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Thierry Lévy

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# ASYMPTOTICS OF BROWNIAN MOTIONS ON CLASSICAL LIE GROUPS, THE MASTER FIELD ON THE PLANE, AND THE MAKEENKO-MIGDAL EQUATIONS

THIERRY LÉVY

ABSTRACT. We study the large  $N$  asymptotics of the Brownian motions on the orthogonal, unitary and symplectic groups, extend the convergence in non-commutative distribution originally obtained by Biane for the unitary Brownian motion to the orthogonal and symplectic cases, and derive explicit estimates for the speed of convergence in non-commutative distribution of arbitrary words in independent Brownian motions.

Using these results, we construct and study the large  $N$  limit of the Yang-Mills measure on the Euclidean plane with orthogonal, unitary and symplectic structure groups. We prove that the Wilson loops admit a deterministic limit, towards which they converge at a speed which is uniform on sets of loops with bounded length and for which we obtain a simple explicit upper bound.

Finally, we establish rigorously, both for finite  $N$  and in the large  $N$  limit, the Schwinger-Dyson equations for the expectations of Wilson loops, which in this context are called the Makeenko-Migdal equations. We study how these equations allow one to compute recursively the expectation of a Wilson loop as a component of the solution of a differential system with respect to the areas of the faces delimited by the loop.

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## INTRODUCTION

The Euclidean two-dimensional Yang-Mills measure is a probability measure which was defined, first by Ambar Sengupta [31] and later in a different way by the author [23, 26], as a mathematically rigorous version of one of the functional integrals considered by physicists in relation to quantum field theory, more precisely in relation to gauge theories. The two-dimensional Yang-Mills measure is specified by the choice of a compact surface endowed with a volume form and the choice of a compact connected Lie group whose Lie algebra is equipped with an invariant scalar product. From a physical perspective, the surface plays the role of space-time, or rather of space, since we are considering a Euclidean theory, and the Lie group characterises the kind of interaction which one wishes to describe. Gauge theories are relevant to the description of three of the four fundamental interactions, namely the electromagnetic, weak and strong interaction, and the corresponding Lie groups are respectively  $U(1)$ ,  $SU(2)$  or  $U(2)$ , and  $SU(3)$ .

In 1974, Gerard 't Hooft, in an attempt to understand quark confinement, considered in [36] gauge theories with larger structure groups, namely the unitary groups  $U(N)$ , and observed that many quantities of interest become simpler in the limit where  $N$  tends to infinity. From there on, the large  $N$  behaviour of gauge theories was extensively studied by physicists (see for example [20, 21, 28, 30]), and the idea emerged that there should be a universal deterministic large  $N$  limit to a broad class of matrix models (see [13] and the references therein). This limit was named the *master field* and it is the main object of study of the present paper.

The master field on the Euclidean plane was first described mathematically at a conjectural level in a visionary paper by I. Singer [34], where in particular the relation between the master field and the theory of free probability was very convincingly sketched. A. Sengupta investigated the problem in [32, 33], and during the preparation of the present paper, M. Anshelevitch and A. Sengupta gave the first construction at a mathematical level of rigour of the master field on the plane [1]. Their approach is based on the use of free white noise and of free stochastic calculus. It differs from the one which we follow here pretty much in the same way A. Sengupta's original construction of the Yang-Mills measure [31] differed from that given by the author in [26].

It is interesting to note that the large  $N$  limit of the two-dimensional Yang-Mills theory was specifically studied by Gross, Taylor and Matytsin [14, 15, 16], but in relation with string theory rather than with the master field. We studied some of the formulas displayed in these papers in our previous work [25], but we do not pursue this investigation in the present paper.

In the rest of this introduction, we present our main results, in an order which is not strictly the same as in the body of the paper, but allows us to emphasise our motivation for the technical results of Part 1.

**The large  $N$  limit of the Brownian motions.** One of the main ingredients of the Yang-Mills measure is the Brownian motion on the structure group. In order to study the large  $N$  limit of the Yang-Mills measure, we naturally start by studying the large  $N$  limit of the Brownian motions on the orthogonal, unitary and symplectic groups  $SO(N)$ ,  $U(N)$  and  $Sp(N)$ .

Let  $G$  be a compact connected Lie group. Endow the Lie algebra  $\mathfrak{g}$  of  $G$  with an invariant scalar product, which we denote by  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ . The Brownian motion on  $G$  is the Markov process on  $G$  issued from the unit element and whose generator is the Laplace-Beltrami operator associated with the bi-invariant Riemannian metric on  $G$  determined by  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ .

For example, for each  $N \geq 1$ , let us endow the Lie algebra  $\mathfrak{u}(N)$  of the unitary group  $U(N)$  with the scalar product  $\langle X, Y \rangle_{\mathfrak{u}(N)} = -N \operatorname{Tr}(XY)$ , where  $\operatorname{Tr}$  denotes the usual trace, so that  $\operatorname{Tr}(I_N) = N$ . Let us denote by  $(U_{N,t})_{t \geq 0}$  the associated Brownian motion on  $U(N)$ . The random matrices  $\{U_{N,t} : t \geq 0\}$  form a collection of elements of the non-commutative probability space

$(L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes M_N(\mathbb{C}), \mathbb{E} \otimes \text{tr})$ , where  $(\Omega, \mathcal{F}, \mathbb{P})$  denotes the underlying probability space and  $\text{tr}$  denotes the normalised trace, so that  $\text{tr}(I_N) = 1$ .

Philippe Biane proved in [4] that the non-commutative distribution of the collection  $\{U_{N,t} : t \geq 0\}$  converges, as  $N$  tends to infinity, to the distribution of a free multiplicative Brownian motion, that is, the distribution of a collection  $\{u_t : t \geq 0\}$  of unitary elements of a non-commutative probability space  $(\mathcal{A}, \tau)$  such that the process  $(u_t)_{t \geq 0}$  has free and stationary increments, and such that these increments have the distribution whose moments are given by (23) and (24).

To say that there is convergence of the non-commutative distributions means that for each integer  $n \geq 1$ , each non-commutative polynomial  $p$  in  $2n$  variables, and each choice of  $n$  non-negative reals  $t_1, \dots, t_n$ , one has the convergence

$$\lim_{N \rightarrow \infty} \mathbb{E}[\text{tr}(p(U_{N,t_1}, U_{N,t_1}^*, \dots, U_{N,t_n}, U_{N,t_n}^*))] = \tau(p(u_{t_1}, u_{t_1}^*, \dots, u_{t_n}, u_{t_n}^*)).$$

The first result which we prove in this paper extends this convergence to the orthogonal and symplectic cases. After establishing some notation and collecting some preliminary information in Section 1, we give in Section 2 a short proof of Biane's result in the unitary case and adapt our argument to prove that, with the correct normalisation of the invariant scalar products on  $\mathfrak{so}(N)$  and  $\mathfrak{sp}(N)$ , the orthogonal and symplectic Brownian motions have the same large  $N$  limit as the unitary Brownian motion (this is Theorem 2.2). Our approach has a combinatorial flavour and aims at proving that the differential systems satisfied by the moments of the limiting distributions are the same as in the unitary case. The main novelty in the orthogonal and symplectic cases is the appearance, fundamentally due to the Schur-Weyl duality, of Brauer diagrams in the combinatorics, in place of permutations. Our analysis shows that the diagrams which are not permutations do not contribute to the large  $N$  limit.

In our treatment, we consistently try to emphasise the similarities between the orthogonal, unitary and symplectic groups, in particular by seeing them respectively as the real, complex and quaternionic unitary groups. Accordingly, we denote them respectively by  $U(N, \mathbb{R}) = \text{SO}(N)$ ,  $U(N, \mathbb{C}) = U(N)$  and  $U(N, \mathbb{H}) = \text{Sp}(N)$ .

**The Yang-Mills field and the master field on the plane.** Let us now jump forward in the paper and discuss the Yang-Mills measure, or Yang-Mills field. Consider again a compact connected Lie group  $G$  whose Lie algebra is endowed with an invariant scalar product. The Yang-Mills field on the Euclidean plane is a collection  $(H_l)_{l \in L_0(\mathbb{R}^2)}$  of  $G$ -valued random variables indexed by the set  $L_0(\mathbb{R}^2)$  of loops with finite length on  $\mathbb{R}^2$  based at the origin (see Section 4.1 for more detail about loops). Let us give a concise characterisation of the distribution of this collection of random variables, by the following five rules. This presentation differs from the more classical way in which we review the Yang-Mills field in Section 4.

YM<sub>1</sub>. For any two loops  $l_1$  and  $l_2$ , one can form the concatenation  $l_1 l_2$  of  $l_1$  and  $l_2$ , and one has  $H_{l_1 l_2} = H_{l_2} H_{l_1}$  almost surely.

YM<sub>2</sub>. If two loops  $l_1$  and  $l_2$  differ by the insertion or the deletion of finitely many sub-loops formed by a path immediately followed by the same path traced backwards, we write  $l_1 \sim l_2$  and in this case,  $H_{l_1} = H_{l_2}$  almost surely.

YM<sub>3</sub>. If a loop  $l$  has no self-intersection and surrounds a domain of area  $t$ , then  $H_l$  has the distribution of the Brownian motion on  $G$  at time  $t$ .

YM<sub>4</sub>. If two loops  $l_1$  and  $l_2$  without self-intersection surround disjoint domains, then  $H_{l_1}$  and  $H_{l_2}$  are independent.

YM<sub>5</sub>. If the loops  $\{l_n : n \geq 1\}$  can be parametrised in such a way that the sequence  $(l_n)_{n \geq 1}$  converges uniformly to a loop  $l$  and if the length of  $l_n$  converges to the length of  $l$ , we say that the

sequence  $(l_n)_{n \geq 1}$  converges to  $l$  and in this case, the sequence  $(H_{l_n})_{n \geq 1}$  converges in probability to  $H_l$ .

It may be useful to think of the collection  $(H_l)_{l \in \mathbf{L}_0(\mathbb{R}^2)}$  as a Brownian motion on  $G$  indexed by the set of loops on  $\mathbb{R}^2$ . The area on the Euclidean plane plays the role of time for this Brownian motion.

For each  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  and each  $N \geq 1$ , the Yang-Mills measure can be constructed with  $G = \mathrm{U}(N, \mathbb{K})$ , the Lie algebra  $\mathfrak{u}(N, \mathbb{K})$  being endowed with the invariant scalar product for which the convergence of classical Brownian motions holds. Let us denote by  $(H_{N,l}^{\mathbb{K}})_{l \in \mathbf{L}_0(\mathbb{R}^2)}$  the corresponding collection of random matrices. Our main result (which is Theorem 5.19) states that the non-commutative distribution of the collection  $\{H_{N,l}^{\mathbb{K}} : l \in \mathbf{L}_0(\mathbb{R}^2)\}$  converges, as  $N$  tends to infinity, to the distribution of a non-commutative process  $(h_l)_{l \in \mathbf{L}_0(\mathbb{R}^2)}$ , defined on a non-commutative probability space  $(\mathcal{A}, \tau)$ , and whose distribution is characterised by the following five rules, which are the free analogues of the five rules which characterise the Yang-Mills field.

MF<sub>1</sub>. For any two loops  $l_1$  and  $l_2$ , one has  $h_{l_1 l_2} = h_{l_2} h_{l_1}$ .

MF<sub>2</sub>. If  $l_1 \sim l_2$ , then  $h_{l_1} = h_{l_2}$ .

MF<sub>3</sub>. If a loop  $l$  has no self-intersection and surrounds a domain of area  $t$ , then  $h_l$  has the distribution of a free multiplicative Brownian motion at time  $t$ .

MF<sub>4</sub>. If two loops  $l_1$  and  $l_2$  without self-intersection surround disjoint domains, then  $h_{l_1}$  and  $h_{l_2}$  are free.

MF<sub>5</sub>. If  $(l_n)_{n \geq 1}$  converges to  $l$ , then the sequence  $(h_{l_n})_{n \geq 1}$  converges to  $h_l$  in  $L^2(\mathcal{A}, \tau)$ .

The process  $(h_l)_{l \in \mathbf{L}_0(\mathbb{R}^2)}$  is what we propose to call the *master field* on the plane.

Just as in the case of the Brownian motion on  $\mathrm{U}(N)$ , this convergence in non-commutative distribution means by definition that for each integer  $n \geq 1$ , each non-commutative polynomial  $p$  in  $2n$  variables, and each choice of  $n$  loops  $l_1, \dots, l_n$ , one has the convergence

$$\lim_{N \rightarrow \infty} \mathbb{E}[\mathrm{tr}(p(H_{N,l_1}^{\mathbb{K}}, (H_{N,l_1}^{\mathbb{K}})^*, \dots, H_{N,l_n}^{\mathbb{K}}, (H_{N,l_n}^{\mathbb{K}})^*))] = \tau(p(h_{l_1}, h_{l_1}^*, \dots, h_{l_n}, h_{l_n}^*)).$$

However, something new happens in this situation thanks to the fact that loops can be multiplied. Thanks to YM<sub>1</sub>, any word in the random matrices  $H_{N,l_1}^{\mathbb{K}}, (H_{N,l_1}^{\mathbb{K}})^*, \dots, H_{N,l_n}^{\mathbb{K}}, (H_{N,l_n}^{\mathbb{K}})^*$  is almost surely equal to the random matrix  $H_l$ , where  $l$  is the same word read backwards in the loops  $l_1, l_1^{-1}, \dots, l_n, l_n^{-1}$ . Here,  $l_1^{-1}$  denotes the loop  $l_1$  traced backwards. A similar statement holds for the non-commutative random variables  $(h_l)_{l \in \mathbf{L}_0(\mathbb{R}^2)}$ . It follows from this observation that the convergence in non-commutative distribution of the Yang-Mills process towards its large  $N$  limit is equivalent to the convergence

$$(1) \quad \forall l \in \mathbf{L}_0(\mathbb{R}^2), \quad \lim_{N \rightarrow \infty} \mathbb{E}[\mathrm{tr}(H_{N,l}^{\mathbb{K}})] = \tau(h_l).$$

**Uniform convergence of expectations of Wilson loops.** The point towards which we focus our attention in the first five sections of this paper is the proof of the convergence of the left-hand side of (1), which then serves as a definition for the right-hand side. The proof of this convergence follows roughly the construction of the Yang-Mills field itself, which is split into two main parts: the first in which one deals with piecewise affine loop, and the second in which one extends the construction to arbitrary loops with finite length.

The first step is mainly combinatorial, and given the convergence results for the classical Brownian motions which we prove in Section 2, the existence of the limit on the left-hand side of (1) when  $l$  is a piecewise affine loop is simply a matter of incorporating into the usual discrete Yang-Mills theory a result of asymptotic freeness of large rotationally invariant matrices. Indeed,

any piecewise affine loop can be written, up the relation  $\sim$  defined in the rule  $YM_2$ , as a word in a finite number of loops which surround disjoint domains of  $\mathbb{R}^2$ . The random matrices associated to these loops are asymptotically free and have a known distribution in the large  $N$  limit. This is what we explain in Sections 5.1 and 5.2.

In the unitary case, the asymptotic freeness result on which this analysis relies is one of the fundamental theorems of free probability theory and was proved by D. Voiculescu [37]. The orthogonal and symplectic analogues of this result are known to hold, thanks to a work of B. Collins and P. Śniady [7]. Nevertheless, we felt that the symplectic case deserved a slightly more explicit treatment than it did receive in this work, and we devoted an appendix to a survey of the proofs of this result in the three cases where we use it.

The second step of the proof of (1), in which we allow  $l$  to be an arbitrary loop with finite length is more delicate. Our strategy is to prove that the convergence (1), which we have established for piecewise affine loops, is uniform on sets of piecewise affine loops with bounded length. To do this, the results of Section 2 do not suffice, for they deal qualitatively with the convergence of one Brownian motion, whereas we find ourselves in need of a quantitative information on the convergence of words in several Brownian motions. In Section 3, we achieve an explicit control of the speed of convergence in non-commutative distribution of a word in independent Brownian motions to its limit (Theorem 3.3). This speed naturally depends on a certain measure of complexity of the word, which we call its Amperean area. It is a real number, which is the sum over each independent Brownian motion appearing in the word of the time at which this Brownian motion is taken, multiplied by the number of times it occurs in the word. For example, if  $(U_{N,i,t})_{t \geq 0}$  with  $i \in \{1, 2, 3\}$  are three independent Brownian motions on  $U(N)$ , then the Amperean area of the word  $U_{N,1,t_1} U_{N,2,t_2} U_{N,1,t_1}^{-1} U_{N,3,t_3}$  is  $4t_1 + t_2 + t_3$ .

In Section 5.4, we show that this estimate is relevant to the study of the large  $N$  limit of the Yang-Mills field by proving that any piecewise affine loop can be expressed as a word of loops surrounding disjoint domains in such a way that the corresponding word of Brownian motions has an Amperean area which is controlled by the length of the loop. Hence, the speed of convergence on the left-hand side of (1) is controlled by the length of  $l$ . It is then a matter of routine to extend (1) to arbitrary loops with finite length.

**A trace on the group of loops.** In the last paragraphs of Section 5, we consider the question of the non-commutative probability space on which the limiting process is defined. Inspired by the point of view of Singer [34], we seek a description of the master field as a trace on the group algebra of the group of loops on  $\mathbb{R}^2$ . But whereas the note of Kobayashi cited by Singer makes use of a hypothetic group of continuous loops endowed with an ill-defined operation of concatenation-reduction, a work of B. Hambly and T. Lyons [18] allows one, among many other things, to make rigorous sense of a group of rectifiable loops based at the origin. In few words, the problem is the following. One says that a loop is *tree-like* if, as a continuous mapping from the circle  $S^1$  to  $\mathbb{R}^2$ , it factorises through a dendrite, the topological space underlying an  $\mathbb{R}$ -tree. One would then like to say that two loops are equivalent if the concatenation of the first loop with the second traced backwards produces a tree-like loop. Unfortunately, when the loops are only assumed to be continuous, it is not known whether this relation is transitive. Hambly and Lyons proved that a loop with finite length is tree-like if and only if its signature, an algebraic object attached to a loop, is trivial. It follows then immediately that the relation is indeed an equivalence relation on the set of loops with finite length.

We discuss this point in Section 5.7 and propose a proof of the transitivity of the relation which is slightly less technical than the original proof of Hambly and Lyons, in that it does not rely on the signature, but rather on the monotone-light decomposition of continuous mappings

introduced by S. Eilenberg in [9], and on Lebesgue's theory of covering dimension of topological spaces (see [19]). Moreover, our proof remains valid for paths with finite  $p$ -variation for  $p < 2$ , a case in which the equivalence for a loop between being tree-like and having a trivial signature is not known.

**The Makeenko-Migdal equations.** In the last section of this paper, we address the problem of the explicit computation of the master field, that is, of the actual computation of either side of (1), when  $l$  is a piecewise affine loop. This is a question to which a brilliant piece of answer was given by Y. Makeenko and A. Migdal [28], but in a very non-rigorous way. The general idea, which also underlies the first part of this paper, is that when  $l$  is a loop traced in a graph, the number  $\tau(h_l)$  should be studied as a function of the areas of the bounded faces of the graph. What Makeenko and Migdal discovered is that some particular linear combinations of the derivatives of  $\tau(h_l)$  have simple expressions, which can be computed graphically. More precisely, they discovered that the alternated sum of the derivatives of  $\tau(h_l)$  with respect to the areas of the four faces surrounding a point of self-intersection of  $l$  is equal to  $\tau(h_{l_1})\tau(h_{l_2})$ , where  $l_1$  and  $l_2$  are the two loops which are formed by changing the way in which the two strands of  $l$  which are incoming at the self-intersection point are connected to the two outgoing strands (see Figure 1).

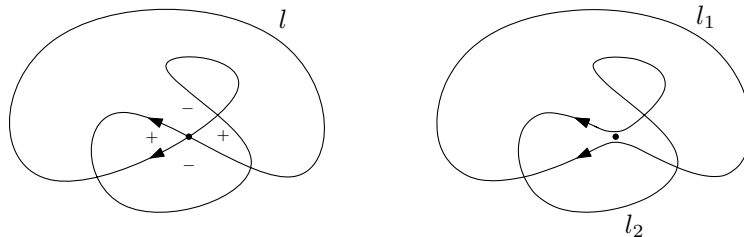


FIGURE 1. A graphical representation of the Makeenko-Migdal equations in the large  $N$  limit. The signs indicate with respect to the areas of which faces the derivatives must be taken, and with which signs.

In Section 6, we give statements and rigorous proofs of formulas which generalise the Makeenko-Migdal equations. We then use these equations to ground a recursive algorithm to compute quantities of the form  $\mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})]$ , and a more efficient one for quantities of the form  $\tau(h_l)$ .

Finally, we discuss a work of Kazakov which sheds some light of this problem of computation, and offer a proof for a couple of statements made at the end of the beautiful paper [20].

Let us emphasise that the idea, on which the proof of the Makeenko-Migdal equation is based, that certain combinatorial features of the unitary Brownian motion can be translated into combinatorial operations on loops, in relation with the computation of expectations of Wilson loops, can be traced back to the work of L. Gross, C. King and A. Sengupta [17]. A related idea was present in our previous work [24].

Before concluding this introduction, we would like to describe the way in which Makeenko and Migdal originally formulated and proved the equation which now bear their names. The striking contrast between the mathematically unorthodox character - to say the less - of the derivation of the equation, and the beauty and simplicity of the equation itself was one of the motivations of the author for undertaking the present study.



Makeenko and Migdal derived their equations (see Propositions 6.16 and 6.17) as particular instances of the Schwinger-Dyson equations, which are the equations which one obtains by formally extending the integration by parts formula to the framework of functional integrals. It may be helpful to start with the following familiar one-dimensional analogue of the Schwinger-Dyson equations: for all smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with bounded derivative, the equality

$$\int_{-\infty}^{+\infty} f'(t)e^{-\frac{t^2}{2}} dt = \int_{-\infty}^{+\infty} tf(t)e^{-\frac{t^2}{2}} dt$$

holds. This equality, which is proved by integration by parts, ultimately relies on the invariance by translation of the Lebesgue measure on  $\mathbb{R}$ .

The Schwinger-Dyson equations are the corresponding statement for functional integrals, which most of the time are ill-defined. A general setting which contains that of Yang-Mills theory is the following. We are given an affine space  $\mathcal{A}$ , possibly infinite dimensional, and a function  $S : \mathcal{A} \rightarrow \mathbb{R}_+$ . In a physical context, the elements of  $\mathcal{A}$  are fields, and  $S$  is an action. We are interested in a certain family of observables, that is, in a certain family of functions  $\psi : \mathcal{A} \rightarrow \mathbb{R}$ , and we are more precisely interested in computing integrals of the form

$$(2) \quad \int_{\mathcal{A}} \psi(A)e^{-\frac{1}{2}S(A)} DA,$$

where  $DA$  is meant to be a properly normalised translation invariant measure on  $\mathcal{A}$ . Such a measure does not exist, at least not with the properties one would expect it to have for interesting physical applications, but we shall choose to leave this point aside and to focus on the algebraic manipulations which one is then able to make with these integrals.

Let  $\eta$  be a vector of the direction of the affine space  $\mathcal{A}$ . The translation invariance of the measure  $DA$  can be written

$$\frac{d}{dt}\Big|_{t=0} \int_{\mathcal{A}} \psi(A+t\eta)e^{-\frac{1}{2}S(A+t\eta)} DA = 0,$$

from which one extracts the Schwinger-Dyson equation

$$(3) \quad \int_{\mathcal{A}} d_A\psi(\eta)e^{-\frac{1}{2}S(A)} DA = \frac{1}{2} \int_{\mathcal{A}} \psi(A)d_A S(\eta)e^{-\frac{1}{2}S(A)} DA.$$

Here,  $d_A\psi(\eta)$  and  $d_A S(\eta)$  denote respectively the differential of  $\psi$  and  $S$  at the point  $A$  in the direction  $\eta$ .

In the case of the two-dimensional Yang-Mills measure, and in some other cases, a rigorous definition can be given for the integral (2) as a whole, at least for a certain class of observables  $\psi : \mathcal{A} \rightarrow \mathbb{R}$ . To each such observable, one is able to associate a number, which physicists usually denote by  $\langle \psi \rangle$  and which we shall here denote by  $\mathbb{E}[\psi]$ , and which can be understood as the rigorous version of (2). With this notation, the Schwinger-Dyson equation writes

$$\mathbb{E}[d\psi(\eta)] = \frac{1}{2}\mathbb{E}[\psi dS(\eta)].$$

Let us describe the context of two-dimensional Yang-Mills theory in which Makeenko and Migdal found a beautiful application of these equations.

Consider a compact Lie group  $G$  and a principal  $G$ -bundle  $P \rightarrow M$  over a surface  $M$ . Let  $\mathcal{A}$  be the space of connections on  $P$ . It is an affine space whose direction is the space  $\Omega^1(M, \text{ad}(P))$  of differential 1-forms on  $M$  with values in the adjoint bundle, which is the vector bundle associated to  $P$  through the adjoint representation of  $G$ . In this paper, we consider the case where  $M = \mathbb{R}^2$  and in this case, after choosing a global section of  $P$ , we can think of  $\mathcal{A}$  as the vector space  $\Omega^1(\mathbb{R}^2, \mathfrak{g})$  of differential 1-forms on  $\mathbb{R}^2$  with values in the Lie algebra of  $G$ .

The curvature of an element  $A$  of  $\mathcal{A}$  is a 2-form  $F$  with values in the vector bundle  $\text{ad}(P)$ , and it is defined by the formula  $F = dA + \frac{1}{2}[A \wedge A]$ . If  $M = \mathbb{R}^2$  and a section of  $P$  is chosen, the form  $F$  can also be seen as a  $\mathfrak{g}$ -valued 2-form on  $\mathbb{R}^2$ , related to  $A$  by the equality  $F(X, Y) = dA(X, Y) + [A(X), A(Y)]$ . The Yang-Mills action  $S : \mathcal{A} \rightarrow \mathbb{R}_+$  is defined by

$$S(A) = \int_M \langle F \wedge *F \rangle_{\mathfrak{g}},$$

where  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  is an invariant scalar product on  $\mathfrak{g}$  and  $*$  is the Hodge operator of a Riemannian metric on  $M$ . Since we use this operator between 2-forms and 0-forms, it depends only on the Riemannian area induced by the Riemannian metric. In the case of  $\mathbb{R}^2$  with the Euclidean metric, if  $A$  writes  $A = A_x dx + A_y dy$ , then

$$S(A) = \int_{\mathbb{R}^2} \|\partial_x A_y - \partial_y A_x + [A_x, A_y]\|_{\mathfrak{g}}^2 dx dy.$$

Let us choose  $\eta \in \Omega^1(M, \text{ad}(P))$  a tangent vector to  $\mathcal{A}$ . A computation whose main difficulty lies in the careful unfolding of the definitions of the objects involved shows that, for all  $A \in \mathcal{A}$ , one has

$$d_A S(\eta) = 2 \int_M \langle \eta \wedge d^A *F \rangle,$$

where  $d^A$  is the covariant exterior differential, defined on  $\Omega^0(M, \text{ad}(P))$  by  $d^A \alpha = d\alpha + [A, \alpha]$ .

