Ensuring the boundedness of the core of games with restricted cooperation

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Abstract

The core of a cooperative game on a set of players N is one of the most popular concepts of solution. When cooperation is restricted (feasible coalitions form a subcollection $\mathcal{F}$ of 2^N), the core may become unbounded, which makes its usage questionable in practice. Our proposal is to make the core bounded by turning some of the inequalities defining the core into equalities (additional efficiency constraints). We address the following mathematical problem: can we find a minimal set of inequalities in the core such that, if turned into equalities, the core becomes bounded? The new core obtained is called the restricted core. We completely solve the question when $\mathcal{F}$ is a distributive lattice, introducing also the notion of restricted Weber set. We show that the case of regular set systems amounts more or less to the case of distributive lattices. We also study the case of weakly union-closed systems and give some results for the general case.

Keywords: cooperative game, core, restricted cooperation, bounded core, Weber set

1 Introduction

In cooperative models, one of the main issues is to define in a rational way the sharing of the total worth of a game among the players, what is usually called the *solution* of the game. The core is perhaps the most popular concept of solution, because it is built on a very simple rationality criterion: no coalition should receive less than that it can earn by itself, thus avoiding any instability in the game (this is often called *coalitional rationality*). The core is a bounded convex polyhedron whenever nonempty, and its properties have been studied in depth (see, e.g., [26, 21, 24]).

The classical setting of cooperative games stipulates that any player can (fully) participate or not participate to the game, and that any coalition can form. This too simplistic framework has been made more flexible in many respects, or more tailored to some special kind of applications by many authors: let us cite on the one hand multichoice games [20, 25], games with multiple alternatives [3, 7] and bicooperative games [4, 22] (participation is gradual, can be positive or negative, or the player has several options), and on the other hand, games with restricted cooperation, where only a limited set of coalitions are allowed to form. A vast literature is devoted to this last category, studying various possibilities for the algebraic structure of the set of *feasible* coalitions: games on antimatroids [2], convex geometries [5], lattices [9, 11, 15], graphs [27, 29], etc.

Our study will concern games with restricted cooperation, and especially the core of such games. Here also, there is a vast literature we will not cite here (see a recent survey by the author on this topic [14]). Indeed, the study of the core in such a general situation becomes much more challenging: since the core is defined by a system of linear inequalities, it is always a polyhedron, however it need not be bounded any more, and it may even have no vertices. As a matter of fact, since the core is supposed to represent a set of payoffs for players, boundedness is perhaps the property one wants to keep in priority (arbitrarily large payoffs cannot exist in reality). Therefore, a natural question arises: *How to make the core bounded in any case, keeping the spirit of its definition?* By "spirit of definition", we mean the essential idea of coalitional rationality. A very simple answer to this question was proposed by Grabisch and Xie [16, 17]: turn some of the inequalities into equalities, which can be seen as adding supplementary *binding* constraints, while preserving coalitional rationality. The authors proposed a systematic way of doing this for games on distributive lattices, according to some interpretation related to the hierarchy of players.

We want to take here a more general and mathematical point of view. Specifically, we address the following question: *Suppose v is a game with restricted cooperation, whatever the structure of its set of feasible coalitions. Can we find a minimal set of inequalities in the core of v such that, if turned into equalities, the core will become bounded?* A second question is: what about the Weber set? Can we define it so that the classical property of inclusion of the core into the Weber set is preserved?

We give a complete answer to these questions for games on distributive lattices, —thus generalizing and simplifying results of Grabisch and Xie, and partial answers for other structures and the general case.

The paper is organized as follows. Section 2 introduces the basic material for the paper: set systems, posets and lattices, etc. We also explain our main idea to make the core bounded. Section 3 studies the case of distributive lattices. It gives an optimal algorithm finding which inequalities must be turned into equalities. Also, it introduces the notion of restricted Weber set, and shows that the classical result of inclusion of the core into the Weber set still holds. Section 4 studies the general case. A first result shows that if rays have a certain form, one can treat an equivalent problem where the set system is a distributive lattice, and therefore benefit from results of Section 3. It is shown that regular set systems fall into this category. However, for weakly union-closed systems, an additional condition on the set system is required. We give also an algorithm to find all extremal rays of the core of a game on a regular set system.

We assume some familiarity of the reader with polyhedra. To avoid a heavy notation, we often omit braces and commas for singletons and sets, writing e.g, $N \setminus i$ instead of $N \setminus \{i\}$, 123 instead of $\{1, 2, 3\}$, etc.

2 Preliminaries

2.1 Games on set systems

We consider $N := \{1, \dots, n\}$ the set of players, agents, etc. A *set system* $\mathcal{F}$ on N is a collection of subsets of N containing N and $\emptyset$. One can think of $\mathcal{F}$ as the collection of *feasible coalitions*, and when $\mathcal{F} \subset 2^N$ we commonly speak of *restricted cooperation*. A *game* on $\mathcal{F}$ is a function $v : \mathcal{F} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

A *payoff vector* x is any vector in $\mathbb{R}^N$, which defines the amount of money given to each player. It is common to use the notation $x(S)$ where $S \in 2^N$, as a shorthand for $\sum_{i \in S} x_i$, with the convention $x(\emptyset) := 0$. The *core* of a game v on $\mathcal{F}$ is the set of payoff vectors being *coalitionally rational*, in the sense that any feasible coalition S gets at least what it can achieve by itself, namely $v(S)$:

$$\mathcal{C}(\mathcal{F}, v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S), \forall S \in \mathcal{F}, x(N) = v(N)\}.$$

The equality $x(N) = v(N)$ is known as the *efficiency condition*. It means that no more than $v(N)$ can be distributed among the players whatsoever, and distributing strictly less would be inefficient (the definition makes sense only if the grand coalition N is the best way to make profit).

By definition, the core is a closed convex polyhedron, however it may be unbounded or even may contain a line, in which case there are no vertices (see in [14] a survey of the properties of the core of games on set systems). We denote by $\mathcal{C}(\mathcal{F}, 0)$ the *recession cone* of $\mathcal{C}(\mathcal{F}, v)$, that is, the cone defined by

$$\mathcal{C}(\mathcal{F}, 0) := \{x \in \mathbb{R}^N \mid x(S) \geq 0, \forall S \in \mathcal{F}, x(N) = 0\}.$$

We need to recall some fundamental facts of the theory of polyhedra (see, e.g., [12]) in the following lemma (valid for any polyhedron, but we express them in our case).

Lemma 1. For any game v on a set system $\mathcal{F}$,

(i) $\mathcal{C}(\mathcal{F}, v)$ has rays (but no line) if and only if $\mathcal{C}(\mathcal{F}, 0)$ is a pointed cone different from $\{0\}$;

(ii) $\mathcal{C}(\mathcal{F}, v)$ is pointed (i.e., has vertices) if and only if $\mathcal{C}(\mathcal{F}, 0)$ does not contain a line, or equivalently, if the system $x(S) = 0, \quad \forall S \in \mathcal{F}$, has 0 as unique solution.

(iii) $\mathcal{C}(\mathcal{F}, v)$ is bounded if and only if $\mathcal{C}(\mathcal{F}, 0) = \{0\}$.

Therefore, if the core is unbounded, the extremal rays and lines of $\mathcal{C}(\mathcal{F}, 0)$ are exactly those of $\mathcal{C}(\mathcal{F}, v)$ for any game v, which shows that we can restrict our study to $\mathcal{C}(\mathcal{F}, 0)$.

2.2 Main families of set systems

Among the numerous families of sets systems, we put emphasis on three of them: distributive lattices of height n, regular set systems, and weakly union-closed systems. These three families are distinct (see [14] for details and other related families).

