Discrete mode observability analysis of switching structured linear systems with unknown input: A graphical approach

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***Abstract*— Switching linear systems are described by a set of continuous state-space models together with conditions (switching law) that decide which model (mode) of this set is valid for the current continuous state. This article deals with the problem of the observability of discrete mode in which the switching law is assumed to be unknown. The formalization of the problem, based on a graph-theoretic approach, is to express sufficient conditions for generic observability of the discrete mode assuming only the knowledge of the system's structure. These conditions allow us to obtain criteria of sensor placement in order to recover the discrete mode observability using properties of the graph associated to the system. We obtain a sensor placement procedure based on classical and well-known graph theory algorithms, which have polynomial complexity orders.**

***Index Terms*— discrete mode observability, state and input observability, switching structured linear systems, graph theory.**

I. INTRODUCTION

Over the past decade, study of hybrid systems has received particular attention in several scientific fields including automation.

In general, this kind of systems can represent, through hybrid systems properties, several physical, technological and biological phenomena. It allows to model complex systems which combine the dynamics of the continuous parts of the system with the dynamics of the logic and discrete parts. Hybrid models are characterized by continuous processes (continuous differential equations) interacting with discrete processes (paradigms from discrete event systems). A hybrid system's structure is illustrated in figure I:

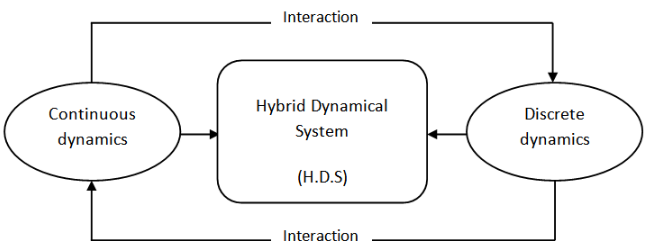

Fig I:Hybrid Dynamical System's structure

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these models have proven their efficiency for the representation of complex systems, unlike the homogeneous approach of continuous linear systems that is insufficient for modeling such systems. The transition from one mode to another one is strongly linked to the global nature of the behavior of the complex system to model. When the mode transition is abrupt, we can define a particular but very important framework of hybrid dynamical systems : switching continuous-time linear systems (SCTLS) or SLS (to simplify the notation). They are composed of a family of subsystems which are linear time-invariant and these subsystems (modes) are orchestrated by a switching law that specifies which subsystem is active. As an extension of the classical linear or affine state-space representations of dynamical systems, this modeling formalism has been thoroughly investigated through several studies.

Knowing that, the focus is on the study of observability property for SLS with unknown input, this property plays a major role in command law synthesis, fault detection and isolation, fault tolerant command law synthesis and also for perturbations rejection.

Many definitions of this property appear in the literature for SLS. For example, we quote [3], [7], [6] devoted to studying observability of hybrid systems where the discrete mode depends on the state trajectory or is associated to discrete outputs. We can also quote [14], [1], [3] for deterministic discrete-time switching linear systems. Knowing the different studies and investigations devoted to SLS, there exist several differences between deterministic discrete-time switching linear systems and deterministic continuous-time switching linear systems mainly proved in [15]. This article focuses on the definition of observability for deterministic continuous-time switching linear systems expressed in [2], [9], [15], [17], [10].

Conventional algebraic and geometric tools which are based on the numerical value of state-space matrices of system's model are needed. However, these variables are subject to parametric uncertainties due to identification processus and so, they are approximatively known.

We consider here a structured switching linear system also in state-space form, knowing that a switching linear system is structured when each entry of the matrices of its state-space form is either a fixed zero or a free parameter. The location of the fixed zeros in these matrices constitutes the structure of the system.

The approach is of interest to investigate many classical properties of structured systems that can be studied in terms of genericity. In this case, properties that are true for almost any value of the free parameters are called generic proprieties.

In order to check generic properties as controllability, observability and so on (see [11]), we can associate in natural way digraphs to structured systems and so verify structural properties by means of graph theoretic terms.

This approach also presents a major advantage. Indeed, through the association of the digraph with the structured system, we can intuitively represent the structural changes on the graph and take into account them when analyzing the property of system. This fact is very interesting knowing that a switch can be related to a change in structure, as for example in the field of electronics [16].

In this paper, our aim is to characterize the discrete mode observability for structured switching linear systems (SSLS). The outline of the paper is as follows. In section II, we expose the problem statement. After that we give some definitions and notations to the graph-theoretic approach in section III, then the main result is given in section IV and we conclude.

II. Problem statement

Consider the state-space form of switching linear systems as follows:

$$\Sigma : \begin{cases} \dot{x}(t) &= A(r_t)x(t) + B(r_t)u(t) \\ y(t) &= C(r_t)x(t) + D(r_t)u(t) \end{cases} \tag{1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ are respectively the state vector, unknown input vector and output vector and matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are of appropriate dimensions. Consider a discrete mode variable (mode sequence) as an exogenous input which is considered to be unobserved and defined by $r_t \ : [0, \infty) \ \rightarrow \ Q \overset{def}{=} \{1, \dots, N\}$. Close to [2], the switching signal is right-continuous and so impulses in state and input of SLS are excluded. The minimum dwell time is considered to avoid zeno behavior which is an undesired phenomenon for the well-definedness of SLS.

In order to have more general framework, a generic study of discrete mode observability is assumed not for all initial conditions and unknown input u but for generic ones. We deal with structured switching linear systems (SSLS) which consider only the structure of modeled system and assume independent all the real parameters of matrices $A(q)$, $B(q)$, $C(q)$, $D(q)$ for each mode $q \in Q$ of SLS.

