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Exponential Stabilization of a Class of Nonlinear Systems : A Generalized Gronwall-Bellman Lemma Approach

Ibrahima N'Doye ^(1,2,3), Michel Zasadzinski ¹, Mohamed Darouach ¹, Nour-Eddine Radhy ², Abdelhaq Bouaziz ³

¹ Nancy Université, Centre de Recherche en Automatique de Nancy (CRAN UMR-7039) CNRS,

IUT de Longwy, 186 rue de Lorraine 54400 Cosnes et Romain, France

³ Université Hassan II Ain Chock - Ecole Supérieure de Technologie de Casablanca

Laboratoire Mécanique, Productique et Génie Industriel (MPGI)

Km 7, Route El Jadida, BP: 8012 Oasis Casablanca, Maroc

² Université Hassan II - Faculté des Sciences Ain Chock de Casablanca

Laboratoire Physique et Matériaux Microélectronique Automatique et Thermique (LP2MAT)

BP: 5366 Maarif Casablanca 20100, Maroc

Ibrahima.Ndoye@iut-longwy.uhp-nancy.fr, Michel.Zasadzinski@iut-longwy.uhp-nancy.fr

Abstract

In this paper, stabilizing control design for a class of nonlinear affine systems is presented by using a new generalized Gronwall-Bellman lemma approach. The nonlinear systems under consideration can be non Lipschitz. Two cases are treated for the exponential stabilization : the static state feedback and the static output feedback. The robustness of the proposed control laws with regards to parameter uncertainties is also studied. A numerical example is given to show the effectiveness of the proposed method.

Keywords Nonlinear affine systems, generalized Gronwall-Bellman lemma, exponential stabilization, static state feedback, static output feedback, robustness.

1 Introduction

Many physical processes may be appropriately modeled as bilinear or more general affine nonlinear systems, specially in biology (distillation columns) and mechanics (motor drives, robot manipulators).

A lot of work has been accorded to the control and state estimation of such systems which are of considerable interest in both theory and applications. In the last decades, control for these systems were developed using the linearization approach and optimal control or Lyapunov theories (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein).

Another approach to stabilize nonlinear systems is based on the Gronwall-Bellman lemma [11, 12, 13]. This lemma has been applied to the exponential stability of nonlinear affine systems in [14], to the nonlinear observer synthesis in [15], to the robust stabilization and observation of nonlinear uncertain systems in [16], to the robust stability of linear systems in [17, 18] and to finite-time stability of linear singular systems in [19]. Some useful generalized Gronwall-Bellman lemma have been proposed in [20, 21, 22, 23].

The aim of this paper is to develop the exponential stabilization of a class of nonlinear systems using a generalized Gronwall-Bellman lemma approach. The nonlinear systems considered are affine in the control, the use of the proposed generalized Gronwall-Bellman lemma allows us to consider nonlinear affine systems which are not necessary Lipschitz. Two cases are presented : the static state feedback control and the static output feedback control. The proposed stabilization methods are simple to implement and are robust with respect to parameter uncertainties.

This paper is organized as follows. The class of nonlinear systems under consideration are described in the beginning of the section 2. The exponential stabilization for the nominal case is treated in section 2.1 by considering two control laws (static state feedback and output state feedback), while the exponential

stabilization with uncertain parameters is given in section 2.2. A numerical example presented in section 3 illustrates the application of the proposed static approach to a multilinear affine nonlinear system with uncertain parameters. The proof of the proposed generalized Gronwall-Bellman lemma is given in the appendix.

Notations. $\|x\| = \sqrt{x^T x}$ and $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ are the Euclidean vector norm and the spectral matrix norm respectively where $\lambda_{\max}(A^T A)$ is the maximal eigenvalue of the symmetric matrix $A^T A$. $(f(\cdot))_i$ stands for the i^{th} component of vector $f(\cdot)$.

2 Feedback stabilization

Consider the following nonlinear affine system

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^m g_i(x(t))u_i(t) + Bu(t) \\ y(t) = Cx(t) \\ x(0) = x_0 \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector and $y(t) \in \mathbb{R}^p$ is the measured output. A , B and C are known constant matrices of appropriate dimensions.

