquad \text{(apply divergence theorem to both terms)}\\ &=\; \int_\Omega \nabla\cdot((\nabla U\cdot g)\nabla V) - \nabla\cdot((\nabla^\perp U\cdot g)\nabla^\perp V),\end{aligned}$$

which proves (63) because $\nabla \cdot \nabla^\perp = 0$. The estimate (64) follows. □

End of the proof of Lemma 4.13.

We apply Lemma 4.13 on $\Omega = \Omega_\varepsilon^i$ to estimate $I_2^\varepsilon, I_3^\varepsilon$ in (61). For I_2^ε, we choose $U = U_i - \epsilon, V = U_0', g = \xi'(0)$ (recall that $\Delta U_0' = 0$) and for I_3^ε, we choose $U = U_i - \epsilon, V = V_j = \partial_j U_i, g = g_j = \xi_j'(0)\xi'(0), j = 1, 2$: here $-\Delta V_j = \partial_j f^+$ or $\partial_j f^-$. Next, we apply the estimate (64) to each of these choices and we are obviously led to estimates independent of ε. For I_4^ε, we make a direct easy estimate. Together also with (60) and using Young inequality we obtain:

$$|e''(0)| \leq C(\|\nabla U_0'\|^2_{L^2(\Omega_0)} + \|\nabla \xi'\|^2_{L^2(\Omega_0)} + \|\xi'\|^2_{L^\infty(\Omega_0)} + \|\xi'\|^2_{L^2(\partial\Omega_0)} + \|\xi''\|_{L^1(\partial\Omega_0)}), \tag{65}$$

$$\text{where } C = C(\|f\|_{L^\infty(\Omega_0)\cap H^1(\Omega_0)}, \|U_i\|_{W^{1,\infty}(\Omega_0)}, \|U_i\|_{H^2(\Omega_0)},\ i = 0, 1, 2). \tag{66}$$

Now, let us write the estimate (65) in terms of v. First note that if $\alpha, \beta \in H^{1/2}(\partial\Omega_0) \cap L^\infty(\partial\Omega)$ then $\alpha\beta \in H^{1/2}(\partial\Omega_0) \cap L^\infty(\partial\Omega_0)$ and

$$\|\alpha\beta\|_{H^{1/2}(\partial\Omega_0)\cap L^\infty(\partial\Omega_0)} \leq C\|\alpha\|_{H^{1/2}(\partial\Omega_0)\cap L^\infty(\partial\Omega_0)}\|\beta\|_{H^{1/2}(\partial\Omega_0)\cap L^\infty(\partial\Omega_0)}.$$

(using the easy fact that $H^1(\Omega_0)\cap L^\infty(\Omega_0)$ is an algebra, and that the $H^{1/2}(\partial\Omega_0)$-norm is equivalent to the $H^1(\Omega_0)$-norm of the harmonic extension in Ω_0). Also, we point out that the transformation $\psi = \psi(r, \theta) := \frac{r}{u_0(\theta)} e^{i\theta}$ is bi-Lipschitz near $\mathbb{T}$ and $\psi(\mathbb{T}) = \partial\Omega_0$. Then $\gamma \in H^{1/2}(\partial\Omega)$ if and only if $\gamma \circ \psi \in H^{1/2}(\mathbb{T})$, and their $H^{1/2}$-norms are equivalent.

Let us remind that, according to the choice of ξ in (52), we have $\xi'(0) = -\frac{v}{u_0^2} e^{i\theta}$, $\xi''(0) = 2\frac{v^2}{u_0^3}$ on $\partial\Omega_0$. Then we obtain, with the same dependence of the various constants C as in (66)

$$\|\nabla U_0'\|^2_{L^2(\Omega_0)} \leq C|\xi' \cdot \nabla U_0|^2_{H^{1/2}(\partial\Omega)} \leq C\|\xi'\|^2_{H^{1/2}(\partial\Omega)\cap L^\infty(\Omega_0)} \leq C\|v\|^2_{H^{1/2}(\mathbb{T})\cap L^\infty(\mathbb{T})}, \tag{67}$$

$$\|\xi'\|_{L^\infty(\partial\Omega_0)} \leq C\|v\|_{L^\infty(\mathbb{T})},\ \ \|\xi'\|^2_{L^2(\partial\Omega_0)} + \|\xi''\|_{L^1(\partial\Omega_0)} \leq C\|v\|^2_{L^2(\mathbb{T})}. \tag{68}$$

All these estimates are valid for all choices of ξ as in (52). Let

$$\mathcal{W} := \{w \in W^{1,\infty}(\Omega_0),\ \ w_{|\partial\Omega_0} = -\frac{v}{u_0^2} e^{i\theta}\ \}.$$