The studied structured state space form is

$$\Sigma_\Lambda : \begin{cases} \dot{x}(t) &= A^\lambda(r_t)x(t) + B^\lambda(r_t)u(t) \\ y(t) &= C^\lambda(r_t)x(t) + D^\lambda(r_t)u(t) \end{cases} \tag{2}$$

Real parameters of this state-space form are either fixed to zero or assumed to be nonzero parameters. In the latter case, they are substituted by free parameters noted λ_i and the set of these parameters forms vector $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_h)^T$ which can take values in $\mathbb{R}^h$. $A^\lambda(q)$, $B^\lambda(q)$, $C^\lambda(q)$ and $D^\lambda(q)$ represent structured matrices for each $q \in Q$ and are obtained by replacing the non zeros parameters of $A(q)$, $B(q)$, $C(q)$, $D(q)$ for each mode $q \in Q$ by free parameters from vector Λ. To address the discrete mode observability problem, some preliminary definitions are useful:

Definition 1: (Mode distinguishability) Two modes $q \in Q$ and $q' \in Q$ (with $q \neq q'$) are distinguishable if at least one of the two following conditions holds:

- there exist an integer $s \geq 0$ and an expression $\Psi_q(y, \dot{y}, \dots, y^{(s)}) = 0$ which is satisfied for mode q but is not satisfied for mode q' for almost all initial conditions x_0 and input u.

- there exist an integer $s' \geq 0$ and an expression $\Psi_{q'}(y, \dot{y}, \dots, y^{(s')}) = 0$ which is satisfied for mode q' but is not satisfied for mode q for almost all initial conditions x_0 and input u.

Here, "for almost all initial conditions x_0 and input u" is to be understood as "for all $(x_0^T, u^T)^T \in \mathbb{R}^{n+m}$ except for the zero set of some polynomials with real coefficients in the $n + m$ initial state and input components". These polynomials can be written down explicitly, *i.e.* we can precisely describe them when the mode distinguishability fails to be true. Obviously $u(t) \equiv 0$ and $x_0 = 0$ are two of these polynomials. The zero set of some polynomial forms a proper algebraic variety of $\mathbb{R}^{n+m}$ which has Lebesgue measure zero. So, by the expression "for almost all initial conditions x_0 and input u" we mean for all initial conditions and input functions except the ones belonging to a proper algebraic variety in the state and input space.

The interpretation of Definition 1 is that q is distinguishable from q' if, for generic initial state x_0 and unknown input u, we can rule out q or q' when observing the output over $[0, T]$. Relatively to the definitions of [2], our notion of distinguishability of q and q' is equivalent to the fact that q is discernible from q' or vice-versa. The mutual mode discernibility, which is a dissymmetric property in [2], is equivalent to have both conditions of Definition 1 satisfied.

Definition 2: (Location observability) SLS (Σ) is location observable if its modes are all distinguishable two-by-two *i.e.* $\forall q \in Q$, $\forall q' \in Q$, with $q \neq q'$, q and q' are distinguishable.

Comparatively with the notion of location observability defined in [9], [10], our definition concerns as well autonomous as non-autonomous systems. In [9], [10], location observability is defined as the ability to reconstruct the mode starting from the knowledge of the input and the output, for any nonzero input value and for all initial conditions. Since we deal with unknown input systems, this definition is not applicable and it cannot be achieved for autonomous systems. In definition 2, we relax this by accepting that the reconstruction of the mode may be possible not for all but for almost all inputs and initial conditions values. To establish the observability of SLS, we have to address, in addition to location observability reduced to the study of the distinguishability of each pair of modes, the state and input observability of each mode as defined classically in [12]. To study location observability, it is pertinent and necessary to highlight the similarities and the differences between the models associated to these modes. Thus, we decompose each

structured matrix into two parts: the first one is common to the two modes and the second one is specific to each mode i.e. for $q \in Q$, $A^\lambda(q) = A_0^\lambda + A_q^\lambda$, $B^\lambda(q) = B_0^\lambda + B_q^\lambda$, $C^\lambda(q) = C_0^\lambda + C_q^\lambda$ and $D^\lambda(q) = D_0^\lambda + D_q^\lambda$ where all nonzeros entries of structured matrices are assumed as free parameters λ_i from vector Λ.
For the sake of simplicity, we consider in the later that we have only two modes $q \in \{1, 2\}$.

III. Graphical representation of structured switching linear systems

In this subsection, our aim is to present a manner of modeling structure of SSLS (Σ_Λ) taking into account different modes of the system. For a such structure, we can associate in a natural way a directed graph noted $\mathcal{G}(\Sigma_\Lambda)$ constituted by a non-empty finite set $\mathcal{V}$ of elements called vertices and a finite set $\mathcal{E}$ of ordered pairs of distinct vertices called edges (directed edges). Notation $\mathcal{G}(\Sigma_\Lambda) = (\mathcal{V}, \mathcal{E})$ means that $\mathcal{V}$ and $\mathcal{E}$ are respectively vertex set and edge set of $\mathcal{G}(\Sigma_\Lambda)$. Vertex set $\mathcal{V}$ defined by $\mathcal{V} = \mathbf{X} \cup \mathbf{U} \cup \mathbf{Y}$ corresponds to the system's variables (inputs $\mathbf{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$, states $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ and outputs $\mathbf{Y} = \{\mathbf{y_1}, \ldots, \mathbf{y_p}\}$) and edge set $\mathcal{E}$ is defined by $\mathcal{E}_0 \cup \mathcal{E}_q$. $\mathcal{E}_0$ represents the common part of both modes of SSLS and $\mathcal{E}_q$ represents the specific part for each mode. They can be respectively defined by $\mathcal{E}_0 = A_0\text{-}edges \cup B_0\text{-}edges \cup C_0\text{-}edges \cup D_0\text{-}edges$, where, $A_0\text{-}edges = \{(\mathbf{x_j}, \mathbf{x_i}) \mid A_0(i,j) \neq 0\}$, $B_0\text{-}edges = \{(\mathbf{u_j}, \mathbf{x_i}) \mid B_0(i,j) \neq 0\}$, $C_0\text{-}edges = \{(\mathbf{x_j}, \mathbf{y_i}) \mid C_0(i,j) \neq 0\}$ and $D_0\text{-}edges = \{(\mathbf{u_j}, \mathbf{y_i}) \mid D_0(i,j) \neq 0\}$ and $\mathcal{E}_q = A_q\text{-}edges \cup B_q\text{-}edges \cup C_q\text{-}edges \cup D_q\text{-}edges$ for each mode $q \in \{1, 2\}$, where, $A_q\text{-}edges = \{(\mathbf{x_j}, \mathbf{x_i}) \mid A_q(i,j) \neq 0\}$, $B_q\text{-}edges = \{(\mathbf{u_j}, \mathbf{x_i}) \mid B_q(i,j) \neq 0\}$, $C_q\text{-}edges = \{(\mathbf{x_j}, \mathbf{y_i}) \mid C_q(i,j) \neq 0\}$ and $D_q\text{-}edges = \{(\mathbf{u_j}, \mathbf{y_i}) \mid D_q(i,j) \neq 0\}$. The existence of free non zero parameters (non-zero entries) of common part $(A_0^\lambda, B_0^\lambda, C_0^\lambda, D_0^\lambda)$ of SSLS is represented by edges $e_0 \in \mathcal{E}_0$ indexed by 0 and the existence of free non-zero parameters of specific part $(A_q^\lambda, B_q^\lambda, C_q^\lambda, D_q^\lambda)$ of SSLS is represented by edges $e_q \in \mathcal{E}_q$ indexed by q for $q \in \{1, 2\}$.