Makeenko and Migdal chose to apply the Schwinger-Dyson equation in this context to a particular observable, namely a Wilson loop. Let  $l : [0, 1] \rightarrow \mathbb{R}^2$  be a loop, that is, a  $C^1$  curve such that  $l(0) = l(1)$ . For each  $A \in \mathcal{A}$ , the holonomy of  $A$  along  $l$  is the element of  $G$  which physicists denote by  $P \exp \oint_l A$  and which is the final value of the solution of the differential equation

$$h(0) = 1 \text{ and } \dot{h}(t) = -h(t)A(\dot{l}(t)), \quad t \in [0, 1]$$

with unknown function  $h : [0, 1] \rightarrow G$ . We shall use the notation  $\text{hol}(A, l) = h(1)$  for the holonomy of  $A$  along  $l$ . It follows from the classical theory of differential equations that the differential of the holonomy is given by

$$d_A (\text{hol}(\cdot, l))(\eta) = \int_0^1 \text{hol}(A, l_{[0,t]}) \eta(\dot{l}(t)) \text{hol}(A, l_{[t,1]}) dt.$$

Makeenko and Migdal consider the case where  $G$  is a matrix group, for example the unitary group  $U(N)$ , with the Lie algebra  $\mathfrak{u}(N)$  endowed with the scalar product  $\langle X, Y \rangle = -N \text{Tr}(XY)$ . This allows them to define a complex-valued observable  $\psi_{l,X}$  by choosing a loop  $l$ , an element  $X$  of  $\mathfrak{u}(N)$  and by setting

$$\psi_{l,X}(A) = \text{tr}(X \text{hol}(A, l)).$$

More precisely, they choose a loop  $l$  with a transverse self-intersection, such that for some  $t_0 \in (0, 1)$ , the equality  $l(t_0) = l(0)$  holds and the vectors  $\dot{l}(0)$  and  $\dot{l}(t_0)$  are not collinear. The vector  $X$  is arbitrary, and meant to be given several special values in a moment. Finally, they choose for  $\eta$  a distributional 1-form, which one could write as

$$\forall m \in M, \forall v \in T_m M, \quad \eta_m(v) = \delta_{m, l(0)} \frac{\det(\dot{l}(0), v)}{\det(\dot{l}(0), \dot{l}(t_0))} X.$$

With this choice of  $\eta$ , the directional derivative of the holonomy is given by

$$d_A (\text{hol}(\cdot, l))(\eta) = \text{hol}(A, l_{[0,t_0]}) X \text{hol}(A, l_{[t_0,1]})$$

and that of the action by  $d_A S(\eta) = 2d^A *F(\dot{l}(0))$ , although there might be a coefficient in front of this expression. Neglecting this, the Schwinger-Dyson equation for the observable  $\psi_{l,X}$  and the derivation in the direction  $\eta$  reads

$$\mathbb{E} [\text{tr} (X \text{hol}(A, l_{[0,t_0]}) X \text{hol}(A, l_{[t_0,1]}))] = -N^2 \mathbb{E} [\text{tr}(X \text{hol}(A, l)) \text{tr}(X d^A *F(\dot{l}(0)))] .$$

Letting  $X$  take all the values of an orthonormal basis of  $\mathfrak{u}(N)$  and adding the equalities, they find

$$\mathbb{E} [\text{tr}(\text{hol}(A, l_0)) \text{tr}(\text{hol}(A, l_1))] = -\mathbb{E} [\text{tr}(\text{hol}(A, l) d^A *F(\dot{l}(0)))] .$$

There remains to understand the right-hand side of this equation. For this, they use the fact that the holonomy around the boundary of a disk of area  $\varepsilon$  around a point  $m$  is close to  $\exp(\varepsilon *F(m))$ . Hence, a term  $*F(l(0))$  in the holonomy would correspond to the adjunction of a small bump to  $l$  at  $l(0)$ . With the covariant differential  $d^A$ , the right-hand side corresponds to the difference between the expectations corresponding to two loops, one with a bump immediately after  $l(0)$ , the other immediately before. It should be apparent on Figure 2 that this can reasonably be interpreted as an alternated sum of the derivatives of the expectation of  $\text{tr}(\text{hol}(A, l))$  with respect to the areas of the four faces which surround  $l(0)$ .

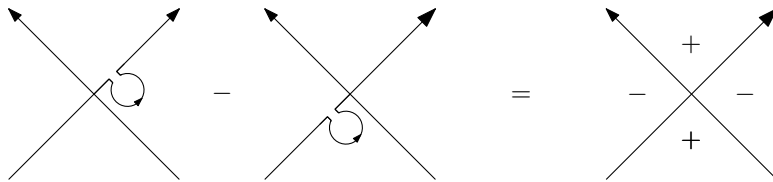


FIGURE 2. The right-hand side of the Schwinger-Dyson equation can be interpreted as the alternated sum of the derivatives with respect to the areas of the faces surrounding the self-intersection point.

### Part 1. Large $N$ limit of Brownian motions.

In the first part of this paper, we study the *large  $N$  limit* of the Brownian motion on a compact matrix group and prove two main convergence results. In the first result, we consider the distribution of the eigenvalues of a matrix taken in a compact matrix group under the heat kernel measure at a fixed time, and prove the convergence of this distribution as the size of the group tends to infinity. By letting the size tend to infinity, we mean that we consider the three series of special orthogonal, unitary and symplectic groups  $\text{SO}(N)$ ,  $\text{U}(N)$  and  $\text{Sp}(N)$ , and let  $N$  tend to infinity. From the point of view of the asymptotic distribution of the eigenvalues, there is no difference between odd and even orthogonal groups.

In the unitary case, the result was proved by P. Biane [4] using harmonic analysis and, with a more combinatorial approach relying on Schur-Weyl duality, by the author in [25]. We recall and slightly improve the latter proof, and extend it to the orthogonal and symplectic cases by showing that the polynomial differential system which characterises the limiting moments of the distribution of the eigenvalues is the same as in the unitary case. In our treatment of this problem, we try to emphasise the similarities between the three series by viewing each of them as the series of unitary groups over one of the three associative real division algebras. We also pay special attention to the symplectic case and to the signs associated to the multiplication of

elements of the Brauer algebra, according to one of the very last sentences<sup>1</sup> of Brauer's original article [6], on which a substantial part of the literature seems ultimately to rely.

Our first main result, combined with a general property of asymptotic freeness for large independent and rotationally invariant matrices, proved by Voiculescu in the unitary case (see [37]) and by Collins and Śniady in the orthogonal and symplectic case (see [7]), implies a convergence result for expected traces of words of independent matrices taken under the heat kernel measures at various times. Our second main result is an explicit estimate of the speed of this convergence in terms of a certain measure of the complexity of the word under consideration and which we call its non-commutative Amperean area. This notion turns out to be very well suited to the study which we develop in the second part of this work of the large  $N$  limit of the Yang-Mills theory on the Euclidean plane.

This first part is divided in three sections. In the first section, we define the Brownian motions which we consider, with the appropriate normalisations, and compute explicitly the Casimir elements of the various Lie algebras involved. Then, the second section is devoted to the proof of our first main theorem and the third and last section to the proof of our second main theorem.

## 1. BROWNIAN MOTIONS ON CLASSICAL GROUPS

In this section, we define the Brownian motion on the orthogonal, unitary, and symplectic groups and establish a concise formula for the expected value of any polynomial function of the entries of a sample of this Brownian motion at a given time. To the extent possible, we treat these three cases on the same footing, by seeing them as the unitary group over the reals, complex numbers, and quaternions. In particular, we avoid as much as possible considering the symplectic group  $\mathrm{Sp}(N)$  as a subgroup of  $\mathrm{U}(2N)$ .

**1.1. Classical groups.** Let  $\mathbb{K}$  be one of the three associative real division algebras  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$ . If  $x \in \mathbb{K}$ , we denote by  $x^*$  the conjugate of  $x$ . If  $M \in M_N(\mathbb{K})$ , the adjoint of  $M$  is the matrix  $M^*$  defined by  $(M^*)_{ab} = (M_{ba})^*$ . We consider the following compact real Lie group, which depend on an integer  $N \geq 1$ :

$$\mathrm{U}(N, \mathbb{K}) = \{M \in M_N(\mathbb{K}) : M^*M = I_N\}^0,$$

where the exponent 0 indicates, for the needs of the real case, that we take the connected component of the unit element. The Lie algebra of this Lie group is the real vector space

$$\mathfrak{u}(N, \mathbb{K}) = \{X \in M_N(\mathbb{K}) : X^* + X = 0\}.$$

We thus have the following table, in which we include the value of classical parameter  $\beta = \dim_{\mathbb{R}} \mathbb{K}$ .

|                  | $\mathrm{U}(N, \mathbb{K})$ | $\mathfrak{u}(N, \mathbb{K})$ | $\beta$ |
|------------------|-----------------------------|-------------------------------|---------|
| (4) $\mathbb{R}$ | $\mathrm{SO}(N)$            | $\mathfrak{so}(N)$            | 1       |
| $\mathbb{C}$     | $\mathrm{U}(N)$             | $\mathfrak{u}(N)$             | 2       |
| $\mathbb{H}$     | $\mathrm{Sp}(N)$            | $\mathfrak{sp}(N)$            | 4       |

Let  $\mathfrak{a}_N$  and  $\mathfrak{s}_N$  denote respectively the linear spaces of skew-symmetric and symmetric real matrices of size  $N$ . Denoting by  $\{1, i, j, k\}$  the standard  $\mathbb{R}$ -basis of  $\mathbb{H}$ , we have the equalities

$$(5) \quad \mathfrak{so}(N) = \mathfrak{a}_N, \quad \mathfrak{u}(N) = \mathfrak{a}_N \oplus i\mathfrak{s}_N, \quad \text{and} \quad \mathfrak{sp}(N) = \mathfrak{a}_N \oplus i\mathfrak{s}_N \oplus j\mathfrak{s}_N \oplus k\mathfrak{s}_N,$$

from which it follows that

$$(6) \quad \dim \mathrm{U}(N, \mathbb{K}) = \frac{N(N-1)}{2} + (\beta-1)\frac{N(N+1)}{2} = \frac{\beta}{2}N^2 + \left(\frac{\beta}{2} - 1\right)N.$$

<sup>1</sup>One has, however, to add a factor  $\varphi(S_1, S_2)$  on the right side, whose value is  $+1$ ,  $-1$  or  $0$ . Brauer does not give the definition of  $\varphi(S_1, S_2)$ .

Let us add to our list the special unitary group  $SU(N) = \{U \in U(N), \det U = 1\}$  whose Lie algebra is  $\mathfrak{su}(N) = \{X \in \mathfrak{u}(N), \operatorname{Tr}(X) = 0\}$ , and which has dimension  $N^2 - 1$ .

**1.2. Invariant scalar products.** The first step in defining a Brownian motion on a compact Lie group is the choice of a scalar product on its Lie algebra invariant under the adjoint action. Excepted the 1-dimensional centre of  $U(N)$ , the Lie groups which we consider are simple, so that their Lie algebras carry, up to a scalar multiplication, a unique invariant scalar product. For a fixed  $N$ , a rescaling of the scalar product corresponds merely to a linear time-change for the Brownian motion. However, since we are going to let  $N$  tend to infinity, the way in which we normalise the scalar products matters.

Let  $\operatorname{Tr} : M_N(\mathbb{K}) \rightarrow \mathbb{K}$  denote the usual trace, so that  $\operatorname{Tr}(I_N) = N$ . We endow our Lie algebras with the following scalar products :

$$(7) \quad \forall X, Y \in \mathfrak{u}(N, \mathbb{K}), \langle X, Y \rangle = \frac{\beta N}{2} \Re \operatorname{Tr}(X^* Y) = -\frac{\beta N}{2} \Re \operatorname{Tr}(XY),$$

and the scalar product on  $\mathfrak{su}(N)$  is the restriction of that on  $\mathfrak{u}(N)$ . The real part is needed only for the quaternionic case, as  $\operatorname{Tr}(X^* Y)$  is real whenever  $X$  and  $Y$  are complex anti-Hermitian.

**1.3. Casimir elements.** Let  $\mathfrak{g} \subset M_N(\mathbb{K})$  be one of our Lie algebras, of dimension  $d$ . Let  $\{X_1, \dots, X_d\}$  be an orthonormal  $\mathbb{R}$ -basis of  $\mathfrak{g}$ . The tensor

$$C_{\mathfrak{g}} = \sum_{k=1}^d X_k \otimes X_k,$$

seen abstractly as an element of  $\mathfrak{g} \otimes \mathfrak{g}$  or more concretely as an element of  $M_N(\mathbb{K}) \otimes_{\mathbb{R}} M_N(\mathbb{K})$ , does not depend on the choice of the orthonormal basis. It is called the Casimir element of  $\mathfrak{g}$ .

Let  $\{E_{ab} : a, b = 1 \dots N\}$  denote the set of elementary matrices in  $M_N(\mathbb{R})$ , defined by  $(E_{ab})_{ij} = \delta_{i,a} \delta_{j,b}$ . Let us define two elements  $T$  and  $W$  of  $M_N(\mathbb{R})^{\otimes 2}$  by

$$(8) \quad T = \sum_{a,b=1}^N E_{ab} \otimes E_{ba} \quad \text{and} \quad W = \sum_{a,b=1}^N E_{ab} \otimes E_{ab}.$$

The letters  $T$  and  $W$  stand respectively for *transposition* and *Weyl contraction*. The operators  $T$  and  $W$  can conveniently be depicted as in Figure 3 below.



FIGURE 3. The operators  $T$  and  $W$ .

On the other hand, set  $l(\mathbb{K}) = \{1, i, j, k\} \cap \mathbb{K}$  and let us define two elements  $\operatorname{Re}^{\mathbb{K}}$  and  $\operatorname{Co}^{\mathbb{K}}$  of  $\mathbb{K} \otimes_{\mathbb{R}} \mathbb{K}$  by

$$(9) \quad \operatorname{Re}^{\mathbb{K}} = \sum_{\gamma \in l(\mathbb{K})} \gamma \otimes \gamma^{-1} \quad \text{and} \quad \operatorname{Co}^{\mathbb{K}} = \sum_{\gamma \in l(\mathbb{K})} \gamma \otimes \gamma.$$

The names  $\operatorname{Re}$  and  $\operatorname{Co}$  stand for *real part* and *conjugation*, with the quaternionic case in mind. Indeed, the following two relations hold, which will prove very useful: for all quaternion  $q$ ,

$$(10) \quad q - iqj - jqj - kqk = 4\Re(q) \quad \text{and} \quad q + iqj + jqj + kqk = -2q^*.$$

In the next lemma, and later in this work, we will use the natural identifications  $M_N(\mathbb{K}) \simeq M_N(\mathbb{R}) \otimes \mathbb{K}$  and  $M_N(\mathbb{K})^{\otimes n} \simeq M_N(\mathbb{R})^{\otimes n} \otimes \mathbb{K}^{\otimes n}$ .

**Lemma 1.1.** *The Casimir element of  $\mathfrak{u}(N, \mathbb{K})$  is given by*

$$(11) \quad C_{\mathfrak{u}(N, \mathbb{K})} = \frac{1}{\beta N} \left( -T \otimes \text{Re}^{\mathbb{K}} + W \otimes \text{Co}^{\mathbb{K}} \right).$$

Moreover,  $C_{\mathfrak{su}(N)} = C_{\mathfrak{u}(N)} - \frac{1}{N^2} iI_N \otimes iI_N$ .

*Proof.* The spaces  $\mathfrak{a}_N$  and  $\mathfrak{s}_N$ , each endowed with the scalar product  $\langle X, Y \rangle = \frac{1}{2} \text{Tr}(X^*Y)$  are Euclidean spaces in which we can compute the sum of the tensor squares of the elements of an orthonormal basis. We find  $C_{\mathfrak{a}_N} = -T + W$  and  $C_{\mathfrak{s}_N} = T + W$ . The result follows from (5) and (7).  $\square$

Because tensor products in (11) are over  $\mathbb{R}$ , the expression in the case of  $U(N)$  is not the most natural one. From now on, let us make the convention that tensor products are on  $\mathbb{R}$  when we deal with orthogonal or symplectic matrices, and over  $\mathbb{C}$  when we deal with unitary ones. Then in particular  $\text{Re}^{\mathbb{C}} = 2$  and  $\text{Co}^{\mathbb{C}} = 0$ . Thus, we have

$$(12) \quad C_{\mathfrak{so}(N)} = -\frac{1}{N}(T - W) \text{ and } C_{\mathfrak{u}(N)} = -\frac{1}{N}T.$$

The explicit expression (11) of the Casimir operators allows us to compute any expression of the form  $\sum_{k=1}^d B(X_k, X_k)$  where  $B$  is an  $\mathbb{R}$ -bilinear map. For example, we can compute the sum of the squares of the elements of an orthonormal basis.

**Lemma 1.2.** *Let  $\mathfrak{g} \subset M_N(\mathbb{K})$  be one of our Lie algebras, of dimension  $d$ . Let  $\{X_1, \dots, X_d\}$  be an orthonormal basis of  $\mathfrak{g}$ . Then  $\sum_{k=1}^d X_k^2 = c_{\mathfrak{g}} I_N$ , where the real constant  $c_{\mathfrak{g}}$  is given by*

$$(13) \quad c_{\mathfrak{u}(N, \mathbb{K})} = -1 + \frac{2 - \beta}{\beta N},$$

and  $c_{\mathfrak{su}(N)} = -1 + \frac{1}{N^2}$ .

*Proof.* This equality follows from Lemma 1.1 and the following facts: the images of  $T$  and  $W$  by the mapping  $X \otimes Y \mapsto XY$  are respectively  $NI_N$  and  $I_N$  (see Figure 4 below for a graphical proof), and the sums  $\sum_{\gamma \in \text{el}(\mathbb{K})} \gamma \gamma^{-1}$  and  $\sum_{\gamma \in \text{el}(\mathbb{K})} \gamma \gamma$  are respectively equal to  $\beta$  and  $2 - \beta$ .  $\square$

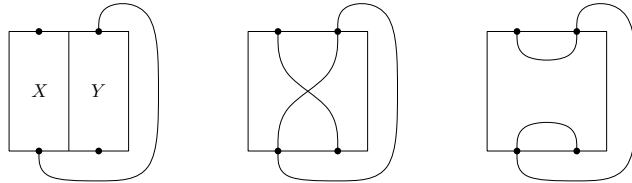


FIGURE 4. The images of the operators  $T$  and  $W$  by the mapping  $X \otimes Y \mapsto XY$  can be computed graphically by joining the top right dot to the bottom left dot of the box. A loop carries a free index and produces a factor  $N$ .

**1.4. Brownian motions.** Let  $\mathfrak{g} \subset M_N(\mathbb{K})$  be one of our Lie algebras and let  $G$  be the corresponding group. Let  $(K_t)_{t \geq 0}$  be the linear Brownian motion in the Euclidean space  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ , that is, the continuous  $\mathfrak{g}$ -valued Gaussian process such that for all  $X, Y \in \mathfrak{g}$  and all  $s, t \geq 0$ , one has

$$\mathbb{E}[\langle X, K_t \rangle \langle Y, K_s \rangle] = \min(s, t) \langle X, Y \rangle.$$

Alternatively,  $K$  can be constructed by picking an orthonormal basis  $(X_k)_{k=1 \dots d}$  of  $\mathfrak{g}$ , a collection  $(B^{(k)})_{k=1 \dots d}$  of independent standard real Brownian motions, and setting

$$(14) \quad K_t = \sum_{k=1}^d B_t^{(k)} X_k.$$

The quadratic variation of  $K$  is easily expressed in terms of the Casimir operator of  $\mathfrak{g}$ : we have

$$(15) \quad dK_t \otimes dK_t = C_{\mathfrak{g}} dt,$$

from which one deduces, in the same way as Lemma 1.2 was deduced from Lemma 1.1,

$$(16) \quad (dKdK)_t = c_{\mathfrak{g}} I_N dt.$$

The Brownian motion on  $G$  is defined as the solution  $(V_t)_{t \geq 0}$  of the following linear Itô stochastic differential equation in  $M_N(\mathbb{K})$ :

$$(17) \quad \begin{cases} dV_t = dK_t V_t + \frac{c_{\mathfrak{g}}}{2} V_t dt, \\ V_0 = I_N. \end{cases}$$

**Lemma 1.3.** *With probability 1, the matrix  $V_t$  belongs to  $G$  for all  $t \geq 0$ .*

*Proof.* One has  $dV_t^* = -V_t^* dK_t + \frac{1}{2} c_{\mathfrak{g}} V_t^* dt$ . Hence, Itô's formula and the expression (16) of the quadratic variation of  $K$  imply that  $d(V_t^* V_t) = 0$ . This proves the assertion, except for  $SU(N)$ . In order to treat this case, write the stochastic differential equation satisfied by the columns of  $V_t$  and deduce an expression of  $d(\det V_t)$ . Using the fact that  $\text{Tr}(dK_t) = 0$  and the fact that  $C_{\mathfrak{su}(N)} = -\frac{1}{N} T + \frac{1}{N^2} I_N \otimes I_N$  in  $M_N(\mathbb{C}) \otimes_{\mathbb{C}} M_N(\mathbb{C})$ , this yields  $d(\det V_t) = 0$ , as expected.  $\square$

We will adopt the following notational convention: the Brownian motions on  $SO(N)$ ,  $U(N)$  and  $Sp(N)$  will respectively be denoted by  $(R_t)_{t \geq 0}$ ,  $(U_t)_{t \geq 0}$ , and  $(S_t)_{t \geq 0}$ .

**1.5. Expected values of polynomials of the entries.** Let  $n \geq 1$  be an integer and  $t \geq 0$  be a real. We give a formula for the expected value of all homogeneous polynomial functions of degree  $n$  in the entries of the Brownian motion on one of our groups at time  $t$ .

For all integers  $i, j$  such that  $1 \leq i < j \leq n$ , let us denote by  $\iota_{i,j} : M_N(\mathbb{K})^{\otimes 2} \rightarrow M_N(\mathbb{K})^{\otimes n}$  the linear mapping defined by

$$(18) \quad \iota_{i,j}(X \otimes Y) = I_N^{\otimes(i-1)} \otimes X \otimes I_N^{\otimes(j-i-1)} \otimes Y \otimes I_N^{\otimes(n-j)}.$$

We will often write  $(X \otimes Y)_{ij}$  instead of  $\iota_{i,j}(X \otimes Y)$

**Proposition 1.4.** *Let  $(V_t)_{t \geq 0}$  be the Brownian motion on one of the groups which we consider with Lie algebra  $\mathfrak{g}$ . Let  $n \geq 1$  be an integer. Let  $t \geq 0$  be a real. We have*

$$(19) \quad \mathbb{E}[V_t^{\otimes n}] = \exp\left(\frac{nc_{\mathfrak{g}}t}{2} + t \sum_{1 \leq i < j \leq n} \iota_{i,j}(C_{\mathfrak{g}})\right).$$

In particular, if  $(R_t)_{t \geq 0}$  denotes the Brownian motion on  $\mathrm{SO}(N)$ , then

$$(20) \quad \mathbb{E} [R_t^{\otimes n}] = \exp \left( -\frac{N-1}{N} \frac{nt}{2} - \frac{t}{N} \sum_{1 \leq i < j \leq n} T_{ij} - W_{ij} \right).$$

If  $(U_t)_{t \geq 0}$  denotes the Brownian motion on  $\mathrm{U}(N)$ , then

$$(21) \quad \mathbb{E} [U_t^{\otimes n}] = \exp \left( -\frac{nt}{2} - \frac{t}{N} \sum_{1 \leq i < j \leq n} T_{ij} \right).$$

Finally, if  $(S_t)_{t \geq 0}$  denotes the Brownian motion on  $\mathrm{Sp}(N)$ , then

$$(22) \quad \mathbb{E} [U_t^{\otimes n}] = \exp \left( -\frac{2N+1}{4N} \frac{nt}{2} - \frac{t}{N} \sum_{1 \leq i < j \leq n} \left( (T \otimes \mathrm{Re}^{\mathbb{H}})_{ij} - (W \otimes \mathrm{Co}^{\mathbb{H}})_{ij} \right) \right).$$

*Proof.* Both sides of (19) are equal to  $I_N^{\otimes n}$  for  $t = 0$ . Moreover, Itô's formula for  $V_t^{\otimes n}$  seen as an element of  $M_N(\mathbb{K})^{\otimes n}$  writes

$$d(V_t^{\otimes n}) = \left( \sum_{i=1}^n I_N^{\otimes(i-1)} \otimes dK_t \otimes I_N^{\otimes(n-i)} + \frac{nc_{\mathfrak{g}}}{2} + \sum_{1 \leq i < j \leq 2} \iota_{i,j}(dK_t \otimes dK_t) \right) V_t^{\otimes n}.$$

Using (15), this implies that the time derivatives of both sides of (19) are equal.  $\square$

## 2. CONVERGENCE RESULTS FOR ONE BROWNIAN MOTION

In this section, we analyse the asymptotic behaviour of the repartition of the eigenvalues of the Brownian motion at time  $t$  on  $\mathrm{U}(N, \mathbb{K})$  as  $N$  tends to infinity, the time  $t$  being fixed. We start by briefly discussing the issue of eigenvalues in the symplectic case.

**2.1. Moments of the empirical spectral measure.** Let  $M$  be a real or complex matrix of size  $N$  with complex eigenvalues  $\lambda_1, \dots, \lambda_N$ . We define the empirical spectral measure of  $M$  by

$$\hat{\mu}_M = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}.$$

The moments of this measure can be expressed as traces of powers of  $M$ . Indeed, for all integer  $n \geq 0$ ,  $\int_{\mathbb{C}} z^n \hat{\mu}_M(dz) = \frac{1}{N} \mathrm{Tr}(M^n) = \mathrm{tr}(M^n)$ , where  $\mathrm{tr}$  denotes the normalised trace, so that  $\mathrm{tr}(I_N) = 1$ . If  $M$  is invertible, then these equalities hold for all  $n \in \mathbb{Z}$ .

For a matrix with quaternionic entries, the very notion of eigenvalue must be handled with care. A matrix  $M \in M_N(\mathbb{H})$  is said to admit the right eigenvalue  $q \in \mathbb{H}$  if there exists a non-zero vector  $X \in \mathbb{H}^N$  such that  $MX = Xq$ . If  $q$  is a right eigenvalue of  $M$ , then any quaternion conjugated to  $q$  is also a right eigenvalue of  $M$ , because for all non-zero quaternion  $u$ , one has  $M(Xu^{-1}) = M(Xu^{-1})uqu^{-1}$ .

It is an elementary property of  $\mathbb{H}$  that two quaternions are conjugated if and only if they have the same real part and the same norm. In particular, each conjugacy class of  $\mathbb{H}$  either consists of a single real element, or meets  $\mathbb{C}$  at exactly two conjugated non-real elements. Thus, a matrix with quaternionic entries determines real eigenvalues, which are to be counted twice, and conjugate pairs of complex eigenvalues.

It is convenient to momentarily see  $\mathbb{H}$  as  $\mathbb{C} \oplus j\mathbb{C}$ , to write any vector  $X \in \mathbb{H}^N$  as  $X = Z + jW$  with  $Z, W \in \mathbb{C}^N$ , and to write any matrix  $M \in M_N(\mathbb{H})$  as  $M = A + jB$  with  $A, B \in M_N(\mathbb{C})$ .



The mappings  $X \mapsto \tilde{X} = \begin{pmatrix} Z \\ W \end{pmatrix}$  and  $M \mapsto \tilde{M} = \begin{pmatrix} A & -\bar{B} \\ B & A \end{pmatrix}$  are respectively an isomorphism of right complex vector spaces  $\mathbb{H}^N \rightarrow \mathbb{C}^{2N}$  and an injective homomorphism of involutive algebras  $M_N(\mathbb{H}) \rightarrow M_{2N}(\mathbb{C})$ . These morphisms are compatible, in that  $\widetilde{MX} = \tilde{M}\tilde{X}$  for all  $M \in M_N(\mathbb{H})$  and  $X \in \mathbb{H}^N$ .

It turns out that the complex eigenvalues of  $\tilde{M}$  are exactly the complex right eigenvalues of  $M$ , counted twice if they are real. Thus,  $M$  admits exactly  $2N$  complex right eigenvalues  $\{\lambda_1, \lambda_1^*, \dots, \lambda_N, \lambda_N^*\}$ . We define the empirical spectral measure of  $M$  as the spectral empirical measure of  $\tilde{M}$ :

$$\hat{\mu}_M = \frac{1}{2N} \sum_{k=1}^N \delta_{\lambda_k} + \delta_{\lambda_k^*}.$$

Observe that the mapping  $M \mapsto \tilde{M}$  does not preserve the trace, since  $\text{Tr}(\tilde{M}) = 2\Re\text{Tr}(M)$ . Hence, the moments of  $\hat{\mu}_M$  are given by  $\int_{\mathbb{C}} z^n \hat{\mu}_M(dz) = \frac{1}{2N} \text{Tr}(\tilde{M}^n) = \Re\text{tr}(M^n)$  for all  $n \geq 0$ , and also for all  $n \in \mathbb{Z}$  if  $M$  is invertible. The situation is thus almost the same as in the real and complex case, the only difference being that the trace is replaced by its real part. One should however keep in mind that, from the point of view of eigenvalues, the natural non-normalised trace on  $M_N(\mathbb{H})$  is *twice* the real part of the usual trace. Indeed, with our way of counting, the eigenvalue 1 of  $I_N \in M_N(\mathbb{H})$  has multiplicity  $2N$ .

Note finally that orthogonal and unitary matrices have eigenvalues of modulus 1. Similarly, symplectic matrices have quaternionic right eigenvalues of norm 1, and in all cases, the empirical spectral measures which we consider are supported by the unit circle of the complex plane, which we denote by  $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ .

**2.2. First main result : convergence of empirical spectral measures.** Let us introduce the limiting measure which appears in our first main result and was first described by P. Biane in the unitary case. It is a one-parameter family of probability measures on  $\mathbb{U}$  which plays for compact matrix groups the role played for Hermitian matrices by the Wigner semi-circle law. The simplest description of this family is through its moments.

For all real  $t \geq 0$  and all integer  $n \geq 0$ , set

$$(23) \quad \mu_n(t) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}.$$

It follows from Biane's result (Theorem 2.1 below) that there exists a probability measure  $\nu_t$  on  $\mathbb{U}$  such that for all integer  $n \geq 0$ , one has

$$(24) \quad \int_{\mathbb{U}} z^n \nu_t(dz) = \int_{\mathbb{U}} z^{-n} \nu_t(dz) = \mu_n(t).$$

Although there is no simple expression for the density of this measure, some information about this measure can be found in [4, 25]. The result in the unitary case is the following.

**Theorem 2.1.** *Let  $(U_{N,t})_{t \geq 0}$  be the Brownian motion on the unitary group  $U(N)$ , or on the special unitary group  $SU(N)$ . Let  $r \geq 1$  be an integer and  $m_1, \dots, m_r \geq 0$  be integers. Let  $t \geq 0$  be a real. Then*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \text{tr}(U_{N,t}^{m_1}) \dots \text{tr}(U_{N,t}^{m_r}) \right] = \mu_{m_1}(t) \dots \mu_{m_r}(t).$$

Our first main result is the following.