Given $w \in \mathcal{W}$, let us choose $\xi(t) := \zeta(t) + t(w - \zeta'(0))$, where ζ is the $W^{1,\infty}$-extension as given in (28), namely

$$\zeta(t)\left(\frac{1}{u_0(\theta)}, \theta\right) = \left(\frac{1}{u_0(\theta) + tv(\theta)} - \frac{1}{u_0(\theta)}\right) e^{i\theta}\eta(r, \theta),\ \eta \in C_0^\infty(\mathbb{R}^2),$$

with $\eta = 0$ (resp. $\eta = 1$) in a neighborhood of the origin (resp. of $\partial\Omega_0$). Then, ξ is as in (52) and $\xi'(0) = w$. Therefore, the estimate (65) together with (67), (68) leads to

$$\forall w \in \mathcal{W},\ |e''(0)| \leq C\left(\|\nabla w\|^2_{L^2(\Omega_0)} + \|v\|^2_{H^{1/2}(\mathbb{T})\cap L^\infty(\mathbb{T})} + \|w\|^2_{L^\infty(\Omega_0)}\right). \tag{69}$$

Let us introduce

$$w_0 \in H^1(\Omega_0),\ \ \Delta w_0 = 0 \text{ on } \Omega_0,\ \ (w_0)_{|\partial\Omega_0} = -\frac{v}{u_0^2} e^{i\theta}\ \ [\text{or } w_0 - \zeta'(0) \in H_0^1(\Omega_0)]\ .$$

Let now δ_n be a sequence of $C_0^\infty(\Omega_0)$-functions converging to $w_0 - \zeta'(0)$ in $H_0^1(\Omega_0)$ and let $w_n := \inf\{\delta_n + \zeta'(0), \|w_0\|_{L^\infty(\Omega_0)}\}$. Then, $w_n \in \mathcal{W}$ and converges in $H^1(\Omega_0)$ to w_0. Applying (69) with w_n in place of w and passing to the limit yields:

$$|e''(0)| \leq C\left(\|\nabla w_0\|^2_{L^2(\Omega_0)} + \|v\|^2_{H^{1/2}(\mathbb{T})\cap L^\infty(\mathbb{T})} + \|w_0\|^2_{L^\infty(\Omega_0)}\right). \tag{70}$$

But, since w_0 is harmonic,

$$\|\nabla w_0\|_{L^2(\Omega_0)} \leq \|w_0\|_{H^{1/2}(\partial\Omega_0)} \leq C\|v\|_{H^{1/2}(\mathbb{T})}, \quad \|w_0\|_{L^\infty(\Omega_0)} \leq \|w_0\|_{L^\infty(\partial\Omega_0)} \leq C\|v\|_{L^\infty(\mathbb{T})}.$$

Finally, the estimate (70) leads to

$$|e''(0)| \leq C\|v\|^2_{H^{1/2}(\mathbb{T})\cap L^\infty(\mathbb{T})}.$$

□

4.3.3 First eigenvalue of the Laplace operator with Dirichlet boundary conditions

The estimate of Proposition 4.11 also holds for $\lambda_1(\Omega_u)$, the first Laplace eigenvalue (see Section 3.3), namely

$$|l_1''(0)| \leq C\|v\|^2 H^{1/2}(\mathbb{T}) \cap L^\infty(\mathbb{T}), \tag{71}$$

where $l_1(t) = \lambda_1(\Omega_{u_0+tv})$. As the computations are very similar, we will only sketch the proof.

Proof of (71).

As for e_f, for $v \in W^{1,\infty}(\mathbb{T})$ fixed and $|t|$ small we consider $l_1(t) := \lambda_1(\Omega_t)$ and U_t, the first eigenvalue and the corresponding eigenfunction of $-\Delta$ in $\Omega_t := \Omega_{u_0+tv}$. As in Lemma 4.12 we can show that

$$l_1''(0) = -\int_{\Omega_0} U_0'\Delta U_0' + U_0\Delta U_0'' =: I_1 + I_2. \tag{72}$$