Example 1: To the system defined by the following structured matrices, we associate the digraph in Figure 1.

$$A_0^\lambda = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_5 & 0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_8 & 0 & 0 & 0 & \lambda_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_0^\lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \lambda_{13} & 0 \\ 0 & 0 \\ 0 & \lambda_{14} \\ 0 & \lambda_{15} \end{pmatrix},$$

$$C_0^\lambda = \begin{pmatrix} \lambda_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{18} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{19} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{20} \end{pmatrix},$$

All the entries of A_1^λ, B_1^λ, B_2^λ are zero except $A_1^\lambda(1,3) = \lambda_{10}$, $B_1^\lambda(10,2) = \lambda_{11}$, $B_2^\lambda(9,2) = \lambda_{12}$. The elements of matrices A_2^λ, C_1^λ, C_2^λ, D_0^λ, D_1^λ and D_2^λ are equal to zero.

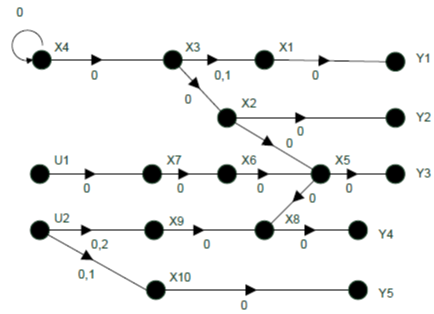

Fig. 1. *Digraph associated to system of Example 1*

A. Notations and definitions

The digraph representing the SSLS is built from the superposition of the digraphs related to each mode. In order to study the properties of the system associated to a specific mode q, we have to restrict the edge set to $\mathcal{E}_0 \cup \mathcal{E}_q$. In this context, many of the functions and specific vertex subsets, defined below, present an index q related to the considered mode.

- Two edges $e_1 = (\mathbf{v_1}, \mathbf{v_1'})$ and $e_2 = (\mathbf{v_2}, \mathbf{v_2'})$ are v-disjoint if $\mathbf{v_1} \neq \mathbf{v_2}$ and $\mathbf{v_1'} \neq \mathbf{v_2'}$.
- Some edges are v-disjoint if they are mutually v-disjoint. In **Example 1**, $(\mathbf{x_4}, \mathbf{x_3})$ and $(\mathbf{x_3}, \mathbf{x_1})$ as well as $(\mathbf{u_1}, \mathbf{x_7})$ and $(\mathbf{x_7}, \mathbf{x_6})$ are v-disjoint. It is not the case for $(\mathbf{u_2}, \mathbf{x_9})$ and $(\mathbf{u_2}, \mathbf{x_{10}})$ which have the same begin vertex and for $(\mathbf{x_2}, \mathbf{x_5})$ and $(\mathbf{x_6}, \mathbf{x_5})$ which have the same end vertex.
- A path P is denoted $P = \mathbf{v_{s_0}} \to \mathbf{v_{s_1}} \to \ldots \to \mathbf{v_{s_i}}$, where $(\mathbf{v_{s_j}}, \mathbf{v_{s_{j+1}}}) \in \mathcal{E}$ for $j = 0, 1, \ldots, i-1$. We say in this case that P covers $\mathbf{v_{s_0}}, \mathbf{v_{s_1}}, \ldots, \mathbf{v_{s_i}}$.
- A path is simple when every vertex occurs only once in this path.
- A cycle is a path of the form $\mathbf{v_{s_0}} \to \mathbf{v_{s_1}} \to \ldots \to \mathbf{v_{s_i}} \to \mathbf{v_{s_0}}$, where $\mathbf{v_{s_0}}, \mathbf{v_{s_1}}, \ldots, \mathbf{v_{s_i}}$ are distinct.
- For $q \in \{1, 2\}$, we say that path P is included in $\mathcal{E}_0 \cup \mathcal{E}_q$ if all its edges are included in $\mathcal{E}_0 \cup \mathcal{E}_q$.
- Some paths (resp. cycles) are disjoint if they have no common vertex.
- A set of disjoint cycles is called a cycle family.
- P is a $\mathbf{Y}$-*topped* path if its end vertex belongs to $\mathbf{Y}$. A

$\mathbf{Y}$-*topped* path family consists of disjoint simple $\mathbf{Y}$-*topped* paths.

• $\mathcal{V}_1$ and $\mathcal{V}_2$ represent two subsets of $\mathcal{V}$. We denote by card$(\cdot)$ the cardinality function and $\mathcal{V}_1 \setminus \mathcal{V}_2$ is the set of elements in $\mathcal{V}_1$ which are not in $\mathcal{V}_2$.

• A path $P = \mathbf{v_{s_0}} \rightarrow \mathbf{v_{s_1}} \rightarrow \ldots \rightarrow \mathbf{v_{s_i}}$ is said a $\mathcal{V}_1$–$\mathcal{V}_2$ path if $\mathbf{v_{s_0}} \in \mathcal{V}_1$ and $\mathbf{v_{s_i}} \in \mathcal{V}_2$. Moreover, if the only vertex of P which belongs to $\mathcal{V}_1$ is $\mathbf{v_{s_0}}$ and the only vertex of P which belongs to $\mathcal{V}_2$ is $\mathbf{v_{s_i}}$, P is called a direct $\mathcal{V}_1$–$\mathcal{V}_2$ path.