**Theorem 2.2.** *Let  $(R_{N,t})_{t \geq 0}$  be the Brownian motion on the special orthogonal group  $\mathrm{SO}(N)$ , and  $(S_{N,t})_{t \geq 0}$  be the Brownian motion on the symplectic group  $\mathrm{Sp}(N)$ . Let  $r \geq 1$  be an integer and  $m_1, \dots, m_r \geq 0$  be integers. Let  $t \geq 0$  be a real. Then*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \mathrm{tr}(R_{N,t}^{m_1}) \dots \mathrm{tr}(R_{N,t}^{m_r}) \right] = \lim_{N \rightarrow \infty} \mathbb{E} \left[ \Re \mathrm{tr}(S_{N,t}^{m_1}) \dots \Re \mathrm{tr}(S_{N,t}^{m_r}) \right] = \mu_{m_1}(t) \dots \mu_{m_r}(t).$$

The rest of this section is devoted to the proof of Theorem 2.2.

**2.3. Characterisation of the moments of the limiting distribution.** Before we jump into the computation of the limiting distribution of the eigenvalues of our Brownian motions, let us say a few words about the disguise under which the moments  $(\mu_n)_{n \geq 0}$  of the limiting distribution will appear.

These moments are defined by (23) and this is the form under which they appear in the original proof of Theorem 2.1 by P. Biane. There are at least two other ways in which they are amenable to appear. The first is purely combinatorial and related to minimal factorisations of an  $n$ -cycle in the symmetric group  $\mathfrak{S}_n$ . Recall the elementary fact that the  $n$ -cycle  $(1 \dots n)$  cannot be written as a product of less than  $n - 1$  transpositions, and the classical fact that the number of ways of writing it as a product of exactly  $n - 1$  transpositions is  $n^{n-2}$ . More generally, the product of  $(1 \dots n)$  and  $k$  transpositions cannot have more than  $k + 1$  cycles. The following result is proved in [27] in a bijective way.

**Proposition 2.3.** *Let  $\mathbb{T}_n$  be the set of transpositions in the symmetric group  $\mathfrak{S}_n$ . Let  $k \geq 0$  be an integer. The set*

$$\left\{ (\tau_1, \dots, \tau_k) \in (\mathbb{T}_n)^k : (1 \dots n)\tau_1 \dots \tau_k \text{ has exactly } k + 1 \text{ cycles} \right\}$$

*is empty if  $k \geq n$  and has otherwise  $n^{k-1} \binom{n}{k+1}$  elements.*

This result, combined with the equality (21), allows one to give a quick proof of Theorem 2.1. It is however a proof which is not easily generalised to the orthogonal and symplectic cases, because it is more difficult to count paths in the set of standard generators of the Brauer algebra than in the symmetric group.

The second way in which the moments  $(\mu_n)_{n \geq 0}$  may appear is the following. Define a sequence of polynomials  $(L_n)_{n \geq 0}$  by setting  $L_0(t) = 1$  and, for all  $n \geq 1$ ,

$$(25) \quad L_n(t) = e^{\frac{nt}{2}} \mu_n(t) = \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}.$$

**Lemma 2.4.** *The sequence  $(L_n)_{n \geq 0}$  is the unique sequence of functions of one real variable such that  $L_0 = 1$  and*

$$(26) \quad \forall n \geq 1, L_n(0) = 1 \text{ and } \dot{L}_n = -\frac{n}{2} \sum_{k=1}^{n-1} L_k L_{n-k}.$$

Despite the relatively simple explicit form of  $L_n$ , this statement seems to resist a direct verification. One way to prove it is to recognise the link between the recurrence relation (26) and the problem of enumeration of paths in the symmetric group solved by Proposition 2.3, but this could hardly be called a simple proof.

*Proof.* The shortest proof I know is to recognise that (26) is equivalent to an easily solved equation in the reciprocal of the generating function of the sequence  $(L_n)_{n \geq 0}$ . Indeed, consider the formal series  $g(t, z) = \sum_{n \geq 1} L_n(t) z^n$ . The recurrence relation (26) is equivalent to the differential

equation  $\partial_t g(t, z) = -zg(t, z)\partial_z g(t, z)$  with initial condition  $g(0, z) = \frac{z}{1-z}$ . This differential equation is in turn equivalent, for the reciprocal formal series  $f(t, z)$ , defined by  $f(t, g(t, z)) = z$ , to the differential equation  $\partial_t f(t, z) = zf(t, z)$ , with the initial condition  $f(0, z) = \frac{z}{1+z}$ . This last equation is solved by  $f(t, z) = \frac{z}{1+z}e^{tz}$  and Lagrange's inversion formula yields the value of the polynomials  $(L_n)_{n \geq 0}$ .  $\square$

The reason why reciprocals of generating functions on one hand and paths of shortest length in the symmetric group on the other hand, although apparently rather remote from each other, allow one to prove Theorem 2.1, is that both are governed by the combinatorics of the lattice of non-crossing partitions of a cycle (see [35, 5]).

**2.4. The unitary case revisited.** The basis of our proof in the orthogonal and symplectic cases is the proof in the unitary case, which we review in this section. We take this opportunity to introduce useful notation, and also to offer what we believe to be a simpler and clearer proof than what can be found in the literature.

Before we start, let us make a short comment on our strategy of exposition. Rather than spending a lot of time introducing from the beginning, and with little motivation, all the tools which will be needed for the three series of groups, we have chosen to introduce the various objects progressively. The drawback of this approach is that many tools will have to be redefined, some more than once, each new definition containing and superseding the previous ones.

*Proof of Theorem 2.1.* Let  $n \geq 1$  be an integer. We denote by  $\mathfrak{S}_n$  the symmetric group of order  $n$ . Let  $\rho : \mathfrak{S}_n \rightarrow \text{GL}((\mathbb{C}^N)^{\otimes n})$  denote the action given by  $\rho(\sigma)(x_1 \otimes \dots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(n)}$ . For all  $\sigma \in \mathfrak{S}_n$ , let us denote by  $\ell(\sigma)$  the number of cycles of  $\sigma$ . To each  $\sigma \in \mathfrak{S}_n$  we associate two complex-valued functions  $P_\sigma$  and  $p_\sigma$  on  $M_N(\mathbb{C})$  by setting

$$P_\sigma(M) = \text{Tr}^{\otimes n}(\rho(\sigma) \circ M^{\otimes n}) \quad \text{and} \quad p_\sigma(M) = N^{-\ell(\sigma)} P_\sigma(M),$$

where by  $\text{Tr}^{\otimes n}(M_1 \otimes \dots \otimes M_n)$  we mean  $\text{Tr}(M_1) \dots \text{Tr}(M_n)$ . If the lengths of the cycles of the permutation  $\sigma$  are  $m_1, \dots, m_{\ell(\sigma)}$ , then these functions can be written in more elementary terms as

$$(27) \quad P_\sigma(M) = \prod_{i=1}^{\ell(\sigma)} \text{Tr}(M^{m_i}) \quad \text{and} \quad p_\sigma(M) = \prod_{i=1}^{\ell(\sigma)} \text{tr}(M^{m_i}).$$

Let  $(U_{N,t})_{t \geq 0}$  be a Brownian motion on the unitary group  $U(N)$ . We are going to study the complex-valued functions  $F_N$  and  $f_N$  defined on  $\mathbb{R}_+ \times \mathfrak{S}_n$  by

$$F_N(t, \sigma) = \mathbb{E}[P_\sigma(U_N(t))] \quad \text{and} \quad f_N(t, \sigma) = \mathbb{E}[p_\sigma(U_N(t))].$$

Let  $\mathbb{T}_n \subset \mathfrak{S}_n$  denote the set of transpositions. An application of Itô's formula and the fact that the Casimir operator is equal to  $-\frac{1}{N}T$ , where  $T$  is the flip operator on  $\mathbb{C}^N \otimes \mathbb{C}^N$  (see (12)), allow us to prove the following fundamental relation : for all  $t \geq 0$  and all  $\sigma \in \mathfrak{S}_n$ , one has

$$(28) \quad \begin{aligned} \frac{\partial}{\partial t} F_N(t, \sigma) &= \mathbb{E} \left[ \text{Tr}^{\otimes n} \left( \rho(\sigma) \circ \left( -\frac{n}{2} - \frac{1}{N} \sum_{1 \leq i < j \leq n} \rho((ij)) \right) \circ U_t^{\otimes n} \right) \right] \\ &= -\frac{n}{2} F_N(t, \sigma) - \frac{1}{N} \sum_{\tau \in \mathbb{T}_n} F_N(t, \sigma\tau). \end{aligned}$$

With the large  $N$  limit in view, it is preferable to work with the function  $f_N$  rather than the function  $F_N$  : for example, one has  $F_N(0, \sigma) = N^{\ell(\sigma)}$  but  $f_N(0, \sigma) = 1$ . When we divide (28) by

$N^{\ell(\sigma)}$ , we must take care about the number of cycles of the permutations  $\sigma\tau$ , which is not the same as that of  $\sigma$ . More precisely, for each  $\tau$ , we have  $\ell(\sigma\tau) \in \{\ell(\sigma) + 1, \ell(\sigma) - 1\}$ . Let us define

$$\mathbb{T}_n^\pm(\sigma) = \{\tau \in \mathbb{T}_n : \ell(\sigma\tau) = \ell(\sigma) \pm 1\}.$$

With this notation, we have

$$(29) \quad \frac{\partial}{\partial t} f_N(t, \sigma) = -\frac{n}{2} f_N(t, \sigma) - \sum_{\tau \in \mathbb{T}_n^+(\sigma)} f_N(t, \sigma\tau) - \frac{1}{N^2} \sum_{\tau \in \mathbb{T}_n^-(\sigma)} f_N(t, \sigma\tau).$$

Let us denote by  $\mathbf{L}_{\mathbb{U}(N)}$  the linear operator on the space  $\mathcal{F}(\mathfrak{S}_n)$  of complex-valued functions on  $\mathfrak{S}_n$  defined by

$$(\mathbf{L}_{\mathbb{U}(N)} f)(\sigma) = -\frac{n}{2} f(\sigma) - \sum_{\tau \in \mathbb{T}_n^+(\sigma)} f(\sigma\tau) - \frac{1}{N^2} \sum_{\tau \in \mathbb{T}_n^-(\sigma)} f(\sigma\tau),$$

and by  $\mathbb{1} \in \mathcal{F}(\mathfrak{S}_n)$  the function identically equal to 1. We have the equality

$$\forall t \geq 0, f_N(t, \cdot) = e^{t\mathbf{L}_{\mathbb{U}(N)}} \mathbb{1}.$$

This expression allows us to let  $N$  tend to infinity very easily. Indeed, if  $\mathbf{L}$  denotes the limit of  $\mathbf{L}_{\mathbb{U}(N)}$  as  $N$  tends to infinity (with  $n$  staying fixed), that is, the operator defined by

$$(30) \quad (\mathbf{L}f)(\sigma) = -\frac{n}{2} f(\sigma) - \sum_{\tau \in \mathbb{T}_n^+(\sigma)} f(\sigma\tau),$$

then it is readily checked that the sequence of functions  $f_N$ , seen as a sequence of functions from  $\mathbb{R}_+$  to  $\mathcal{F}(\mathfrak{S}_n)$ , converges uniformly on every compact subset of  $\mathbb{R}_+$  towards the function  $f(t, \cdot)$  defined by

$$(31) \quad \forall t \geq 0, f(t, \cdot) = e^{t\mathbf{L}} \mathbb{1}.$$

In order to compute this exponential, let us make the Ansatz that  $f(t, \sigma)$  factorises with respect to the lengths of the cycles of  $\sigma$ , that is, that there exists a sequence  $(\tilde{L}_n)_{n \geq 0}$  of functions such that for all  $t \geq 0$  and all permutation  $\sigma$  with cycles of lengths  $m_1, \dots, m_r$ , we have  $f(t, \sigma) = e^{-\frac{nt}{2}} \tilde{L}_{m_1}(t) \dots \tilde{L}_{m_r}(t)$ . A little computation shows that (31) is equivalent to the recurrence relation (26) for the sequence  $(\tilde{L}_n)_{n \geq 0}$ , of which we know that the sequence  $(L_n)_{n \geq 0}$  defined by (25) is the unique solution. This proves the theorem.  $\square$

**2.5. The Brauer algebra I.** In the orthogonal and symplectic cases, the role played by the symmetric group will be held by an algebra known as the Brauer algebra, which we now describe.

The integer  $n \geq 1$  being fixed, let  $\mathfrak{B}_n$  be the set of partitions of the set  $\{1, \dots, 2n\}$  by pairs. Let  $\lambda$  be a real number. The real Brauer algebra  $B_{n, \lambda}$  admits, as a real vector space, a basis which is in one-to-one correspondence with  $\mathfrak{B}_n$  and which we identify with it. For example,  $B_{2, \lambda}$  has dimension 3 and the basis  $\mathfrak{B}_2$  consists in the three pairings  $\{\{1, 2\}, \{3, 4\}\}$ ,  $\{\{1, 3\}, \{2, 4\}\}$  and  $\{\{1, 4\}, \{2, 3\}\}$ .

An element of  $\mathfrak{B}_n$  can be represented by a horizontal box with  $n$  dots on its bottom edge labelled from 1 to  $n$  and  $n$  dots on its top edge labelled from  $n+1$  to  $2n$ , both from left to right, the appropriate pairs of dots being joined by lines inside the box. The product  $\pi_1\pi_2$  of two elements  $\pi_1$  and  $\pi_2$  of  $\mathfrak{B}_n$  is computed by putting the box representing  $\pi_1$  on the top of the box representing  $\pi_2$ . This produces a new pairing  $\pi$  between the points on the bottom of the box representing  $\pi_2$  and those on the top of the box representing  $\pi_1$ . The superposition of two boxes may moreover lead to the formation of loops inside the box. If  $r$  loops appear in the process, then we set  $\pi_1\pi_2 = \lambda^r \pi$  (see Figure 5 for an example).

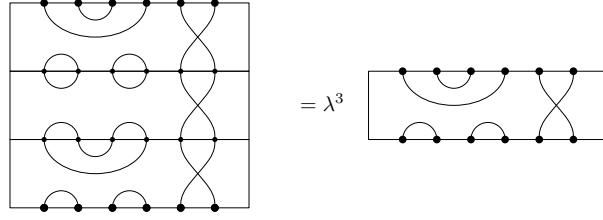


FIGURE 5. With  $\pi_1 = \{\{1, 2\}, \{3, 4\}, \{5, 12\}, \{6, 11\}, \{7, 10\}, \{8, 9\}\}$  and  $\pi_2 = \{\{1, 2\}, \{3, 4\}, \{5, 12\}, \{6, 11\}, \{7, 8\}, \{9, 10\}\}$ , we have  $\pi_1\pi_2\pi_1 = \lambda^3\pi_1$ .

Let  $\mathfrak{S}_n$  denote the symmetric group of order  $n$ . There is a natural inclusion  $\mathfrak{S}_n \subset \mathfrak{B}_n$  which to a permutation  $\sigma \in \mathfrak{S}_n$  associates the pairing  $\{\{i, \sigma(i) + n\} : i \in \{1, \dots, n\}\}$ . Since the multiplication of pairings associated with permutations does never make loops appear, this correspondence determines an injective homomorphism of algebras  $\mathbb{R}[\mathfrak{S}_n] \hookrightarrow \mathbb{B}_{n,\lambda}$ , regardless of the value of  $\lambda$ .

For all integers  $r, s$  such that  $1 \leq r < s \leq n$ , we denote by  $(rs)$  the element of  $\mathfrak{B}_n$  corresponding to the transposition which exchanges  $r$  and  $s$ . We also denote by  $\langle rs \rangle$  the partition of  $\{1, \dots, 2n\}$  which consists of the pairs  $\{k, k + n\}$  for  $k \in \{1, \dots, n\} \setminus \{r, s\}$ , and the two pairs  $\{r, s\}$  and  $\{r + n, s + n\}$ . We call this pairing a Weyl contraction. We denote by  $\mathsf{T}_n$  the set of all transpositions and by  $\mathsf{W}_n$  the subset of  $\mathfrak{B}_n$  which consists of all contractions. Note that the algebra  $\mathbb{B}_{n,\lambda}$  is generated by  $\mathsf{T}_n \cup \mathsf{W}_n$ .

For the needs of the orthogonal case, let us define an action of the Brauer algebra  $\mathbb{B}_{n,N}$  on  $(\mathbb{R}^N)^{\otimes n}$ , that is, a morphism of algebras  $\rho : \mathbb{B}_{n,N} \rightarrow M_N(\mathbb{R})^{\otimes n}$ . Let  $(e_1, \dots, e_N)$  denote the canonical basis of  $\mathbb{R}^N$ . Let  $\pi \in \mathfrak{B}_n$  be a basis vector of  $\mathbb{B}_{n,N}$ , which we identify with the partition in pairs of  $\{1, \dots, 2n\}$  which labels it. We set

$$(32) \quad \rho(\pi) = \sum_{i_1, \dots, i_{2n} \in \{1, \dots, N\}} \left( \prod_{\{k, l\} \in \pi} \delta_{i_k, i_l} \right) E_{i_{n+1}, i_1} \otimes \dots \otimes E_{i_{2n}, i_n}.$$

Consider two elements  $\pi_1, \pi_2 \in \mathfrak{B}_n$ . In the product  $\rho(\pi_1)\rho(\pi_2)$ , the only non-zero contributions come from the terms in which the  $n$  bottom indices of  $\pi_1$  are equal to the  $n$  top indices of  $\pi_2$ . Moreover, any loop carries a free index which runs from 1 to  $N$  and thus produces a factor  $N$ . Hence, if  $r$  loops are formed in the product of  $\pi_1$  and  $\pi_2$ , then  $\rho(\pi_1\pi_2) = N^r \rho(\pi_1)\rho(\pi_2)$ . This shows that the unique linear extension of  $\rho$  to  $\mathbb{B}_{n,N}$  is a homomorphism of algebras  $\rho : \mathbb{B}_{n,N} \rightarrow \text{End}((\mathbb{R}^N)^{\otimes n})$ .

The restriction of  $\rho$  to the subalgebra  $\mathbb{C}[\mathfrak{S}_n]$  coincides with the action of the symmetric group which we considered in the unitary case.

**2.6. The orthogonal case.** On the orthogonal group  $SO(N)$ , the Casimir operator is equal to

$$(33) \quad C_{\mathfrak{so}(N)} = -\frac{1}{N}(T - W)$$

so that for all  $i, j$  such that  $1 \leq i < j \leq n$ , we have

$$(34) \quad \iota_{i,j}(C_{\mathfrak{so}(N)}) = -\frac{1}{N}(\rho(\langle ij \rangle) - \rho(\langle i j \rangle)).$$

Because of the presence of  $W$ , the orthogonal analogues of the functions  $(t \mapsto F_N(t, \sigma))_{\sigma \in \mathfrak{S}_n}$  do not satisfy a closed differential system anymore. We must therefore introduce new functions, which are naturally indexed by the elements of the Brauer algebra.

*Proof of Theorem 2.2 in the orthogonal case.* Let  $n \geq 1$  be an integer. To each element  $\pi \in \mathfrak{B}_n$  we associate the function  $P_\pi$  on  $M_N(\mathbb{R})$  by setting

$$P_\pi(M) = \text{Tr}^{\otimes n}(\rho(\pi) \circ M^{\otimes n}).$$

For example, if  $\pi$  is the element of  $\mathfrak{B}_6$  depicted on the right-hand side of Figure 5, then  $P_\pi(M) = \text{Tr}(M^t M M^t M) \text{Tr}(M^2)$ . Note that when it is restricted to the orthogonal group, the function  $M \mapsto P_\pi(M)$  can be a polynomial in the entries of  $M$  of degree strictly smaller than  $n$ . It is possible, but unnecessary at this stage, to give for the function  $P_\pi$  an expression similar to (27). Our treatment of the symplectic case will however require such a formula, and it may be instructive to take a glance at (39).

The correct definition of the normalised function  $p_\pi$  requires an appropriate definition of the number of cycles of  $\pi$ . The simplest way to define this number is through the equality  $P_\pi(I_N) = N^{\ell(\pi)}$ . Alternatively, it is the number of loops formed after completing the diagram of  $\pi$  by the  $n$  vertical lines which join  $k$  to  $n+k$  for all  $k$  between 1 and  $n$ . We set, as in the unitary case,

$$(35) \quad p_\pi(M) = N^{-\ell(\pi)} P_\pi(M).$$

We extend the definitions of  $P_\pi$  and  $p_\pi$  by linearity to any  $b \in B_{n,N}$ . Note however that the function  $\ell$  is only defined on the elements of  $\mathfrak{B}_n$ . We need to extend it, for future use, to multiple of basis elements by setting  $\ell(c\pi) = \ell(\pi)$  for all complex number  $c \neq 0$ .

Let  $(R_{N,t})_{t \geq 0}$  be a Brownian motion on the orthogonal group  $\text{SO}(N)$ . As in the unitary case, we are going to study the functions  $F_N$  and  $f_N$  defined on  $\mathbb{R}_+ \times B_{n,N}$  by

$$F_N(t, b) = \mathbb{E}[P_b(R_{N,t})] \quad \text{and} \quad f_N(t, b) = \mathbb{E}[p_b(R_{N,t})].$$

The normalisation has been chosen such that  $f_N(0, \pi) = 1$  for all  $\pi \in \mathfrak{B}_n$ . With these definitions and considering the stochastic differential equation which defines the Brownian motion on  $\text{SO}(N)$ , an application of Itô's formula yields the following fundamental relation : for all  $t \geq 0$  and all  $b \in B_{n,N}$ , one has

$$(36) \quad \frac{\partial}{\partial t} F_N(t, b) = -\frac{n(N-1)}{2N} F_N(t, b) - \frac{1}{N} \sum_{\tau \in \mathfrak{T}_n} F_N(t, b\tau) + \frac{1}{N} \sum_{\kappa \in \mathfrak{W}_n} F_N(t, b\kappa),$$

from which it follows immediately that for all  $\pi \in \mathfrak{B}_n$ ,

$$(37) \quad \frac{\partial}{\partial t} f_N(t, \pi) = -\frac{n(N-1)}{2N} f_N(t, \pi) - \sum_{\tau \in \mathfrak{T}_n} N^{\ell(\pi\tau) - \ell(\pi) - 1} f_N(t, \pi\tau) + \sum_{\kappa \in \mathfrak{W}_n} N^{\ell(\pi\kappa) - \ell(\pi) - 1} f_N(t, \pi\kappa).$$

Note that in this equation,  $\pi\tau$  and  $\pi\kappa$  might be non-trivial scalar multiples of basis elements, thus possibly introducing extra powers of  $N$  in the expression. Note also that, for the same reason, we are using the extended definition of the function  $\ell$ .

In fact, the only case where a loop is formed is for the product  $\pi\kappa$  when  $\kappa = \langle ij \rangle$  and the pair  $\{i, j\}$  belongs to  $\pi$ . Moreover, in this case,  $\pi\kappa = N\pi$ .

Let us denote by  $\mathbf{L}_{\text{SO}(N)}$  the linear operator on the dual space  $B_{n,N}^*$  of linear forms on  $B_{n,N}$  characterised by the fact that for all  $\pi \in \mathfrak{B}_n$ ,

$$(\mathbf{L}_{\text{SO}(N)} f)(\pi) = -\frac{n(N-1)}{2N} f(\pi) - \sum_{\tau \in \mathfrak{T}_n} N^{\ell(\pi\tau) - \ell(\pi) - 1} f(\pi\tau) + \sum_{\kappa \in \mathfrak{W}_n} N^{\ell(\pi\kappa) - \ell(\pi) - 1} f(\pi\kappa).$$

We also denote by  $\mathbb{1} \in B_{n,N}^*$  the linear form equal to 1 on each basis vector. Then we have the equality

$$\forall t \geq 0, f_N(t, \cdot) = e^{t\mathbf{L}_{\text{SO}(N)}} \mathbb{1}.$$

Let us now determine which powers of  $N$  appear in  $\mathbf{L}_{\mathrm{SO}(N)}$ . First of all, the observation which we made just after (37) and an elementary verification show that  $\mathbf{L}_{\mathrm{SO}(N)}$  is a polynomial of degree at most 2 in  $N^{-1}$ .

Now comes the crucial argument, namely the observation that multiplying a permutation by a Weyl contraction does never create a loop nor increase the number of cycles. The first assertion is a consequence of the fact that for all  $\pi \in \mathfrak{B}_n$  and all  $i, j \in \{1, \dots, n\}$  with  $i < j$ , the product  $\pi \langle i j \rangle$  involves a loop if and only if the pair  $\{i, j\}$  belongs to  $\pi$ . If  $\pi$  is a permutation, this never happens. Moreover, one checks, depending on whether  $i$  and  $j$  belong to the same cycle of  $\sigma$  or not, that  $\ell(\pi \langle i j \rangle)$  belongs to  $\{\ell(\pi) - 1, \ell(\pi)\}$ . We also know already that if  $\pi$  is a permutation, then  $\ell(\pi\tau)$  belongs to  $\{\ell(\pi) - 1, \ell(\pi) + 1\}$ . These observations imply that when  $\pi$  is a permutation, the last term of (37) is dominated by  $N^{-1}$ .

Recall the definition of the operator  $\mathbf{L}$  from the unitary case (see (30)). The previous discussion shows that, in the basis of  $\mathbf{B}_{n,N}^*$  dual to  $\mathfrak{B}_n$ , split into dual permutations on one hand and the other dual basis elements on the other hand, the matrix of  $\mathbf{L}_{\mathrm{SO}(N)}$  is

$$\mathbf{L}_{\mathrm{SO}(N)} = \left( \begin{array}{c|c} \mathbf{L}_{\mathrm{U}(N)} + \frac{n}{2N} I_n! & * \\ \hline O(N^{-1}) & * \end{array} \right),$$

where the second column of this block matrix is a polynomial of degree 2 in  $N^{-1}$ . In particular,  $\mathbf{L}_{\mathrm{SO}(N)}$  admits a limit as  $N$  tends to infinity and this limit is of the form

$$\lim_{N \rightarrow \infty} \mathbf{L}_{\mathrm{SO}(N)} = \left( \begin{array}{c|c} \mathbf{L} & * \\ \hline 0 & * \end{array} \right).$$

Ignoring the second column of this matrix, we conclude that the sequence of functions  $(f_N)_{n \geq 1}$ , restricted to  $\mathbb{R}_+ \times \mathfrak{S}_n$ , converges uniformly on every compact subset of  $\mathbb{R}_+$  towards the function  $f(t, \cdot)$  defined by

$$(38) \quad \forall t \geq 0, f(t, \cdot) = e^{t\mathbf{L}} \mathbb{1}.$$

We recognise here the equation (31). □

**2.7. The Brauer algebra II.** In the treatment of the symplectic case, we will consider a homomorphism of algebras  $\rho_{\mathbb{H}} : \mathbf{B}_{n,-2N} \rightarrow M_N(\mathbb{H})^{\otimes n}$ . This homomorphism will be constructed as the tensor product of the homomorphism  $\rho$  considered in the orthogonal case and another homomorphism  $\gamma : \mathbf{B}_{n,-2} \rightarrow \mathbb{H}^{\otimes n}$ , which we define and study in this section.

In order to define  $\gamma$ , we need to discuss a cyclic structure on  $\{1, \dots, 2n\}$  associated to each element of  $\mathfrak{B}_n$ . We have already implicitly considered this cyclic structure in the definition of  $\ell(\pi)$  just before (35).

Let us consider a pairing  $\pi$  of  $\{1, \dots, 2n\}$ . Let us consider the usual graph associated with  $\pi$ , with vertices  $\{1, \dots, 2n\}$  and  $n$  edges, one joining  $i$  and  $j$  for each pair  $\{i, j\} \in \pi$ . We call these  $n$  edges the primary edges. Let us add to this graph  $n$  other edges, one joining  $i$  to  $i+n$  for each  $i \in \{1, \dots, n\}$ . We call these edges the secondary edges. We get a graph in which each vertex has degree 2, being adjacent to one primary and one secondary edge. This graph is thus a union of disjoint cycles of even length, for which  $\pi$  provides no canonical orientation. We decide to orient each of these cycles by declaring that the primary edge adjacent to the smallest element of each cycle is outgoing at this vertex. In this way, we get a partition of  $\{1, \dots, 2n\}$  by oriented cycles, that is, a permutation of  $\{1, \dots, 2n\}$ , which we denote by  $\Sigma_\pi$ . For an example of this construction, see Figure 6.