In the sequel of the paper, we use the following assumption.

Assumption 1. *The nonlinear affine system (1) satisfies the following conditions.*

1. *The functions $g_i(x(t))$ are bounded and measurable with $g_i(0) = 0$ (for $i = 1, \dots, m$).*
2. *For $i = 1, \dots, m$, there exists an integer $q \geq 1$, such that*

$$\|g_i(x(t))\| \leq \mu_i \|x(t)\|^q \quad (2)$$

where μ_i are given positive constants.

3. *The pair (A, B) is stabilizable.*
4. *The pair (C, A) is detectable. □*

In the sequel, we define $\mu = \sum_{i=1}^m \mu_i$.

2.1 Exponential stabilization in the nominal case

First, we consider the state feedback control, and we assume that the nonlinear system (1) satisfies Assumption 1. The following theorem gives the exponential state feedback stabilization of system (1).

Theorem 1. *Under Assumption 1, the system (1) controlled by the following state feedback*

$$u(t) = Lx(t) \quad (3)$$

is exponentially stable if all eigenvalues of matrix $A + BL$ have a strictly negative real part and if

$$0 < \|x_0\| \leq \varepsilon_0, \quad (4)$$

$$\varepsilon_0^q < \beta = \frac{|\omega|}{\mu M^{q+1} \|L\|}, \quad (5)$$

where scalars $\varepsilon_0 > 0$, $M > 0$ and $\omega < 0$ are given scalars satisfying

$$\left\| e^{(A+BL)t} \right\| < M e^{\omega t} \quad \forall t \geq 0. \quad (6)$$

Moreover, there exists a strictly positive real ε_1 such that the state $x(t)$ is bounded as follows

$$\|x(t)\| \leq \frac{\varepsilon_1 M e^{\omega t} \|x_0\|}{\left(1 - \frac{M^{q+1} \mu \|L\| \varepsilon_1^q \|x_0\|^q}{|\omega|}\right)^{\frac{1}{q}}}. \quad (7)$$

Proof. With the feedback control (3), the solution of (1) is given by

$$x(t) = e^{(A+BL)t} x_0 + \int_0^t e^{(A+BL)(t-s)} \sum_{i=1}^m g_i(x(s))(Lx(s))_i \, ds \quad (8)$$

where $(Lx(t))_i$ stands for the i^{th} component of vector $Lx(t)$.

Under item 3 in Assumption 1, the gain matrix L can be chosen such that all eigenvalues of matrix $A + BL$ have a strictly negative real part.

Then there exist two reals $M > 0$ and $\omega < 0$ such that relation (6) holds and, under Assumption 1, $x(t)$ can be bounded as

$$\begin{aligned} \|x(t)\| &\leq M e^{\omega t} \|x_0\| + M e^{\omega t} \int_0^t \mu e^{-\omega s} \|L\| \|x(s)\|^{q+1} \, ds \\ &\leq M e^{\omega t} \varepsilon_0 + M e^{\omega t} \int_0^t \mu e^{-\omega s} \|L\| \|x(s)\|^{q+1} \, ds \end{aligned} \quad (9)$$

where ε_0 is defined in relation (4), or equivalently as

$$\|x(t)\| e^{-\omega t} \leq M \varepsilon_0 + \mu M \|L\| \int_0^t e^{q\omega s} \|x(s)\|^{q+1} e^{-(q+1)\omega s} \, ds. \quad (10)$$

Since, $\forall t > 0$, we have

$$h(t) = 1 - q\mu M^{q+1} \|L\| \varepsilon_0^q \int_0^t e^{q\omega s} \, ds = 1 - \frac{\mu M^{q+1} \|L\| \varepsilon_0^q}{|\omega|} (1 - e^{\omega t}) \geq 1 - \frac{\mu M^{q+1} \|L\| \varepsilon_0^q}{|\omega|} \quad (11)$$

and there exists a strictly positive real ε_2 such that

$$0 < \varepsilon_2 \leq h(t). \quad (12)$$

if inequality (5) holds.