• For $q = \{1, 2\}$, $\rho_q\big[\mathcal{V}_1, \mathcal{V}_2\big]$ is the maximal number of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths included in $\mathcal{E}_0 \cup \mathcal{E}_q$. Moreover, a set of $\rho_q\big[\mathcal{V}_1, \mathcal{V}_2\big]$ disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths included in $\mathcal{E}_0 \cup \mathcal{E}_q$ is a maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings in $\mathcal{E}_0 \cup \mathcal{E}_q$.

• For $q \in \{1, 2\}$, $\mu_q\big[\mathcal{V}_1, \mathcal{V}_2\big]$ denotes the minimal number of vertices of $\mathbf{U} \cup \mathbf{X} \cup \mathbf{Y}$ belonging to a maximum $\mathcal{V}_1$–$\mathcal{V}_2$ linking included in $\mathcal{E}_0 \cup \mathcal{E}_q$.

• For $q \in \{1, 2\}$, $V_{ess,q}\big[\mathcal{V}_1, \mathcal{V}_2\big]$ is the vertex subset including the vertices present in all the maximum $\mathcal{V}_1$–$\mathcal{V}_2$ linkings included in $\mathcal{E}_0 \cup \mathcal{E}_q$.

• For $q \in \{1, 2\}$, there exists a unique vertex subset noted $S_q^o\big[\mathcal{V}_1, \mathcal{V}_2\big]$ and called minimum output separator which is the set of begin vertices of all direct $V_{ess,q}\big[\mathcal{V}_1, \mathcal{V}_2\big]$–$\mathcal{V}_2$ paths included in $\mathcal{E}_0 \cup \mathcal{E}_q$.

IV. Results

A. Preliminaries

First of all, we begin to introduce some existing results, in graphical terms, for continuous state and input observability of SSLS. Whole of these results are based on several works in ([5], [11], [8]). It is characterized the generic dimension of the observability subspace related to the degeneration of pencil matrix for each mode $q \in \{1, 2\}$ (due often to invariant properties [13] such as invariant zeros). In this paper, our aim, through a subdivision close to [5], [4], is to express propositions to assess the observability of the discrete mode for SSLS by using some subsets emerged from subdivision of SSLS into two distinct parts. Towards this end, the following definitions are useful.

Definition 3: Consider SSLS (Σ_Λ) associated to digraph $\mathcal{G}(\Sigma_\Lambda)$. The following vertex subsets emerge from SSLS subdivision :

• $\mathbf{X_{1,q}} \overset{def}{=} \big\{\mathbf{x_i} \,|\, \rho_q\big[\mathbf{U} \cup \{\mathbf{x_i}\}, \mathbf{Y}\big] > \rho_q\big[\mathbf{U}, \mathbf{Y}\big]\big\}$;

• $\mathbf{Y_{0,q}} \overset{def}{=} \mathbf{Y} \cap V_{ess,q}\big[\mathbf{U}, \mathbf{Y}\big]$;

• $\mathbf{Y_{1,q}} \overset{def}{=} \mathbf{Y} \setminus \mathbf{Y_{0,q}}$;

In order to rule on discrete mode observability of SSLS, we should be able to express an algebraic equation linking only output components of $\mathbf{Y_{1,q}}$ and their derivatives. This equation has to be satisfied by only one of the two modes $q \in \{1, 2\}$.
The particularities of each subset proposed above are detailed in [4].

Definition 4: Consider SSLS (Σ_Λ) associated to digraph $\mathcal{G}(\Sigma_\Lambda)$ for $q \in \{1, 2\}$. We associate integer $\beta_q(\mathbf{Y})$ defined by $\mu_q\big[\mathbf{U}, (S_q^o\big[\mathbf{U}, \mathbf{Y}\big] \cap \mathbf{X}) \cup \mathbf{Y_{0,q}}\big] - \rho_q\big[\mathbf{U}, (S_q^o\big[\mathbf{U}, \mathbf{Y}\big] \cap \mathbf{X}) \cup \mathbf{Y_{0,q}}\big]$ plus the maximal number of vertices of $\mathbf{X_{1,q}} \cup S_q^o\big[\mathbf{U}, \mathbf{Y}\big]$ covered by a disjoint union of :

- a $S_q^o\big[\mathbf{U}, \mathbf{Y}\big]$–$\mathbf{Y_{1,q}}(\mathbf{Y})$ linking of maximal size;
- a $\mathbf{Y_{1,q}}$-*topped* path family ;
- a cycle family covering only elements of $\mathbf{X_{1,q}}$.

As expressed in Lemma 3 of [4], $\beta_q(\mathbf{Y})$ is equal to the generic dimension of the observable subspace in the extended state and input space (x^T, u^T) for each $q \in \{1, 2\}$.

Definition 5: (strongly connected component)
Two vertices $\mathbf{v_i}$ and $\mathbf{v_j}$ are said to be strongly connected if it exists a path from $\mathbf{v_i}$ to $\mathbf{v_j}$ and a path from $\mathbf{v_j}$ to $\mathbf{v_i}$.It is assumed that a vertex is strongly connected to itself. The relation " is strongly connected to " is an equivalence relation and we can define its equivalence classes. Each equivalent class is called a strongly connected component

Definition 6: (maximal elements subset)
Let's " $\leq$ " a partial order relation on S, $\{a, b\} \in S$ and $a \leq b$. a is a maximal element of S if $a = b$. A set of maximal elements of S is denoted by $max(S)$. A same definition can be formulated for a set of minimal elements of S denoted by $min(S)$.

B. Discrete mode observability of SSLS

Hypothesis 1: SSLS is assumed to be continuous state and input observable for each mode $q \in \{1, 2\}$.
In this part, we will treat the observability of the discrete mode for SSLS with unknown input and so based only on the measurements given by the output set $\mathbf{Y}$.