We are now going to use the permutation  $\Sigma_\pi \in \mathfrak{S}_{2n}$  to define a permutation  $\sigma_\pi \in \mathfrak{S}_n$  and to attach a sign to each integer  $\{1, \dots, n\}$ . Let us start with the signs. For each  $i \in \{1, \dots, n\}$ , we set  $\varepsilon_\pi(i) = 1$  if  $\{i, \Sigma_\pi(i)\}$  is a primary edge and  $\varepsilon_\pi(i) = -1$  otherwise. If  $(i, n+i)$  is a cycle of  $\Sigma_\pi$ , then  $\varepsilon_\pi(i) = 1$ . Then, we define  $\sigma_\pi$  as the permutation of  $\{1, \dots, n\}$  obtained by removing the integers  $\{n+1, \dots, 2n\}$  from their cycles in  $\Sigma_\pi$ . Note that  $\Sigma_\pi$ , and hence  $\sigma_\pi$ , have exactly  $\ell(\pi)$  cycles. For example, if  $\pi$  is a permutation, then  $\varepsilon_\pi(i) = 1$  for all  $i \in \{1, \dots, n\}$  and  $\sigma_\pi = \pi$ .

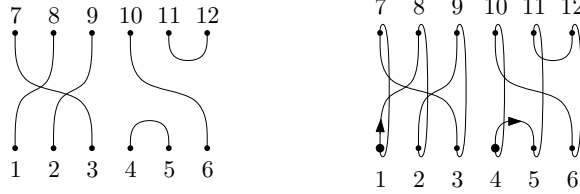


FIGURE 6. Consider  $\pi = \{\{1, 8\}, \{2, 9\}, \{3, 7\}, \{4, 5\}, \{6, 10\}, \{11, 12\}\} \in \mathfrak{B}_6$ . The primary edges are represented on the left and the full graph on the right. There are two cycles with respective smallest element 1 and 4. We thus have  $\Sigma_\pi = (1\ 8\ 2\ 9\ 3\ 7)(4\ 5\ 11\ 12\ 6\ 10)$  and  $\sigma_\pi = (1\ 2\ 3)(4\ 5\ 6)$ . For each  $i \in \{1, \dots, 6\}$ ,  $\varepsilon_\pi(i)$  equals 1 if  $i$  is traversed upwards and  $-1$  if  $i$  is traversed downwards. Here,  $\varepsilon_\pi(5) = -1$  and the other signs are 1.

The signification of the permutation  $\sigma_\pi$  and the signs  $\varepsilon_\pi(1), \dots, \varepsilon_\pi(n)$  is given by the following formula. Recall the definition of  $\rho$  from (32).

**Proposition 2.5.** *Let  $\pi$  be an element of  $\mathfrak{B}_n$ . Let  $R_1, \dots, R_n$  be elements of  $\text{SO}(N)$ . Let us write  $(i_1 \dots i_s) \preceq \sigma_\pi$  if  $(i_1 \dots i_s)$  is a cycle of  $\sigma_\pi$ . Then*

$$(39) \quad \text{Tr}^{\otimes n}(\rho(\pi) \circ R_1 \otimes \dots \otimes R_n) = \prod_{(i_1 \dots i_s) \preceq \sigma_\pi} \text{Tr}(R_{i_s}^{\varepsilon_\pi(i_s)} \dots R_{i_1}^{\varepsilon_\pi(i_1)}).$$

The same identity holds with arbitrary matrices provided inverse matrices are replaced by transposed ones.

Although one might say that this equality results from a direct computation, we give a short proof.

*Proof.* If  $\pi$  is a permutation, then a direct computation shows that the formula holds. Now, let us pick an arbitrary pairing  $\pi \in \mathfrak{B}_n$ , an integer  $i \in \{1, \dots, n\}$  and let us consider the pairing  $\pi'$  obtained by exchanging  $i$  and  $n+i$  in the pairs to which they belong in  $\pi$ . We have  $\sigma_{\pi'} = \sigma_\pi$ ,  $\varepsilon_{\pi'}(i) = -\varepsilon_\pi(i)$  and  $\varepsilon_{\pi'}(j) = \varepsilon_\pi(j)$  for all  $j \neq i$ . Moreover,

$$\text{Tr}^{\otimes n}(\rho(\pi') \circ R_1 \otimes \dots \otimes R_n) = \text{Tr}^{\otimes n}(\rho(\pi) \circ R_1 \otimes \dots \otimes {}^t R_i \otimes \dots \otimes R_n).$$

Hence, if (39) holds for  $\pi$ , it also holds for  $\pi'$ . It only remains to convince oneself that any pairing can be turned into a permutation by a finite succession of exchanges of the sort which we have just considered.  $\square$

Through the mapping  $\pi \rightarrow (\sigma_\pi, \varepsilon_\pi)$ , we associate to each element of  $\mathfrak{B}_n$  an element of  $\mathfrak{S}_n$  and an element of  $(\mathbb{Z}/2\mathbb{Z})^n$ , that is, an element of the hyperoctahedral group  $H_n = \mathfrak{S}_n \times (\mathbb{Z}/2\mathbb{Z})^n \subset \mathfrak{S}_{2n}$ . Since  $\mathfrak{B}_n$ , seen as the set of fixed point free involutions of  $\{1, \dots, 2n\}$ , is isomorphic to the quotient  $\mathfrak{S}_{2n}/H_n$ , it would be natural to expect a neater definition of the pair  $(\sigma_\pi, \varepsilon_\pi)$ , but I was not able to find it.



Let us now turn to the definition of the mapping  $\gamma$ . Recall that  $\mathbf{l}(\mathbb{H})$  denotes the subset  $\{1, i, j, k\}$  of  $\mathbb{H}$ . For each pairing  $\pi \in \mathfrak{B}_n$ , set

$$(40) \quad \gamma(\pi) = \frac{1}{(-2)^n} \sum_{\gamma_1, \dots, \gamma_n \in \mathbf{l}(\mathbb{H})} \left( \prod_{(i_1 \dots i_s) \preceq \sigma_\pi} (-2\Re)(\gamma_{i_s} \dots \gamma_{i_1}) \right) \gamma_1^{-\varepsilon_\pi(1)} I_N \otimes \dots \otimes \gamma_n^{-\varepsilon_\pi(n)} I_N.$$

If  $\pi$  is the pairing corresponding to the identity permutation, then  $\gamma(\pi) = I_N^{\otimes n}$ . We set

$$(41) \quad \rho_{\mathbb{H}}(\pi) = \rho(\pi) \otimes \gamma(\pi)$$

and will sometimes use the lighter notation  $\rho_{\mathbb{H}}\pi$ .

Recall (9) and observe that  $\gamma(\langle 12 \rangle) = -\frac{1}{2}\text{Re}^{\mathbb{H}}$ ,  $\gamma(\langle 1 \bar{2} \rangle) = -\frac{1}{2}\text{Im}^{\mathbb{H}}$ , so that

$$\rho_{\mathbb{H}}\langle 12 \rangle = -\frac{1}{2}T \otimes \text{Re}^{\mathbb{H}} \text{ and } \rho_{\mathbb{H}}\langle 1 \bar{2} \rangle = -\frac{1}{2}W \otimes \text{Im}^{\mathbb{H}},$$

and by comparing with (11), we have for all  $i, j$  with  $1 \leq i < j \leq n$  the equality

$$(42) \quad \iota_{i,j}(C_{\text{sp}(N)}) = -\frac{1}{-2N} (\rho_{\mathbb{H}}(ij) - \rho_{\mathbb{H}}\langle ij \rangle).$$

This is a first piece of a justification for our arguably strange definition of  $\gamma$ . A second piece of justification is given by the following lemma. By analogy with the real and complex cases, we denote by  $\circ$  the product in the algebra  $M_N(\mathbb{H})^{\otimes n}$ , but we would like to emphasise that the natural action of this algebra on  $(\mathbb{H}^N)^{\otimes n}$  which is implicit in this notation is the action of a real algebra on the tensor product over  $\mathbb{R}$  of real linear spaces. The trace denote by  $\text{Tr}$  on the other hand is still the usual trace on  $M_n(\mathbb{H})$ .

**Lemma 2.6.** *For all  $n \geq 1$ , all  $\pi \in \mathfrak{B}_n$  and all  $S_1, \dots, S_n \in \text{Sp}(N)$ , we have*

$$(-2\Re\text{Tr})^{\otimes n}(\rho_{\mathbb{H}}(\pi) \circ S_1 \otimes \dots \otimes S_n) = \prod_{(i_1 \dots i_s) \preceq \sigma_\pi} (-2\Re\text{Tr})(S_{i_s}^{\varepsilon_\pi(i_s)} \dots S_{i_1}^{\varepsilon_\pi(i_1)}).$$

*Proof.* Relabelling the matrices  $S_1, \dots, S_n$  if necessary and using the fact that both  $\rho(\pi)$  and  $\gamma(\pi)$  factorise according to the cycles of  $\sigma_\pi$ , we may reduce the problem to the case where  $\sigma_\pi$  has a single cycle, and we may choose the cycle  $(n \dots 1)$ . In this case, after developing the traces, the equality results from the following identity, valid for all quaternions  $q_1, \dots, q_n$ :

$$\sum_{\gamma_1, \dots, \gamma_n \in \mathbf{l}(\mathbb{H})} \gamma_1 \dots \gamma_n \Re(\gamma_1^{-\varepsilon_1} q_1) \dots \Re(\gamma_n^{-\varepsilon_n} q_n) = q_1^{*1} \dots q_n^{*n},$$

where we set  $q_i^{*i} = q_i$  if  $\varepsilon_i = 1$  and  $q_i^{*i} = q_i^*$  if  $\varepsilon_i = -1$ . □

The main property of  $\gamma$  is the following, which determined its definition.

**Proposition 2.7.** *The unique extension of  $\gamma$  to a linear mapping  $\mathfrak{B}_{n,-2} \rightarrow \mathbb{H}^{\otimes n}$  is a homomorphism of algebras.*

*Proof.* Since the algebra  $\mathfrak{B}_{n,-2}$  is generated by  $\mathsf{T}_n \cup \mathsf{W}_n$ , it suffices to prove that for all pairing  $\pi \in \mathfrak{B}_n$  and all  $i, j$  with  $1 \leq i < j \leq n$ , we have  $\gamma(\pi(ij)) = \gamma(\pi)\gamma(\langle ij \rangle)$  and  $\gamma(\pi\langle ij \rangle) = \gamma(\pi)\gamma(\langle ij \rangle)$ . For each equality, there are three cases to consider: the case where  $i$  and  $j$  do not belong to the same cycle of  $\sigma_\pi$ , then the case where they do, which itself is subdivided into the

sub-cases  $\varepsilon_\pi(i) = \varepsilon_\pi(j)$  and  $\varepsilon_\pi(i) = -\varepsilon_\pi(j)$ . In each of the six cases, the key of the result is one of the following elementary identities, valid for all  $q_1, q_2 \in \mathbb{H}$ :

$$\begin{aligned}
\text{(I)} \quad & \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\gamma_1 \gamma_2) (-2\Re)(\gamma_1^{-1} q_1) (-2\Re)(\gamma_2^{-1} q_2) = (-2\Re)(q_1 q_2), \\
\text{(II)} \quad & \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\gamma_1 \gamma_2) (-2\Re)(\gamma_1^{-1} q_1) (-2\Re)(\gamma_2 q_2) = (-2\Re)(q_1 q_2^*), \\
\text{(III)} \quad & \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\gamma_1 \gamma_2) (-2\Re)(\gamma_1^{-1} q_1 \gamma_2^{-1} q_2) = (-2\Re)(q_1) (-2\Re)(q_2), \\
\text{(IV)} \quad & \frac{1}{4} \sum_{\gamma_1, \gamma_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\gamma_1 \gamma_2) (-2\Re)(\gamma_1^{-1} q_1 \gamma_2 q_2) = (-2\Re)(q_1 q_2^*).
\end{aligned}$$

The first equality is the multiplication rule in  $\mathbb{H}$  and the second follows by replacing  $q_2$  by  $q_2^*$ . The third and fourth equality follow from the identities (10).

Let us give the details of the proof of the equality  $\gamma(\pi)\gamma(\langle i j \rangle) = \gamma(\pi \langle i j \rangle)$  in the case where  $i$  and  $j$  belong to the same cycle of  $\sigma_\pi$  and  $\varepsilon_\pi(i) = -\varepsilon_\pi(j)$ . Recall the notation  $\iota_{i,j}$  from (18). To start with, we have

$$\gamma(\langle i j \rangle) = \frac{1}{4} \sum_{\alpha_1, \alpha_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\alpha_1 \alpha_2) \iota_{i,j}(\alpha_1^{-1} \otimes \alpha_2).$$

Let us write  $(i i_1 \dots i_s j j_1 \dots j_t)$  the cycle of  $\sigma_\pi$  which contains  $i$  and  $j$ . Reversing the orientation of this cycle if necessary, we may assume that  $\varepsilon_\pi(i) = 1$  and  $\varepsilon_\pi(j) = -1$ . In the expression of  $\gamma(\pi)\gamma(\langle i j \rangle)$ , we have the sum over all possible values of  $\gamma_1, \dots, \gamma_n, \alpha_1, \alpha_2$  in  $\mathbb{I}(\mathbb{H})$  of the product of a term

$$\frac{1}{4} (-2\Re)(\alpha_1 \alpha_2) (-2\Re)(\gamma_{j_t} \dots \gamma_{j_1} \gamma_j \gamma_{i_s} \dots \gamma_{i_1} \gamma_i) \dots$$

and a term

$$\dots \otimes \gamma_i^{-1} \alpha_1^{-1} \otimes \dots \otimes \gamma_j \alpha_2 \otimes \dots$$

In this sum, we would like to perform a change of variables and to replace  $\gamma_i$  by  $\alpha_1^{-1} \gamma_i$  and  $\gamma_j$  by  $\gamma_j \alpha_2^{-1}$ . This would however introduce troublesome minus signs. The neatest way to do this is to allow temporarily our variables to vary in the set  $\mathbb{I}(\mathbb{H}) \cup -\mathbb{I}(\mathbb{H})$  instead of  $\mathbb{I}(\mathbb{H})$ , to the price of a factor  $\frac{1}{2}$  for each variable. This does not affect the sum otherwise, because each variable appears exactly twice. The advantage is that  $\mathbb{I}(\mathbb{H}) \cup -\mathbb{I}(\mathbb{H})$  is a subgroup of  $\mathbb{H}$ , so that the change of variables is justified. After this change of variables, the two terms which we are considering are replaced respectively by

$$\frac{1}{4} (-2\Re)(\alpha_1 \alpha_2) (-2\Re)(\gamma_{j_t} \dots \gamma_{j_1} \gamma_j \alpha_2^{-1} \gamma_{i_s} \dots \gamma_{i_1} \alpha_1^{-1} \gamma_i) \dots$$

and

$$\dots \otimes \gamma_i^{-1} \otimes \dots \otimes \gamma_j \otimes \dots$$

Thanks to the third of the four elementary identities mentioned above, summing over  $\alpha_1$  and  $\alpha_2$  transforms the first term into

$$(-2\Re)(\gamma_{j_t} \dots \gamma_{j_1} \gamma_j \gamma_i) (-2\Re)(\gamma_{i_s} \dots \gamma_{i_1}) \dots$$

On the other hand, the cycles of  $\sigma_{\pi \langle i j \rangle}$  are the same as those of  $\sigma_\pi$ , except for  $(i i_1 \dots i_s j j_1 \dots j_t)$  which is replaced by  $(i j j_1 \dots j_s)(i_1 \dots i_s)$ . Moreover, for all  $k \in \{1, \dots, n\}$ , we have  $\varepsilon_{\pi \langle i j \rangle}(k) = \varepsilon_\pi(k)$ .

Finally, it may happen that  $s = 0$ , in which case the cycle  $(i_1 \dots i_s)$  is absent in  $\pi \langle i j \rangle$ . In this case, the fact that  $\sigma_\pi(i) = j$ ,  $\varepsilon_\pi(i) = 1$  and  $\varepsilon_\pi(j) = -1$  imposes that  $\{i, j\}$  is a pair of  $\pi$ . Since we are working in  $B_{n,-2}$ , the appearance of a loop in the multiplication of  $\pi$  and  $\langle i j \rangle$  brings the missing factor  $-2$ . In fact, this is the only case in the whole proof where a loop is formed and where the parameter of the Brauer algebra plays a role.

Let us indicate what differs in the proof of  $\gamma(\pi)\gamma(\langle i j \rangle) = \gamma(\pi \langle i j \rangle)$  in the same case, when  $i$  and  $j$  belong to the same cycle of  $\sigma_\pi$  and  $\varepsilon_\pi(i) = -\varepsilon_\pi(j)$ . With the same notation, using

$$\gamma(\langle i j \rangle) = \frac{1}{4} \sum_{\alpha_1, \alpha_2 \in \mathbb{I}(\mathbb{H})} (-2\Re)(\alpha_1 \alpha_2) \iota_{i,j}(\alpha_1^{-1} \otimes \alpha_2^{-1})$$

and performing exactly the same steps, only applying the fourth elementary equality instead of the third, we end up with a term

$$(-2\Re)(\gamma_{j_t} \dots \gamma_{j_1} \gamma_j \gamma_{i_1}^{-1} \dots \gamma_{i_s}^{-1} \gamma_i) \dots$$

A second change of variables is needed at this point, and justified as the first, by which we replace  $\gamma_{i_1}, \dots, \gamma_{i_s}$  by their inverses. This comes in agreement with the fact that not only  $\sigma_\pi \langle i j \rangle$  has  $(i i_s \dots i_1 j j_1 \dots j_s)$  as a cycle, but  $\varepsilon_{\pi \langle i j \rangle}(i_k) = -\varepsilon_\pi(i_k)$  for all  $k \in \{1, \dots, s\}$ , the other signs being unchanged.

Nothing new is needed to check the four other cases and we spare the reader a detailed account of them.  $\square$

It follows from this result and from our earlier study of  $\rho$  that the linear extension  $\rho_{\mathbb{H}} : B_{n,-2N} \rightarrow M_N(\mathbb{H})^{\otimes n}$  is a homomorphism of algebras.

At this point, we can uniformise our definitions of the representations  $\rho$  and  $\rho_{\mathbb{H}}$ . Indeed, we have defined, for each  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ , with the corresponding value of  $\beta = \dim_{\mathbb{R}} \mathbb{K}$ , a representation

$$(43) \quad \rho_{\mathbb{K}} : B_{n,(2-\beta)N} \rightarrow M_N(\mathbb{K})^{\otimes n}.$$

In the case  $\mathbb{K} = \mathbb{C}$ , we need to set  $\rho_{\mathbb{C}}(\pi) = 0$  whenever  $\pi \in \mathfrak{B}_n$  is not a permutation. We shall henceforward use the notation  $\rho_{\mathbb{K}}$ , that is, in particular,  $\rho_{\mathbb{R}}$  instead of  $\rho$ .

We can now proceed to the proof of our first main theorem in the symplectic case.

**2.8. The symplectic case.** The symplectic case is similar to the orthogonal case, but more complicated, since there is no expression of the Casimir operator which is really simpler than (11). One possibility would be to work through the embedding  $\mathrm{Sp}(N) \rightarrow \mathrm{U}(2N)$ , but this is not the approach which we choose.

*Proof of Theorem 2.2 in the symplectic case.* To each element  $\pi \in \mathfrak{B}_n$ , we associate the function  $P_\pi$  on  $M_N(\mathbb{H})$  by setting

$$P_\pi(M) = (-2\Re \mathrm{Tr})^{\otimes n} (\rho_{\mathbb{H}} \pi \circ M^{\otimes n}),$$

and the function  $p_\pi(M) = (-2N)^{-\ell(\pi)} P_\pi(M)$ . By Lemma 2.6, we have  $p_\pi(I_N) = 1$ .

Let  $(S_{N,t})_{t \geq 0}$  be a Brownian motion on the symplectic group  $\mathrm{Sp}(N)$ . We define the functions  $F_N$  and  $f_N$  defined on  $\mathbb{R}_+ \times \mathfrak{B}_n$  by

$$F_N(t, \pi) = \mathbb{E} [P_\pi(S_{N,t})] \quad \text{and} \quad f_N(t, \pi) = \mathbb{E} [p_\pi(S_{N,t})],$$

and extend them by linearity to  $\mathbb{R}_+ \times B_{n,-2N}$ . The normalisation has been chosen such that  $f_N(0, \pi) = 1$  for all  $\pi \in \mathfrak{B}_n$ .

Let us apply Itô's formula in this new context. Thanks to (42) and Proposition (2.7), we have, for all  $t \geq 0$  and all  $b \in \mathbf{B}_{n,-2N}$ ,

$$(44) \quad \begin{aligned} \frac{\partial}{\partial t} F_N(t, b) &= \mathbb{E} \left[ \rho_{\mathbb{H}}(b) \circ \left( -\frac{2N+1}{2N} \frac{n}{2} + \frac{1}{2N} \sum_{1 \leq i < j \leq n} (\rho_{\mathbb{H}}(ij) - \rho_H \langle ij \rangle) \right) \circ S_t^{\otimes n} \right] \\ &= -\frac{n(2N+1)}{4N} F_N(t, b) - \frac{1}{-2N} \sum_{\tau \in \mathbf{T}_n} F_N(t, b\tau) + \frac{1}{-2N} \sum_{\kappa \in \mathbf{W}_n} F_N(t, b\kappa), \end{aligned}$$

which is the symplectic version of (36). From this equality, we deduce that for all  $\pi \in \mathfrak{B}_n$ ,

$$(45) \quad \begin{aligned} \frac{\partial}{\partial t} f_N(t, \pi) &= -\frac{n(2N+1)}{4N} f_N(t, \pi) - \sum_{\tau \in \mathbf{T}_n} (-2N)^{\ell(\pi\tau) - \ell(\pi) - 1} f_N(t, \pi\tau) \\ &\quad + \sum_{\kappa \in \mathbf{W}_n} (-2N)^{\ell(\pi\kappa) - \ell(\pi) - 1} f_N(t, \pi\kappa). \end{aligned}$$

Recall that in (45),  $\pi\tau$  and  $\pi\kappa$  can be scalar multiples of basis elements. Just as in the orthogonal case, a loop is formed in the product  $\pi\kappa$  only when  $\kappa = \langle ij \rangle$  and the pair  $\{i, j\}$  belongs to  $\pi$ , and in this case, we have  $\pi\kappa = N\pi$ .

Let us denote by  $\mathbf{L}_{\mathrm{Sp}(N)}$  the linear operator on  $\mathbf{B}_{n,-2N}^*$  defined by the following equality, valid for all  $\pi \in \mathfrak{B}_n$ :

$$\begin{aligned} (\mathbf{L}_{\mathrm{Sp}(N)} f)(\pi) &= -\frac{n(2N+1)}{4N} f(\pi) - \sum_{\tau \in \mathbf{T}_n} (-2N)^{\ell(\pi\tau) - \ell(\pi) - 1} f(\pi\tau) \\ &\quad + \sum_{\kappa \in \mathbf{W}_n} (-2N)^{\ell(\pi\kappa) - \ell(\pi) - 1} f(\pi\kappa). \end{aligned}$$

We also denote by  $\mathbb{1} \in \mathbf{B}_{n,-2N}^*$  the linear form equal to 1 on each element of  $\mathfrak{B}_n$ . Then we have the equality

$$\forall t \geq 0, f_N(t, \cdot) = e^{t\mathbf{L}_{\mathrm{Sp}(N)}} \mathbb{1}.$$

Our discussion of the powers of  $N$  involved in the operator  $\mathbf{L}_{\mathrm{SO}(N)}$  did not depend on the signs of the coefficients, or of factors independent of  $N$ . It remains thus entirely valid for the operator  $\mathbf{L}_{\mathrm{Sp}(N)}$ . Thus, in the basis of  $\mathbf{B}_{n,-2N}^*$  dual to  $\mathfrak{B}_n$ , split as in the orthogonal case, the matrix of  $\mathbf{L}_{\mathrm{Sp}(N)}$  is again

$$\mathbf{L}_{\mathrm{Sp}(N)} = \left( \begin{array}{c|c} \mathbf{L}_{\mathrm{U}(N)} + \frac{n}{2(-2N)} I_n! & * \\ \hline O(N^{-1}) & * \end{array} \right),$$

where as in the orthogonal case, the second column is a polynomial of degree 2 in  $N^{-1}$ . In fact, we have, formally, the equality  $\mathbf{L}_{\mathrm{Sp}(N)} = \mathbf{L}_{\mathrm{SO}(-2N)}$ .

In particular,  $\mathbf{L}_{\mathrm{Sp}(N)}$  admits a limit as  $N$  tends to infinity and this limit is of the form

$$\lim_{N \rightarrow \infty} \mathbf{L}_{\mathrm{Sp}(N)} = \left( \begin{array}{c|c} \mathbf{L} & * \\ \hline 0 & * \end{array} \right).$$

We can conclude the proof as in the orthogonal case.  $\square$

### 3. SPEED OF CONVERGENCE FOR WORDS OF INDEPENDENT BROWNIAN MOTIONS

Theorems 2.1 and 2.2, together with a classical result of Voiculescu and its extension to the orthogonal and symplectic cases by Collins and Śniady, allow one to determine the limit of expected traces of arbitrary words in independent Brownian motions on  $U(N, \mathbb{K})$  as  $N$  tends to infinity. Our second main result provides a quantitative estimate of the rate of convergence of such expected traces, in terms of a certain measure of the complexity of the word considered.

Let us start by recalling how the results of Voiculescu and Collins-Śniady apply in the present context.

**3.1. Free limits.** We do not recall the main definitions of free probability theory. Instead, we refer the reader to [29].

Recall from (24) the definition of the measures  $(\nu_t)_{t \geq 0}$ . A *free multiplicative Brownian motion* is a family  $(u_t)_{t \geq 0}$  of unitary elements of a non-commutative probability space  $(\mathcal{A}, \tau)$  such that for all  $0 \leq t_1 \leq \dots \leq t_n$ , the increments  $u_{t_2} u_{t_1}^*, \dots, u_{t_n} u_{t_{n-1}}^*$  are free and have respectively the distributions  $\nu_{t_2-t_1}, \dots, \nu_{t_n-t_{n-1}}$ . Free multiplicative Brownian motions exist and can be realised as the large  $N$  limit of the Brownian motion on the unitary group.

**Theorem 3.1** (Biane, [4]). *For each  $N \geq 1$ , let  $(U_{N,t})_{t \geq 0}$  be a Brownian motion on  $U(N)$  issued from  $I_N$ , associated with the scalar product  $\langle X, Y \rangle = N \text{Tr}(X^*Y)$  on  $\mathfrak{u}(N)$ , defined on a probability space  $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$ . Then the collection  $\{U_{N,t} : t \geq 0\}$  of elements of the non-commutative probability space  $(L^\infty(\Omega_N, \mathcal{A}_N, \mathbb{P}_N) \otimes M_N(\mathbb{C}), \mathbb{E} \otimes \text{tr})$  converges in non-commutative distribution as  $N$  tends to infinity to a free unitary Brownian motion. Moreover, independent Brownian motions converge to free unitary Brownian motions which are mutually free.*

It follows from our study of the orthogonal and symplectic case, and from a result of Collins and Śniady [7, Thm. 5.2] that a similar result holds for orthogonal and symplectic Brownian motions.

There is a small complication due to the fact that we do not regard symplectic matrices as complex matrices. Indeed, the algebra  $L^\infty(\Omega_N, \mathcal{A}_N, \mathbb{P}_N) \otimes M_N(\mathbb{H})$  is a real algebra and not a complex one, and we are slightly outside the usual framework of non-commutative probability theory. Here is the short argument which we need to go back to it.

Consider a real involutive unital algebra  $\mathcal{A}$  endowed with a linear form  $\tau$  such that  $\tau(1)$  and for all  $a \in \mathcal{A}$ , one has  $\tau(aa^*) \geq 0$ . We shall call such a pair  $(\mathcal{A}, \tau)$  a real non-commutative probability space. It is straightforward to check that the complexified algebra  $\mathcal{A} \otimes \mathbb{C}$  endowed with the involution  $(a \otimes z)^* = a^* \otimes \bar{z}$  and the linear form  $\tau \otimes \text{id}_{\mathbb{C}}$  is a non-commutative probability space in the usual sense. Moreover, for all  $a \in \mathcal{A}$ , the moments of  $a \otimes 1$  in  $(\mathcal{A} \otimes \mathbb{C}, \tau \otimes \text{id}_{\mathbb{C}})$  are the same as those of  $a$  in  $(\mathcal{A}, \tau)$ .

This being said, we take the liberty of using the language of free probability in a real non-commutative probability space.

**Theorem 3.2.** *For each  $N \geq 1$ , let  $(R_{N,t})_{t \geq 0}$  be a Brownian motion on  $SO(N)$  issued from  $I_N$ , associated with the scalar product  $\langle X, Y \rangle = \frac{N}{2} \text{Tr}(XY)$  on  $\mathfrak{so}(N)$ , defined on a probability space  $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$ . Then the collection  $\{R_{N,t} : t \geq 0\}$  of elements of the non-commutative probability space  $(L^\infty(\Omega_N, \mathcal{A}_N, \mathbb{P}_N) \otimes M_N(\mathbb{R}), \mathbb{E} \otimes \text{tr})$  converges in non-commutative distribution as  $N$  tends to infinity to a free unitary Brownian motion. Moreover, independent Brownian motions converge to free unitary Brownian motions which are mutually free.*

*For each  $N \geq 1$ , let  $(S_{N,t})_{t \geq 0}$  be a Brownian motion on  $Sp(N)$  issued from  $I_N$ , associated with the scalar product  $\langle X, Y \rangle = 2N \Re \text{Tr}(X^*Y)$  on  $\mathfrak{sp}(N)$ , defined on a probability space  $(\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$ . Then the collection  $\{S_{N,t} : t \geq 0\}$  of elements of the non-commutative probability space  $(L^\infty(\Omega_N, \mathcal{A}_N, \mathbb{P}_N) \otimes M_N(\mathbb{H}), \mathbb{E} \otimes \Re \text{tr})$  converges in non-commutative distribution as  $N$*

tends to infinity to a free unitary Brownian motion. Moreover, independent Brownian motions converge to free unitary Brownian motions which are mutually free.