Hence, using (11) and (12), the generalized Gronwall-Bellman given in Lemma 2 yields the following inequality

$$\|x(t)\| e^{-\omega t} \leq \frac{M \varepsilon_0}{\left(1 - q M^{q+1} \mu \|L\| \varepsilon_0^q \int_0^t e^{q\omega s} \, ds\right)^{\frac{1}{q}}}. \quad (13)$$

Using (11) and inequality (13), the following inequality

$$\|x(t)\| \leq \frac{M e^{\omega t} \varepsilon_0}{\left(1 - \frac{M^{q+1} \mu \|L\| \varepsilon_0^q}{|\omega|}\right)^{\frac{1}{q}}} = \frac{\varepsilon_1 M e^{\omega t} \|x_0\|}{\left(1 - \frac{M^{q+1} \mu \|L\| \varepsilon_1^q \|x_0\|^q}{|\omega|}\right)^{\frac{1}{q}}} \quad (14)$$

holds, where ε_1 is given a strictly positive real such that $\varepsilon_0 = \varepsilon_1 \|x_0\|$. \square

Now the case where the state of the nonlinear system is partially available through a measurement equation is considered.

Notice that item 3 in Assumption 1 is a necessary and sufficient condition for the existence of a gain matrix L such that all the eigenvalues of the matrix $A + BL$ have a strictly negative real part. But, items

3 and 4 in Assumption 1 give only necessary conditions for the existence of a gain matrix K such that all the eigenvalues of the matrix $A + BKC$ have a strictly negative real part.

As shown in [24], the design of an output feedback gain K such that all eigenvalues of matrix $A + BKC$ have a strictly negative real part is difficult to solve and leads to non convex optimization problems : there do not exist necessary and sufficient conditions on given matrices A , B and C such that there exists a gain matrix K given stable eigenvalues for the matrix $A + BKC$ (see [25] for a survey). In the literature, many authors have proposed sufficient conditions for the static output feedback stabilization problem for linear systems without guaranteeing the existence of a solution if their algorithm fails (for example, see [26, 27, 28, 29, 30, 31, 32]).

The exponential static output feedback stabilization of system (1) is given by the following theorem which is a direct extension of Theorem 1.

Theorem 2. *Under Assumption 1, the system (1) controlled by the following output feedback*

$$u(t) = Ky(t) \quad (15)$$

is exponentially stable if all eigenvalues of matrix $A + BKC$ have a strictly negative real part and if

$$0 < \|x_0\| \leq \varepsilon_0, \quad (16)$$

$$\varepsilon_0^q < \beta = \frac{|\omega|}{\mu M^{q+1} \|KC\|}, \quad (17)$$

where scalars $\varepsilon_0 > 0$, $M > 0$ and $\omega < 0$ are given reals satisfying

$$\left\| e^{(A+BKC)t} \right\| < Me^{\omega t} \quad \forall t \geq 0. \quad (18)$$

Moreover, there exists a strictly positive real ε_1 such that the state $x(t)$ is bounded as follows

$$\|x(t)\| \leq \frac{\varepsilon_1 M e^{\omega t} \|x_0\|}{\left(1 - \frac{M^{q+1} \mu \|KC\| \varepsilon_1^q \|x_0\|^q}{|\omega|}\right)^{\frac{1}{q}}}. \quad (19)$$

Proof. With the feedback control (15), the solution of (1) is given by

$$x(t) = e^{(A+BKC)t} x_0 + \int_0^t e^{(A+BKC)(t-s)} \sum_{i=1}^m g_i(x(s)) (KCx(s))_i ds \quad (20)$$

where $(KCx(t))_i$ stands for the i^{th} component of vector $KCx(t)$.

Notice that items 3 and 4 in Assumption 1 are necessary but not sufficient conditions for the existence of a gain matrix K such that all eigenvalues of matrix $A + BKC$ have a strictly negative real part, but we assume that this matrix K exists.