We begin by the simplest family. A set system $\mathcal{F}$ is *weakly union-closed* if for any $S_1, S_2 \in \mathcal{F}$ such that $S_1 \cap S_2 \neq \emptyset$, we have $S_1 \cup S_2 \in \mathcal{F}$ (see [10] and also [1] where weakly union-closed systems are called union stable structures).

A set system is *regular* if all its maximal chains from $\emptyset$ to N are of length n (see [18, 19, 23] for works dealing with regular set systems). We recall that a maximal chain from $\emptyset$ to N of length n is a sequence of $n+1$ strictly nested subsets of N: $\emptyset \subset S_1 \subset \cdots \subset S_n = N$. Any set system closed under union and intersection of height n is regular, but the converse is not true.

A set system $\mathcal{F}$ is a *lattice* if for any $S, T \in \mathcal{F}$, their supremum $S \vee T$ and infimum $S \wedge T$ exist. The lattice is *distributive* if $\vee, \wedge$ obey distributivity. Although $\vee, \wedge$ do not necessarily coincide with the usual $\cup, \cap$, we will consider only this case here. Therefore, these set systems are closed under union and intersection. In addition, we restrict to lattices of height n, where the height of a lattice is defined as the length of a longest maximal chain. From the above remark, it follows that such distributive lattices are regular set systems.

The fundamental property of such distributive lattices is that they can be generated by a partial order on N. We need some definitions to establish this. Let us first denote this partially ordered set (*poset* for short) by $(N, \leq)$. Considering $J \subseteq N$, J is a *downset* if $i \in J$ and $j \leq i$ imply $j \in J$. Any element $i \in N$ generates a downset, defined by $\downarrow i = \{j \in N \mid j \leq i\}$. We denote by $\mathcal{O}(N)$ the collection of all downsets in $(N, \leq)$. Now, $\mathcal{O}(N)$ endowed with inclusion is a distributive lattice with $\cup, \cap$ as supremum and infimum, with height n.

Conversely, consider such a distributive lattice $\mathcal{F}$ on N. A nonempty set $S \in \mathcal{F}$ is said to be a *join-irreducible element* if it cannot be expressed as a supremum (union) of other sets in $\mathcal{F}$; equivalently it covers only one set in $\mathcal{F}$ (i.e., it has only one predecessor in the Hasse diagram). Denote by $\mathcal{J}(\mathcal{F})$ the set of join-irreducible elements of $\mathcal{F}$ (there are n join-irreducible elements when the height is n). Then Birkhoff's theorem [6] says that $\mathcal{F}$ can be reconstructed solely from $\mathcal{J}(\mathcal{F})$: one has $\mathcal{F} = \mathcal{O}(\mathcal{J}(\mathcal{F}))$. Finally, assign $i \in N$ to J_i, the smallest join-irreducible element containing i (by construction, $J_i = \downarrow i$). Doing so, the poset $(\mathcal{J}(\mathcal{F}), \subseteq)$ is isomorphic to $(N, \leq)$, the poset on N generating $\mathcal{F}$. Figure 1 illustrates this fundamental result.

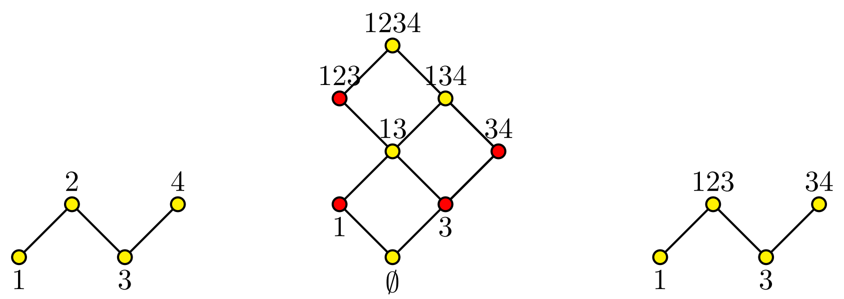


Figure 1: Left: a poset $(N, \leq)$ on N. Center: the distributive lattice $\mathcal{F}$ generated by $(N, \leq)$. Join-irreducible elements are in dark. Right: the poset of join-irreducible elements, isomorphic to $(N, \leq)$

For subsequent development, we need also the following notions for the generating poset $(N, \leq)$. For $i, j \in N$, we write $i \prec j$ if $i < j$ and there is no k such that $i < k < j$, and we say that j *covers* i. An element $i \in N$ is *minimal* if there is no $j \in N$ such that $j < i$. A *chain* from i to j in N is any sequence $i, i_1, \dots, i_p, j$ of elements of N such that $i < i_1 < \cdots < i_p < j$. The chain is *maximal* if no other chain from i to j contains it, i.e., if $i \prec i_1 \prec \cdots \prec i_p \prec j$. The *length* of a chain is its number of elements minus 1. The *height* of $i \in N$, denoted by $h(i)$, is the length of a longest chain from a minimal element to i. Elements of same height l form *level* $l+1$. Hence, level 1 (denoted by L_1) is the set of all minimal elements, level 2 (denoted by L_2) is the set of minimal elements of $N \setminus L_1$, etc. The height of N, denoted by $h(N)$, is the maximum of $h(i)$ taken over all elements of N.

Remark 1. (i) The poset $(N, \leq)$ can be thought of as a hierarchy on players or a precedence order. This was considered by Faigle and Kern [11] under the name of games with precedence constraints.

(ii) We discard from the analysis distributive lattices of height smaller than n: essentially, it amounts to redefining the set of players as N' with $|N'| = h(L)$, where some of the players of N have been regrouped into "macro-players".

2.3 How to make the core bounded

Given a set system $\mathcal{F}$, our main goal is to modify the definition of the core to make it bounded for any game v. We make the assumption that $\mathcal{C}(\mathcal{F}, 0)$ is pointed (i.e., it contains no lines). From Lemma 1 (ii), this means that $\mathcal{F}$ must contain at least n independent subsets, i.e., there is a collection $\mathcal{F}_0$ of n subsets in $\mathcal{F}$ such that the system $x(S) = 0$, $S \in \mathcal{F}_0$, is nonsingular. Following [9], we call such set systems *nondegenerate*. This is the case for distributive lattices of height n (just take $\mathcal{F}_0$ as the set of join-irreducible elements), and for regular set systems (take any maximal chain from $\emptyset$ to N). However, this is generally not true for weakly-union closed systems, and we will have to make additional assumptions.

Supposing that $\mathcal{F}$ is nondegenerate, we make the core bounded by turning some of the inequalities $x(S) \geq v(S)$ into equalities. Two remarks are noteworthy:

(i) This process always works: it suffices to do this for the n sets in the above defined collection $\mathcal{F}_0$. Then, $\mathcal{C}(\mathcal{F}, 0)$ reduces to $\{0\}$, and by Lemma 1 (iii), this guarantees that the core becomes bounded. We will see however that we can achieve this with much less than n sets.

(ii) This preserves the coalitional rationality principle, and it can be interpreted as adding "local efficiency constraints" or "binding constraints" for some specified subcoalitions of the grand coalition, i.e., players in those coalitions have a binding agreement.

Definition 1. Let v be a game on a nondegenerate set system $\mathcal{F}$.

(i) We call *normal collection* $\mathcal{N} := \{N_1, \dots, N_q\}$ a collection of sets in $\mathcal{F}$ (called *normal sets*) such that turning inequalities $x(N_i) \geq v(N_i)$ into equalities, for $i = 1, \dots, q$, makes the core bounded. We make the convention that N is *not* an element of $\mathcal{N}$.

(ii) We call the core with these additional equalities the *core restricted by the normal collection* $\mathcal{N}$, or if no ambiguity occurs, the *restricted core*, and denote it by $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$.