Proposition 1: A sufficient condition for location observability of SSLS represented by digraph $\mathcal{G}(\Sigma_\Lambda)$ is:

• There exists $e_\kappa = (\mathbf{v_i}, \mathbf{v_j})$ a specific edge of one mode $q \in \{1, 2\}$ and $\mathbf{y_i} \in \mathbf{Y_{1,q}}$ such that there exists $\mathbf{y_i} - topped$ path of length strictly greater than $d_q(\mathbf{y_i}) = \beta_q(\mathbf{y}) - \beta_q(\mathbf{y} \setminus \{\mathbf{y_i}\})$ which covers e_κ ended by vertex $\mathbf{v_j}$.

• $\mathbf{v_j}$ belongs to a direct $s_q^o[max(\mathbf{u}, \mathbf{x_{1,q}}), \mathbf{y_{1,q}}]$-$\mathbf{y_i}$ path included in $\mathcal{E}_0 \cup \mathcal{E}_q$.

Proof:
Sufficiency:
The fact that, for some q, $\mathbf{y_i}$ belongs to $\mathbf{Y_{1,q}}$ implies that there exists a vertex subset $\mathbf{Y_u} \subseteq \mathbf{Y_{1,q}} \setminus \{\mathbf{y_i}\}$ such that $\rho_q\big[S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_u}\big] = \text{card}(max(\mathbf{U}, \mathbf{X_{1,q}}))$. This implies, from Lemma 2 of [4], that there exist a matrix G, a function φ and an integer $\nu \leq n_1$ such that the dynamics equation of subsystem $(\Sigma_{1,q})$ can be put on form:

$$\begin{aligned}\dot{X}_{1,q} &= \big(A_{1,1} + (A_{1,s},\, B_{1,1})G\big)X_{1,q} \\ &\quad + \varphi_x(Y_u,\, \dot{Y}_u,\, \ldots, Y_u^{(\nu)}) \\ &\overset{def}{=} \tilde{A}X_{1,q} + \varphi_x(Y_u,\, \dot{Y}_u,\, \ldots, Y_u^{(\nu)}) \\ Y_{1,q} &= \big(C_{1,1} + (C_{1,s},\, D_{1,1})G\big)X_{1,q} \\ &\quad + \varphi_y(Y_u,\, \dot{Y}_u,\, \ldots, Y_u^{(\nu)}) \\ &\overset{def}{=} \tilde{C}X_{1,q} + \varphi_y(Y_u,\, \dot{Y}_u,\, \ldots, Y_u^{(\nu)}) \qquad (3)\end{aligned}$$

Moreover, by definition of $d_q(\mathbf{y_i})$, we have that, there exists a minimal subset $\tilde{\mathbf{Y}} \subseteq \mathbf{Y_{1,q}}(\mathbf{Y}) \setminus (\mathbf{Y_u} \cup \{\mathbf{y_i}\})$, such that $\forall k \geq d_q(\mathbf{y_i})$,

$$y_i^{(k)} = \sum_{s<\tilde{k}_i} \alpha_{i,s} y_i^{(s)} + \sum_{l \mid \mathbf{y}_l \in \tilde{\mathbf{Y}}} \sum_{s=0}^{n_1} \alpha_{l,s} y_l^{(s)} + v(Y_u, \ldots, Y_u^{(n_1)}) \tag{4}$$

where $n_1 = \text{card}(\mathbf{X_{1,q}})$. Since subset $\tilde{\mathbf{Y}}$ is minimal *i.e.* $\forall \mathbf{y_j} \in \tilde{\mathbf{Y}}, \beta_q\big((\tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\} \cup \mathbf{Y_u}) \setminus \{\mathbf{y_j}\}\big) - \beta_q(\mathbf{Y_u} \cup \tilde{\mathbf{Y}} \setminus \{\mathbf{y_j}\}) > \tilde{k}_i$, then in relation (4), all the components of $\tilde{\mathbf{Y}}$ are present. Let us denote by $max(\mathbf{U}, \mathbf{X_{1,q}}) = \mathbf{x_j}$ the begin vertex of the so-called path P satisfying condition of Proposition 1 (*i.e.* P is a $\mathbf{y_i}$-*topped* path of length $k+1$ strictly greater than $d_q(\mathbf{y_i})$ and covering e_s) and e_j the j^{th} Euclidean vector. Relation (4) can be written as:

$$\tilde{C}_i \tilde{A}^k \mathrm{e}_j = \left(\sum_{s<d_q(\mathbf{y_i})} \alpha_{i,s} \tilde{C}_i \tilde{A}^s + \sum_{\mathbf{y}_l \in \tilde{\mathbf{Y}}} \sum_{s=0}^{n_1} \alpha_{l,s} \tilde{C}_l \tilde{A}^s + v(Y_u, \ldots, Y_u^{(n_1)}) \right) \mathrm{e}_j \tag{5}$$