The sequel of the proof is omitted since it can be obtained by the relations (9) to (14) by replacing the matrix L by matrix KC and equations (4) to (6) by relations (16) to (18). \square

The proposed generalized Gronwall-Bellman lemma allows to obtain simple designs to exponentially stabilize a class of affine nonlinear systems. Indeed, unlike in the literature where the most kinds of affine nonlinear systems are Lipschitz, the nonlinear affine systems under consideration are non Lipschitz when the parameter q in item 2 of Assumption 1 is strictly superior to one. For example, this class includes multilinear affine systems in the form

$$\dot{x}(t) = Ax(t) + \underbrace{\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n A_{i,j,k} x_j(t) x_k(t)}_{g_i(x(t))} u_i(t) + Bu(t) \quad (21)$$

where $x_j(t)$ is the j^{th} component of vector $x(t)$. In this case, $q = 3$ and the scalars μ_i in relation (2) can be done as follows

$$\|g_i(x(t))\| \leq \underbrace{\| [A_{i,1,1} \ \dots \ A_{i,n,1} \ \dots \ A_{i,1,n} \ \dots \ A_{i,n,n}] \|}_{\mu_i} \|x(t)\|^3 \quad (22)$$

since

$$g_i(x(t)) = P [A_{i,1,1} \ \dots \ A_{i,n,1} \ \dots \ A_{i,1,n} \ \dots \ A_{i,n,n}] (x(t) \otimes x(t) \otimes x(t)) \quad (23)$$

where P is a given selection matrix of appropriate dimension of full row rank satisfying $\|P\| = 1$ with all components are equal to zero except one component equal to one in each row and with at most one component equal to one in each column and where \otimes is the Kronecker product satisfying $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ and $\|A \otimes B\| = \|A\| \|B\|$ (see [33], p. 408 and p. 439).

In Theorems 1 and 2, one can design the gain matrix L or K in order to maximizing the radius β of the ball defining the domain of admissible initial conditions given in relations (4) and (5) or in relations (16) and (17). This can be done by minimizing $\|L\|$ or $\|K\|$ and by imposing some constraints to obtain a sufficiently large $|\omega|$ (see (6) or (18)).

To minimizing $\|L\|$ (or $\|K\|$), one can add the following LMI optimization in the control design procedure

$$\min \alpha \quad \text{such that} \quad \begin{bmatrix} \alpha I & L \\ L & \alpha I \end{bmatrix} > 0.$$

Since, it is well known that there exists M such that $\|e^{At}\| \leq M e^{\omega t}$, $\forall t > 0$, where $\omega < 0$ is the largest real part of the eigenvalues of matrix A , we can maximize $|\omega|$ by solving the following LMI (see [34], p. 66))

$$\max \kappa \quad \text{such that} \quad P = P^T > 0 \quad \text{and} \quad A^T P + P A + 2\kappa P \leq 0$$

where $\kappa = -\omega$. The previous LMI can be included in the design procedure by replacing A by $A + BL$ or $A + BKC$.

2.2 Exponential stabilization for the uncertain systems

In this section, we show that the two stabilization theorems given in section 2.1 are intrinsically robust with respect to parameter uncertainties. We consider only the case of static output feedback, *i.e.* the case where the state of the nonlinear system is partially available through a measurement equation as in Theorem 2. Notice that the developments given in this section can be easily extended to the state feedback case as in Theorem 1.

Consider the nonlinear uncertain affine system described by a differential equation of the following form

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta_A)x(t) + \sum_{i=1}^m g_{i\Delta}(x(t))u_i(t) + (B_0 + \Delta_B)u(t) \\ y(t) = (C_0 + \Delta_C)x(t) \\ x(0) = x_0 \end{cases} \quad (24)$$

where the vectors $x(t)$, $u(t)$, $y(t)$, the constant matrices A , B and C have been defined in relation (1).

The uncertain matrices Δ_A , Δ_B and Δ_C are constant and can be rewritten as follows

$$\begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & 0 \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \end{bmatrix} \Delta [E_x \ E_u] \quad (25)$$

where Δ is a constant unknown matrix satisfying $\|\Delta\| \leq 1$ and where N_x , N_y , E_x and E_u are known given constant matrices of appropriate dimensions.