As mentioned in the introduction, Grabisch and Xie have proposed a particular way for defining a normal collection when $\mathcal{F}$ is a distributive lattice. Suppose $\mathcal{F}$ is a distributive lattice of height n, with generating poset $(N, \leq)$. As mentioned in Section 2.2, the height function on $(N, \leq)$ induces a partition of N into levels $L_1, \ldots, L_p$. Then the normal collection of Grabisch and Xie is simply $(L_1, L_1 \cup L_2, \ldots, L_1 \cup \cdots \cup L_{p-1})$. Note that the obtained normal collection is *nested*, i.e., it forms a chain in $\mathcal{F}$.

3 Case of distributive lattices of height n

As said in Section 2.2, distributive lattices of height n are closed under union and intersection, they possess n join-irreducible elements, and they are generated by a poset $(N, \leq)$ (i.e., $\mathcal{F} = \mathcal{O}(N, \leq)$). We recall that $i \prec j$ means that $i < j$ and there is no $k \in N$ such that $i < k < j$. Also, $\downarrow i := \{j \in N \mid j \leq i\}$ is the downset in $(N, \leq)$ generated by i. We begin with the study of normal collections.

3.1 Normal collections

For distributive lattices of height n, we know the following result from Tomizawa [28]. We denote by J_i, $i \in N$, the join-irreducible element of $\mathcal{F}$ induced by i, that is simply, $J_i = \downarrow i$. Also we use the notation 1_i for the vector of $\mathbb{R}^N$ having component i equal to 1 and 0 otherwise, and similarly for $(1_j, -1_i)$, etc.

Theorem 1. Let $\mathcal{F} = \mathcal{O}(N)$ be a distributive lattice of height n. The extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ are of the form $(1_j, -1_i)$, with $i \in N$ such that $|J_i| > 1$, $j \in J_i$ and $j \prec i$.

Recall that $\mathcal{C}(\mathcal{F}, v)$ will become bounded if there are no more extremal rays in $\mathcal{C}(\mathcal{F}, 0)$. Therefore, we must study how inequalities turned into equalities can "kill" extremal rays of $\mathcal{C}(\mathcal{F}, 0)$. Formally, the extremal ray $r \in \mathcal{C}(\mathcal{F}, 0)$ is *killed* by equality $x(F) = 0$ if $r \notin \mathcal{C}_{\{F\}}(\mathcal{F}, 0)$, where

$$\mathcal{C}_{\{F\}}(\mathcal{F}, 0) := \{x \in \mathcal{C}(\mathcal{F}, 0) \mid x(F) = 0\}.$$

The following can be proved.

Lemma 2. Let $\mathcal{F} = \mathcal{O}(N)$ be a distributive lattice of height n, and consider an extremal ray $(1_j, -1_i)$ of $\mathcal{C}(\mathcal{F}, 0)$ (that is, $i \in N$ such that $|J_i| > 1$, $j \in J_i$ and $j \prec i$). The extremal ray $(1_j, -1_i)$ is killed by equality $x(F) = 0$ if and only if $j \in F$ and $i \notin F$.

Proof. ($\Leftarrow$) Suppose that $j \in F$ and $i \notin F$. Then, if $x \in \mathcal{C}(\mathcal{F}, 0)$ satisfies $x(F) = 0$, we have

$$x(F) = x_j + \sum_{k \in F \setminus j} x_k = 0. \tag{1}$$

Consider now $x' := x + \alpha(1_j, -1_i)$, $\alpha > 0$. Then x' does not satisfy equality $x'(F) = 0$ since

$$x'(F) = x_j + \alpha + \sum_{k \in F \setminus j} x_k = \alpha.$$

Therefore, $(1_j, -1_i)$ is no more a ray.

($\Rightarrow$) Let $x \in \mathcal{C}(\mathcal{F}, 0)$ and satisfy the additional equality $x(F) = 0$ for some $F \in \mathcal{F}$. Suppose that for some $\alpha > 0$, $x' := x + \alpha(1_j, -1_i)$ does not belong to $\mathcal{C}(\mathcal{F}, v) \cap \{x(F) = 0\}$. It means that

$$\sum_{k \in F} x'_k = \sum_{k \in K} x_k + \alpha\delta_F(j) - \alpha\delta_F(i) \neq 0,$$

where $\delta_F(k) = 1$ if $k \in F$ and 0 otherwise. This implies $\delta_F(i) \neq \delta_F(j)$, therefore either i or j belongs to F, but not both. Because $j \prec i$ and since any set $F \in \mathcal{F}$ corresponds to a downset in $(N, \leq)$, it must be $j \in F$ and $i \notin F$. $\square$

Lemma 3. Let $\mathcal{F} = \mathcal{O}(N)$ be a distributive lattice of height n. The minimal size of a normal collection is $h(N)$, the height of the poset $(N, \leq)$ generating $\mathcal{F}$.

Proof. Let us assume that all rays are killed. By definition of the height, there exists a maximal chain in $(N, \leq)$ of length $h(N)$ going from a minimal element to a maximal element, say $i_0, i_1, \ldots, i_{h(N)}$. Then by Theorem 1, $(1_{i_0}, -1_{i_1})$, $(1_{i_1}, -1_{i_2})$, $\ldots$, $(1_{i_{h(N)-1}}, -1_{i_{h(N)}})$ are extremal rays. Because $(1_{i_0}, -1_{i_1})$ is killed, by Lemma 2 there must be an equality $x(K_1) = 0$ such that $i_0 \in K_1$ and $i_1 \notin K_1$. Moreover, since K_1 must be a downset, $i_2, \ldots, i_{h(N)}$ cannot belong to K_1. Similarly, there must exist an equality $x(K_2) = 0$ killing ray $(1_{i_1}, -1_{i_2})$ such that $i_1 \in K_2$ and $i_2, \ldots, i_{h(N)} \notin K_2$. Therefore, $K_1 \neq K_2$. Continuing this process we construct a sequence of distinct $h(N)$ subsets $K_1, K_2, \ldots, K_{h(N)}$, the last one killing ray $(1_{i_{h(N)-1}}, -1_{i_{h(N)}})$. Therefore, at least $h(N)$ equalities are needed. $\square$

The next algorithm shows an optimal way to define equalities killing all extremal rays. It is optimal in the sense that it uses only $h(N)$ equalities and each equality is the "smallest" possible (in the number of terms, or equivalently, in the size of F).

ALGO 1

Step 0 Initialization. Set $M = N$, endow it with the same order $\leq$ as for N. Set the normal collection to be empty: $\mathcal{N} = \emptyset$.

Step 1 Remove all disconnected elements from $(M, \leq)$ (i.e., those elements which are both minimal and maximal). If $M = \emptyset$, then STOP. Otherwise, go to Step 2.

Step 2 Build M_0 the set of all minimal elements of M, and set $\mathcal{N} \leftarrow \mathcal{N} \cup \{\downarrow M_0\}$, i.e., set equality $x(\downarrow M_0) = 0$ in the core, where $\downarrow M_0$ is the downset generated by M_0 in $(N, \leq)$.

Step 3 Set $M \leftarrow M \setminus M_0$, and go to Step 1.

Theorem 2. Under the assumption that $\mathcal{F} = \mathcal{O}(N)$, ALGO 1 kills all extremal rays and is optimal.

Proof. Steps 1 and 2 build subsets of the level sets of $(N, \leq)$, except the last $h(N)$th level, because in Step 2, all maximal elements of N are suppressed. Therefore, the algorithm necessarily finishes in exactly $h(N)$ iterations, and builds $h(N)$ equalities. By Lemma 3, this number is optimal.

Consider the first occurrence of Step 2, where M_0 is the set of minimal elements of N (minus those disconnected). Clearly, the equality $x(M_0) = 0$ kills all rays of the form $(1_j, -1_i)$, where j is a minimal element and i is a successor of j (i.e., $j \prec i$). Therefore, all such i's belong to the level 2. Taking a proper subset of M_0 will necessarily leave some rays of this form, and subsequent iterations will not kill them. This proves that in each step M_0 has a minimal size.