Here U_0' and U_0'' satisfy

$$\begin{aligned}
-\Delta U_0' = l_1 U_0' + l_1' U_0 \quad \text{in } \Omega_0, \qquad U_0' = -\xi'(0)\cdot\nabla U_0 \quad \text{on} \quad \partial\Omega_0, \qquad \int_{\Omega_0} U_0 U_0' = 0,\\
-\Delta U_0'' = l_1 U_0'' + 2l_1' U_0' + l_1'' U_0 \quad \text{in} \quad \Omega_0, \qquad \int_{\Omega_0} |U_0'|^2 + U_0 U_0'' = 0,\\
U_0'' = \hat{U}_0'' - \left(2\xi'(0)\cdot\nabla U_0' + \xi'(0)\cdot D^2 U_0\cdot\xi'(0) + \xi''(0)\cdot\nabla U_0\right) \quad \text{in} \quad \Omega_0,
\end{aligned}$$

where $l_1 = l_1(0)$, $l_1' = l_1'(0)$, $l_1'' = l_1''(0)$. Then considering $\Omega_\varepsilon = \{x \in \Omega_0, U_0 > \varepsilon\}$ as in the proof of Proposition 4.11 (note that $U_0 > 0$ on Ω_0 here), we have :

$$I_1 = \int_{\Omega_0} U_0'(l_1 U_0' + l_1' U_0) = l_1 \int_{\Omega_0} |U_0'|^2, \tag{73}$$

$$\begin{aligned}
I_2 &= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} U_0\Delta U_0'' = -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \varepsilon\Delta U_0'' + (U_0-\varepsilon)\Delta U_0'' = -\lim_{\varepsilon\to 0}\int_{\Omega_\epsilon} (U_0-\varepsilon)\Delta U_0''\\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} U_0''(-\Delta U_0) - \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} (U_0-\varepsilon)\partial_{\nu_\varepsilon} U_0'' - U_0''\partial_{\nu_\varepsilon} U_0\\
&= -l_1\int_{\Omega_0} |U_0'|^2 + \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} U_0''\partial_{\nu_\varepsilon} U_0.
\end{aligned} \tag{74}$$

Combining (72) with the last two equalities gives

$$l_1''(0) = \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} U_0'' \partial_{\nu_\varepsilon} U_0.$$

Then we proceed exactly as in Proposition 4.11, and obtain for $l_1''(0)$ an estimate exactly similar to (65).

Next, we prove that $\|\nabla U_0'\|_{L^2(\Omega_0)} \leq C\|\xi'(0)\|_{H^1(\Omega_0)}$. As $U_0' = \hat{U}_0' - \xi'(0)\cdot\nabla U_0$ it is enough to prove $\|\nabla \hat{U}_0'\|_{L^2(\Omega_0)} \leq C\|\xi'(0)\|_{H^1(\Omega_0)}$. One can verify that $\hat{U}_0'$ satisfies

$$\hat{U}_0' \in H_0^1(\Omega_0)\cap H^2(\Omega_0), \quad \Delta\hat{U}_0' + l_1\hat{U}_0' = 2\mathrm{trace}([\nabla\xi']\cdot[D^2U_0]) - l_1'U_0, \quad \int_{\Omega_0}\hat{U}_0'U_0 = 0.$$

Using the convexity of Ω_0 and Fredholm alternative theorem, we can prove that the operator

$$V \in (H_0^1(\Omega_0)\cap H^2(\Omega_0))\backslash \mathrm{span}\{U_0\} \mapsto \Delta V + l_1 V \in L^2(\Omega_0)\cap\{h,\ \int_{\Omega_0} hU_0 = 0\},$$

defines an isomorphism (see for example [8]), which together with the formula for $l_1'(0)$ provides the required estimate for $\hat{U}_0'$. Therefore, as for $e''(0)$, for all ξ as in (52), we have

$$|l_1''(0)| \leq C\left(\|\nabla\xi'(0)\|^2_{L^2(\Omega_0)} + \|v\|^2_{H^{1/2}(\mathbb{T})\cap L^\infty(\mathbb{T})} + \|\xi'(0)\|_{L^\infty(\Omega_0)}\right).$$

Then we complete the proof as in Proposition 4.11. □

5 Remarks and perspectives

5.1 Localization of our two approaches

As explained in the introduction, the approaches leading to our two families of results are very "local" with respect to the boundary of the optimal shape. Indeed, each proof uses test functions $v \in W^{1,\infty}$ whose support may be as small as we want and only covers the portion of the boundary that we want to analyze. To show how this can be exploited, we give now -without proof- , an example of a result which can be reached by the same two methods when applied locally.