**3.2. Second main result : speed of convergence.** In this section, we state our second main result, firstly in its most natural form and then in the form under which we will prove it.

Let  $\mathbb{K}$  be one of our three division algebras. We denote generically by  $(V_{N,s})_{s \geq 0}$  a Brownian motion on  $U(N, \mathbb{K})$  as defined in Section 1.4. We are going to consider several independent copies of this Brownian motion, with which we are going to form a word, of which in turn we will estimate the expected trace. The number of independent copies which we use to form our word will not appear in our final estimates, and this is one of their main strengths. We will nevertheless fix this number and denote it by  $q$ . Let us thus choose an integer  $q \geq 1$ , which will stay fixed until the end of Section 3.

We shall denote by  $F_q$  be the free group on  $q$  letters  $x_1, \dots, x_q$ . Let  $w$  be an element of  $F_q$ . Let us write  $w$  in reduced form as  $x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$ , where  $r \geq 0$  is the length of  $w$ , and  $\varepsilon_1, \dots, \varepsilon_r$  belong to  $\{-1, 1\}$ . If  $u_1, \dots, u_q$  are invertible elements of an algebra, we denote by  $w(u_1, \dots, u_q)$  the element  $u_{i_1}^{\varepsilon_1} \dots u_{i_r}^{\varepsilon_r}$  of this algebra. We shall use this notation for matrices and for elements of non-commutative probability spaces. The following notation will also be useful later: if  $U_1, \dots, U_q$  belong to  $U(N, \mathbb{K})$ , we shall denote by  $w_{\otimes}(U_1, \dots, U_q)$  the element  $U_{i_1}^{\varepsilon_1} \otimes \dots \otimes U_{i_r}^{\varepsilon_r}$  of  $M_N(\mathbb{K})^{\otimes r}$ . Observe that this second definition depends crucially on the fact that  $w$  was written in reduced form.

We will use the free group  $F_q$  to produce a non-commutative probability space in the usual way. Let  $\mathbb{C}[F_q]$  be the complex algebra of the group  $F_q$ . It is isomorphic to the algebra  $\mathbb{C}\langle x_1, x_1^{-1}, \dots, x_q, x_q^{-1} \rangle$  of Laurent polynomials in  $q$  non-commuting indeterminates. It carries an involution characterised by the equality  $(\lambda x_i)^* = \bar{\lambda} x_i^{-1}$ , valid for all  $i \in \{1, \dots, q\}$  and all  $\lambda \in \mathbb{C}$ .

Let us fix an integer  $N \geq 1$ . Let  $(V_{N,1,s})_{s \geq 0}, \dots, (V_{N,q,s})_{s \geq 0}$  be  $q$  independent Brownian motions on the group  $U(N, \mathbb{K})$ . Let also  $(u_{1,s})_{s \geq 0}, \dots, (u_{q,s})_{s \geq 0}$  be  $q$  free unitary Brownian motions which are mutually free, carried by a non-commutative probability space  $(\mathcal{A}, \varphi)$ .

In the words which we shall consider, each of our  $q$  Brownian motions will always be evaluated at the same time. Since the increments of a Brownian motion are independent, and since the number of independent Brownian motions which we consider does not affect our estimates, this does not entail any loss of generality. The times at which we evaluate our Brownian motions are of course important, and we put them into a vector  $t = (t_1, \dots, t_q) \in \mathbb{R}_+^q$ .

Let us define a state  $\tau_t^{\mathbb{K}, N}$  on  $\mathbb{C}[F_q]$  by setting, for all  $w \in F_q$ ,

$$\tau_t^{\mathbb{K}, N}(w) = \begin{cases} \mathbb{E} [\operatorname{tr} (w(V_{N,1,t_1}, \dots, V_{N,q,t_q}))] & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}, \\ \mathbb{E} [\Re \operatorname{tr} (w(V_{N,1,t_1}, \dots, V_{N,q,t_q}))] & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

Theorems 3.1 and 3.2 assert that, as  $N$  tends to infinity,  $\tau_t^{\mathbb{K}, N}$  converges pointwise to the state  $\tau_t$  defined by

$$\tau_t(w) = \varphi(w(u_{1,t_1}, \dots, u_{q,t_q})).$$

The main result of this section gives an explicit bound on  $|\tau_t^{\mathbb{K}, N}(w) - \tau_t(w)|$ . This bound must of course depend on the word  $w$ . It does so through a certain non-negative real which we assign to each pair  $(w, t) \in F_q \times \mathbb{R}_+^q$ , and which we call its Amperean area, for a reason which shall become clear in the second part of this work.

Let us define  $q$  functions  $n_1, \dots, n_q : F_q \rightarrow \mathbb{N}$ , which could be called partial lengths, as follows. For all element  $w$  of  $F_q$ , written in reduced form as  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$ , and all  $k \in \{1, \dots, q\}$ , we define  $n_k(w)$  as the total number of occurrences of the letter  $x_k$  in  $w$ , that is,

$$(46) \quad n_k(w) = \#\{j \in \{1, \dots, r\} : i_j = k\}.$$

We do not make any distinction between  $x_k$  or  $x_k^{-1}$ . For example,  $\mathfrak{n}_3(x_3x_1x_3^{-1}) = 2$ . Using these partial lengths, we define the Amperean area of the word  $w$  relative to  $t$  as the real number

$$(47) \quad \bar{A}_t(w) = \sum_{k=1}^q t_k \mathfrak{n}_k(w)^2.$$

Let us emphasise that this number does not really depend on  $q$ . We could see the word  $w$  as a word in infinitely many letters, and  $t$  as a infinite vector with only finitely many non-zero components. The main estimate is the following.

**Theorem 3.3.** *For all  $w \in \mathbb{F}_q$  and all  $N \geq 1$ , the following inequality holds:*

$$(48) \quad |\tau_t^{\mathbb{K},N}(w) - \tau_t(w)| \leq \begin{cases} \frac{1}{N^2} \bar{A}_t(w) e^{\bar{A}_t(w)} & \text{if } \mathbb{K} = \mathbb{C}, \\ \frac{1}{N} \bar{A}_t(w) e^{\bar{A}_t(w)} & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{H}. \end{cases}$$

We will in fact prove a more general result, which asserts that the same bounds hold for quantities which are built from the word  $w$  but which are more general than  $\tau_t^{\mathbb{K},N}(w)$  and  $\tau_t(w)$ . Just as in the proofs of Theorems 2.1 and 2.2, this generalisation is meant to provide us with a finite set of functions of  $t = (t_1, \dots, t_q)$  which satisfies an autonomous differential system. The quantities which we will consider are very similar to the functions  $f_N(t, \pi)$  considered in these proofs. In particular, we will need a larger set of quantities in the orthogonal and symplectic cases as in the unitary case.

Let us start by the unitary case. For this, let us consider again an element  $w$  of  $\mathbb{F}_q$ , written in reduced form as  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$ . Let us consider a permutation  $\sigma \in \mathfrak{S}_r$ . We write  $(j_1 \dots j_s) \preccurlyeq \sigma$  to indicate that  $(j_1 \dots j_s)$  is a cycle of  $\sigma$ . Recall from the beginning of Section 2.4 that we defined  $\rho_{\mathbb{C}}(\sigma) \in M_N(\mathbb{C})^{\otimes r}$ . Recall also, from the beginning of the current section, the notation  $w_{\otimes}(U_{N,1,t_1}, \dots, U_{N,q,t_q})$ . In accordance with the convention made at the end of Section 1.4, we denote respectively by  $R$ ,  $U$  and  $S$  the orthogonal, unitary and symplectic Brownian motions. With all this preparation, we set

$$\begin{aligned} p_t^{\mathbb{C},N}(w, \sigma) &= N^{-\ell(\sigma)} \mathbb{E} [\text{Tr}^{\otimes r}(\rho_{\mathbb{C}}(\sigma) \circ w_{\otimes}(U_{N,1,t_1}, \dots, U_{N,q,t_q}))] \\ &= \mathbb{E} \left[ \prod_{(j_1 \dots j_s) \preccurlyeq \sigma} \text{tr} \left( U_{N,i_{j_1},t_{i_{j_1}}}^{\varepsilon_{j_1}} \dots U_{N,i_{j_s},t_{i_{j_s}}}^{\varepsilon_{j_s}} \right) \right], \end{aligned}$$

and

$$p_t(w, \sigma) = \prod_{(j_1 \dots j_s) \preccurlyeq \sigma} \varphi \left( u_{i_{j_1},t_{j_1}}^{\varepsilon_{j_1}} \dots u_{i_{j_s},t_{j_s}}^{\varepsilon_{j_s}} \right).$$

As usual, we extend these definitions by linearity with respect to  $\sigma$ , so as to allow an arbitrary element of  $\mathbb{C}[\mathfrak{S}_n]$  to replace  $\sigma$ .

In the orthogonal and symplectic cases, we introduce the analogous functions indexed by pairings. Let  $r \geq 1$  be an integer. Let  $\pi \in \mathfrak{B}_r$  be a pairing of  $\{1, \dots, 2r\}$ . Recall the construction of the permutation  $\sigma_{\pi} \in \mathfrak{S}_r$  and the signs  $\varepsilon_{\pi}(1), \dots, \varepsilon_{\pi}(r)$  made at the beginning of Section 2.8. The following definitions imitate the equation (39). We define, in the orthogonal case,

$$\begin{aligned} p_t^{\mathbb{R},N}(w, \pi) &= N^{-\ell(\pi)} \mathbb{E} [\text{Tr}^{\otimes r}(\rho_{\mathbb{R}}(\pi) \circ w_{\otimes}(R_{N,1,t_1}, \dots, R_{N,q,t_q}))] \\ &= \mathbb{E} \left[ \prod_{(j_1 \dots j_s) \preccurlyeq \sigma_{\pi}} \text{tr} \left( \left( R_{N,i_{j_s},t_{i_{j_s}}}^{\varepsilon_{j_s}} \right)^{\varepsilon_{\pi}(s)} \dots \left( R_{N,i_{j_1},t_{i_{j_1}}}^{\varepsilon_{j_1}} \right)^{\varepsilon_{\pi}(1)} \right) \right], \end{aligned}$$

and, in the symplectic case,

$$\begin{aligned} p_t^{\mathbb{H},N}(w, \pi) &= (-2N)^{-\ell(\pi)} \mathbb{E} \left[ (-2\Re\mathrm{Tr})^{\otimes r} (\rho_{\mathbb{H}}(\pi) \circ w_{\otimes}(S_{N,1,t_1}, \dots, S_{N,q,t_q})) \right] \\ &= \mathbb{E} \left[ \prod_{(j_1 \dots j_s) \preceq \sigma_{\pi}} \Re\mathrm{tr} \left( \left( S_{N,i_{j_s},t_{i_{j_s}}}^{\varepsilon_{j_s}} \right)^{\varepsilon_{\pi(s)}} \cdots \left( S_{N,i_{j_1},t_{i_{j_1}}}^{\varepsilon_{j_1}} \right)^{\varepsilon_{\pi(1)}} \right) \right]. \end{aligned}$$

We have left the case  $r = 0$  aside. In this case,  $w$  is the empty word, the unit element of  $F_q$ , and  $\pi$  the empty pairing of the empty set. For the sake of this case, we define  $p_t^{\mathbb{K},N}(1, \emptyset) = 1$ .

Let us also define, for both the orthogonal and symplectic cases,  $p_t(1, \emptyset) = 1$  and

$$p_t(w, \pi) = \prod_{(j_1 \dots j_s) \preceq \sigma} \varphi \left( \left( u_{i_{j_s},t_{j_s}}^{\varepsilon_{j_s}} \right)^{\varepsilon_{\pi(s)}} \cdots \left( u_{i_{j_1},t_{j_1}}^{\varepsilon_{j_1}} \right)^{\varepsilon_{\pi(1)}} \right).$$

We extend these definitions by linearity with respect to  $\pi$ , in order to be able to replace  $\pi$  by an arbitrary element of  $B_{r,N}$  in the orthogonal case, or  $B_{r,-2N}$  in the symplectic case.

If we apply these new definitions with  $\sigma = (1 \dots r)$ , we find  $p_t^{\mathbb{K},N}(w, (1 \dots r)) = \tau_t^{\mathbb{K},N}(w)$  and  $p_t(w, (1 \dots n)) = \tau_t(w)$ , so that the following proposition implies Theorem 3.3.

**Proposition 3.4.** *Let  $w \in F_q$  be an element whose reduced form is a word of length  $r \geq 0$ . Let  $N \geq 1$  be an integer. The following inequalities hold:*

$$\max_{\sigma \in \mathfrak{S}_r} |p_t^{\mathbb{C},N}(w, \sigma) - p_t(w, \sigma)| \leq \frac{1}{2N^2} \bar{A}_t(w) e^{\frac{1}{2}\bar{A}_t(w)},$$

and, for  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{H}$ ,

$$\max_{\pi \in \mathfrak{B}_r} |p_t^{\mathbb{K},N}(w, \pi) - p_t(w, \pi)| \leq \frac{1}{N} \bar{A}_t(w) e^{\bar{A}_t(w)}.$$

We will moreover get the following information from the proof of this proposition.

**Proposition 3.5.** *For  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ , the expected trace of any word in independent Brownian motions on  $U(N, \mathbb{K})$  is real.*

**3.3. Itô's equation for words.** With our present notation, Section 2 was devoted to the study of quantities of the form  $p_t^{\mathbb{K},N}(w, \pi)$  when  $w$  is a non-negative power of a single letter. In the present setting, we need to extend this study in two respects : firstly, we need to allow more than one letter to appear in  $w$  and secondly, we need to allow negative powers of letters to appear. The treatment of the latter issue requires the introduction of some new notation, which is forced upon us by Itô's formula. Let us see how.

Let  $w \in F_q$  be a word of length  $r \geq 0$ . In this paragraph, we will write Itô's formula for  $w_{\otimes}(V_{N,1,t_1}, \dots, V_{N,q,t_q})$  when among the times  $t_1, \dots, t_q$ , all but one are fixed. The integer  $N$  is fixed in this section and we will omit it in the notation. The first fundamental relation is the stochastic differential equation satisfied by  $V_t^*$ , namely

$$dV_t^* = -V_t^* dK_t + \frac{c_{\mathbf{u}(N,\mathbb{K})}}{2} V_t^* dt.$$

The algebra  $M_N(\mathbb{K})^{\otimes r}$  is both a left and right  $M_N(\mathbb{K})$ -module in  $r$  different ways: for each  $i \in \{1, \dots, r\}$  and all  $X, M_1, \dots, M_r \in M_N(\mathbb{K})$ , we define

$$\begin{aligned} \theta_i^+(X) \cdot M_1 \otimes \dots \otimes M_r &= M_1 \otimes \dots \otimes X M_i \otimes \dots \otimes M_r, \\ \theta_i^-(X) \cdot M_1 \otimes \dots \otimes M_r &= M_1 \otimes \dots \otimes M_i X \otimes \dots \otimes M_r. \end{aligned} \tag{49}$$



With this notation, we unify Itô's formulas for  $V_t$  and  $V_t^{-1} = V_t^*$ . Indeed, for all  $\varepsilon \in \{-1, 1\}$ ,

$$(50) \quad dV_t^\varepsilon = \varepsilon \theta_1^\varepsilon(dK_t) \cdot V_t^\varepsilon + \frac{c_{\mathfrak{u}(N, \mathbb{K})}}{2} V_t^\varepsilon dt.$$

Here and thereafter, we identify the sets  $\{-1, 1\}$  and  $\{-, +\}$  in the obvious way without further comment.

Note that  $\theta_i^+$  and  $\theta_i^-$  satisfy the following relation of adjunction: for all  $\xi_1, \xi_2 \in M_N(\mathbb{K})^{\otimes r}$ ,

$$(51) \quad \mathrm{Tr}^{\otimes r} \left( (\theta_i^\pm(X) \cdot \xi_1) \xi_2 \right) = \mathrm{Tr}^{\otimes r} \left( \xi_1 (\theta_i^\mp(X) \cdot \xi_2) \right).$$

Let us write  $w$  in its reduced form  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$ . For each  $k \in \{1, \dots, q\}$ , let us record the positions of  $x_k$  and  $x_k^{-1}$  in  $w$  by defining

$$(52) \quad X_k(w) = \{j \in \{1, \dots, r\} : i_j = k\}.$$

For example, if  $w = x_2 x_1^{-1} x_3 x_1^2 x_2$ , then  $X_1(w) = \{2, 4, 5\}$ . Recall that  $\mathfrak{n}_k(w)$  is the cardinal of  $X_k(w)$ .

**Lemma 3.6.** *Choose  $k \in \{1, \dots, q\}$ . Choose  $q - 1$  reals  $t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_q \geq 0$ . Itô's formula for the process  $(w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}))_{t_k \geq 0}$  reads*

$$(53) \quad \begin{aligned} d_{t_k} w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}) &= \sum_{l \in X_k(w)} \varepsilon_l \theta_l^{\varepsilon_l}(dK_{t_k}) \cdot w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}) \\ &+ \frac{\mathfrak{n}_k(w) c_{\mathfrak{u}(N, \mathbb{K})}}{2} w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}) dt_k \\ &+ \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m (\theta_l^{\varepsilon_l} \otimes \theta_m^{\varepsilon_m})(C_{\mathfrak{g}}) \cdot w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}) dt_k. \end{aligned}$$

In particular, for all  $\pi \in \mathfrak{B}_r$ , or all  $\pi \in \mathfrak{S}_r$  if  $\mathbb{K} = \mathbb{C}$ ,

$$(54) \quad \begin{aligned} \frac{\partial}{\partial t_k} p_t^{\mathbb{K}, N}(w, \pi) &= \frac{\mathfrak{n}_k(w) c_{\mathfrak{u}(N, \mathbb{K})}}{2} p_t^{\mathbb{K}, N}(w, \pi) \\ &+ \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m N^{-\ell(\pi)} \mathbb{E} \left[ \mathrm{Tr}^{\otimes r} \left[ \left( (\theta_l^{-\varepsilon_l} \otimes \theta_m^{-\varepsilon_m})(C_{\mathfrak{u}(N, \mathbb{K})}) \cdot \rho_{\mathbb{K}}(\pi) \right) w_\otimes(V_{1,t_1}, \dots, V_{q,t_q}) \right] \right], \end{aligned}$$

or the same equality with  $N$  replaced by  $(-2N)$ ,  $\mathrm{Tr}$  by  $(-2\Re\mathrm{Tr})$  if  $\mathbb{K} = \mathbb{H}$ .

*Proof.* The equality (53) is only a matter of notation. We apply Itô's formula in its most usual form to  $w_\otimes(V_{1,t_1}, \dots, V_{q,t_q})$ , using Itô's formula for a single Brownian motion as written in (50) and with the help of the operators  $\theta_i^\pm$  defined by (49). The Casimir operator appears thanks to the expression (15) of the quadratic variation of  $(K_t)_{t \geq 0}$ .

Equation (54) follows from (53), the definition of  $p_t^{\mathbb{K}, N}(w, \pi)$  given earlier in this section, and the adjunction relation (51).  $\square$

**3.4. The Brauer algebra III.** It appears in (54) that we need to compute  $(\theta_l^{\varepsilon_l} \otimes \theta_m^{\varepsilon_m})(C_{\mathfrak{u}(N, \mathbb{K})}) \cdot \rho(\pi)$  when  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and  $(\theta_l^{\varepsilon_l} \otimes \theta_m^{\varepsilon_m})(C_{\mathfrak{u}(N, \mathbb{H})}) \cdot \rho_{\mathbb{H}}(\pi)$ . We are already familiar with this quantity when  $\varepsilon_l = \varepsilon_m = -1$ , since in this case it is simply  $\rho(\pi)(C_{\mathfrak{u}(N, \mathbb{K})})_{lm}$  or  $\rho_{\mathbb{H}}(\pi)(C_{\mathfrak{u}(N, \mathbb{H})})_{lm}$ .

Our aim in this last section devoted to the Brauer algebra is to describe these quantities for all values of  $\varepsilon_l$  and  $\varepsilon_m$ . For this, we will introduce six linear operations on the Brauer algebra  $\mathfrak{B}_{n, \lambda}$  which generalise the operations which we have already encountered of left and right multiplication by transpositions and contractions. Note that we consider the Brauer algebra of order  $n$ , although in the context of Section 3, we take  $n$  to be the length of our word  $w$ , which we denote by  $r$ .

Let us choose  $n \geq 1$  and two distinct integers  $a, b$  in  $\{1, \dots, 2n\}$ . Let us start by describing two simple linear operations associated to  $a$  and  $b$  on the Brauer algebra  $B_{n,\lambda}$ . For this, let us choose a pairing  $\pi \in \mathfrak{B}_n$ . Let  $\{a, a'\}$  and  $\{b, b'\}$  be the pairs of  $\pi$  which contain  $a$  and  $b$ . These pairs must not be distinct. The first operation which we define is the swap of  $a$  and  $b$ : we set

$$S_{a,b}(\pi) = (\pi \setminus \{\{a, a'\}, \{b, b'\}\}) \cup \{\{a, b'\}, \{b, a'\}\}.$$

The second operation is the forcing of the pair  $\{a, b\}$ : we set

$$F_{a,b}(\pi) = \begin{cases} (\pi \setminus \{\{a, a'\}, \{b, b'\}\}) \cup \{\{a, b\}, \{a', b'\}\}. & \text{if } \{a, b\} \notin \pi, \\ \lambda\pi & \text{if } \{a, b\} \in \pi. \end{cases}$$

The factor  $\lambda$  in this definition can be understood as follows: applying  $F_{a,b}$  consists in adding twice the pair  $\{a, b\}$ , once to form the pair itself, and once to form, by contiguity with the pairs  $\{a, a'\}$  and  $\{b, b'\}$ , the pair  $\{a', b'\}$ . If the pair  $\{a, b\}$  is already present in  $\pi$ , then this procedure forms a loop, hence the factor  $\lambda$ .

We can now define the six operations which we are interested in. Let us choose two distinct integers  $l, m$  in  $\{1, \dots, n\}$ . We define six linear endomorphisms of  $B_{n,\lambda}$ , which we denote by  $T_{lm}^{++}$ ,  $T_{lm}^{--}$ ,  $T_{lm}^{+-}$ ,  $W_{lm}^{++}$ ,  $W_{lm}^{--}$  and  $W_{lm}^{+-}$ , according to the following table, where the second row defines the first.

|               |               |               |               |               |               |
|---------------|---------------|---------------|---------------|---------------|---------------|
| $T_{lm}^{++}$ | $T_{lm}^{--}$ | $T_{lm}^{+-}$ | $W_{lm}^{++}$ | $W_{lm}^{--}$ | $W_{lm}^{+-}$ |
| $S_{l,m}$     | $S_{n+l,n+m}$ | $F_{l,n+m}$   | $F_{l,m}$     | $F_{n+l,n+m}$ | $S_{l,n+m}$   |

We complete these definitions by setting  $T_{lm}^{-+} = T_{ml}^{+-}$  and  $W_{lm}^{-+} = W_{ml}^{+-}$ . One checks that if  $\{l, m\} \cap \{i, j\} = \emptyset$ , then with all possible choices of signs, the following commutation relations hold:

$$(55) \quad [T_{lm}^{**}, T_{ij}^{**}] = [T_{lm}^{**}, W_{ij}^{**}] = [W_{lm}^{**}, W_{ij}^{**}] = 0.$$

It follows immediately from the definitions that the following equalities hold:

$$T_{lm}^{++}(\pi) = \pi(lm), \quad T_{lm}^{--}(\pi) = (lm)\pi, \quad W_{lm}^{++}(\pi) = \pi\langle lm \rangle, \quad W_{lm}^{--}(\pi) = \langle lm \rangle\pi.$$

The definitions of  $T_{lm}^{+-}$  and  $W_{lm}^{+-}$  may look inconsistent with the previous ones, but the following lemma explains why we chose them in this way.

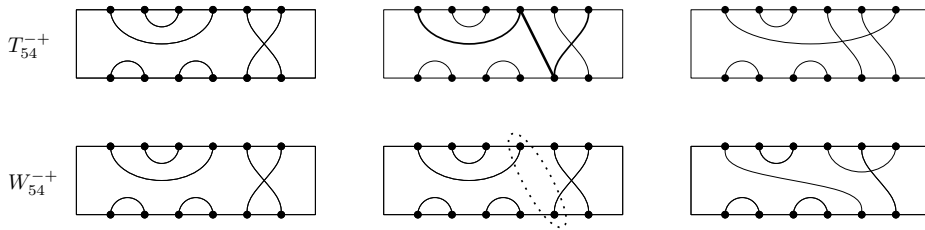


FIGURE 7. In the first line, the operation  $T_{54}^{-+}$  is applied to the pairing represented on the left. The result is represented on the right. The second line is a similar representation of the operation  $W_{54}^{-+}$ .

**Lemma 3.7.** *Let  $\pi \in \mathfrak{B}_n$  be a pairing. Let  $l, m$  be distinct integers between 1 and  $n$ . Let  $\varepsilon_l, \varepsilon_m$  be two elements of  $\{-1, 1\}$ . The following equalities hold in  $B_{n,N}$ :*

$$(\theta_l^{-\varepsilon_l} \otimes \theta_m^{-\varepsilon_m})(T) \cdot \rho(\pi) = \rho(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \quad \text{and} \quad (\theta_l^{-\varepsilon_l} \otimes \theta_m^{-\varepsilon_m})(W) \cdot \rho(\pi) = \rho(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)).$$

*Proof.* In the case where  $\varepsilon_l = \varepsilon_m = -1$ , the first equality follows from the identity  $(\theta_l^+ \otimes \theta_m^+)(T) \cdot \rho(\pi) = \rho(lm)\rho(\pi)$  and the fact that  $\rho : B_{n,N} \rightarrow M_N(\mathbb{R})^{\otimes r}$  is a homomorphism of algebras (see Section 2.5). The same arguments apply to the second equality, as well as to both equalities in the case where  $\varepsilon_l = \varepsilon_m = 1$ .

Let us compute  $(\theta_l^- \otimes \theta_m^+)(T) \cdot \rho(\pi)$ . We will make the assumption that  $l < m$  but this plays no role in the computation. We find

$$\begin{aligned} (\theta_l^- \otimes \theta_m^+)(T) \cdot \rho(\pi) &= \sum_{i_1, \dots, i_{2n}, a, b=1}^N \left( \prod_{\{u,v\} \in \pi} \delta_{i_u, i_v} \right) \dots \otimes E_{i_{n+l}, i_l} E_{ab} \otimes \dots \otimes E_{ba} E_{i_{n+m}, i_m} \otimes \dots \\ &= \sum_{i_1, \dots, i_{2n}, b=1}^N \left( \delta_{i_l, i_{n+m}} \prod_{\{u,v\} \in \pi} \delta_{i_u, i_v} \right) \dots \otimes E_{i_{n+l}, b} \otimes \dots \otimes E_{b, i_m} \otimes \dots \end{aligned}$$

If  $\{l, n+m\}$  is a pair of  $\pi$ , then the factor  $\delta_{i_l, i_{n+m}}$  is already present in the product over the pairs of  $\pi$ , the matrices  $E_{i_{n+l}, b}$  and  $E_{b, i_m}$  can be replaced by  $E_{i_{n+l}, i_l}$  and  $E_{i_{n+m}, i_m}$ , and we recover  $\rho(\pi)$ , multiplied by the factor  $N$  due to the now superfluous index  $b$ . If  $\{l, n+m\}$  is not a pair of  $\pi$ , then we perform the summation over  $i_l$  and  $i_{n+m}$  which do not appear in the tensor product anymore. We have the partial sum  $\sum_{i_l, i_{n+m}} \delta_{i_l, i_l} \delta_{i_l, i_{n+m}} \delta_{i_{n+m}, i_{n+m}} = \delta_{i_l, i_{n+m}}$ . We finally use the index  $b$  to reintroduce  $i_l$  and  $i_{n+m}$ , according to the relation

$$\sum_b \dots \otimes E_{i_{n+l}, b} \otimes \dots \otimes E_{b, i_m} \otimes \dots = \sum_{i_l, i_{n+m}} \delta_{i_l, i_{n+m}} \dots \otimes E_{i_{n+l}, i_l} \otimes \dots \otimes E_{i_{n+m}, i_m} \otimes \dots,$$

and find ourselves left with the very definition of  $\rho(T_{lm}^{+-}(\pi))$ .