For each iteration, it is not necessary to keep elements i which have no successors (i.e., they are maximal), because there cannot exist rays of the form $(1_i, -1_k)$. Therefore those elements are suppressed in Step 1. All other elements are necessary since they have a successor and therefore generate a ray. This proves that in any iteration, M_0 has the minimal size, and so $\downarrow M_0$ too. □

We call the normal collection $\mathcal{N}$ found by ALGO 1 the collection of *irredundant normal sets* or *irredundant (normal) collection.* We introduce another one, which we call the collection of *Weber normal sets* or the *Weber (normal) collection* (reasons for this will be clear after). Supposing $\mathcal{N} = \{N_1, \dots, N_{h(N)}\}$ is the irredundant collection, the Weber collection is $\{N_1, N_1 \cup N_2, N_1 \cup N_2 \cup N_3, \dots, N_1 \cup \cdots \cup N_{h(N)}\}$.

Lemma 4. The Weber collection is a normal collection which is a chain in $\mathcal{F}$.

Proof. Lemma 2 shows that the collection is normal (only elements below those in the irredundant sets are added). The second assertion is obvious by construction. □

Recall that the normal collection introduced by Grabisch and Xie is $(L_1, L_1 \cup L_2, \dots, L_1 \cup \cdots \cup L_{p-1})$, where $L_1, \dots, L_p$ are the level sets of $(N, \leq)$. By construction, $N_1 \subseteq L_1$, $N_2 \subseteq L_1 \cup L_2$, etc., with proper inclusion in general. This shows that in general the three normal collections introduced so far differ.

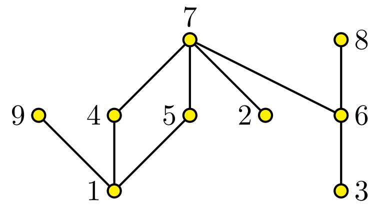
When a normal collection forms a chain, we say that the collection is *nested.* Note that the Weber collection is the "smallest" nested collection, in the sense that no other nested collection can contain proper subsets of the Weber collection. Indeed, it is built from the irredundant normal collection by adding the minimum number of elements to make the collection a chain.

Interestingly, the normal collection of Grabisch and Xie is also nested, and it is the "largest" nested collection[1], in the sense that no other nested collection can contain supersets of this normal collection. Indeed, since a normal set is built from the union of all level sets up to a given height, adding a new element i means adding an element from a higher level. Then $(1_k, -1_i)$ for some $k \prec i$ is an extremal ray, which will not be killed if i is incorporated into the normal set. Consequently, any nested collection (with optimal number of normal sets) is comprised between the Weber collection and the Grabisch-Xie collection.

The following example illustrates that the three normal collections differ.

[1]Note that this collection is still optimal in number of normal sets. "Largest" applies here for the size of the normal sets.

Example 1. Consider the poset $(N, \leq)$ of 9 elements depicted below.



Level 1 is $\{1, 2, 3\}$, level 2 is $\{4, 5, 6, 9\}$ and level 3 is $\{7, 8\}$. Extremal rays are

$$(1_1, -1_9), (1_1, -1_4), (1_1, -1_5), (1_3, -1_6), (1_4, -1_7), (1_5, -1_7), (1_2, -1_7), (1_6, -1_7), (1_6, -1_8).$$

The two irredundant normal sets built by Algo 1 are 123 and 13456, the two Weber normal sets are 123 and 123456, and the Grabisch-Xie normal sets are 123 and 1234569.

3.2 The Weber set

Let us denote by $\mathcal{C}$ the set of all maximal chains from $\emptyset$ to N in $\mathcal{F}$. Consider any maximal chain $C \in \mathcal{C}$ and its associated permutation σ on N, i.e.,

$$C = \{\emptyset, S_1, S_2, \ldots, S_n = N\},$$

with $S_i := \{\sigma(1), \ldots, \sigma(i)\}$, $i = 1, \ldots, n$. Considering a game v on $\mathcal{F}$, the *marginal vector* $x^C \in \mathbb{R}^N$ associated to C is the payoff vector defined by

$$x^C_{\sigma(i)} := v(S_i) - v(S_{i-1}), \quad i \in N.$$

The *Weber set* is the convex hull of all marginal vectors:

$$\mathcal{W}(\mathcal{F}, v) := \mathrm{conv}(x^C \mid C \in \mathcal{C}).$$

In the classical case $\mathcal{F} = 2^N$, it is well known that for any game v, $\mathcal{C}(\mathcal{F}, v) \subseteq \mathcal{W}(\mathcal{F}, v)$ holds , with equality if and only if v is convex. In our general case, this inclusion cannot hold any more since the core is unbounded in general. We propose a restricted version of the Weber set so that the classical results still hold.

Consider a nested normal collection (like the Weber collection or the Grabisch-Xie one) $\mathcal{N} = \{N_1, \ldots, N_{h(N)}\}$. A *restricted maximal chain* (w.r.t. $\mathcal{N}$) is a maximal chain from $\emptyset$ to N in $\mathcal{F}$ containing $\mathcal{N}$. A *restricted marginal vector* is a (classical) marginal vector whose underlying maximal chain is restricted. The *(restricted) Weber set* $\mathcal{W}_{\mathcal{N}}(\mathcal{F}, v)$ is the convex hull of all restricted marginal vectors w.r.t. $\mathcal{N}$. The (unrestricted) Weber set corresponds to the situation $\mathcal{N} = \emptyset$.

Lemma 5. For any restricted maximal chain C in $\mathcal{F}$, its associated restricted marginal vector x^C coincides with v on C, i.e., $x^C(S) = v(S)$ for all $S \in C$.

The proof of this Lemma follows directly from the definition of restricted marginal vectors. We recall the following result (see Fujishige and Tomizawa [13, 12]).

Theorem 3. Let v be a game on a distributive lattice $\mathcal{F}$ of height n. Then $\mathcal{C}(\mathcal{F}, v) = \mathcal{W}(\mathcal{F}, v)$ if and only if v is convex.

The following theorems generalize results of [17] and provide more elegant proofs.

Theorem 4. Consider $\mathcal{N}$ a nested normal collection on a distributive lattice $\mathcal{F}$ of height n. Then for every game v on $\mathcal{F}$, $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \subseteq \mathcal{W}_{\mathcal{N}}(\mathcal{F}, v)$.

Proof. We put $\mathcal{N} := \{N_1, \dots, N_q\}$. We prove the result by the separation theorem, proceeding as in [8]. Suppose there exists $x \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) \setminus \mathcal{W}_{\mathcal{N}}(\mathcal{F}, v)$. Then it exists $y \in \mathbb{R}^n$ such that $\langle w, y\rangle > \langle x, y\rangle$ for all $w \in \mathcal{W}_{\mathcal{N}}(\mathcal{F}, v)$.

Let π be a permutation on N such that $y_{\pi(1)} \geq y_{\pi(2)} \geq \cdots \geq y_{\pi(n)}$. Let us build a permutation π' from π so that π' corresponds to a restricted maximal chain as follows:

> Order the elements of N_1 according to the π order; then order the elements of N_2 according to the π order and put them after ; etc. Lastly, put the remaining elements (in $N \setminus (N_1 \cup \cdots \cup N_q)$) according to the π order.

Note that $\pi' = \pi$ if π corresponds to a restricted maximal chain. With Example 1 and the Weber collection, taking $\pi = 1, 4, 5, 2, 9, 3, 6, 7, 8$ leads to $\pi' = 1, 2, 3, 4, 5, 6, 9, 7, 8$.