In place of items 1 and 2 in Assumption 1, it is assumed that the m functions $g_{i\Delta}(x(t))$ with unknown parameters satisfy $g_{i\Delta}(0) = 0$ and are bounded as follows

$$\|g_{i\Delta}(x(t))\| \leq \mu_{i\Delta} \|x(t)\|^q \quad (26)$$

where $q \geq 1$ is a known integer and μ_{i_Δ} are known given positive constants with $\mu_\Delta = \sum_{i=1}^m \mu_{i_\Delta}$. In addition, matrices A , B and C in items 3 and 4 in Assumption 1 are replaced by matrices A_0 , B_0 and C_0 , respectively.

The uncertain system considered in (24) and (25) with $g_{i_\Delta}(x(t)) = 0$ ($i = 1, \dots, m$) is equivalent to the following system

$$\begin{cases} \dot{x}(t) = A_0x(t) + B_0u(t) + \lambda N_x \widehat{w}(t) \\ y(t) = C_0x(t) + \lambda N_y \widehat{w}(t) \\ \widehat{z}(t) = \lambda^{-1} E_x x(t) + \lambda^{-1} E_u u(t) \\ x(0) = x_0 \end{cases} \quad (27)$$

connected with

$$\widehat{w}(t) = \Delta \widehat{z}(t) \quad (28)$$

for any given strictly positive real λ .

Then, there exists an output feedback gain K such that the uncertain systems given by (24) and (25), with $g_{i_\Delta}(x(t)) = 0$ ($i = 1, \dots, m$), is quadratically stable if and only if there exists a scaling parameter $\lambda > 0$ such that the system (27) connected with the feedback control law $u(t) = Ky(t)$ is quadratically stable with an \mathcal{H}_∞ -norm bound inferior or equal to 1 from $\widehat{w}(t)$ to $\widehat{z}(t)$ (see Corollary 3 in [35]). This quadratic stabilization implies that there exist two scalars $M > 0$ and $\omega < 0$, and an output feedback gain matrix K such that

$$\left\| e^{(A+BKC)t} \right\| < M e^{\omega t} \quad \forall t \geq 0, \quad (29)$$

where $A = A_0 + \Delta_A$, $B = B_0 + \Delta_B$ and $C = C_0 + \Delta_C$.

Using (25), we have

$$\|B\| \leq \|B_0\| + \|N_x\| \|E_u\| = \rho_B \quad \text{and} \quad \|C\| \leq \|C_0\| + \|N_y\| \|E_x\| = \rho_C. \quad (30)$$

Then, using the above developments, we can apply Theorem 2 to the uncertain system (24)-(25) by replacing μ , $\|B\|$ and $\|KC\|$ by μ_Δ , ρ_B and $\|K\| \rho_C$, respectively, and by using system (27) to design the feedback gain K .

3 Illustrative example

Consider the following uncertain nonlinear system

$$\begin{cases} \dot{x} = \left(\begin{bmatrix} -0.1 & -1 \\ -0.1763 & -1.197 \end{bmatrix} + \Delta_A \right) x + \sum_{i=1}^2 g_{i_\Delta}(x) u_i + \left(\begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix} + \Delta_B \right) u \\ y = \left(\begin{bmatrix} 0.7 & 0 \\ 0 & 0.01 \end{bmatrix} + \Delta_C \right) x \\ x(0) = x_0 \end{cases} \quad (31)$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \Delta_A = \begin{bmatrix} 0 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad \Delta_B = \begin{bmatrix} \delta_2 \\ 0 \end{bmatrix}, \quad \Delta_C = \begin{bmatrix} \delta_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad g_{1_\Delta}(x) = \begin{bmatrix} x_1^2(1 + \delta_4) \\ 0 \end{bmatrix}, \quad g_{2_\Delta}(x) = \begin{bmatrix} 0 \\ -x_1 x_2(1 + \delta_5) \end{bmatrix},$$

where $|\delta_1| \leq 0.3$, $|\delta_2| \leq 0.25$, $|\delta_3| \leq 0.2$, $|\delta_4| \leq 0.2$ and $|\delta_5| \leq 0.4$.