The computation of  $(\theta_l^- \otimes \theta_m^+)(W) \cdot \rho(\pi)$  is similar, but the difference is significant enough for us to deem it necessary to give some details. We have

$$\begin{aligned} (\theta_l^- \otimes \theta_m^+)(W) \cdot \rho(\pi) &= \sum_{i_1, \dots, i_{2n}, a, b=1}^N \left( \prod_{\{u,v\} \in \pi} \delta_{i_u, i_v} \right) \dots \otimes E_{i_{n+l}, i_l} E_{ab} \otimes \dots \otimes E_{ab} E_{i_{n+m}, i_m} \otimes \dots \\ &= \sum_{i_1, \dots, i_{2n}, a, b=1}^N \left( \delta_{i_l, a} \delta_{i_{n+m}, b} \prod_{\{u,v\} \in \pi} \delta_{i_u, i_v} \right) \dots \otimes E_{i_{n+l}, b} \otimes \dots \otimes E_{a, i_m} \otimes \dots \end{aligned}$$

If  $\{l, n+m\}$  is a pair of  $\pi$ , then the only non-zero contributions come from the terms where  $a = b = i_l = i_{n+m}$  and the last expression is equal to  $\rho(\pi)$ . Otherwise, we can sum over  $i_{n+m}$  and  $i_l$  thanks to  $\sum_{i_{n+m}, i_l} \delta_{a, i_l} \delta_{i_l, i_{n+m}} \delta_{b, i_{n+m}} \delta_{i_{n+m}, i_m} = \delta_{a, i_l} \delta_{b, i_m}$ , and use the same formula in the reverse direction, only exchanging  $i_l$  and  $i_{n+m}$ , thus replacing  $\delta_{a, i_l} \delta_{b, i_m}$  by  $\sum_{i_{n+m}, i_l} \delta_{a, i_{n+m}} \delta_{i_{n+m}, i_l} \delta_{b, i_l} \delta_{i_l, i_m}$ . If we finally replace  $a$  by  $i_{n+m}$  and  $b$  by  $i_l$ , we find  $\rho(W_{lm}^{+-}(\pi))$ .  $\square$

In the unitary case, we are going to apply Lemma 3.7 only when  $\pi$  is a permutation and considering only the actions derived from  $T$ . It follows from the observation made just after their definition that  $T_{lm}^{++}$  and  $T_{lm}^{--}$  leave the subspace  $\mathbb{C}[\mathfrak{S}_n]$  of  $B_{n,N}$  invariant. The next lemma asserts the same of  $T_{lm}^{+-}$ .

**Lemma 3.8.** *Let  $\lambda$  be a complex number. The linear subspace  $\mathbb{C}[\mathfrak{S}_n]$  of  $B_{n,\lambda}$  is stable by  $T_{lm}^{+-}$ . More precisely, let  $\sigma$  be an element of  $\mathfrak{S}_n$ . For all distinct integers  $l, m$  between 1 and  $n$ , we have the following equality in  $\mathbb{C}[\mathfrak{S}_n]$  :*

$$T_{lm}^{+-}(\sigma) = \lambda^{\delta_{\sigma(l), m}}(\sigma(l) m)\sigma.$$

*Proof.* The pair  $\{l, n + m\}$  belongs to the pairing associated to  $\sigma$  if and only if  $\sigma(l) = m$ . The formula is thus true in this case. Let us assume that  $\sigma(l) \neq m$ . Then  $T_{lm}^{+-}(\sigma)$  is the pairing associated to  $\sigma$  in which the pairs  $\{l, n + \sigma(l)\}$  and  $\{\sigma^{-1}(m), n + m\}$  have been replaced by  $\{\sigma^{-1}(m), n + \sigma(m)\}$  and  $\{l, n + m\}$ . It is the pairing associated to a permutation  $\tilde{\sigma}$ , which satisfies  $\tilde{\sigma}(i) = \sigma(i)$  for all  $i \in \{1, \dots, n\} \setminus \{\sigma^{-1}(m), l\}$ ,  $\tilde{\sigma}(\sigma^{-1}(m)) = \sigma(l)$  and  $\tilde{\sigma}(l) = m$ . Thus,  $\tilde{\sigma} = (\sigma(l) m)\sigma$ , as expected.  $\square$

In the symplectic case, we will need, in addition to the tools developed for the unitary and orthogonal cases, a description similar to that given by Lemma 3.7 of the behaviour of the homomorphism  $\gamma$  defined by (40) with respect to the operations  $T_{lm}^{+-}$  and  $W_{lm}^{+-}$ . Recall from (9) the definition of  $\text{Re}^{\mathbb{H}}$  and  $\text{Co}^{\mathbb{H}}$ .

**Proposition 3.9.** *Let  $\pi \in \mathfrak{B}_n$  be a pairing. Choose two distinct integers  $l, m$  in  $\{1, \dots, n\}$ . Then*

$$\begin{aligned} (\theta_l^{-\varepsilon_l} \otimes \theta_m^{-\varepsilon_m}) \left( \text{Re}^{\mathbb{H}} \right) \cdot \gamma(\pi) &= \gamma(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)), \\ (\theta_l^{-\varepsilon_l} \otimes \theta_m^{-\varepsilon_m}) \left( \text{Co}^{\mathbb{H}} \right) \cdot \gamma(\pi) &= \gamma(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)). \end{aligned}$$

*Proof.* When  $\varepsilon_l = \varepsilon_m = -1$ , the two assertions are a consequence of Proposition 2.7. The other cases are treated exactly in the same way as we proved Proposition 2.7. We summarise in Figure 8 the information which is needed to prove each equality on the model of the computation given extensively in the proof of Proposition 2.7. This table contains in fact all cases, including those of Proposition 2.7 itself.  $\square$

|                                   |                      | $T_{lm}^{++}$ | $T_{lm}^{--}$ | $T_{lm}^{+-}$ | $W_{lm}^{++}$ | $W_{lm}^{--}$    | $W_{lm}^{+-}$ |
|-----------------------------------|----------------------|---------------|---------------|---------------|---------------|------------------|---------------|
| Same cycle<br>$\varepsilon_l = 1$ | $\varepsilon_m = 1$  | 1             | 1<br>(III)    | 0* or 1       | 0             | 0<br>(IV)        | 0             |
|                                   | $\varepsilon_m = -1$ | 0             | 0<br>(IV)     | 0             | 0* or 1       | 0† or 1<br>(III) | 1             |
| Different cycles                  |                      | -1            | -1<br>(I)     | -1            | -1            | -1<br>(II)       | -1            |

If and only if  ${}^* \sigma_\pi(l) = m$  or  ${}^\dagger \sigma_\pi(m) = l$ .  
 ${}^*, \dagger$  A factor  $\lambda$  is produced.

FIGURE 8. THE NON-PERIODIC TABLE OF THE SIX OPERATIONS ON THE BRAUER ALGEBRA  $\mathbb{B}_{n,\lambda}$ . The table is read as follows. Consider a pairing  $\pi \in \mathfrak{B}_n$ . Choose  $l, m$  distinct integers between 1 and  $n$ . Whether  $l$  and  $m$  are in the same cycle of  $\sigma_\pi$  or not, and if they are, whether  $\varepsilon_\pi(l)\varepsilon_\pi(m) = 1$  or  $-1$  determines which row of the table we must look at. When  $l$  and  $m$  are in the same cycle, we orient this cycle in such a way that  $\varepsilon_l = 1$ . The entry of the table corresponding to the operation we are interested in tells us how the number of cycles of  $\pi$  will be affected by this operation, if it will produce a factor  $\lambda$  (the parameter of the Brauer algebra), and which of the four identities (I) - (IV) is used in the proof of the corresponding part of Proposition 3.9.

**3.5. The unitary case.** We now turn to the proof of Proposition 3.4 in the unitary case. Just as in the proof of the first main result, the strategy is to differentiate with respect to  $t = (t_1, \dots, t_q)$ , to show that  $p_t^{c,N}(w, \sigma)$  and  $p_t(w, \sigma)$  satisfy differential relations which are not very different.

The difference with the first main result is that we will quantify the difference between the differential systems and draw quantitative conclusions on the difference between  $p_t^{c,N}(w, \sigma)$  and  $p_t(w, \sigma)$ . The following elementary and well-known fact will be instrumental.

**Lemma 3.10.** *Let  $d \geq 1$  be an integer. Let  $\|\cdot\|$  be a norm of algebra on  $M_d(\mathbb{C})$ . Let  $A, B$  be two elements of  $M_d(\mathbb{C})$ . Then*

$$(56) \quad \|e^{A+B} - e^A\| \leq \|B\| e^{\max(\|A+B\|, \|A\|)}.$$

*Proof.* We simply write

$$\begin{aligned} \|e^{A+B} - e^A\| &= \left\| \int_0^1 \frac{d}{dt} \left[ e^{t(A+B)} e^{(1-t)A} \right] dt \right\| \leq \int_0^1 \left\| e^{t(A+B)} B e^{(1-t)A} \right\| dt \\ &\leq \|B\| \int_0^1 e^{t\|A+B\| + (1-t)\|A\|} dt \leq \|B\| e^{\max(\|A+B\|, \|A\|)}, \end{aligned}$$

and find the expected inequality.  $\square$

We will apply this result with the norm on  $M_d(\mathbb{C})$  associated to the  $\ell^\infty$  norm on  $\mathbb{C}^d$ . It matters for us that this norm is given explicitly, for a matrix  $A = (A_{ij})_{i,j=1\dots d}$ , by

$$(57) \quad \|A\| = \max_{i=1\dots d} \sum_{j=1}^d |A_{ij}|.$$

*Proof of Proposition 3.4 in the unitary case.* Let  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$  be a an element of  $F_q$  written in reduced form as a word of length  $r$ . Let  $\sigma \in \mathfrak{S}_r$  be a permutation. We start from the result of Lemma 3.6 and more specifically from (54), applied to the word  $w$ , the pairing  $\pi = \sigma$ , and an integer  $k \in \{1, \dots, q\}$ .

Let us apply Lemmas 3.7 and 3.8. Thanks to the expression (12) of  $C_{u(N)}$ , we find that  $\frac{d}{dt_k} p_t^{c,N}(w, \sigma) + \frac{n_k(w)}{2} p_t^{c,N}(w, \sigma)$  is equal to

$$- \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m N^{-\ell(\sigma)-1} \mathbb{E} \left[ \text{Tr}^{\otimes r} \left( \rho(T_{lm}^{\varepsilon_l \varepsilon_m}(\sigma)) \circ w_{\otimes} (U_{N,1,t_1}, \dots, U_{N,q,t_q}) \right) \right].$$

Let us write  $l \overset{\sigma}{\sim} m$  if  $l$  and  $m$  are in the same cycle of  $\sigma$ , and  $l \not\overset{\sigma}{\sim} m$  otherwise. We find

$$(58) \quad \frac{d}{dt_k} p_t^{c,N}(w, \sigma) = -\frac{n_k(w)}{2} p_t^{c,N}(w, \sigma) - \sum_{\substack{l, m \in X_k(w) \\ l < m}} \left( \mathbb{1}_{l \overset{\sigma}{\sim} m} + \frac{1}{N^2} \mathbb{1}_{l \not\overset{\sigma}{\sim} m} \right) \varepsilon_l \varepsilon_m p_t^{c,N}(w, T_{lm}^{\varepsilon_l \varepsilon_m}(\sigma))$$

Let us write (58) in its integral form

$$p_t^{c,N}(w, \sigma) = 1 + \int_0^t (\text{r.h.s. of (58) at } t = s) ds.$$

As  $N$  tends to infinity, the pointwise convergence of  $p_t^{c,N}(w, \sigma)$  towards  $p_t(w, \sigma)$ , the fact that  $|p_t^{c,N}(w, \sigma)| \leq 1$  and the dominated convergence theorem imply that

$$p_t(w, \sigma) = 1 + \int_0^t (\text{r.h.s. of (58) at } N = \infty \text{ and } t = s) ds.$$

Hence, the family of functions  $\{p_t(w, \sigma) : \sigma \in \mathfrak{S}_r\}$  satisfies the following differential system: for all  $\sigma \in \mathfrak{S}_r$ ,

$$\frac{d}{dt_k} p_t(w, \sigma) = -\frac{\mathbf{n}_k(w)}{2} p_t(w, \sigma) - \sum_{\substack{l, m \in X_k(w) \\ l < m, l \not\sim m}} \varepsilon_l \varepsilon_m p_t(w, T_{lm}^{\varepsilon_l \varepsilon_m}(\sigma)).$$

To the word  $w$ , and for each  $k \in \{1, \dots, q\}$ , we may thus associate two real  $r! \times r!$  matrices  $A_k$  and  $C_k$ , as follows. We define, for all  $\sigma, \sigma' \in \mathfrak{S}_r$ ,

$$(A_k)_{\sigma, \sigma'} = -\frac{\mathbf{n}_k(w)}{2} \delta_{\sigma, \sigma'} - \sum_{\substack{l, m \in X_k(w) \\ l < m, l \not\sim m}} \varepsilon_l \varepsilon_m \delta_{T_{lm}^{\varepsilon_l \varepsilon_m}(\sigma), \sigma'}$$

and

$$(C_k)_{\sigma, \sigma'} = - \sum_{\substack{l, m \in X_k(w) \\ l < m, l \not\sim m}} \varepsilon_l \varepsilon_m \delta_{T_{lm}^{\varepsilon_l \varepsilon_m}(\sigma), \sigma'}.$$

For all distinct  $k_1, k_2 \in \{1, \dots, q\}$ , the sets  $X_{k_1}(w)$  and  $X_{k_2}(w)$  are disjoint, so that (55) implies the commutation relations

$$(59) \quad [A_{k_1}, A_{k_2}] = [A_{k_1}, C_{k_2}] = [C_{k_1}, C_{k_2}] = 0.$$

Let us define the vector  $p_t^{\mathbb{C}, N}(w) = (p_t^{\mathbb{C}, N}(w, \sigma))_{\sigma \in \mathfrak{S}_r}$ . Let us write explicitly the dependence of  $p_t^{\mathbb{C}, N}(w)$  on  $t_1, \dots, t_q$ . We have

$$p_{(t_1, \dots, t_q)}^{\mathbb{C}, N}(w) = e^{t_1(A_1 + \frac{1}{N^2} C_1)} p_{(0, t_2, \dots, t_q)}^{\mathbb{C}, N}(w).$$

We have  $p_{(0, \dots, 0)}^{\mathbb{C}, N}(w) = \mathbb{1}$ , the vector of  $\mathbb{C}^{r!}$  whose components are all equal to 1. Thus, we have

$$(60) \quad p_{(t_1, \dots, t_q)}^{\mathbb{C}, N}(w) = \left( \prod_{k=1}^q e^{t_k (A_k + \frac{1}{N^2} C_k)} \right) \mathbb{1},$$

where the order in this product is irrelevant, thanks to (59). Similarly, defining  $p_t(w) = (p_t(w, \sigma))_{\sigma \in \mathfrak{S}_r}$ , we have

$$(61) \quad p_{(t_1, \dots, t_q)}(w) = \left( \prod_{k=1}^q e^{t_k A_k} \right) \mathbb{1}.$$

We can express the  $\ell^\infty$  norm of the difference:

$$\begin{aligned} \|p_{(t_1, \dots, t_q)}^{\mathbb{C}, N}(w) - p_{(t_1, \dots, t_q)}(w)\|_\infty &= \left\| \left( \prod_{k=1}^q e^{t_k (A_k + \frac{1}{N^2} C_k)} - \prod_{k=1}^q e^{t_k A_k} \right) \mathbb{1} \right\|_\infty \\ &\leq \sum_{l=1}^q \left\| \prod_{k=l+1}^q e^{t_k (A_k + \frac{1}{N^2} C_k)} \left( e^{t_l (A_l + \frac{1}{N^2} C_l)} - e^{t_l A_l} \right) \prod_{k=1}^{l-1} e^{t_k A_k} \mathbb{1} \right\|_\infty \\ &\leq \sum_{l=1}^q \prod_{k=l+1}^q e^{t_k \|A_k + \frac{1}{N^2} C_k\|} \left\| e^{t_l (A_l + \frac{1}{N^2} C_l)} - e^{t_l A_l} \right\| \prod_{k=1}^{l-1} e^{t_k \|A_k\|}. \end{aligned}$$

Recall from (57) the definition of the norm which we are using on  $M_d(\mathbb{C})$ . It is easy to check that the following inequalities hold for all  $N \geq 1$ :

$$(62) \quad \|A_k\| \leq \frac{\mathfrak{n}_k(w)^2}{2}, \quad \|C_k\| \leq \frac{\mathfrak{n}_k(w)^2}{2}, \quad \|A_k + \frac{1}{N^2}C_k\| \leq \frac{\mathfrak{n}_k(w)^2}{2}.$$

Now, applying (56) and thanks to (62), we find

$$\begin{aligned} \|p_{(t_1, \dots, t_q)}^{\mathbb{C}, N}(w) - p_{(t_1, \dots, t_q)}(w)\|_\infty &\leq \sum_{l=1}^q \prod_{k=l+1}^q e^{\frac{1}{2}t_k \mathfrak{n}_k(w)^2} \frac{t_l \mathfrak{n}_l(w)^2}{2N^2} e^{\frac{1}{2}t_l \mathfrak{n}_l(w)^2} \prod_{k=1}^{l-1} e^{\frac{1}{2}t_k \mathfrak{n}_k(w)^2} \\ &= \frac{1}{2N^2} \bar{A}(w) e^{\frac{1}{2}\bar{A}(w)}, \end{aligned}$$

which is the expected inequality.  $\square$

**3.6. The orthogonal case.** The proof in the orthogonal case follows the same pattern as in the unitary case.

*Proof of Proposition 3.4 in the orthogonal case.* Let  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$  be an element of  $F_q$  of length  $r$ . Let  $\pi \in \mathfrak{B}_r$  be a pairing of  $\{1, \dots, 2r\}$ . We start again from (54), applied to  $w, \pi$  and an integer  $k \in \{1, \dots, q\}$ . We find, thanks to Lemma 3.7 and (34), that  $\frac{d}{dt_k} p_t^{\mathbb{R}, N}(w, \pi) + \frac{\mathfrak{n}_k(w)(N-1)}{2N} p_t^{\mathbb{R}, N}(w, \pi)$  is equal to

$$- \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m N^{-\ell(\pi)-1} \mathbb{E} \left[ \text{Tr}^{\otimes r} \left( \rho(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi) - W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \circ w_\otimes(R_{N,1,t_1}, \dots, R_{N,q,t_q}) \right) \right].$$

From this expression, we deduce

$$(63) \quad \begin{aligned} \frac{d}{dt_k} p_t^{\mathbb{R}, N}(w, \pi) &= - \frac{\mathfrak{n}_k(w)(N-1)}{2N} p_t^{\mathbb{R}, N}(w, \pi) \\ &\quad - \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m N^{\ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) - \ell(\pi) - 1} p_t^{\mathbb{R}, N}(w, T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \\ &\quad + \sum_{\substack{l, m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m N^{\ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) - \ell(\pi) - 1} p_t^{\mathbb{R}, N}(w, W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)). \end{aligned}$$

We claim that the only exponents of  $N$  which can appear in this sum are 0,  $-1$  and  $-2$ . For the terms where  $\varepsilon_l = \varepsilon_m$ , this is something which we already discussed in the one-matrix case. Recall in particular from the proof of Theorem 2.2 in the orthogonal case, which we gave in Section 2.6, that in the case where  $\langle lm \rangle \pi = N\pi$  or  $\pi \langle lm \rangle = N\pi$ , we get a term of order  $N^0$ . The situation is the same for  $T_{lm}^{+-}(\pi)$  and  $W_{lm}^{+-}(\pi)$ : both  $\ell(T_{lm}^{+-}(\pi)) - \ell(\pi)$  and  $\ell(W_{lm}^{+-}(\pi)) - \ell(\pi)$  belong to  $\{-1, 0, 1\}$ . Moreover, in the case where  $T_{lm}^{+-}(\pi) = N\pi$ , we get a term of order  $N^0$ .

As in the unitary case, the integral form  $p_t^{\mathbb{R}, N}(w, \pi) = 1 + \int_0^t (\text{r.h.s. of (63) at } t=s) ds$  of (63) converges, as  $N$  tends to infinity, to  $p_t(w, \pi) = 1 + \int_0^t (\text{r.h.s. of (63) at } N=\infty \text{ and } t=s) ds$ .

Hence, the family of functions  $\{p_t(w, \pi) : \pi \in \mathfrak{B}_r\}$  satisfies the following differential system: for all  $\pi \in \mathfrak{B}_r$ ,

$$(64) \quad \begin{aligned} \frac{d}{dt_k} p_t(w, \pi) = & -\frac{\mathfrak{n}_k(w)}{2} p_t(w, \pi) - \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) + 1}} \varepsilon_l \varepsilon_m p_t(w, T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \\ & + \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) + 1}} \varepsilon_l \varepsilon_m p_t(w, W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)). \end{aligned}$$

Let  $(2r)!! = \prod_{k=1}^r (2k-1)$  denote the cardinal of  $\mathfrak{B}_r$ . To the word  $w$ , and for each  $k \in \{1, \dots, q\}$ , we associate three matrices  $A_k$ ,  $B_k$  and  $C_k$  in  $M_{(2r)!!}(\mathbb{R})$ , as follows. We define, for all  $\pi, \pi' \in \mathfrak{B}_r$ ,

$$(65) \quad \begin{aligned} (A_k)_{\pi, \pi'} = & -\frac{\mathfrak{n}_k(w)}{2} \delta_{\pi, \pi'} - \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) + 1}} \varepsilon_l \varepsilon_m \delta_{T_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'} + \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) + 1}} \varepsilon_l \varepsilon_m \delta_{W_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'}, \\ (B_k)_{\pi, \pi'} = & \frac{\mathfrak{n}_k(w)}{2} \delta_{\pi, \pi'} - \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi)}} \varepsilon_l \varepsilon_m \delta_{T_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'} + \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi)}} \varepsilon_l \varepsilon_m \delta_{W_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'}, \\ (C_k)_{\pi, \pi'} = & - \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) - 1}} \varepsilon_l \varepsilon_m \delta_{T_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'} + \sum_{\substack{l, m \in X_k(w), l < m \\ \ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) = \ell(\pi) - 1}} \varepsilon_l \varepsilon_m \delta_{W_{lm}^{\varepsilon_l \varepsilon_m}(\pi), \pi'}, \end{aligned}$$

which satisfy commutation relations analogous to (59). Setting  $p_t^{\mathbb{R}, N}(w) = (p_t^{\mathbb{R}, N}(w, \pi))_{\pi \in \mathfrak{B}_r}$ , we have  $p_{(0, \dots, 0)}^{\mathbb{R}, N}(w) = \mathbb{1}$ , the vector of  $\mathbb{R}^{(2r)!!}$  whose components are all equal to 1, and

$$(66) \quad p_{(t_1, \dots, t_q)}^{\mathbb{R}, N}(w) = \left( \prod_{k=1}^q e^{t_k (A_k + \frac{1}{N} B_k + \frac{1}{N^2} C_k)} \right) \mathbb{1}.$$

Similarly, if we define  $p_t(w) = (p_t(w, \pi))_{\pi \in \mathfrak{B}_r}$ , we have

$$(67) \quad p_{(t_1, \dots, t_q)}(w) = \left( \prod_{k=1}^q e^{t_k A_k} \right) \mathbb{1}.$$

The same computation as in the unitary case shows that  $\|p_t^{\mathbb{R}, N}(w) - p_t(w)\|_\infty$  is smaller than

$$\sum_{l=1}^q \prod_{k=l+1}^q e^{t_k \|A_k + \frac{1}{N} B_k + \frac{1}{N^2} C_k\|} \left\| e^{t_l (A_l + \frac{1}{N} B_l + \frac{1}{N^2} C_l)} - e^{t_l A_l} \right\| \prod_{k=1}^{l-1} e^{t_k \|A_k\|}.$$

It is easy to check that the following inequalities hold for all  $N \geq 1$ :

$$(68) \quad \|A_k\| \leq \mathfrak{n}_k(w)^2, \quad \left\| A_k + \frac{1}{N} B_k + \frac{1}{N^2} C_k \right\| \leq \mathfrak{n}_k(w)^2, \quad \left\| \frac{1}{N} B_k + \frac{1}{N^2} C_k \right\| \leq \frac{\mathfrak{n}_k(w)^2}{N}.$$

Now, applying (56) and thanks to (68), we find

$$\begin{aligned} \|p_{(t_1, \dots, t_q)}^{\mathbb{R}, N}(w) - p_{(t_1, \dots, t_q)}(w)\|_\infty & \leq \sum_{l=1}^q \prod_{k=l+1}^q e^{t_k \mathfrak{n}_k(w)^2} \frac{t_l \mathfrak{n}_l(w)^2}{N} e^{t_l \mathfrak{n}_l(w)^2} \prod_{k=1}^{l-1} e^{t_k \mathfrak{n}_k(w)^2} \\ & = \frac{1}{N} \bar{A}(w) e^{\bar{A}(w)}, \end{aligned}$$



which is the expected inequality.  $\square$

### 3.7. The symplectic case.

*Proof of Proposition 3.4 in the symplectic case.* Let  $w = x_{i_1}^{\varepsilon_1} \dots x_{i_r}^{\varepsilon_r}$  be an element of  $F_q$  of length  $r$ . Let  $\pi \in \mathfrak{B}_r$  be a pairing of  $\{1, \dots, 2r\}$ . By (54), Lemma 3.7, Proposition 3.9 and (42),  $\frac{d}{dt_k} p_t^{\mathbb{H},N}(w, \pi) + \frac{\mathfrak{n}_k(w)(2N+1)}{4N} p_t^{\mathbb{H},N}(w, \pi)$  is equal to

$$\begin{aligned} & - \sum_{\substack{l,m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m (-2N)^{-\ell(\pi)-1} \mathbb{E} \left[ (-2\Re\mathrm{Tr})^{\otimes r} (T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \circ w_{\otimes} (S_{N,1,t_1}, \dots, S_{N,q,t_q}) \right] \\ & + \sum_{\substack{l,m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m (-2N)^{-\ell(\pi)-1} \mathbb{E} \left[ (-2\Re\mathrm{Tr})^{\otimes r} (W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \circ w_{\otimes} (S_{N,1,t_1}, \dots, S_{N,q,t_q}) \right]. \end{aligned}$$

From this expression, we deduce

$$\begin{aligned} (69) \quad \frac{d}{dt_k} p_t^{\mathbb{H},N}(w, \pi) &= - \frac{\mathfrak{n}_k(w)(2N+1)}{4N} p_t^{\mathbb{H},N}(w, \pi) \\ & - \sum_{\substack{l,m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m (-2N)^{\ell(T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) - \ell(\pi) - 1} p_t^{\mathbb{H},N}(w, T_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) \\ & + \sum_{\substack{l,m \in X_k(w) \\ l < m}} \varepsilon_l \varepsilon_m (-2N)^{\ell(W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)) - \ell(\pi) - 1} p_t^{\mathbb{H},N}(w, W_{lm}^{\varepsilon_l \varepsilon_m}(\pi)). \end{aligned}$$

For the same reason as in the orthogonal case, the only exponents of  $N$  which can appear in this sum are 0,  $-1$  and  $-2$ . Still as in the unitary and orthogonal cases, the integral form  $p_t^{\mathbb{R},N}(w, \pi) = 1 + \int_0^t (\text{r.h.s. of (69) at } t=s) ds$  converges, as  $N$  tends to infinity, to  $p_t(w, \pi) = 1 + \int_0^t (\text{r.h.s. of (69) at } N=\infty \text{ and } t=s) ds$ . We recover, in the limit, the differential system (64).

To the word  $w$ , and for each  $k \in \{1, \dots, q\}$ , we associate the same matrices  $A_k$  and  $C_k$  in  $M_{(2r)!!}(\mathbb{R})$  defined by (65), and a matrix  $B'_k$ , which differs from  $B_k$  only by its diagonal terms, to compensate the difference between  $c_{\mathfrak{so}(N)}$  and  $c_{\mathfrak{sp}(N)}$ : we define, for all  $\pi, \pi' \in \mathfrak{B}_r$ ,

$$(B'_k)_{\pi, \pi'} = (B_k)_{\pi, \pi'} - \frac{3\mathfrak{n}_k(w)}{4} \delta_{\pi, \pi'}.$$

Setting  $p_t^{\mathbb{H},N}(w) = (p_t^{\mathbb{H},N}(w, \pi))_{\pi \in \mathfrak{B}_r}$ , we have  $p_{(0, \dots, 0)}^{\mathbb{H},N}(w) = \mathbb{1}$ , the vector of  $\mathbb{R}^{(2r)!!}$  whose components are all equal to 1, and

$$(70) \quad p_{(t_1, \dots, t_q)}^{\mathbb{H},N}(w) = \left( \prod_{k=1}^q e^{t_k \left( A_k + \frac{1}{(-2N)} B'_k + \frac{1}{(-2N)^2} C_k \right)} \right) \mathbb{1}.$$

By (67) and the same computation as in the other cases,  $\|p_t^{\mathbb{H},N}(w) - p_t(w)\|_{\infty}$  is smaller than

$$\sum_{l=1}^q \prod_{k=l+1}^q e^{t_k \|A_k - \frac{1}{2N} B'_k + \frac{1}{4N^2} C_k\|} \left\| e^{t_l \left( A_l - \frac{1}{2N} B'_l + \frac{1}{4N^2} C_l \right)} - e^{t_l A_l} \right\| \prod_{k=1}^{l-1} e^{t_k \|A_k\|}.$$

It is easy to check that the following inequalities hold for all  $N \geq 1$ :

$$(71) \quad \|A_k\| \leq \mathfrak{n}_k(w)^2, \quad \left\| A_k - \frac{1}{2N} B_k + \frac{1}{4N^2} C_k \right\| \leq \mathfrak{n}_k(w)^2, \quad \left\| -\frac{1}{2N} B_k + \frac{1}{4N^2} C_k \right\| \leq \frac{\mathfrak{n}_k(w)^2}{N}.$$

Now, applying (56) and thanks to (71), we find

$$\begin{aligned} \|P_{(t_1, \dots, t_q)}^{\mathbb{H}, N}(w) - p_{(t_1, \dots, t_q)}(w)\|_\infty &\leq \sum_{l=1}^q \prod_{k=l+1}^q e^{t_k n_k(w)^2} \frac{t_l n_l(w)^2}{N} e^{t_l n_l(w)^2} \prod_{k=1}^{l-1} e^{t_k n_k(w)^2} \\ &= \frac{1}{N} \bar{A}(w) e^{\bar{A}(w)}, \end{aligned}$$

which is the expected inequality.  $\square$

## Part 2. The master field on the plane

In the second part of this work, we apply the results of the first part to the Yang-Mills measure on the plane and, specifically, to its large  $N$  limit.