Denoting by $m^{\pi'}$ the marginal vector associated to π' we have

$$\begin{aligned}
\langle m^{\pi'}, y\rangle &= \sum_{i=1}^{n} y_{\pi'(i)}\big(v(\{\pi'(1), \dots, \pi'(i)\}) - v(\{\pi'(1), \dots, \pi'(i-1)\})\big) \\
&= y_{\pi'(n)} v(N) + \sum_{i=1}^{n-1} (y_{\pi'(i)} - y_{\pi'(i+1)}) v(\{\pi'(1), \dots, \pi'(i)\}).
\end{aligned}$$

We claim that if $y_{\pi'(i)} - y_{\pi'(i+1)} < 0$ then $\{\pi'(1), \dots, \pi'(i)\}$ is a normal set. Indeed, by construction of π', the situation $y_{\pi'(i)} - y_{\pi'(i+1)} < 0$ can arise only if $\pi'(i) \in N_j$ for some j and $\pi'(i+1) \in N_{j+1}$. But then by construction again $N_j = \{\pi'(1), \dots, \pi'(i)\}$, which proves the claim.

Therefore since $x \in \mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$ we have

$$\begin{aligned}
\langle m^{\pi'}, y\rangle &\leq y_{\pi'(n)} x(N) + \sum_{i=1}^{n-1} (y_{\pi'(i)} - y_{\pi'(i+1)}) x(\{\pi'(1), \dots, \pi'(i)\}) \\
&= \sum_{i=1}^{n} y_{\pi'(i)} x(\{\pi'(1), \dots, \pi'(i)\}) - \sum_{i=2}^{n} y_{\pi'(i)} x(\{\pi'(1), \dots, \pi'(i-1)\}) \\
&= \sum_{i=1}^{n} y_{\pi'(i)} x_{\pi'(i)} = \langle y, x\rangle,
\end{aligned}$$

a contradiction with the assumption. □

Theorem 5. Consider $\mathcal{N}$ a nested normal collection on a distributive lattice $\mathcal{F}$ of height n. If v is convex on $\mathcal{F}$, then $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v) = \mathcal{W}_{\mathcal{N}}(\mathcal{F}, v)$.

Proof. By Theorem 4, it suffices to show that any restricted marginal vector is a vertex of $\mathcal{C}_{\mathcal{N}}(\mathcal{F}, v)$. We know already from Theorem 3 that it is a vertex of $\mathcal{C}(\mathcal{F}, v)$. It remains to show that any marginal vector satisfies the normality conditions $x(N_i) = v(N_i)$, $i = 1, \dots, q$, but this is established in Lemma 5. □

4 The general case

We suppose now that $\mathcal{F}$ is an arbitrary set system. We introduce $\widetilde{\mathcal{F}}$ the closure of $\mathcal{F}$ under union and intersection, i.e., the smallest set system closed under union and intersection containing $\mathcal{F}$. It is obtained by iteratively augmenting $\mathcal{F}$ with unions and intersections of pairs of subsets of the current set system (starting with $\mathcal{F}$), till there is no more change in the set system. As in Section 3, we assume that $\widetilde{\mathcal{F}}$ has height n (i.e., it has n join-irreducible elements). Note that, as shown by Example 2, this does not guarantee that the system is nondegenerate, however the converse is true, as shown by the next Lemma.

Lemma 6. *Let $\mathcal{F}$ be a nondegenerate set system. Then its closure $\widetilde{\mathcal{F}}$ under union and intersection has height n.*

Proof. Suppose that the height of $\widetilde{\mathcal{F}}$ is smaller than n. Then it is generated by a poset which is a partition of N, different from the collection of singletons. It implies that there exists at least two elements $i, j \in N$ which always appear together in any set of $\widetilde{\mathcal{F}}$, hence in any set of $\mathcal{F}$. This implies that in any subsystem of equalities $x(S) = 0$ for some sets S in $\mathcal{F}$, we can deduce at best that $x_i + x_j = 0$ but not that $x_i = x_j = 0$. Hence $\mathcal{F}$ is degenerate. $\square$

Theorem 6. *Consider a nondegenerate set system $\mathcal{F}$ (hence its closure $\widetilde{\mathcal{F}}$ has height n). Denote by $\mathcal{C}(\mathcal{F}, 0)$ and $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$ the recession cones generated by $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Then $\mathcal{C}(\mathcal{F}, 0)$ and $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$ have the same extremal rays (i.e., $\mathcal{C}(\mathcal{F}, 0) = \mathcal{C}(\widetilde{\mathcal{F}}, 0)$) if and only if all extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ are of the form $(1_j, -1_i)$, for some $i, j \in N$.*

Proof. The "only if" part is obvious from Theorem 1. Let us prove the "if" part. Suppose r is an extremal ray of $\mathcal{C}(\mathcal{F}, 0)$. By hypothesis, it has the form $(1_j, -1_i)$ for some $i, j \in N$. Also, by definition, it satisfies the system $r(S) \geq 0$ for all $S \in \mathcal{F}$, which gives $1_S(j) - 1_S(i) \geq 0$ for all $S \in \mathcal{F}$, which implies that there is no $S \in \mathcal{F}$ such that $S \ni i$ and $S \not\ni j$. Therefore it suffices to show that no such S exists in $\widetilde{\mathcal{F}}$. We show this by induction since $\widetilde{\mathcal{F}}$ is obtained iteratively from $\mathcal{F}$. We first prove that the union or intersection of two sets S_1, S_2 of $\mathcal{F}$ cannot at the same time contain i and not j. For intersection, if $S_1 \cap S_2 \ni i$, then S_1, S_2 too, so they cannot contain j, which implies $S_1 \cap S_2 \not\ni j$. Now, suppose that $S_1 \cup S_2$ does not contain j, which implies that neither S_1 nor S_2 contain j. If $i \in S_1 \cup S_2$, then i belongs at least to one of the sets S_1, S_2, which contradicts the hypothesis. Assume now that the hypothesis holds up to some step in the iteration process. Clearly, the same reasoning applies again, which proves that r is a ray of $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$. Hence we have proved $\mathcal{C}(\mathcal{F}, 0) \subseteq \mathcal{C}(\widetilde{\mathcal{F}}, 0)$.

Conversely, suppose r is an extremal ray of $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$, hence of the form $(1_j, -1_i)$ by Theorem 1. Then it satisfies the system $r(S) \geq 0$ for all $S \in \widetilde{\mathcal{F}}$, and $r(N) = 0$. Hence in particular it satisfies the system $r(S) \geq 0$ for all $S \in \mathcal{F}$ and $r(N) = 0$, and therefore r is a ray of $\mathcal{C}(\mathcal{F}, 0)$. Therefore $\mathcal{C}(\widetilde{\mathcal{F}}, 0) \subseteq \mathcal{C}(\mathcal{F}, 0)$. Hence, we have proved $\mathcal{C}(\mathcal{F}, 0) = \mathcal{C}(\widetilde{\mathcal{F}}, 0)$ and so extremal rays of $\mathcal{C}(\mathcal{F}, v)$ and $\mathcal{C}(\widetilde{\mathcal{F}}, v)$ are identical. $\square$

Unfortunately, not all set systems $\mathcal{F}$, even if $\widetilde{\mathcal{F}}$ has height n, induce extremal rays of the form $(1_j, -1_i)$, as shown by the next example.