where each non-zero component of $\tilde{C}_l \tilde{A}^s$ is associated to the paths arriving to $\mathbf{y}_l \in \tilde{\mathbf{Y}}$ of length $s+1$. Since all the $\{\mathbf{x_j}\}$-$\tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}$ paths starting from $\mathbf{x_j}$ cover, by definition, $S_q^o\big[\{\mathbf{x_j}\}, \tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}\big] \overset{def}{=} \{\mathbf{x_r}\}$, then there exist k_r and k' such that $k_r + k' = k$ and $\tilde{C}_i \tilde{A}^k \mathrm{e}_j = \tilde{C}_i A^{k_r} \Delta_r \tilde{A}^{k'} \mathrm{e}_j$ where Δ_r is a diagonal matrix which has only one non-zero element $\Delta_r(r,r) = 1$. We can do the same reasoning for each term $\tilde{C}_l \tilde{A}^s \mathrm{e}_j$ and so there exist s_r and s' such that $s_r + s' = s$ and $\tilde{C}_l \tilde{A}^s \mathrm{e}_j = \tilde{C}_l \tilde{A}^{s_r} \Delta_r \tilde{A}^{s'} \mathrm{e}_j$. The fact that end vertex of e_κ i.e. $\mathbf{x}_\ell$ belongs to a direct $S_q^o[\{\mathbf{x_j}\}, \mathbf{Y_{1,q}}]$-$\mathbf{y_i}$ path implies that specific edge $e_\kappa \in \mathcal{E}_q$ belongs to a $S_q^o\big[\mathbf{v_P}, \tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}\big]$–$\tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}$ path. This means that edge e_κ appears in only some $S_q^o\big[max(\mathbf{U}, \mathbf{X_{1,q}}), \tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}\big]$–$\tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}$ paths. Thus, some terms $\tilde{C}_i A^{k_r}$ and $\tilde{C}_l \tilde{A}^{s_r}$, but not all, contain the non-zero parameter corresponding to edge e_κ, which is specific to mode q. Denoting by $C_r = \mathrm{e}_r^T$, where e_r is the r^{th} Euclidean vector, we have that $\tilde{C}_i \tilde{A}^k \mathrm{e}_j = \tilde{C}_i A^{k_r} \Delta_r \tilde{A}^{k'} \mathrm{e}_j = \alpha' C_r \tilde{A}^{k'} \mathrm{e}_j$ and $\tilde{C}_l \tilde{A}^s \mathrm{e}_j = \tilde{C}_l \tilde{A}^{s_r} \Delta_r \tilde{A}^{s'} \mathrm{e}_j = \alpha'_{l,s} C_r \tilde{A}^{s'} \mathrm{e}_j$. Thus, after substitution of the previous terms in relation (**??**),

$$\alpha' C_r \tilde{A}^{k'} \mathrm{e}_j = \Big(\sum_{s_r \leq s < d_q(\mathbf{y_i})} \alpha'_{i,s} \alpha_{i,s} C_r \tilde{A}^{s-s_r} + \sum_{l \mid \mathbf{y}_l \in \tilde{\mathbf{Y}}} \sum_{s=s_r}^{n_1} \alpha'_{l,s} \alpha_{l,s} C_r \tilde{A}^{s-s_r} + v(Y_u, \dot{Y}_u, \ldots, Y_u^{(n)}) \Big) \mathrm{e}_j \tag{6}$$

where some coefficients α' and $\alpha'_{l,s}$ but not all depend on the weight of e_κ. This weight cannot be factorized and simplified because all the coefficients do not depend on it (some $S_q^o\big[max(\mathbf{U}, \mathbf{X_{1,q}}), \tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}\big]$–$\tilde{\mathbf{Y}} \cup \{\mathbf{y_i}\}$ paths do not contain edge e_κ). Therefore, equality (6) is valid only if some of the coefficients α, $\alpha_{i,s}$ and $\alpha_{l,s}$ depend also on the weight λ_κ of e_κ. Thus, by means of equation (4) in which appear coefficients $\alpha_{i,s}$ and $\alpha_{l,s}$, we obtain an algebraic relation depending on λ_κ and satisfied only when the discrete mode variable is equal to q.□

In **Example 1** above, a partial order of strongly connected components for both modes $q \in \{1, 2\}$ is:

- $\mathbf{y_1} \leq \mathbf{x_1} \leq \mathbf{x_3} \leq \mathbf{x_4}$.
- $\mathbf{y_2} \leq \mathbf{x_2} \leq \mathbf{x_3} \leq \mathbf{x_4}$.
- $\mathbf{y_3} \leq \mathbf{x_5} \leq \mathbf{x_2} \leq \mathbf{x_3} \leq \mathbf{x_4}$.
- $\mathbf{y_4} \leq \mathbf{x_5} \leq \mathbf{x_2} \leq \mathbf{x_3} \leq \mathbf{x_4}$.
- $\mathbf{y_3} \leq \mathbf{x_5} \leq \mathbf{x_6} \leq \mathbf{x_7} \leq \mathbf{u_1}$.
- $\mathbf{y_4} \leq \mathbf{x_8} \leq \mathbf{x_5} \leq \mathbf{x_6} \leq \mathbf{x_7} \leq \mathbf{u_1}$.
- $\mathbf{y_4} \leq \mathbf{x_8} \leq \mathbf{x_9} \leq \mathbf{u_2}$.
- $\mathbf{y_5} \leq \mathbf{x_{10}} \leq \mathbf{u_2}$.

and then we have that $\mathbf{X_{1,1}} = \mathbf{X_{1,2}} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_4}, \mathbf{x_8}, \mathbf{x_9}, \mathbf{x_{10}}\}$, $\mathbf{Y_{0,1}} = \mathbf{Y_{0,2}} = 0$, $\mathbf{Y_{1,1}} = \mathbf{Y_{1,2}} = Y$ and $max(\mathbf{U}, \mathbf{X_{1,q}}) = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{x_4}\}$ thus $S_q^o\big[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_1}\big] = \{\mathbf{u_2}, \mathbf{x_3}, \mathbf{x_5}\}$.

As example, $(\mathbf{x_3}, \mathbf{x_1})$ is a specific edge to mode 1. Since, for both modes, $S_q^o\big[\mathbf{U}, \mathbf{Y}\big] \cap \mathbf{X} = \{x_5\}$, we can then calculate $\mu_q\big[\mathbf{U}, (S_q^o\big[\mathbf{U}, \mathbf{Y}\big] \cap \mathbf{X}) \cup \mathbf{Y_{0,q}}\big] - \rho_q\big[\mathbf{U}, (S_q^o\big[\mathbf{U}, \mathbf{Y}\big] \cap \mathbf{X}) \cup \mathbf{Y_{0,q}}\big] = 4 - 1 = 3$. In this case, we have that $d_1(\mathbf{y_1}) = \beta_1\big(\mathbf{Y}\big) - \beta_1(\mathbf{Y} \setminus \{\mathbf{y_1}\}) = 12 - 11 = 1$, this implies that output y_1 allows us to observe $d_1(\mathbf{y_1}) = 1$ new directions.