The matrix A_0 is unstable since its two eigenvalues are given by $\lambda_1 = 0.0423$, and $\lambda_2 = -1.3393$. The pair (A_0, B_0) is controllable and the pair (C_0, A_0) is observable.

Matrices N_x , N_y , E_x and E_u defined in relation (25) are given by

$$\begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & 0 \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \end{bmatrix} \Delta \begin{bmatrix} E_x & E_u \end{bmatrix} = \begin{bmatrix} 0 & 0.15 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ 1 & 0 & | & 0 \end{bmatrix}$$

where Δ is a constant unknown matrix satisfying $\|\Delta\| \leq 1$.

The parameters ρ_B and ρ_C defined in (30) are given by

$$\rho_B = \|B_0\| + \|N_x\| \|E_u\| = 1.0062 \quad \text{and} \quad \rho_C = \|C_0\| + \|N_y\| \|E_x\| = 0.8.$$

We have $g_{1\Delta}(0) = g_{2\Delta}(0) = 0$ and using the approach developed in (21), (22) and (23), we obtain

$$g_{1\Delta}(x) = \begin{bmatrix} x_1^2(1 + \delta_4) \\ 0 \end{bmatrix} = \begin{bmatrix} (1 + \delta_4) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (x \otimes x),$$

$$g_{2\Delta}(x) = \begin{bmatrix} 0 \\ -x_1x_2(1 + \delta_5) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(1 + \delta_5) & 0 & 0 \end{bmatrix} (x \otimes x),$$

and

$$\|g_{1\Delta}(x)\| \leq \left\| \begin{bmatrix} (1 + \delta_4) & 0 \\ 0 & 0 \end{bmatrix} \right\| \|x\|^2 \leq \sqrt{1 + |\delta_4|} \|x\|^2 = \sqrt{1.2} \|x\|^2 = \mu_{1\Delta} \|x\|^2,$$

$$\|g_{2\Delta}(x)\| \leq \left\| \begin{bmatrix} 0 & 0 \\ 0 & -(1 + \delta_5) \end{bmatrix} \right\| \|x\|^2 \leq \sqrt{1 + |\delta_5|} \|x\|^2 = \sqrt{1.4} \|x\|^2 = \mu_{2\Delta} \|x\|^2.$$

Then Assumption 1 holds where functions $g_{1\Delta}(x)$ and $g_{2\Delta}(x)$, and matrices A_0 , B_0 and C_0 are used.

The nonlinear system (31) with control $u(t) = 0$ in the nominal case is given in figure 1, which shows that it is not stable and its states do not converge to zero.

Using Theorem 2, the uncertain nonlinear system (31) controlled by the static output feedback $u(t) = Ky(t)$ is exponentially stable, where the gain matrix

$$K = [-1.8908 \quad -57.2962]$$

has been obtained by solving the “ W -problem” formulated in Theorem 2 of [30] for the system (27) with $\lambda = 1$ and an \mathcal{H}_∞ -norm inferior or equal to one. Notice that the eigenvalues of $A_0 + B_0KC_0$ are $(-0.5340, -1.8792)$. Choosing $\delta_1 = 0.15$, $\delta_2 = 0.20$, $\delta_3 = 0.18$, $\delta_4 = 0.12$ and $\delta_5 = 0.38$. Then, the time response of the uncertain system (31) controlled with $u = Ky$ is shown in figures 2 and 3.

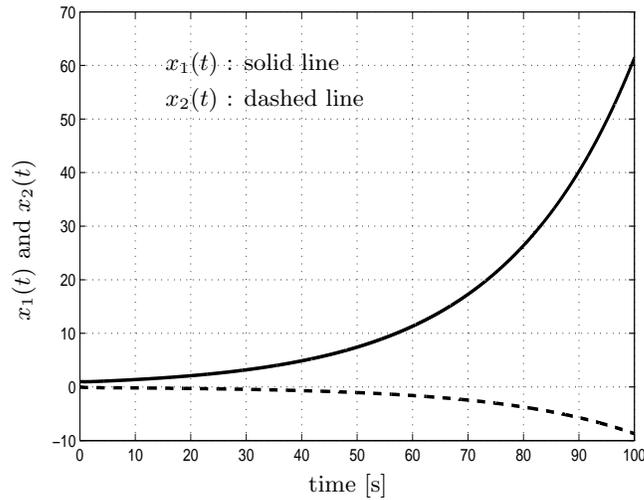


Figure 1: Time response of the system with $u(t) = 0$ in the nominal case.