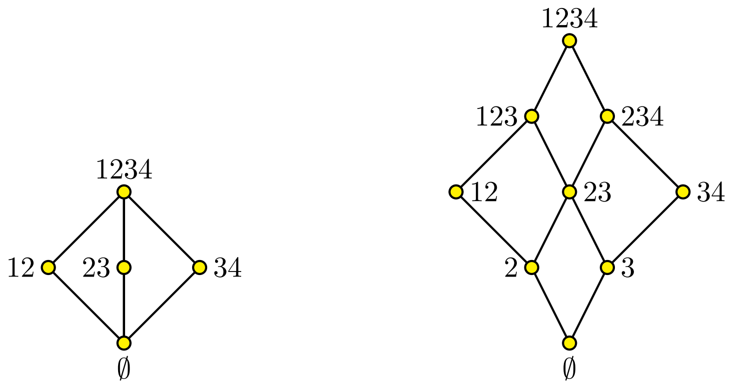


Figure 2: Set system $\mathcal{F}$ (left) and its closure under union and intersection $\widetilde{\mathcal{F}}$ (right)

Example 2. Consider $N = \{1, 2, 3, 4\}$, the set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$ depicted in Figure 2. The extremal rays of $\mathcal{F}$ are $(1, -1, 1, -1)$, $(-1, 1, -1, 1)$ and $(0, 0, 1, -1)$, while the extremal rays of $\widetilde{\mathcal{F}}$ are $(-1, 1, 0, 0)$ and $(0, 0, 1, -1)$. Note that the first two rays of $\mathcal{F}$ in fact define a line (hence $\mathcal{F}$ is degenerate), and that $\mathcal{F}$ is neither regular nor weakly union-closed.

Suppose now that $\mathcal{F}$ has rays of the form $(1_j, -1_i)$. How to kill them? Lemma 2 tells us how to kill rays of $\mathcal{F}$, by considering the equality $x(F) = 0$ with $j \in F$ and $i \notin F$. Therefore, the only thing we have to prove is that in any case, such a set F exists in $\mathcal{F}$.

Lemma 7. Let $\mathcal{F}$ be a (nondegenerate) set system such that all extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ are of the form $(1_j, -1_i)$. Then for each extremal ray $(1_j, -1_i)$, there exists a set $F \in \mathcal{F}$ such that $j \in F$ and $i \notin F$.

Proof. We consider the ray $(1_j, -1_i)$. We know that in $\widetilde{\mathcal{F}}$ it exists F_0 such that $j \in F_0$ and $i \notin F_0$. Suppose that no such F exists in $\mathcal{F}$ and show that in this case F_0 cannot exist in $\widetilde{\mathcal{F}}$. We suppose therefore that in $\mathcal{F}$ all sets satisfy either $F \not\ni j$ or $F \ni i$ and we consider two sets F_1, F_2. Observe that we have four possible situations: 1) $F_1 \not\ni j$ and $F_2 \not\ni j$, 2) $F_1 \ni i, j$ and $F_2 \ni i, j$ 3) $F_1 \not\ni j$ and $F_2 \ni i, j$, and 4) $F_1 \ni i, j$ and $F_2 \not\ni j$. In all four situations, we cannot have both $F_1 \cup F_2 \ni j$ and $F_1 \cup F_2 \not\ni i$, and the same is true for $F_1 \cap F_2$. Therefore, after one iteration, the set system has the same property than $\mathcal{F}$, and so by successive iterations, F_0 cannot be built. $\square$

The above lemma tells us that it is possible to kill rays for such set systems by turning at most r inequalities to equalities, if r is the number of rays. Is it possible to give a better answer by using results from Section 3.1 on $\widetilde{\mathcal{F}}$? Unfortunately, it does not seem possible to give a general answer here, even for regular set systems. This is because the irredundant normal sets found by ALGO 1 or the Weber normal collection of $\widetilde{\mathcal{F}}$ need not belong to $\mathcal{F}$, as the following simple example shows.

Example 3. Consider $N = \{1, 2, 3, 4\}$, the regular set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$ depicted in Figure 3. The unique ray of $\mathcal{C}(\mathcal{F}, 0)$ is (0,0,1,-1). Application of ALGO 1 on $\widetilde{\mathcal{F}}$ gives as normal set 3 (the Weber normal set is therefore the same). However, 3 does not belong to $\mathcal{F}$. Either 13 or 23 can be taken instead. Note that the Grabisch-Xie normal set is 123, which does not belong either to $\mathcal{F}$.

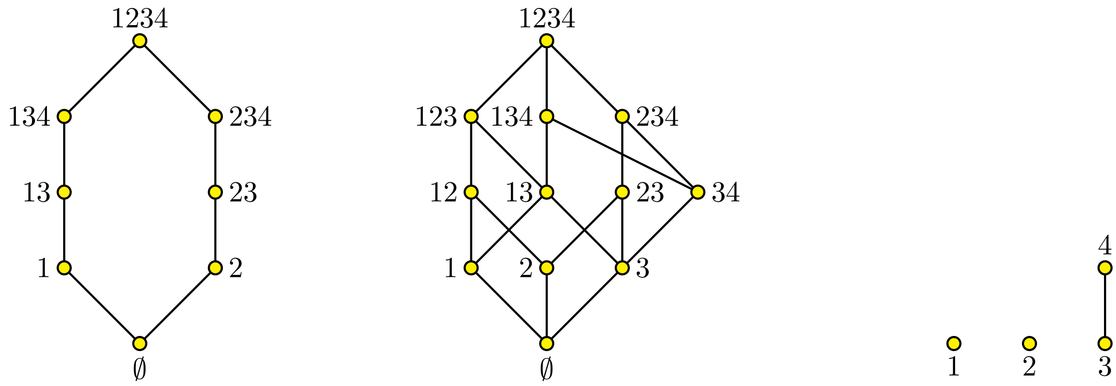


Figure 3: Set system $\mathcal{F}$ (left), its closure under union and intersection $\widetilde{\mathcal{F}}$ (center), and the generating poset $(N, \leq)$ (right)

Hence, the only thing which can be done is to build $\widetilde{\mathcal{F}}$, apply ALGO 1 or compute the Weber normal collection. If some normal sets do not belong to $\mathcal{F}$, take the smallest ones of $\mathcal{F}$ containing them and obeying Lemma 2. It is not guaranted however that we do not need more normal sets than for $\widetilde{\mathcal{F}}$ (but we do not have an example for this).

In the rest of the paper, we study two particular types of sets systems, namely regular set systems and weakly union-closed set systems, which both generalize systems closed under union and intersection, and where the above results can be applied.

4.1 The case of regular set systems

As noted in Section 2.3, any regular set system is nondegenerate. Let $\mathcal{C}$ be the set of all maximal chains from $\emptyset$ to N in $\mathcal{F}$.

Recall that any maximal chain in $\mathcal{C}$ induces a total order (permutation) on N, and therefore giving a regular set system $\mathcal{F}$ is equivalent to giving a set of (permitted) total orders on N.

Theorem 7. Suppose $\mathcal{F}$ is a regular set system. Then all extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ have the form $(1_l, -1_m)$ for some $l, m \in N$.

Proof. Consider a particular chain in $\mathcal{C}$, say $\emptyset, \{i\}, \{i,j\}, \{i,j,k\}, \ldots, N$, inducing the total order $i, j, k, \ldots,$ on N, and let us construct an extremal ray r.