Let us search a $\mathbf{y_1} - topped$ path P which length is strictly greater than $d_1(\mathbf{y_1})$ and including specific edge $(\mathbf{x_3}, \mathbf{x_1})$, we can choose $P = \mathbf{x_3} \to \mathbf{x_1} \to \mathbf{y_1}$, whose length is equal to 2.

We also have that the ending vertex $\mathbf{x_1}$ of the specific edge $(\mathbf{x_3}, \mathbf{x_1})$ for mode 1 belongs to a $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_1}$ path. So, Conditions of proposition 1 are satisfied.

In the same manner, if we take $(\mathbf{u_2}, \mathbf{x_9})$ as a specific edge of mode 2, we have $d_1(\mathbf{y_4}) = \beta_1\big(\mathbf{Y}\big) - \beta_1(\mathbf{Y} \setminus \{\mathbf{y_4}\}) = 12 - 10 = 2$. Let us find a path P of length strictly greater than $d_1(\mathbf{y_4})$ and including specific edge $(\mathbf{u_2}, \mathbf{x_9})$, we can choose $P = \mathbf{u_2} \to \mathbf{x_9} \to \mathbf{x_8} \to \mathbf{y_4}$ which length is equal to 3. We have also that the ending vertex $\mathbf{x_9}$ of the specific edge $(\mathbf{u_2}, \mathbf{x_9})$ belongs to a $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_4}$ path. (So, conditions of proposition 1 are satisfied).

When the proposition 1 is not satisfied, an additional sensor is required to recover discrete mode observability of SSLS. To do so, we define a new output vector $\mathbf{Z}$ representing the additional sensors collecting new measurements $z(t) = H_x^\lambda(r_t)x(t) + H_u^\lambda(r_t)u(t)$. The completed system is denoted by Σ_Λ^c and a structured state-space form is as following:

$$\Sigma_\Lambda^c : \begin{cases} \dot{x}(t) &= A^\lambda(r_t)x(t) + B^\lambda(r_t)u(t) \\ y(t) &= C^\lambda(r_t)x(t) + D^\lambda(r_t)u(t) \\ z(t) &= H_x^\lambda(r_t)x(t) + H_u^\lambda(r_t)u(t) \end{cases} \tag{7}$$

The additional sensor components can be represented by vertex set Z and edge subsets $H_x - edges$ and $H_u - edges$.

Proposition 2: Consider SSLS (Σ_Λ) associated to digraph $\mathcal{G}(\Sigma_\Lambda)$, when conditions of Proposition 1 are not satisfied. Let $\mathbf{v_i} \in \mathcal{V}_i$ be the ending vertex of a specific edge associated to one mode $q \in \{1, 2\}$ which belongs to any direct $max(\mathbf{U}, \mathbf{X_{1,q}}) - S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$ path. In order to ensure that $\mathbf{v_i}$ belongs to any direct $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}] - \mathbf{Y_{1,q}}$ path, an additional sensor

$\mathbf{z_i} \in Z$ has to be placed on vertex $\mathbf{v_i}$ or on $\mathbf{v_j} \in \mathcal{V}_j$ such that $\mathcal{V}_i \leq \mathcal{V}_j$.
After sensor placement, the length condition has to be checked in order to satisfy **Proposition 1**.

Proof.

Let $\mathbf{v_i} \in \mathcal{V}_i$ be the ending vertex of a specific edge for one mode $q \in \{1, 2\}$ and $\mathbf{v_j} \in \mathcal{V}_j$ such that $\mathbf{v_i}$ and $\mathbf{v_j}$ belong to any direct $max(\mathbf{U}, \mathbf{X_{1,q}}) - S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$ path and $\mathcal{V}_i \leq \mathcal{V}_j$.
A sensor placement $\mathbf{z_i} \in Z$ on $\mathbf{v_i} \in \mathcal{V}_i$ such that $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}] \leq \mathcal{V}_i$ makes elements from $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$ non essential. it is due to the addition of a new direct $max(\mathbf{U}, \mathbf{X_{1,q}}) - \{\mathbf{z_i}\}$ path which covers only $\mathbf{v_i}$ (it can be the latest essential vertex in $max(\mathbf{U}, \mathbf{X_{1,q}}) - \{\mathbf{z_i}\}$) and then $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}] \leq S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}} \cup \{\mathbf{z_i}\}]$, so, $\mathbf{v_i}$ belongs to a direct $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}} \cup \{\mathbf{z_i}\}]$-$\mathbf{Y_{1,q}}$ path.
The same reasoning can be done when an additional sensor $\mathbf{z_j} \in Z$ is placed to measure $\mathbf{v_j} \in \mathcal{V}_j$ such that $\mathcal{V}_i \leq \mathcal{V}_j$.

Comments and interpretation

An additional sensor does not add a path from input $\mathbf{U}$ to output $\mathbf{Y}$ and so the cardinality of minimal output separator does not increase. The aim of proposition 2 is to formalize sensor placement problem in order to have the ending vertex of specific edge belonging to a direct $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_i}$ path included in $\mathcal{E}_0 \cup \mathcal{E}_q$ under constraint that the length condition have to be checked after this placement.
A length condition can be satisfied after a finite sensor placement iterations which is estimated as a number of vertices located at distance of $\mathbf{y_i}$, this distance is equal to the length of $max(\mathbf{U}, \mathbf{X_{1,q}}) - \mathbf{y_i}$ which covers specific edge.□
Consider **Example 2** which keeps the same digraph structure of example 1 and consider new entries $A_1^\lambda(3, 4) = \lambda_{21}$ for mode 1 and $A_2^\lambda(6, 7) = \lambda_{22}$ for mode 2. The figure below shows the digraph's structure of example 2. Note that the

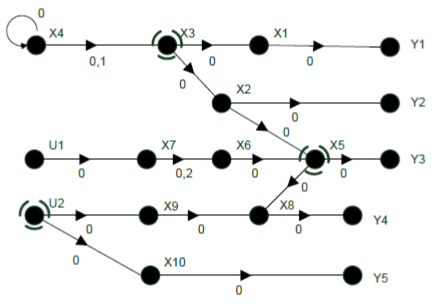

Fig. 2. *Digraph associated to system of Example 2*

partial order of strongly connected components is the same for the examples 1 and 2.
Ending vertex $\mathbf{x_3}$ of specific edge $(\mathbf{x_4}, \mathbf{x_3})$ of mode 1 and ending vertex $\mathbf{x_6}$ of a specific edge $(\mathbf{x_7}, \mathbf{x_6})$ of mode 2 do not belong respectively to $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_1}$ path and $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_4}$ path for both modes $q \in \{1, 2\}$. So the condition of **Proposition 1** is not satisfied.