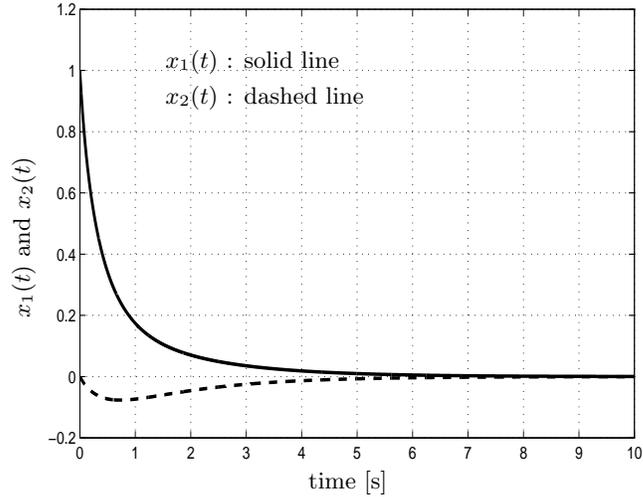


Figure 2: Time response of the system with $u(t) = Ky(t)$.

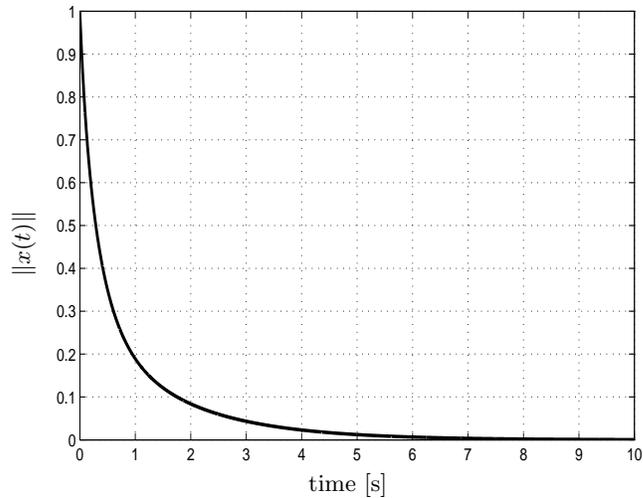


Figure 3: Time response of the system with $u(t) = Ky(t)$.

4 Conclusion

In this paper, the exponential stabilization of a class of nonlinear affine systems has been derived from the application of a generalized Gronwall-Bellman lemma which has been proved in the appendix. Two cases are studied : first, a state feedback control law is designed if all the state of the nonlinear system is available, secondly, a static output control law is proposed if the state of the nonlinear system is partially available through a measurement equation. These control laws are simple to implement. The robustness of these two exponential stabilization techniques with respect to parameter uncertainties has been studied. It should be stressed that the use of the generalized Gronwall-Bellman lemma makes it possible to consider nonlinear affine systems which can be non Lipschitz. A numerical example shows the effectiveness of the proposed approach.

Appendix : Generalized Gronwall-Bellman lemma

The “stantard” Gronwall-Bellman lemma is given in Lemma 1.

Lemma 1 (Gronwall-Bellman lemma). [13] (p 292) [12] (p 252) *Let*

- i) f, g and $k, \mathbb{R}^+ \mapsto \mathbb{R}$ and locally integrable,

ii) $g \geq 0, k \geq 0,$

iii) $g \in \mathcal{L}_\infty,$

iv) gk is locally integrable on $\mathbb{R}^+.$

If $u : \mathbb{R}^+ \mapsto \mathbb{R}$ satisfies

$$u(t) \leq f(t) + g(t) \int_0^t k(\tau)u(\tau) d\tau, \quad \forall t \geq 0 \quad (32)$$

then

$$u(t) \leq f(t) + g(t) \int_0^t k(\tau)f(\tau) \exp\left(\int_\tau^t k(s)g(s) ds\right) d\tau, \quad \forall t \geq 0. \quad (33)$$

Corollary 1 is a special case of this lemma.