### 4. THE YANG-MILLS MEASURE ON THE PLANE

Let us start by recalling the definition of the Yang-Mills measure on the plane. For a more detailed presentation, we refer the reader to [23], although strictly speaking the case of the plane was not treated there.

We consider the plane  $\mathbb{R}^2$  endowed with the usual Euclidean metric and the Lebesgue measure. It will be useful at a later stage to allow the measure to vary and to become a measure with a bounded density with respect to the Lebesgue measure, but for the moment we ignore this refinement.

Let us choose a connected compact Lie group  $G$  which will stay fixed in this section. The examples which we have in mind are of course the special orthogonal, unitary and symplectic groups which we studied in the first part of this work, but for the purposes of the definition of the Yang-Mills measure, we do not need to specify  $G$ . We denote the Lie algebra of  $G$  by  $\mathfrak{g}$  and we endow it with a scalar product invariant by the adjoint action of  $G$ , which we denote by  $\langle \cdot, \cdot \rangle$ . For example, one can think of  $G$  being  $U(N)$  for some  $N \geq 1$ , so that  $\mathfrak{g} = \mathfrak{u}(N)$ , and the scalar product on  $\mathfrak{g}$  being given by  $\langle X, Y \rangle = N \text{Tr}(X^* Y)$ .

The Yang-Mills measure, or rather, the Yang-Mills process, is a collection of random variables with values in the group  $G$ , one for each loop with finite length on  $\mathbb{R}^2$ . In order to construct this collection, one proceeds by discrete approximation, considering at first only loops which are traced in a fixed graph. We start by recalling the main aspects of this discrete theory.

**4.1. Discrete Yang-Mills field.** Let us start by giving precise definitions of the sets of paths which we will consider. A *parametrized path* on  $\mathbb{R}^2$  is a Lipschitz continuous mapping  $c : [0, 1] \rightarrow \mathbb{R}^2$  which is either constant or such that its speed is bounded below by a positive constant. A *path* is a parametrized path taken up to bi-Lipschitz increasing reparametrization. The set of paths on  $\mathbb{R}^2$  is denoted by  $\mathcal{P}(\mathbb{R}^2)$ .

The endpoints of a path are denoted respectively by  $\underline{c} = c(0)$  and  $\bar{c} = c(1)$ . Two paths  $c_1$  and  $c_2$  such that  $\bar{c}_1 = \underline{c}_2$  can be concatenated to form a new path denoted by  $c_1 c_2$ . This partially defined operation on  $\mathcal{P}(\mathbb{R}^2)$  is associative. For each path  $c$  we define the path  $c^{-1}$  which is the class of  $t \mapsto c(1 - t)$ , the path  $c$  traced backwards.

A path whose endpoints coincide is called a *loop*. The set of loops on  $\mathbb{R}^2$  is denoted by  $\mathcal{L}(\mathbb{R}^2)$ . A loop whose restriction to  $[0, 1)$  is injective is called a *simple loop*. The set of loops starting, and hence finishing, at a point  $m \in \mathbb{R}^2$  is denoted by  $\mathcal{L}_m(\mathbb{R}^2)$ . For all  $m \in \mathbb{R}^2$ , the set  $\mathcal{L}_m(\mathbb{R}^2)$  endowed with the operation of concatenation is a monoid. We shall explain later (see Section 5.7) that there is a natural, though not easy to define, equivalence relation on this monoid such that the quotient is actually a group.

For the time being, let us turn to graphs. An *edge* is a path which is either injective or a simple loop. Note that an edge traced backwards is still an edge, though distinct from the original one. A *graph* is a triple  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  such that the following properties are satisfied.

1. The set  $\mathbb{E}$  is a finite subset of  $\mathbb{P}(\mathbb{R}^2)$  consisting of edges. For all edge  $e \in \mathbb{E}$ , the edge  $e^{-1}$  belongs to  $\mathbb{E}$ . Any two edges of  $\mathbb{E}$  which are distinct and not each other's inverse meet, if at all, only at some of their endpoints.
2. The set  $\mathbb{V}$  is the set of endpoints of the elements of  $\mathbb{E}$ .
3. The set  $\mathbb{F}$  is the set of connected components of the complement in  $\mathbb{R}^2$  of the *skeleton* of  $\mathbb{G}$ , which is the subset  $\mathbf{Sk}(\mathbb{G}) = \bigcup_{e \in \mathbb{E}} e([0, 1])$ .
4. Each element of  $\mathbb{F}$  is either a bounded subset of  $\mathbb{R}^2$  homeomorphic to the open unit disk of  $\mathbb{R}^2$ , or an unbounded subset of  $\mathbb{R}^2$  homeomorphic to the complement of the origin in  $\mathbb{R}^2$ .

The elements of  $\mathbb{V}, \mathbb{E}, \mathbb{F}$  are called respectively the vertices, edges, and faces of  $\mathbb{G}$ . The fourth condition is equivalent to the fact that the skeleton of the graph is connected (see [23, Proposition 1.3.10]). All faces of a graph are bounded but one, which we naturally call the unbounded face and which we usually denote by  $F_\infty$ . We shall use the notation  $\mathbb{F}^b = \mathbb{F} \setminus \{F_\infty\}$  for the set of bounded faces. For each bounded face  $F$ , we denote by  $|F|$  the area of  $F$ .

Let  $\mathbb{G}$  be a graph. The set of paths which can be formed by concatenating edges of  $\mathbb{G}$  is denoted by  $\mathbb{P}(\mathbb{G})$ . The subset of  $\mathbb{P}(\mathbb{G})$  consisting of loops is denoted by  $\mathbb{L}(\mathbb{G})$ . Each bounded face of  $\mathbb{G}$  is positively bounded by a loop which we call its boundary and which is ill-defined because it has no preferred base point. Nevertheless, we denote by  $\partial F$  the boundary of the face  $F$ , keeping in mind that this is not properly speaking a loop, but rather a collection of loops which differ only by their starting point.

The discrete Yang-Mills measure associated with  $\mathbb{G}$  is a probability measure on a space which can be described in several equivalent and equally useful ways. Let  $P$  be a subset of  $\mathbb{P}(\mathbb{R}^2)$ . A function  $h : P \rightarrow G$  is said to be *multiplicative* if for any  $c \in P$  such that  $c^{-1} \in P$  one has  $h(c^{-1}) = h(c)^{-1}$ , and for any two paths  $c_1$  and  $c_2$  in  $P$  such that  $\overline{c_1} = \underline{c_2}$  and  $c_1 c_2 \in P$  one has

$$(72) \quad h(c_1 c_2) = h(c_2) h(c_1).$$

We denote the set of multiplicative functions from  $P$  to  $G$  by  $\mathcal{M}(P, G)$ . The discrete Yang-Mills measure on  $\mathbb{G}$  shall be defined as a probability measure on  $\mathcal{M}(\mathbb{P}(\mathbb{G}), G)$ .

Since any path traced in  $\mathbb{G}$  is a concatenation of edges, a multiplicative function on  $\mathbb{P}(\mathbb{G})$  is completely determined by its restriction to the set of edges. Actually, one needs only to know its value on one element of each pair  $\{e, e^{-1}\}$ , where  $e$  spans the set of edges. We call *orientation* of the edges of  $\mathbb{G}$  a subset  $\mathbb{E}^+$  of  $\mathbb{E}$  which contains exactly one element in each pair  $\{e, e^{-1}\}$ ,  $e \in \mathbb{E}$ . An orientation of the edges of  $\mathbb{G}$  being chosen, we have the following identifications

$$(73) \quad \mathcal{M}(\mathbb{P}(\mathbb{G}), G) \simeq \mathcal{M}(\mathbb{E}, G) \simeq \mathcal{M}(\mathbb{E}^+, G) \simeq G^{\mathbb{E}^+}.$$

The last identification expresses the fact that any function from  $\mathbb{E}^+$  to  $G$  is multiplicative, since the concatenation of two edges is never an edge. We call any of these spaces the *configuration space* of the discrete theory and denote it by  $\mathcal{C}_G^{\mathbb{G}}$ , or simply  $\mathcal{C}^{\mathbb{G}}$  as long as the group  $G$  is kept fixed. The reader who feels uncomfortable with such a row of identifications can take  $\mathcal{C}^{\mathbb{G}} = G^{\mathbb{E}^+}$  as an efficient definition.

As announced, the discrete Yang-Mills measure is a Borel probability measure on  $\mathcal{C}^{\mathbb{G}}$ , which is naturally a compact topological space. The normalised Haar measure on the compact group  $G$  determines, through the identifications above, a reference probability measure on  $\mathcal{C}^{\mathbb{G}}$  which we denote by  $dh = \bigotimes_{e \in \mathbb{E}^+} dg_{e^+}$ . The Yang-Mills measure has a density with respect to this uniform measure and in order to define it, we must introduce the heat kernel on  $G$ , which is a one-parameter family of smooth positive functions on  $G$ , namely the fundamental solution of the

heat equation. If  $G$  is one of the groups which we studied in the first part, then this function is also the time-dependent density of the distribution of the Brownian motion on the group.

The Lie algebra  $\mathfrak{g}$  of  $G$  is the space of left-invariant first-order differential operators on  $G$ : to each element  $X \in \mathfrak{g}$ , one associates the differential operator  $\mathcal{L}_X$  defined by the equality, valid for all differentiable function  $f : G \rightarrow \mathbb{R}$  and all  $g \in G$ ,  $(\mathcal{L}_X f)(g) = \frac{d}{dt}|_{t=0} f(ge^{tX})$ .

Let  $d$  denote the dimension of  $G$ . Given an orthonormal basis  $(X_1, \dots, X_d)$  of  $\mathfrak{g}$  with respect to the invariant scalar product which we have chosen on  $\mathfrak{g}$ , we can form the second-order differential operator  $\sum_{k=1}^d \mathcal{L}_{X_k}^2$ . This operator does not depend on the choice of the orthonormal basis, it is called the Laplace operator on  $G$  and we denote it by  $\Delta$ .

The heat kernel on  $G$  is the unique function  $Q : \mathbb{R}_+^* \times G \rightarrow \mathbb{R}_+^*$  such that  $(\partial_t - \frac{1}{2}\Delta)Q = 0$  and the measure  $Q(t, g) dg$  converges weakly, as  $t$  tends to 0, to the Dirac measure at the unit element of  $G$ . The measure  $Q(t, g) dg$  is simply the distribution of the Brownian motion on  $G$  at time  $t$ .

We will denote the number  $Q(t, g)$  by  $Q_t(g)$ , thus seeing  $Q$  as a one-parameter family of functions on  $G$ . A crucial property of these functions is that they are invariant by conjugation: they satisfy, for all  $t > 0$  and all  $x, y \in G$ , the equality  $Q_t(yxy^{-1}) = Q_t(x)$ . This is a consequence of the fact that the Laplace operator belongs to the centre of the algebra of left-invariant differential operators on  $G$ .

We can now define the Yang-Mills measure  $\text{YM}_{\mathbb{G}}^{\mathbb{G}}$ , or simply  $\text{YM}^{\mathbb{G}}$ , on  $\mathcal{C}^{\mathbb{G}}$ . We need only to make a last observation: if  $F$  is a face of a graph  $\mathbb{G}$ , and if  $h$  is a multiplicative function on  $\text{P}(\mathbb{G})$ , then for all  $t > 0$  the number  $Q_t(h(\partial F))$  does not depend on the particular choice of the origin of the loop  $\partial F$ . Indeed, changing the origin of  $\partial F$  alters  $h(\partial F)$  by conjugating it in  $G$ , and this does not change the value of  $Q_t$ . The following expression is thus well defined:

$$(74) \quad \text{YM}^{\mathbb{G}}(dh) = \prod_{F \in \mathbb{F}^b} Q_{|F|}(h(\partial F)) dh.$$

This is indeed a probability measure, as one verifies by successively integrating over all edges using the convolution property of the heat kernel, according to which  $\int_G Q_t(xy^{-1})Q_s(y) dy = Q_{t+s}(x)$ , and finally the fact that  $\int_G Q_t(x) dx = 1$ . Note that the product in this definition is over the set of bounded faces of  $\mathbb{G}$ . In fact,  $Q_t$  converges uniformly and exponentially fast to 1 as  $t$  tends to infinity, and we could just as well include the unbounded face in the product, provided we make the very natural convention  $Q_{\infty} = 1$ .

With this definition, the Borel probability space  $(\mathcal{C}^{\mathbb{G}}, \text{YM}^{\mathbb{G}})$  is essentially the canonical space of the stochastic process  $(H_c)_{c \in \text{P}(\mathbb{G})}$ , which is defined simply by  $H_c(h) = h(c)$  for all  $c \in \text{P}(\mathbb{G})$ . The fact that we are working with multiplicative functions implies that the stochastic process  $H$  is trajectoryally multiplicative. This means that if  $c_1$  and  $c_2$  can be concatenated, then the functions  $H_{c_1}H_{c_2}$  and  $H_{c_2c_1} : \mathcal{C}^{\mathbb{G}} \rightarrow G$  are the same.

To conclude this section, let us observe that in the definition (74), the Lebesgue measure, which is used to define the area of each bounded face of the graph, could be replaced by an arbitrary Borel measure on  $\mathbb{R}^2$ , again with the convention  $Q_{\infty} = 1$ . In Section 6, we shall use this possibility of replacing the Lebesgue measure by a measure which has a smooth positive density with respect to it.

**4.2. Continuous Yang-Mills field.** The single most important property of the discrete Yang-Mills field is that it is consistent with respect to the subdivision, or refinement, of the underlying graph. The meaning of this assertion is the following. If  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are two graphs, we say that  $\mathbb{G}_2$  is finer than  $\mathbb{G}_1$  if  $\text{P}(\mathbb{G}_1)$  is a subset of  $\text{P}(\mathbb{G}_2)$ . In this case, there is a natural mapping of restriction  $\mathcal{M}(\text{P}(\mathbb{G}_2), G) \rightarrow \mathcal{M}(\text{P}(\mathbb{G}_1), G)$  and the invariance of the Yang-Mills measure under

refinement of the graph is the fact that the image of the measure  $\text{YM}^{\mathbb{G}_2}$  under this mapping is  $\text{YM}^{\mathbb{G}_1}$ .

The practical consequence of this invariance is that if a certain set  $P$  of paths belongs to  $\mathcal{P}(\mathbb{G}_1)$  and  $\mathcal{P}(\mathbb{G}_2)$  for two graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  such that one is finer than the other, then the distribution of the family of random variables  $(H_c)_{c \in P}$  is the same when computed under  $\text{YM}^{\mathbb{G}_1}$  or under  $\text{YM}^{\mathbb{G}_2}$ . The same conclusion holds if  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are both finer than a third graph. Unfortunately, such a third graph does not always exist.

The invariance under subdivision also allows one to take an inverse limit of the discrete construction which we have recalled as the graphs become arbitrarily fine. However, the fact that for two arbitrary graphs there does not always exist a third graph which is finer than both of them spoils the projective structure of the set of graphs and forces the use of an approximation procedure in the construction of the Yang-Mills field. It is thus necessary to consider an appropriate topology on the set of paths, which we now describe.

Let  $c_1$  and  $c_2$  be two paths. We denote by  $\ell(c_1)$  and  $\ell(c_2)$  respectively the lengths of  $c_1$  and  $c_2$ . The uniform distance between  $c_1$  and  $c_2$  is  $d_\infty(c_1, c_2) = \inf_{\varphi_1, \varphi_2} \sup\{|c_1(\varphi_1(t)) - c_2(\varphi_2(t))| : t \in [0, 1]\}$ , where the infimum is taken over all pairs of increasing bi-Lipschitz homeomorphisms of  $[0, 1]$ . We define two distances between  $c_1$  and  $c_2$  by setting

$$d_1(c_1, c_2) = |c_1(0) - c_2(0)| + \int_0^1 |\dot{c}_1(t) - \dot{c}_2(t)| dt$$

and

$$d_\ell(c_1, c_2) = d_\infty(c_1, c_2) + |\ell(c_1) - \ell(c_2)|.$$

The first distance is the distance in 1-variation and the second we call the length distance. Although the first is complete on  $\mathcal{P}(\mathbb{R}^2)$  and the second is not, it can be shown that these distances determine the same topology. We thus simply speak of convergence of paths, without mentioning a distance. We shall frequently use the notion of convergence with fixed endpoints of a sequence of paths, where all the paths of the sequence are required to have the same endpoints as the limiting path.

The main theorem is the following.

**Theorem 4.1.** *There exists on the space  $\mathcal{M}(\mathcal{P}(\mathbb{R}^2), G)$  endowed with the cylinder  $\sigma$ -algebra a unique probability measure  $\text{YM}_G$  such that the following two properties are satisfied.*

1. *For all graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ , the family of random variables  $(H_c)_{c \in \mathcal{P}(\mathbb{G})}$  has the same distribution under  $\text{YM}_G$  as under  $\text{YM}_G^{\mathbb{G}}$ .*
2. *For all path  $c \in \mathcal{P}(\mathbb{R}^2)$  and all sequence  $(c_n)_{n \geq 1}$  of paths converging with fixed endpoints to  $c$ , the sequence  $(H_{c_n})_{n \geq 1}$  converges in probability to  $H_c$ .*

In [26], this theorem is proved on a compact Riemannian surface rather than on the plane  $\mathbb{R}^2$ . The proof is however valid without any modification.

**4.3. The group of loops in a graph.** In the next section, we shall undertake the proof of the central result of the second part of this work, which asserts the existence of a limit for the Yang-Mills process on the plane when the group  $G$  is one of the groups  $U(N, \mathbb{K})$  which we have considered earlier and when  $N$  tends to infinity.

In a first step, we shall study the large  $N$  limit of the discrete Yang-Mills measure associated with a graph on the plane, using a very explicit description of this measure in terms of a collection of independent random matrices, some uniform and some distributed according to the heat kernel measure. In preparation for this, we need to understand fairly concretely the structure of the set of paths and loops on the graph  $\mathbb{G}$ , and this is what we explain now. What we have to say

in the present section is still valid for any compact connected Lie group  $G$ . The details of what is explained below can be found in [26, Sec. 2.4].

Let  $\mathbb{G}$  be a graph on  $\mathbb{R}^2$ . There is a very natural equivalence relation on the set  $\mathcal{P}(\mathbb{G})$ , which declares two paths equivalent if one can transform one into the other by a finite sequence of insertions or erasures of paths of the form  $ee^{-1}$ , where  $e$  is an edge. For example, the paths  $e_0e_1e_2e_3e_3^{-1}e_2^{-1}$  and  $e_0e_2^{-1}e_2e_1e_3e_4^{-1}e_4e_3^{-1}$  are equivalent. One proves that in each equivalence class for this relation there is a unique path of shortest combinatorial length, that is, a unique path which traverses a minimal number of edges. It is characterised by the fact that it is *reduced*, which means that it does not contain any sub-path of the form  $ee^{-1}$ . In the example above, none of the two paths are reduced, and the unique reduced path to which they are equivalent is  $e_0e_1$ . The equivalence relation thus defined preserves the endpoints and is compatible with concatenation. For all vertex  $v \in \mathbb{V}$ , the quotient of the set of loops  $\mathcal{L}_v(\mathbb{G})$  based at  $v$  by this equivalence relation becomes a group for the operation of concatenation. The unit element is the class of the constant loop at  $v$ . Equivalently, one can consider the set  $\mathcal{RL}_v(\mathbb{G})$  of reduced loops based at  $v$ , endowed with the operation of concatenation-reduction.

If  $v$  and  $w$  are two vertices of  $\mathbb{G}$ , and if  $c$  is a path in  $\mathbb{G}$  which joins  $v$  to  $w$ , then the mapping  $l \mapsto clc^{-1}$  induces an isomorphism of groups between  $\mathcal{RL}_w(\mathbb{G})$  and  $\mathcal{RL}_v(\mathbb{G})$ . A crucial fact is that for all  $v \in \mathbb{V}$ , the group  $\mathcal{RL}_v(\mathbb{G})$  is a free group of rank equal to the number of bounded faces of  $\mathbb{G}$ . It is a very useful fact for the purposes of the discrete Yang-Mills theory that this group possesses bases, indeed many bases, which are naturally indexed by the bounded faces of  $\mathbb{G}$ . Let us explain how to associate such a basis to each choice of a spanning tree of  $\mathbb{G}$ , or rather of its dual graph.

The *dual graph* of  $\mathbb{G}$  is a graph which is not exactly of the same nature as  $\mathbb{G}$  insofar its edges are not concretely embedded in the plane. Technically, it is a rooted ribbon graph. It is a quadruple  $\widehat{\mathbb{G}} = (\widehat{\mathbb{V}}, \widehat{\mathbb{E}}, s, t)$ , where  $\widehat{\mathbb{V}} = \mathbb{F}$  is the set of faces of  $\mathbb{G}$ ,  $\widehat{\mathbb{E}}$  is the set of triples  $(F_0, e, F_1) \in \mathbb{F} \times \mathbb{E} \times \mathbb{F}$  such that the edge  $e$  bounds  $F_0$  positively and  $F_1$  negatively, and  $s, t : \widehat{\mathbb{E}} \rightarrow \widehat{\mathbb{V}}$  are the two mappings defined by  $s(F_0, e, F_1) = F_0$  and  $t(F_0, e, F_1) = F_1$ . We call respectively the elements of  $\widehat{\mathbb{V}}$  and  $\widehat{\mathbb{E}}$  *dual vertices* and *dual edges*. Each edge  $e$  appears in a unique dual edge  $(F_0, e, F_1)$  which we denote by  $\widehat{e}$ . We define the inverse of the dual edge  $\widehat{e} = (F_0, e, F_1)$  by  $\widehat{e}^{-1} = (F_1, e^{-1}, F_0)$ , so that  $\widehat{e}^{-1} = \widehat{e}^{-1}$ . Observe that the equality  $F_0 = F_1$  is not excluded.

The unbounded face of  $\mathbb{G}$  determines a particular dual vertex which we denote by  $\widehat{v}_\infty$  and call the dual root. For each dual vertex  $\widehat{v} \in \widehat{\mathbb{V}}$ , there is a cyclic order on the set  $\{\widehat{e} \in \widehat{\mathbb{E}} : s(\widehat{e}) = \widehat{v}\}$  of dual edges issued from this dual vertex, which is the order of the corresponding edges in the positively oriented boundary of the corresponding face of  $\mathbb{G}$ . Note that the boundary of the unbounded face runs clockwise around the skeleton of the graph. Among all the dual edges issued from the dual root, we select one and, breaking the cyclic order, call it the first edge issued from  $\widehat{v}_\infty$  (see Figure 9).

In order to construct a basis of the group of loops, we choose a spanning tree of  $\widehat{\mathbb{G}}$ , that is, a subset  $\widehat{\mathbb{T}} \subset \widehat{\mathbb{E}}$  which is the set of edges, taken each with both orientations, of a connected subgraph of  $\widehat{\mathbb{G}}$  without cycle and which contains every vertex. The choice of the spanning tree  $\widehat{\mathbb{T}}$ , the order structure at each vertex and the total order on the edges issued from the dual root determine for each vertex of  $\widehat{\mathbb{T}}$ , that is, for each face of  $\mathbb{G}$ , a label which is a word of integers : the empty word for the dual root, a single integer for its neighbours, two integers for their neighbours other than the dual root, and so on. Thus, listing the dual vertices in the lexicographical order amounts to listing them in the order of first discovery by an explorer of the tree who starts from the root and walks along the tree, keeping it on its left hand side.

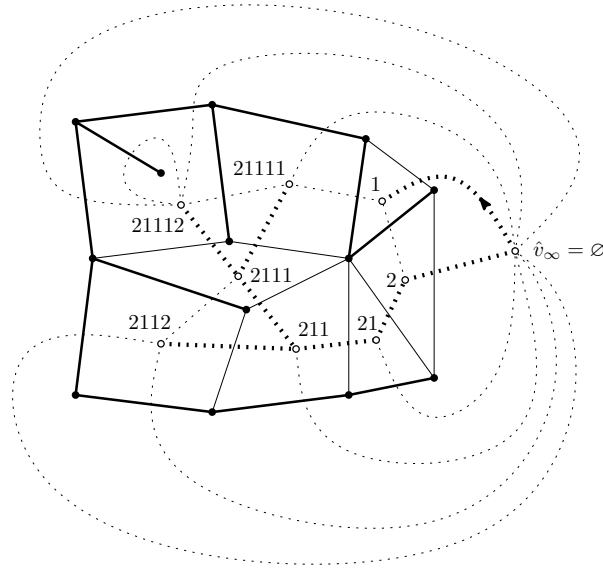


FIGURE 9. A graph and its dual. The white vertices are the dual vertices and the dotted lines are the dual edges. The edge carrying an arrow is the first edge issued from the root. The heavier edges are those of the spanning trees.

The set  $T = \{e : \hat{e} \notin \widehat{T}\}$  is a spanning tree of  $\mathbb{G}$  itself. Note that it is equivalent to choose  $\widehat{T}$  or  $T$  first, as each of them completely determines the other. For any two vertices  $v, w$  of  $\mathbb{G}$ , we denote by  $[v, w]_T$  the unique reduced path which goes from  $v$  to  $w$  using only edges of  $T$ .

Let  $v_0 \in \mathbb{V}$  be a vertex. To each bounded face of  $\mathbb{G}$  we associate a loop based at  $v_0$  as follows. Let  $F$  be a face, which we see as a dual vertex. Consider the dual edge  $\hat{e}$  of  $\widehat{T}$  which starts from this dual vertex in the direction of the dual root. Among all the vertices located on the boundary of  $F$ , we distinguish the starting point of the edge  $e$ , which is directly on the right of the dual edge  $\hat{e}$ . We define the loop  $\lambda_F$  as the loop obtained by reducing the loop which starts from  $v_0$ , goes along  $T$  to the distinguished vertex on the boundary of  $F$ , turns once positively around  $F$  and goes back to  $v_0$  along the same path (see Figure 10).

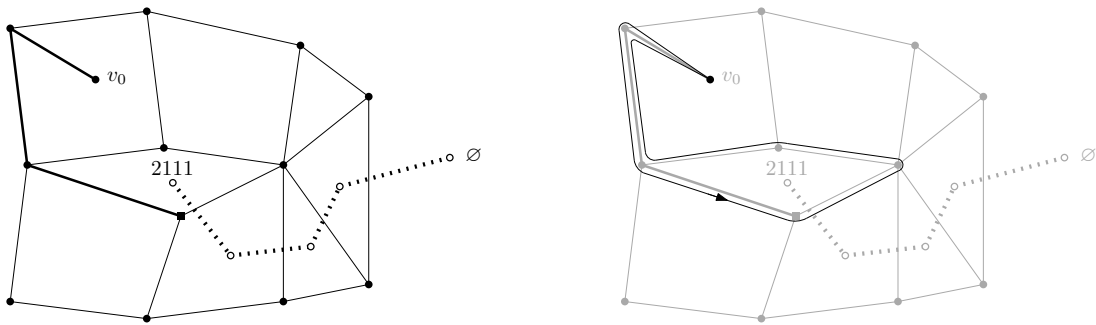


FIGURE 10. The basis element of  $RL_{v_0}(\mathbb{G})$  associated with the face 2111 of the graph of Figure 9. The distinguished vertex on the boundary of this face is indicated with a square.

The following result summarises and completes our discussion.