In order to satisfy these conditions to recover the discrete mode observability of SSLS of example 2, spreading procedure of **Proposition 2** is needed.

- For mode 1, if an additional sensor $\mathbf{z_1}$ is placed to measure state vertex $\mathbf{x_4}$ then $\mathbf{x_3}$ belongs to a $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_1}$ path. After that , the length condition should be verified. We have that $d_1(\mathbf{y_1}) = \beta_1(\mathbf{Y} \cup \{\mathbf{z_1}\}) - \beta_1(\mathbf{Y} \setminus \{\mathbf{y_1}\}) = 12 - 11 = 1$. We choose then a $\mathbf{y_1} - topped$ path $P = \mathbf{x_4} \rightarrow \mathbf{x_3} \rightarrow \mathbf{x_1} \rightarrow \mathbf{y_1}$ which covers specific edge $(\mathbf{x_4}, \mathbf{x_3})$ of mode 1 and its length is greater than 2. Condition of **Proposition 1** is then satisfied.
- For mode 2, if an additional sensor $\mathbf{z_2}$ is placed to measure state vertex $\mathbf{x_7}$ or $\mathbf{u_1}$ then $\mathbf{x_6}$ belongs to a $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_4}$ path. After that , the length condition should be verified. We have that $d_2(\mathbf{y_4}) = \beta_2(\mathbf{Y} \cup \{\mathbf{z_2}\}) - \beta_2(\mathbf{Y} \setminus \{\mathbf{y_4}\}) = 12 - 10 = 2$. We choose then a $\mathbf{y_4} - topped$ path $P = \mathbf{x_7} \rightarrow \mathbf{x_6} \rightarrow \mathbf{x_5} \rightarrow \mathbf{x_8} \rightarrow \mathbf{y_4}$ which covers specific edge $(\mathbf{x_7}, \mathbf{x_6})$ of mode 2 and its length is greater than 3. The condition of **Proposition 1** is then satisfied. Figure 3 illustrates sensors placement of example 2 to recover the discrete mode discernability.

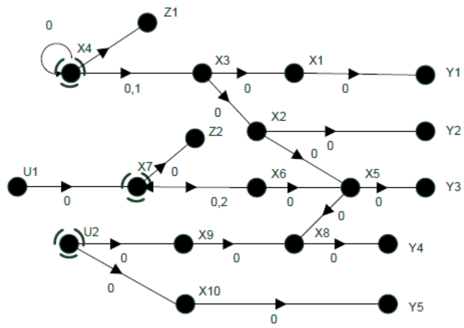

Fig. 3. *Sensor placement recovering discrete mode observability of SSLS*

Note that after sensor placement, the output separator (illustrated by dashed line circles) is $S_q^o\left[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_1} \cup \{\mathbf{z_1}, \mathbf{z_2}\}\right] = \{\mathbf{u_2}, \mathbf{x_4}, \mathbf{x_7}\}$.

C. *Algorithmic aspect and complexity*

The evaluation of the complexity of an algorithm plays a key role in its effectiveness in treating data.The efficiency of an algorithm is strongly linked to the analysis of the amount of time taken by algorithm to run (time complexity) independently of any hardware or software support or storage locations (space complexity) in order to estimate memory space required for implementation.
Estimating complexity of the algorithms proposed to verify discrete mode observability is estimated as follows:

Step 0:Identification of strongly connected components of $\mathcal{G}(\Sigma_\Lambda)$ using Tarjan's algorithm with $O(|V| + |E|)$ time complexity ($|V|$ and $|E|$ denote respectively cardinality of vertex set and edge set of digraph).

Step 1:Scheduling strongly connected components of $\mathcal{G}(\Sigma_\Lambda)$ by selection sort algorithm with $O(n^2)$

time complexity (n: number of strongly connected components in digraph of SSLS).

Step 2: Identification of output separator by Denic, Edmonts and Karp algorithm (D.E.K) with $O(|V|^2.|E|)$ time complexity.

Step 3: Evaluating the location of ending vertices of specific edges for each mode relative to the output separator specified in proposition 1 and proposition 2, the evaluation can be done by Depth-first search (DFS) with $O(|V|+|E|)$ time complexity.

Step 4: If ending vertex of specific edge of mode $q \in \{1,2\}$ belongs to a direct $S_q^o[max(\mathbf{U}, \mathbf{X_{1,q}}), \mathbf{Y_{1,q}}]$-$\mathbf{y_i}$ path for some $\mathbf{y_i} \in \mathbf{Y_{1,q}}$ then go to **Step 5**, else sensor placement procedure from **Proposition 2** is needed and go to **Step 1**. For this step, we use Dijkstra's algorithm whose complexity order equals to $O(|V|^2)$.

Step 5: If the length condition is satisfied then **Proposition 1** is satisfied else a iterative sensor placement procedure is needed until satisfying the length condition. The evaluation of length condition requires an algorithm of complexity order $O(|V|^6)$ [4].

Therefore all the algorithms proposed above have a polynomial complexity and thus the conditions mentioned in this article can be implemented using a polynomial complexity order algorithm $O(|V|^6)$.

V. Conclusion

This paper proposes, through an intuitive graphical approach, a sufficient condition to investigate discrete mode observability of switching structured linear systems with unknown input. Under assumption that continuous state and input of SSLS are observable as it is widely treated in [5], we propose a graphical criterion of sensor placement to recover the discrete mode observability of SSLS when this property is not satisfied.

Rule on this property, only through the knowledge of the structure of the system, makes our approach interesting for the analysis of large scale systems using graph-theoretical techniques. Finally, two examples have been presented to illustrate the usefulness of the theoretic results.

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