Suppose $r_i > 0$, hence w.l.o.g. we can set $r_i = 1$. By the condition $r(N) = 0$, there must be at least one $\ell \in N \setminus i$ such that $r_\ell < 0$. Select ℓ such that ℓ is ranked after i in every maximal chain in $\mathcal{C}$. Observe that $(1_i, -1_\ell)$ is a solution of the system $r(S) \geq 0$ for all $S \in \mathcal{F}$ and $r(N) = 0$ (i.e., it is a ray of $\mathcal{C}(\mathcal{F}, 0)$) if and only if ℓ has the above property, because any $S \ni \ell$ contains also i. If no such ℓ exists, then set $r_i = 0$, which gives a new system of inequalities where r_i has disappeared, and consider the next element j and do the same (note that if exhausting all elements $i, j, k, \ldots$ without finding ℓ, is equivalent to the fact that there is no ray, a situation which happens for example if all orders exist, i.e., $\mathcal{F} = 2^N$). Suppose now that there exist several ℓ ranked after i in every maximal chain, say $\ell_1, \ldots, \ell_q$. Then for every $\alpha_1, \ldots, \alpha_q \geq 0$ such that $\sum_{p=1}^{q} \alpha_p = 1$, the vector $(1_i, -\alpha_1 1_{\ell_1}, \ldots, -\alpha_q 1_{\ell_q})$ is a ray. But each $(1_i, -1_{\ell_p})$, $p = 1, \ldots, q$ is also a ray, and $(1_i, -\alpha_1 1_{\ell_1}, \ldots, -\alpha_q 1_{\ell_q})$ can be expressed as a convex combination of these rays, proving that it is not extremal. Therefore extremal rays are necessarily of the form $(1_i, -1_\ell)$. In addition, if ℓ_2 is ranked after ℓ_1 in every order, then $(1_{\ell_1}, -1_{\ell_2})$ is a ray, therefore $(1_i, -1_{\ell_2})$ is not extremal since it can be obtained as $(1_i, -1_{\ell_1}) + (1_{\ell_1}, -1_{\ell_2})$ (and similarly for the others). $\square$

By Theorem 6, we deduce immediately:

Corollary 1. If $\mathcal{F}$ is a regular set system, then $\mathcal{C}(\mathcal{F}, 0) = \mathcal{C}(\widetilde{\mathcal{F}}, 0)$.

We can also deduce Theorem 1 from the above, and therefore derive an alternative proof of it:

Corollary 2. If $\mathcal{F}$ is regular and union and intersection closed, then the extremal rays are $(1_j, -1_i)$ with $i \in N$ such that $|J_i| > 1$ and $j \in J_i$, $j \prec i$ (see above Theorem 1 for notation).

Proof. Under the hypothesis, $\mathcal{F}$ is generated by a poset $(N, \leq)$, and the set of total orders generated by the maximal chains are those orders compatible with the partial order $\leq$ on N. Then it is easy to see from the proof of Theorem 7 that we obtain the desired extremal rays. $\square$

The proof of Theorem 7 being constructive, we can propose the following simple algorithm to produce all extremal rays of a regular set system.

ALGO 2

Step 0 Initialization. Select a maximal chain C in $\mathcal{C}$, and denote for simplicity by $1, 2, \ldots, n$ the order induced by C. Put $L = \emptyset$.

For $i = 1$ to $n - 1$ **do**:

For $j = i + 1$ to n **do**:

If j is ranked after i in every chain in $\mathcal{C}$, **then**

- Put $(1_i, -1_j)$ in L
  % this is a candidate for being an extremal ray
- **For** $k < i$, check if $(1_k, -1_i)$ and $(1_k, -1_j)$ both exist in L. **If** yes, remove $(1_k, -1_j)$ from L
  % it can be obtained as the sum of $(1_k, -1_i)$ *and* $(1_i, -1_j)$

Final step: output list L of extremal rays.

Example 4. Let us apply ALGO 2 on the regular set system of Fig. 4 (left). The four orders induced by the maximal chains are:

$$1 - 4 - 2 - 3 - 5$$
$$2 - 4 - 1 - 3 - 5$$
$$2 - 4 - 3 - 5 - 1$$
$$2 - 4 - 3 - 1 - 5$$

Let us take the first order for running the algorithm. Taking $i = 1$, we see that no j can be found. Therefore, we take $i = 4$, then $j = 3$ and 5 are possible, so we put in L

the rays $(0,0,-1,1,0)$ and $(0,0,0,1,-1)$. Let us take now $i=2$, then $j=3$ and 5 are possible, so we add in L the two rays $(0,1,-1,0,0)$ and $(0,1,0,0-1)$. Next, we take $i=3$ and see that $j=5$ is possible, therefore we put $(0,0,1,0,-1)$ in L. However, we have to remove $(0,0,0,1,0,-1)$ and $(0,1,0,0,-1)$ from L. The extremal rays are therefore $(0,0,-1,1,0)$, $(0,1,-1,0,0)$ and $(0,0,1,0,-1)$. This result is confirmed by the PORTA software.

We end this section by addressing the definition of the Weber set. Since $\mathcal{F}$ is regular, marginal vectors can be defined as usual and therefore it makes sense to speak of the Weber set. Suppose we have found a normal nested collection of sets $\mathcal{N}$, then the restricted Weber set $\mathcal{W}_{\mathcal{N}}(\mathcal{F},v)$ for v defined on $\mathcal{F}$ can be defined as before. The question is then to compare $\mathcal{W}_{\mathcal{N}}(\mathcal{F},v)$ with $\mathcal{C}_{\mathcal{N}}(\mathcal{F},v)$ and also $\widetilde{\mathcal{W}}_{\mathcal{N}'}(v)$, the restricted Weber set on $\widetilde{\mathcal{F}}$, with $\mathcal{N}'$ the Weber normal collection of $\widetilde{\mathcal{F}}$. Little can be said in general if one does not have $\mathcal{N}'=\mathcal{N}$. Suppose then that this is the case. Because of regularity, any restricted maximal chain in $\mathcal{F}$ is a restricted maximal chain in $\widetilde{\mathcal{F}}$, so that we have $\mathcal{W}_{\mathcal{N}}(\mathcal{F},v)\subseteq\widetilde{\mathcal{W}}_{\mathcal{N}}(v)$. Recall also that $\mathcal{C}_{\mathcal{N}}(\mathcal{F},v)\supseteq\widetilde{\mathcal{C}}_{\mathcal{N}}(v)$, hence the question whether $\mathcal{C}_{\mathcal{N}}(\mathcal{F},v)\subseteq\mathcal{W}_{\mathcal{N}}(\mathcal{F},v)$ remains. An examination of the proof of Theorem 4 reveals that the technique of the proof cannot extend to this case. Indeed, the following example shows that this is not true in general.

Example 5. Consider $N=\{1,2,3,4,5\}$, the regular set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$ depicted in Figure 4. ALGO 1 applied on $\widetilde{\mathcal{F}}$ gives 24 and 234 as normal sets, which is

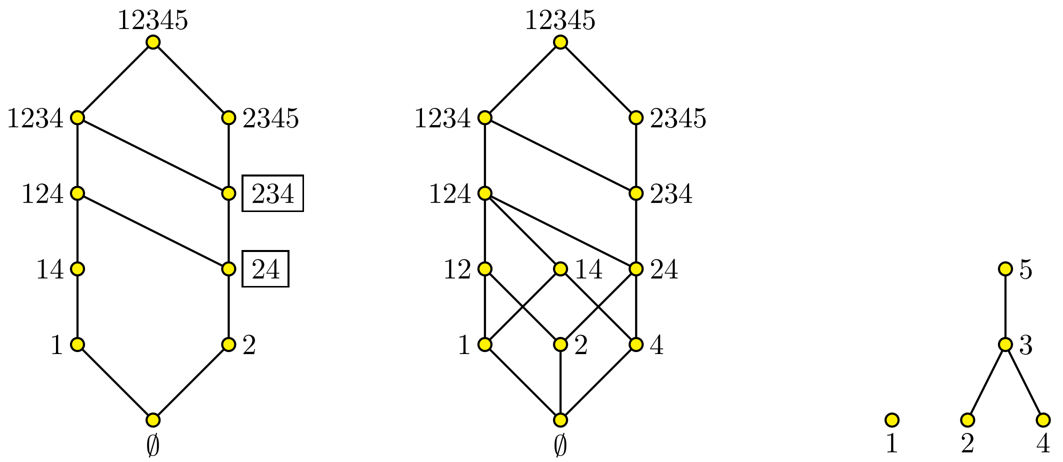


Figure 4: Set system $\mathcal{F}$ (left), its closure under union and intersection $\widetilde{\mathcal{F}}$ (center), and the generating poset $(N,\leq)$ (right)

also the Weber collection. These sets belong to $\mathcal{F}$, therefore the restricted Weber set can be defined with the Weber collection. There are only two restricted maximal chains on $\mathcal{F}$, namely $\emptyset,2,24,234,2345,N$ and $\emptyset,2,24,234,1234,N$, inducing the two vertices of $\mathcal{W}_{\mathcal{N}}(\mathcal{F},v)$:

$$\begin{aligned} w_1 &= (v(N)-v(2345),v(2),v(234)-v(24),v(24)-v(2),v(2345)-v(234)) \\ w_2 &= (v(1234)-v(234),v(2),v(234)-v(24),v(24)-v(2),v(N)-v(1234)). \end{aligned}$$