Let us consider the following optimization problem where $G : (\theta, u, q) \in \mathbb{T}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be of class C^2 and $a, b \in (0,\infty)$:

$$\begin{cases} u_0 \in W^{1,\infty}(\mathbb{T}),\ \ j(u_0) = \min\{j(u),\ u'' + u \geq 0,\ \ a \leq u \leq b\}, \\ \text{where } j(u) = \displaystyle\int_{\mathbb{T}} G\left(\theta, u(\theta), u'(\theta)\right) d\theta. \end{cases} \tag{75}$$

We define $\mathbb{T}_{in}$ as in (13) and we introduce the partition $\mathbb{T}_{in} = \mathbb{T}_+ \cup \mathbb{T}_0 \cup \mathbb{T}_-$ where
$\mathbb{T}_+ := \{\theta \in \mathbb{T}_{in}\ ;\ \widetilde{G_{qq}}(\theta) \in (0,\infty)\}$, (recall $\widetilde{G_{qq}(\theta)} = \int_0^1 G_{qq}\left(\theta, u_0(\theta), tu_0'(\theta^+) + (1-t)u_0'(\theta^-)\right) dt$,
$\mathbb{T}_- := \{\theta \in \mathbb{T}_{in},\ [G_{qq}\left(\theta, u_0(\theta), u_0'(\theta^-)\right), G_{qq}\left(\theta, u_0(\theta), u_0'(\theta^+)\right)] \subset (-\infty, 0)\}$,
$\mathbb{T}_0 := \mathbb{T}_{in} \setminus (\mathbb{T}_+ \cup \mathbb{T}_-)$.
Then
(i) $\mathbb{T}_+$ is open and $u_0'' \in L^\infty_{loc}(\mathbb{T}_+)$, so that $u_0 \in W^{2,\infty}_{loc}(\mathbb{T}^+)$,
(ii) There is no accumulation point of $\mathrm{Supp}(u_0'' + u_0)$ in the open set $\mathbb{T}_-$; in other words, $[\theta \in \mathbb{T}_- \to \partial\Omega_{u_0}(\theta)]$ is locally polygonal.
The situation on $\mathbb{T}_0$ requires a complementary study specific to each functional.

5.2 Very singular optimal shapes

In this paper, we gave some sufficient conditions on the shape functional so that an optimal shape be smooth or polygonal. But there exist convex sets which are not of this type, and in a certain sense have "intermediate regularity". Namely, there are convex sets which are singular in the sense that they do not have corners (they are C^1), but their curvature is zero almost everywhere. As an example, one may consider any convex set such that $u'' + u$ is a Radon measure, without mass, but singular with respect the Lebesgue measure.

Let us mention a shape optimization problem whose solution is neither regular nor polygonal (see [17] for an analysis of this problem). Let Ω_0 be a convex set, $V_0 = |\Omega_0|$, $P_0 = P(\Omega_0)$ and $D = (\Omega_0)_T = \{x \in \mathbb{R}^2,\ d(x, \Omega_0) < T\}$. Then Theorem 8 in [17] states that:

$$J(\Omega_0) = \min\{J(\Omega) \ / \ \Omega \subset D \text{ convex such that } P(\Omega) = P_0,\ |\Omega| = V_0\}, \tag{76}$$

where J is the distance functional:

$$J(\Omega) := \int_D d(x, \Omega) dx.$$

Since Ω is *any convex set*, one cannot expect any geometrical property for a minimizer of (76) without extra conditions on D, V_0 and P_0. Remark also that the box $D = (\Omega_0)_T$ is $C^{1,1}$ here.

5.3 Problem without perimeter

An interesting problem, which has not been analyzed in this paper, is the following (we use the notation of Section 4.3):

$$\max\{E_f(\Omega), |\Omega| = V_0, \Omega \text{ convex } \subset D\}. \tag{77}$$

It is easy to prove the existence of an optimal shape Ω_0. In this situation, we expect the term $E_f(\Omega)$ to be leading over $|\Omega|$ (whereas the perimeter was the stronger term in the examples solved in this paper). So we are naturally led to the following question : do there exist $\alpha > 0$, $\beta, \gamma \geq 0$ such that

$$\forall v \in W^{1,\infty}(\mathbb{T}), \quad e''(0) \geq \alpha |v|^2_{H^{1/2}} - \gamma |v|_{H^{1/2}} \|v\|_{L^2} - \beta \|v\|^2_{L^2} \ ? \tag{78}$$

A consequence of such an estimate, would be that any solution of (77) is locally polygonal inside D (the same strategy as in the proof of Theorem 2.9 would provide the result, we just need to adapt Lemma 4.1 to $H^{1/2}$-norms). It is easy to prove that (78) holds if v is supported by a subset of $\mathbb{T}$ which parametrizes a C^2 strictly convex part of $\partial\Omega_0$. Therefore, with the same proof as for Theorem 2.9, we are in position to deduce that $\partial\Omega_0 \cap D$ is nowhere C^2 with a positive curvature. But it is not clear whether estimate (78) remains valid in a more general situation and, consequently, whether $\partial\Omega_0 \cap D$ is a polygon or not.

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