Corollary 1. [13] (p 236) [12] (p 252) Let $k : \mathbb{R}^+ \mapsto \mathbb{R},$ locally integrable on \mathbb{R}^+ et $k \geq 0.$

If $u : \mathbb{R}^+ \mapsto \mathbb{R}$ satisfies

$$u(t) \leq c + \int_0^t k(\tau)u(\tau) d\tau, \quad \forall t \geq 0 \quad (34)$$

then

$$u(t) \leq c \exp\left(\int_0^t k(\tau) d\tau\right), \quad \forall t \geq 0. \quad (35)$$

The following generalized Gronwall-Bellman lemma is an extension of the works of El Alami [22, 23] and Pachpatte [11, 20, 21].

Lemma 2 (Generalized Gronwall-Bellman lemma). [23] Let

i) $a, b, k \in \mathbb{R}, 0 \leq a < b, k > 0$ and an integer $\ell > 1,$

ii) $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ an integrable function such that, $\forall \alpha, \beta \in [a, b], (0 \leq \alpha < \beta),$ we have

$$\int_\alpha^\beta f(s) ds > 0,$$

iii) $x : [a, b] \mapsto \mathbb{R}^+$ an essential bounded function such that

$$x(t) \leq k + \int_a^t f(s)(x(s))^\ell ds, \quad \forall t \in [a, b]. \quad (36)$$

If the following inequality

$$1 - (\ell - 1)k^{\ell-1} \int_a^b f(s) ds > 0, \quad (37)$$

holds, then

$$x(t) \leq \frac{k}{\left(1 - (\ell - 1)k^{\ell-1} \int_a^t f(s) ds\right)^{\frac{1}{\ell-1}}} \quad \forall t \in [a, b]. \quad (38)$$

Proof. Relation (36) can be written as follows

$$x(t) \leq k + \int_a^t \left(f(s)(x(s))^{\ell-1}\right) x(s) ds, \quad \forall t \in [a, b],$$

and, from the Corollary 1, we obtain

$$x(t) \leq k \exp\left(\int_a^t f(s)(x(s))^{\ell-1} ds\right)$$

or equivalently

$$(x(t))^{\ell-1} \leq k^{\ell-1} \exp\left((\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right). \quad (39)$$

Multiplying the above inequality by $-(\ell-1)f(t)$ gives

$$-(\ell-1)f(t)(x(t))^{\ell-1} \geq -(\ell-1)k^{\ell-1}f(t) \exp\left((\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right)$$

or equivalently

$$-(\ell-1)f(t)(x(t))^{\ell-1} \exp\left(-(\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right) \geq -(\ell-1)k^{\ell-1}f(t)$$

Using the primitive of the exponential function, the above inequality becomes

$$\frac{d}{dt} \left(\exp\left(-(\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right) \right) \geq -(\ell-1)k^{\ell-1}f(t)$$

and integrating from a to t , we obtain

$$\exp\left(-(\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right) \geq 1 - (\ell-1)k^{\ell-1} \int_a^t f(s) ds.$$

Notice that the constant in the above integration is equal to 1 (this can be shown with $t = a$).

If the inequality (37) holds, we have

$$\exp\left((\ell-1) \int_a^t f(s)(x(s))^{\ell-1} ds\right) \leq \frac{1}{1 - (\ell-1)k^{\ell-1} \int_a^t f(s) ds}. \quad (40)$$

Inequalities (39) and (40) imply

$$(x(t))^{\ell-1} k^{-(\ell-1)} \leq \frac{1}{1 - (\ell-1)k^{\ell-1} \int_a^t f(s) ds}$$

or equivalently

$$x(t) \leq \frac{k}{\left(1 - (\ell-1)k^{\ell-1} \int_a^t f(s) ds\right)^{\frac{1}{\ell-1}}}. \quad (41)$$

□

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