The restricted core is defined by the system:

$$\begin{aligned} x_1 &\geq v(1) \\ x_2 &\geq v(2) \\ x_1 + x_4 &\geq v(14) \\ x_2 + x_4 &= v(24) \\ x_1 + x_2 + x_4 &\geq v(124) \\ x_2 + x_3 + x_4 &= v(234) \\ x_1 + x_2 + x_3 + x_4 &\geq v(1234) \\ x_2 + x_3 + x_4 + x_5 &\geq v(2345) \\ x_1 + x_2 + x_3 + x_4 + x_5 &= v(N) \end{aligned}$$

Let us take the game defined by $v(N) = 3$, $v(1234) = v(2345) = 2$, $v(234) = 1$, $v(124) = 2$, $v(24) = v(14) = 1$, $v(2) = v(1) = 0$. Then the two vertices of the Weber set are $(1, 0, 0, 1, 1)$ and $(1, 0, 0, 1, 1)$, which makes the Weber set a singleton. However, the vector $(1, 1, 0, 0, 1)$ is an element of the restricted core, which forbids the core to be included into the Weber set.

4.2 The case of weakly union-closed systems

We begin by showing that weakly union-closed systems whose closure has height n are nondegenerate.

Lemma 8. Let $\mathcal{F}$ be a weakly union-closed set system, and consider its closure $\widetilde{\mathcal{F}}$ under union and intersection. Then $\mathcal{F}$ is nondegenerate if and only if its closure has height n.

Proof. ($\Rightarrow$) This is Lemma 6.

($\Leftarrow$) If $\widetilde{\mathcal{F}}$ has height n, any singleton i can be obtained by

$$\{i\} = \downarrow i \setminus \Big(\bigcup_{j \prec i} \downarrow j \Big) \tag{2}$$

as it can be easily checked from the Birkhoff theorem (see Section 2.2). Observe that $\downarrow i, \downarrow j$ are all sets in $\widetilde{\mathcal{F}}$, hence they can be obtained by union and intersection of sets in $\mathcal{F}$.

Now, since $\mathcal{F}$ is weakly union-closed, for any $S, T \in \mathcal{F}$, either they are disjoint, and the characteristic function $1_{S \cup T}$ is simply $1_S + 1_T$, or they intersect, and $S \cup T \in \mathcal{F}$. Also, $1_{S \setminus T} = 1_S - 1_T$ if $T \subseteq S$. It follows that for any $i \in N$, 1_i can be obtained by linear combinations of characteristic functions 1_S with $S \in \mathcal{F}$, using (2), which proves nondegeneracy. □

The following theorem gives a sufficient condition for the equality of $\mathcal{C}(\mathcal{F}, 0)$ and $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$.

Theorem 8. Assume that $\mathcal{F}$ is a nondegenerate weakly union-closed system, and denote by $\widetilde{\mathcal{F}}$ its closure under union and intersection. Then the extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ and $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$ are the same if for any $S \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$, it is either a union of disjoint sets of $\mathcal{F}$, or there exist $S_1, S_2 \in \mathcal{F}$ such that $S = S_1 \cap S_2$, and there exists a covering in $\mathcal{F}$ of $N \setminus (S_1 \cup S_2)$.

Proof. We consider the set of inequalities of $\mathcal{C}(\mathcal{F}, 0)$, i.e., $x(S) \geq 0$ for all $S \in \mathcal{F}$ and $x(N) = 0$. We will prove that any additional inequality $x(F) \geq 0$ with $F \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$ is redundant. By the Farkas lemma, we know that this amounts to prove that $x(F) \geq 0$ can be obtained by a positive linear combination of the inequalities $x(S) \geq 0$, $S \in \mathcal{F}$ and $x(N) = 0$.

We consider $S \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$. Assume first that S is a disjoint union of sets in $\mathcal{F}$, say $S = S_1 \cup \cdots \cup S_k$. Then obviously $x(S) \geq 0$ is implied by equalities $x(S_i) \geq 0$, $i = 1, \ldots, q$, since it can be obtained as their sum. Suppose on the contrary that S is not a disjoint union of sets in $\mathcal{F}$. By hypothesis, there exists $S_1, S_2 \in \mathcal{F}$ such that $S_1 \cap S_2 = S$ and there exists a partition $\{T_1, \ldots, T_k\}$ of $N \setminus (S_1 \cup S_2)$. Let us write the following system of inequalities:

$$
\begin{array}{rl}
x(S_1) \geq 0 & \qquad\qquad (a_1) \\
x(S_2) \geq 0 & \qquad\qquad (a_2) \\
x(T_1) \geq 0 & \qquad\qquad (b_1) \\
\vdots & \qquad\qquad \vdots \\
x(T_k) \geq 0 & \qquad\qquad (b_k) \\
-x(N) \geq 0 & \qquad\qquad (c),
\end{array}
$$

the last one coming from $x(N) = 0$. Then the inequality $x(S) \geq 0$ is obtained by $(a_1) + (a_2) + (b_1) + \cdots (b_k) + (c)$, which proves that $x(S) \geq 0$ is redundant. $\square$

Remark 2. By definition of weakly union-closed systems, note that the covering of $N \setminus (S_1 \cup S_2)$ is a partition.

The next example illustrates the case where this condition is not satisfied.

Example 6. Take $N = \{1, 2, 3, 4\}$ and consider the nondegenerate weakly union-closed set system $\mathcal{F}$ and its closure $\widetilde{\mathcal{F}}$ depicted in Figure 5. The required condition fails: take

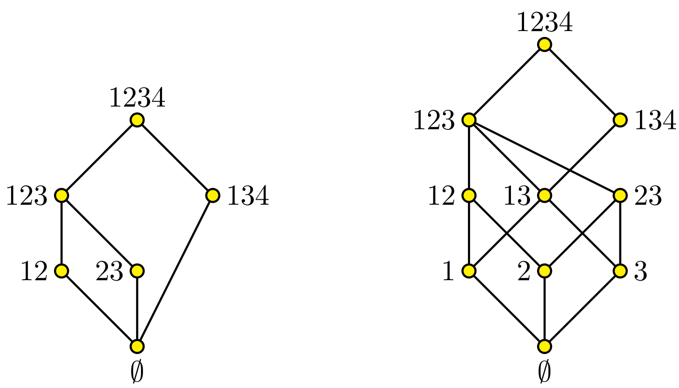


Figure 5: Set system $\mathcal{F}$ (left) and its closure under union and intersection $\widetilde{\mathcal{F}}$ (right)

$S = 2$, then it can obtained only by the intersection of 12 and 23. But $N \setminus 123 = 4$ is not a subset of $\mathcal{F}$. The extremal rays of $\mathcal{C}(\mathcal{F}, 0)$ are $(0, 0, 1, -1)$, $(1, 0, 0, -1)$ and $(1, -1, 1, -1)$, but $\mathcal{C}(\widetilde{\mathcal{F}}, 0)$ has only the two first rays as extremal rays.

References