**Proposition 4.2.** *Let  $\mathbb{G}$  be a graph with dual graph  $\widehat{\mathbb{G}}$ .*

1. *Spanning trees of  $\mathbb{G}$  are in one-to-one correspondence with spanning trees of  $\widehat{\mathbb{G}}$ , by the formulas  $\mathbb{T} = \{e : \hat{e} \notin \widehat{\mathbb{T}}\}$  and  $\widehat{\mathbb{T}} = \{\hat{e} : e \notin \mathbb{T}\}$ .*

*Let  $v_0$  be a vertex of  $\mathbb{G}$ . Assume that an edge issued from the dual root has been chosen in  $\widehat{\mathbb{G}}$ .*

2. *For all spanning tree  $\mathbb{T}$  of  $\mathbb{G}$ , the collection of loops  $\Lambda_{\mathbb{T}} = \{\lambda_F : F \in \mathbb{F}^b\}$  constructed as above is a basis of the free group  $\mathbf{RL}_{v_0}(\mathbb{G})$ .*

3. *For any path  $c$  in  $\mathbb{G}$ , there exists a unique sequence of faces  $F_1, \dots, F_n \in \mathbb{F}^b$  and a unique sequence of signs  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ , of the same length, possibly empty and such that for all  $k \in \{1, \dots, n-1\}$  one has  $F_k \neq F_{k+1}$  or  $\varepsilon_k = \varepsilon_{k+1}$ , such that  $c$  is equivalent to the path  $[\underline{c}, v_0]_{\mathbb{T}} \lambda_{F_1}^{\varepsilon_1} \dots \lambda_{F_n}^{\varepsilon_n} [v_0, \bar{c}]_{\mathbb{T}}$ .*

We shall call *lassos* the loops of the form  $\lambda_F$ , and *lasso basis* associated to  $\mathbb{T}$  or to  $\widehat{\mathbb{T}}$  the basis  $\Lambda_{\mathbb{T}}$ , which we also denote by  $\Lambda_{\widehat{\mathbb{T}}}$ .

We are thus able to write any path as a word in a certain alphabet of elementary paths. The number of these elementary paths is the number of edges of a spanning tree plus the number of bounded faces. Let us denote by  $v, e, f$  the numbers of vertices, unoriented edges and bounded faces of  $\mathbb{G}$ . Here, by the number of unoriented edges, we mean the half of the number of elements of  $\mathbb{E}$ . There are  $v - 1$  unoriented edges in  $\mathbb{T}$ , so that the number of elementary paths is  $v + f - 1$ . On the other hand, Euler's relation for  $\mathbb{G}$  reads  $v - e + f = 1$ , hence  $v + f - 1$  is the number of edges of  $\mathbb{G}$ . Let us choose an orientation  $\mathbb{E}^+$  of the edges of  $\mathbb{G}$  and set  $\mathbb{T}^+ = \mathbb{T} \cap \mathbb{E}^+$ . We can add a new identification

$$(75) \quad \mathcal{M}(\mathbf{P}(\mathbb{G}), G) \simeq G^{\mathbb{E}^+} \simeq G^{\Lambda_{\mathbb{T}}} \times G^{\mathbb{T}^+}$$

to the row (73). The last isomorphism may be fairly complicated and it essentially encodes the geometry of the graph (see Figure 11 for an example).

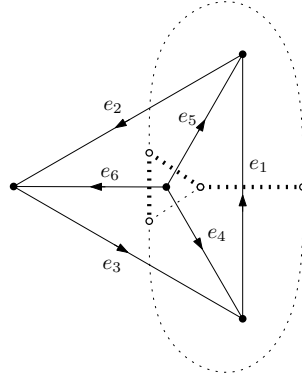


FIGURE 11. In this example, the last identification of (75) is the following :

$$(g_1, g_2, g_3, g_4, g_5, g_6) \mapsto (g_4^{-1} g_5^{-1} g_1, g_4^{-1} g_6^{-1} g_2 g_5 g_6, g_3 g_6 g_4, g_2, g_3, g_4).$$

It may be an instructive exercise to compute its inverse.

The interest of the last description of the configuration space of the discrete Yang-Mills theory is that it allows a very pleasant description of the probability measure  $\mathbf{YM}_{\mathbb{G}}^{\mathbb{G}}$ .



**Proposition 4.3.** *Under the identification  $\mathcal{M}(\mathbb{P}(\mathbb{G}), G) \simeq G^{\Lambda_{\mathbb{T}}} \times G^{\mathbb{T}^+}$ , the discrete Yang-Mills measure  $\mathbf{YM}_{\mathbb{G}}^{\mathbb{G}}$  corresponds to the measure*

$$\bigotimes_{F \in \mathbb{F}^b} Q_{|F|}(g) dg \otimes \bigotimes_{e \in \mathbb{T}^+} dg.$$

*In other words, under  $\mathbf{YM}_{\mathbb{G}}^{\mathbb{G}}$ , the random variables  $\{H_{\lambda_F} : F \in \mathbb{F}^b\} \cup \{H_e : e \in \mathbb{T}^+\}$  are independent, each  $H_{\lambda_F}$  distributed according to the heat kernel measure at time  $|F|$  on  $G$  and each  $H_e$  distributed according to the Haar measure on  $G$ .*

With this description in hand, we can safely turn to the study of the large  $N$  limit of the Yang-Mills field.

## 5. THE MASTER FIELD ON THE PLANE

As announced at the beginning of the previous section, we turn to the proof of the main result of the second part of this work, indeed the main motivation for this whole work. Our goal is to describe the large  $N$  limit of the Yang-Mills field with structure group  $U(N, \mathbb{K})$  for  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ .

The study of this limit follows the construction of the field itself. We shall start by applying the results of Section 2 to the discrete theory, on a graph ; then take an easy step and assemble the results for a large family of graphs in order to be able to treat all piecewise affine loops at once ; and finally, apply the results of Section 3 in order to tackle the approximation procedure involved in the construction of the Yang-Mills field, and succeed in obtaining the limit for all rectifiable loops.

**5.1. Large  $N$  limit of the Yang-Mills field on a graph.** Let us choose one of the three division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  and denote it by  $\mathbb{K}$ , as we did in the first part. For each  $N \geq 1$ , let us consider the Yang-Mills field on  $\mathbb{R}^2$  with structure group  $U(N, \mathbb{K})$ , associated with the scalar product given by (7). We denote by  $(H_{N,c}^{\mathbb{K}})_{c \in \mathbb{P}(\mathbb{R}^2)}$  the corresponding process.

For each  $N \geq 1$ , the random variables  $(H_{N,c}^{\mathbb{K}})_{c \in \mathbb{P}(\mathbb{G})}$  form a family of non-commutative random variables in the non-commutative probability space  $(L^\infty(\mathcal{C}_{U(N,\mathbb{K})}^{\mathbb{G}}, \mathbf{YM}_{U(N,\mathbb{K})}^{\mathbb{G}}) \otimes M_N(\mathbb{K}), \mathbb{E} \otimes \text{tr})$ , where  $\text{tr}$  must be replaced by  $\Re \text{tr}$  when  $\mathbb{K} = \mathbb{H}$ . When  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{H}$ , this is a real non-commutative probability space, in the sense described just before the statement of Theorem 3.2. The convergence result in the discrete setting is the following. Recall that a unitary element  $u$  of a non-commutative probability space  $(\mathcal{A}, \tau)$  is said to be a Haar unitary if  $\tau(u^n) = \delta_{n,0}$  for all  $n \in \mathbb{Z}$ .

**Theorem 5.1.** *Let  $\mathbb{G}$  be a graph. The collection of random matrices  $\{H_{N,c}^{\mathbb{K}} : c \in \mathbb{P}(\mathbb{G})\}$  has a limit in non-commutative distribution as  $N$  tends to infinity. This limit can be described as follows.*

*Let  $v_0$  be a vertex of  $\mathbb{G}$ . Let  $\mathbb{T}$  be a spanning tree of  $\mathbb{G}$  and  $\{\lambda_F : F \in \mathbb{F}^b\}$  the corresponding lasso basis of  $\mathbf{RL}_{v_0}(\mathbb{G})$ . Let  $(\mathcal{A}, \tau)$  be a non-commutative probability space such that  $\mathcal{A}$  contains a free family of unitary elements  $\{h_{\lambda_F} : F \in \mathbb{F}^b\} \cup \{h_e : e \in \mathbb{T}^+\}$  such that for each  $F \in \mathbb{F}^b$ , the distribution of  $h_{\lambda_F}$  is  $\nu_{|F|}$  and for each  $e \in \mathbb{T}^+$ , the element  $h_e$  is a Haar unitary. Then, as  $N$  tends to infinity, the random variables  $\{H_{N,\lambda_F}^{\mathbb{K}} : F \in \mathbb{F}^b\} \cup \{H_{N,e}^{\mathbb{K}} : e \in \mathbb{T}^+\}$  converge in non-commutative distribution to  $\{h_{\lambda_F} : F \in \mathbb{F}^b\} \cup \{h_e : e \in \mathbb{T}^+\}$ .*

*Proof.* The second part of the theorem is a direct consequence of Proposition 4.3 and the results of asymptotic freeness which we recalled in Section 3.1. The first part of the theorem follows

since, by the third part of Proposition 4.2, any path in  $\mathbb{G}$  can be written as a word in the edges of  $\mathbb{T}^+$  and the lassos  $\lambda_F, F \in \mathbb{F}^b$ .  $\square$

Theorem 5.1 provides us with the distribution of a family  $\{h_{\lambda_F} : F \in \mathbb{F}^b\} \cup \{h_e : e \in \mathbb{T}^+\}$  of non-commutative random variables, from which we can form a family  $\{h_c : c \in \mathbb{P}(\mathbb{G})\}$  by using the fact that the set of equivalence classes of paths in  $\mathbb{P}(\mathbb{G})$  is generated as a groupoid by  $\Lambda_{\mathbb{T}} \cup \mathbb{T}^+$ . Concretely, this means that each  $h_c$  is a certain word in the variables  $\{h_{\lambda_F} : F \in \mathbb{F}^b\} \cup \{h_e : e \in \mathbb{T}^+\}$ . The canonical space of the family  $\{h_c : c \in \mathbb{P}(\mathbb{G})\}$  is, as explained in Section 3.1, the algebra of non-commutative polynomials  $\mathbb{C}\langle X_c, X_c^{-1} : c \in \mathbb{P}(\mathbb{G}) \rangle$  endowed with the linear form

$$(76) \quad \tau(X_{c_1}^{\varepsilon_1} \dots X_{c_n}^{\varepsilon_n}) = \lim_{N \rightarrow \infty} \mathbb{E} \left[ \text{tr} \left( (H_{N,c_1}^{\mathbb{C}})^{\varepsilon_1} \dots (H_{N,c_n}^{\mathbb{C}})^{\varepsilon_n} \right) \right],$$

for all  $c_1, \dots, c_n \in \mathbb{P}(\mathbb{G})$  and all  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ .

In proceeding this way, we are however oblivious of the multiplicativity of the family  $\{h_c : c \in \mathbb{P}(\mathbb{G})\}$ , that is, of the fact that  $h_{c_1 c_2} = h_{c_2} h_{c_1}$  whenever this makes sense. To the cost of reducing the family of paths to the set of loops based at a certain vertex, or better, to the group of reduced loops based at this vertex, we can gain a much nicer formulation. This restriction on the set of paths is not a severe one, since  $h_c$  is always a Haar unitary if  $c$  has distinct endpoints, and the collection  $\{h_l : l \in \text{RL}_{v_1}(\mathbb{G})\}$  is conjugated to the collection  $\{h_l : l \in \text{RL}_{v_0}(\mathbb{G})\}$  for all vertices  $v_0, v_1$ .

Let us choose a vertex  $v_0$  and consider the complex unital algebra  $\mathbb{C}[\text{RL}_{v_0}(\mathbb{G})]$  endowed with the usual involution for the algebra of a group, namely the involution given by

$$\left( \sum_{l \in \text{RL}_{v_0}(\mathbb{G})} \lambda_l l \right)^* = \sum_{l \in \text{RL}_{v_0}(\mathbb{G})} \bar{\lambda}_l l^{-1}.$$

For each  $N \geq 1$  and each  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ , let us define a linear form  $\Phi_N^{\mathbb{G}, \mathbb{K}}$  on  $\mathbb{C}[\text{RL}_{v_0}(\mathbb{G})]$  by setting, for all  $l \in \text{RL}_{v_0}(\mathbb{G})$ ,

$$\Phi_N^{\mathbb{G}, \mathbb{K}}(l) = \mathbb{E} \left[ \text{tr} \left( H_{N,l}^{\mathbb{K}} \right) \right],$$

with  $\text{tr}$  replaced by  $\Re \text{tr}$  if  $\mathbb{K} = \mathbb{H}$ . Finally, let us define the collection  $\{h_l : l \in \text{L}_{v_0}(\mathbb{G})\}$  of elements of  $\mathbb{C}[\text{RL}_{v_0}(\mathbb{G})]$  by letting  $h_l$  be equal to the unique reduced loop equivalent to  $l$ .

We can reformulate the convergence expressed by Theorem 5.1 as follows.

**Proposition 5.2.** *Let  $\mathbb{G}$  be a graph. Let  $v_0$  be a vertex of  $\mathbb{G}$ . On the complex unital involutive algebra  $\mathbb{C}[\text{RL}_{v_0}(\mathbb{G})]$ , the sequence of linear forms  $(\Phi_N^{\mathbb{G}, \mathbb{K}})_{N \geq 1}$  converges pointwise to a linear form  $\Phi^{\mathbb{G}}$  which is a state.*

*Moreover, as  $N$  tends to infinity, and regardless of the value of  $\mathbb{K}$ , the collection of random matrices  $(H_{N,l}^{\mathbb{K}})_{l \in \text{L}_{v_0}(\mathbb{G})}$  converges in non-commutative distribution to the family  $(h_l)_{l \in \text{L}_{v_0}(\mathbb{G})}$ .*

*Proof.* Both assertions are straightforward consequences of Theorem 5.1. For the second, we need only add that for all  $\mathbb{K}$  and all  $N \geq 1$ , and all loops  $l_1, l_2$  which are equivalent, one has the equality of random variables  $H_{N,l_1}^{\mathbb{K}} = H_{N,l_2}^{\mathbb{K}}$ .  $\square$

The state  $\Phi^{\mathbb{G}}$  on the involutive algebra  $\mathbb{C}[\text{RL}_{v_0}(\mathbb{G})]$  is the discrete counterpart of what we shall call the master field on the plane.

**5.2. Large  $N$  limit for piecewise affine loops.** Having understood the large  $N$  limit of the theory on a graph, it is easy to go one step beyond and to consider several graphs at once. As in the construction of the Yang-Mills field itself, we can however not consider all graphs at once but we must restrict ourselves to a class of graphs where any two graphs are dominated in the partial order of fineness by a third one. Graphs with piecewise affine edges are such a class.

Consider two graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  such that  $\mathbb{G}_2$  is finer than  $\mathbb{G}_1$ . Let  $v_0$  be a vertex of  $\mathbb{G}_1$ , hence of  $\mathbb{G}_2$ . The inclusion  $L(\mathbb{G}_1) \subset L(\mathbb{G}_2)$  is of course compatible with the equivalence of paths, for two loops in  $\mathbb{G}_1$  which are equivalent in  $\mathbb{G}_1$  are also equivalent in  $\mathbb{G}_2$ .

**Lemma 5.3.** *Consider two graphs  $\mathbb{G}_1$  and  $\mathbb{G}_2$  such that  $\mathbb{G}_2$  is finer than  $\mathbb{G}_1$ . If two elements of  $L(\mathbb{G}_1)$  are equivalent in  $L(\mathbb{G}_2)$ , then they are equivalent in  $L(\mathbb{G}_1)$ .*

*Proof.* It suffices to show that if a loop traced in  $\mathbb{G}_1$  is not reduced in  $L(\mathbb{G}_2)$ , it is not reduced in  $L(\mathbb{G}_1)$  either. Let  $l \in L(\mathbb{G}_1)$  which is not reduced in  $L(\mathbb{G}_2)$ . Thus, the decomposition of  $l$  as a product of edges contains a sequence of the form  $ee^{-1}$ . The final point of  $e$  must be a vertex of  $\mathbb{G}_1$ , otherwise  $l$  could not backtrack at this point. Hence,  $e$  is the last segment in  $\mathbb{G}_2$  of a well-determined edge  $e'$  of  $\mathbb{G}_1$ , and the decomposition of  $l$  as a product of edges of  $\mathbb{G}_1$  must contain the sequence  $e'e'^{-1}$ .  $\square$

According to this lemma, we have an inclusion  $RL_{v_0}(\mathbb{G}_1) \subset RL_{v_0}(\mathbb{G}_2)$ . The invariance under refinement of the Yang-Mills measure can be expressed by saying that this inclusion is compatible with the states  $\Phi_N^{\mathbb{G}_1, \mathbb{K}}$  and  $\Phi_N^{\mathbb{G}_2, \mathbb{K}}$ .

Note that a consequence of Lemma 5.3 is that any loop which can be traced in a graph has a unique reduced representant, which is defined independently of the choice of a graph in which the loop is traced. This is in particular the case for piecewise affine loops.

The collection of graphs with piecewise affine edges and which have the origin as a vertex, ordered by the relation of fineness, is a directed set. The direct limit of the groups  $RL_0(\mathbb{G})$  along this directed set is naturally isomorphic to the group  $\text{RAff}_0(\mathbb{R}^2)$  of piecewise affine reduced loops based at the origin, endowed with the operation of concatenation-reduction.

Thus, the direct limit of the family of non-commutative probability spaces  $(\mathbb{C}[RL_0(\mathbb{G})], \Phi_N^{\mathbb{G}, \mathbb{K}})$  along the set of graphs with piecewise affine edges and the origin as a vertex is the non-commutative probability space  $(\mathbb{C}[\text{RAff}_0(\mathbb{R}^2)], \Phi_N^{\text{Aff}, \mathbb{K}})$ , where  $\Phi_N^{\text{Aff}, \mathbb{K}}$  is defined by

$$\forall l \in \text{RAff}_0(\mathbb{G}), \Phi_N^{\text{Aff}, \mathbb{K}}(l) = \mathbb{E} \left[ \text{tr} \left( H_{N,l}^{\mathbb{K}} \right) \right],$$

where as usual  $\text{tr}$  must be replaced by  $\Re \text{tr}$  if  $\mathbb{K} = \mathbb{H}$ .

The following result is still a consequence of Theorem 5.1. We denote by  $\text{Aff}_0(\mathbb{R}^2)$  the set of piecewise affine loops on  $\mathbb{R}^2$  based at the origin.

**Theorem 5.4.** *On the algebra  $\mathbb{C}[\text{RAff}_0(\mathbb{R}^2)]$ , regardless of  $\mathbb{K}$ , the sequence of states  $(\Phi_N^{\text{Aff}, \mathbb{K}})_{N \geq 1}$  converges pointwise to a state  $\Phi^{\text{Aff}}$ .*

*Define the collection  $\{h_l : l \in \text{Aff}_0(\mathbb{R}^2)\}$  of random variables on  $(\mathbb{C}[\text{RAff}_0(\mathbb{R}^2)], \Phi^{\text{Aff}})$  by setting  $h_l$  equal to the unique reduced loop equivalent to  $l$ , seen as an element of  $\mathbb{C}[\text{RAff}_0(\mathbb{R}^2)]$ . As  $N$  tends to infinity, and regardless of the value of  $\mathbb{K}$ , the collection of random matrices  $(H_{N,l}^{\mathbb{K}})_{l \in \text{Aff}_0(\mathbb{R}^2)}$  converges in non-commutative distribution to the family  $(h_l)_{l \in \text{Aff}_0(\mathbb{R}^2)}$ .*

In their recent work [1], M. Anshelevitch and A. Sengupta prove a theorem similar to Theorem 5.4, and provide a model for the limiting distribution which is in a sense more natural than ours. The authors consider a slightly different class of paths, which they call *basic loops*, and which are finite concatenations of radial segments and paths which can be parametrised in polar coordinates under the form  $\theta \mapsto (r(\theta), \theta)$ . In the context of axial gauge fixing in which they work, this class of paths plays essentially the role of our class of piecewise affine edges. Using free stochastic calculus, the authors achieve the construction of a free process indexed by the set of basic loops on the algebra of bounded operators on the full Fock space on  $L^2(\mathbb{R}^2) \otimes \mathfrak{u}(N)$ . This is in very suggestive agreement with the informal description of the Yang-Mills measure by means of a functional integral, which through an appropriate choice of gauge, becomes a Gaussian measure

on the Hilbert space  $L^2(\mathbb{R}^2) \otimes \mathfrak{u}(N)$ . The transition from a commutative Gaussian setting to a non-commutative semi-circular setting is thus naturally reflected in the transition from the symmetric Fock space to the full Fock space, although the former usually stays hidden behind the probabilistically more amenable white noise. M. Anshelevitch and A. Sengupta worked with the unitary group, but given the results of Section 2 of the present work, it should be possible to extend their results to the orthogonal and symplectic cases.

**5.3. Uniformity of the convergence towards the master field (statement).** The proof of Theorem 5.1, from which we have now exhausted the algebraic consequences, consists in blending the notion of freeness with a combinatorial description of the set of paths in a graph. We are now going to enter more deeply into the convergence that it expresses, in order to prove that this convergence has a property of uniformity on sets of paths with bounded length. This is the crucial result which will allow us to take the last step in the construction of the master field and extend the state  $\Phi^{\text{Aff}}$  to an algebra constructed from all rectifiable loops.

Not all Lipschitz continuous paths belong to  $\mathcal{P}(\mathbb{G})$  for some graph  $\mathbb{G}$ . Indeed, the complement of the range of a path in a graph must have finitely many connected components and this is not always the case for a path with finite length. On the other hand, any piecewise affine path belongs to  $\mathcal{P}(\mathbb{G})$  for some graph  $\mathbb{G}$ . Moreover, the set of piecewise affine paths is large enough to be dense in the set of all paths for the topology that we have introduced in Section 4.2.

It will however be convenient to consider a slightly more restricted class of loops. Let us call *elementary loop* a loop which can be traced in a graph with affine edges, using at most once an edge of each pair  $\{e, e^{-1}\}$ . We denote by  $\text{EL}(\mathbb{R}^2)$  as the set of elementary loops. Elementary loops are in particular piecewise affine, and reduced in any graph where they are traced.

Recall that we denote the Euclidean length of a loop  $l$  by  $\ell(l)$ . Our main result of uniformity is the following.

**Theorem 5.5.** *Let  $l$  be an elementary loop. Let  $\mathbb{G}$  a graph such that  $l \in \mathcal{L}(\mathbb{G})$ . Then, for all  $\mathbb{K}$  and all  $N \geq 1$ , one has the inequalities*

$$\left| \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})] - \lim_{N \rightarrow \infty} \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})] \right| = \left| \Phi_N^{\text{Aff}, \mathbb{K}}(l) - \Phi^{\text{Aff}}(l) \right| \leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2}$$

and

$$\text{Var}(\text{tr}(H_{N,l}^{\mathbb{K}})) \leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2},$$

where  $\text{tr}$  must be replaced by  $\Re \text{tr}$  if  $\mathbb{K} = \mathbb{H}$ . Moreover, when  $\mathbb{K} = \mathbb{C}$ , the inequalities hold with the factor  $\frac{1}{N}$  replaced by  $\frac{1}{N^2}$ .

This result will be deduced from the main result of the first part of the present work, which is Theorem 3.3. We know from Proposition 4.2 that a loop in a graph can be expressed as a word in certain paths and loops in this graph, essentially one loop for each bounded face. To each of these faces is associated an independent random element of  $U(N, \mathbb{K})$  and we are thus in a situation where Theorem 3.3 provides us with an explicit estimate. However, we need to bound the Amperean area of the word involved in terms of the loop which we consider. The main step of the proof of Theorem 5.5 consists in proving that the length of the loop allows us to control the Amperean area of the word.

**5.4. Maximal Amperean area.** In this paragraph, we prove a quantitative version of Proposition 4.2 by establishing that a loop in a graph can be written as a word in the facial lassos of this graph in such a way that the Amperean area of the word is controlled by the length of the loop (see Proposition 5.8). To this end, we introduce a third quantity which we call the maximal Amperean area of the loop, which allows us to relate the two quantities which we want

to compare. We start by defining the maximal Amperean area of a loop and comparing it to its length.

Let  $l$  be a loop traced in a graph  $\mathbb{G}$ . Consider the edges of  $\mathbb{G}$  traversed by  $l$ . Since the range of  $l$  is connected, these edges and their inverses form a graph. Indeed, looking back in Section 4.1 at the definition of a graph, we see that the only condition which is not obviously satisfied is the topological condition on the faces, which was then called assumption 4, but of which we mentioned already that it is equivalent to the fact that the skeleton of the graph is connected. We denote this graph by  $\mathbb{G}(l)$ . Let us make the additional assumption that  $l$  is an elementary loop. Then each vertex of  $\mathbb{G}(l)$  has even degree, that is, each vertex is adjacent to an even number of pairs  $\{e, e^{-1}\}$  of edges. In what follows, we will often identify the loop  $l$  with its range.

Recall that the winding number of  $l$  is the function  $\mathfrak{n}_l : \mathbb{R}^2 \setminus l \rightarrow \mathbb{Z}$  defined on the complement of  $l$  and which to each point  $x \in \mathbb{R}^2$  associates the index of  $l$  with respect to  $x$ . It is integer-valued, locally constant outside the range of  $l$ , and it has compact support. The Banchoff-Pohl inequality (see [3]), which generalises the isoperimetric inequality in this context, reads

$$\int_{\mathbb{R}^2} \mathfrak{n}_l(x)^2 dx \leq \frac{1}{4\pi} \ell(l)^2.$$

The left-hand side of this inequality is sometimes called the Amperean area of the loop  $l$ , and we shall denote it by  $\mathbf{A}(l)$ . It can be understood as the energy of the magnetic field induced by a unit current flowing along  $l$ .

Let us introduce another integer-valued function  $\bar{\mathfrak{n}}_l : \mathbb{R}^2 \setminus l \rightarrow \mathbb{N}$ , this time with non-negative values. In words, for all  $x \in \mathbb{R}^2 \setminus l$ ,  $\bar{\mathfrak{n}}_l(x)$  is the minimal number of crossings between a path which joins  $x$  to infinity and the loop  $l$ . Formally, let us consider the dual graph of the graph  $\mathbb{G}(l)$ . Recall that  $\widehat{\mathbb{G}(l)}$  has a distinguished vertex  $\hat{v}_\infty$  which corresponds to the unbounded face of  $\mathbb{R}^2 \setminus l$ . We define, for all  $x \in \mathbb{R}^2 \setminus l$ , the dual vertex  $\hat{v}(x)$  as the vertex of  $\widehat{\mathbb{G}(l)}$  corresponding to the face of  $\mathbb{G}$  which contains  $x$ . Finally, we denote by  $\hat{d}$  the graph distance in  $\widehat{\mathbb{G}(l)}$ . We define the function  $\bar{\mathfrak{n}}_l$  by setting

$$\forall x \in \mathbb{R}^2 \setminus l, \bar{\mathfrak{n}}_l(x) = \hat{d}(\hat{v}(x), \hat{v}_\infty).$$

We call the function  $\bar{\mathfrak{n}}_l$  the maximal winding number of  $l$ . Note that it depends on  $l$  only through  $\mathbb{G}(l)$ , and in particular not on the direction in which  $l$  traverses the edges of  $\mathbb{G}(l)$ . The maximal Amperean area of  $l$ , denoted by  $\bar{\mathbf{A}}(l)$ , is defined by replacing  $\mathfrak{n}_l$  by  $\bar{\mathfrak{n}}_l$  in the definition of the Amperean area:

$$(77) \quad \bar{\mathbf{A}}(l) = \int_{\mathbb{R}^2} \bar{\mathfrak{n}}_l(x)^2 dx.$$

For our purposes, the first main property of the maximal Amperean area is the following.

**Proposition 5.6.** *The maximal Amperean area of an elementary loop satisfies the Banchoff-Pohl inequality. By this we mean that for all  $l \in \mathbf{EL}(\mathbb{R}^2)$ ,*

$$\bar{\mathbf{A}}(l) \leq \frac{1}{4\pi} \ell(l)^2.$$

This proposition follows at once from the following result, which explains the name of the maximal Amperean area.

**Lemma 5.7.** *Let  $l \in \mathbf{EL}(\mathbb{R}^2)$  be an elementary loop.*

1. *The inequality  $|\mathfrak{n}_l| \leq \bar{\mathfrak{n}}_l$  holds on  $\mathbb{R}^2 \setminus l$ . In particular,  $\mathbf{A}(l) \leq \bar{\mathbf{A}}(l)$ .*
2. *There exists  $\bar{l} \in \mathbf{EL}(\mathbb{R}^2)$  with the same range and length as  $l$  such that  $\mathfrak{n}_{\bar{l}} = \bar{\mathfrak{n}}_l$ .*

That this lemma implies Proposition 5.6 is straightforward. Indeed, if  $l$  is an elementary loop and  $\bar{l}$  is given by the second assertion of this lemma, then  $\bar{\mathbf{A}}(l) = \mathbf{A}(\bar{l}) \leq \frac{1}{4\pi} \ell(\bar{l})^2 = \frac{1}{4\pi} \ell(l)^2$ .

*Proof.* 1. Consider an edge of  $\mathbb{G}(l)$ . Since the faces located on either side of this edge correspond to neighbouring vertices in the dual graph  $\widehat{\mathbb{G}(l)}$ , the values of  $\bar{n}_l = \hat{d}(\cdot, \hat{v}_\infty)$  on both sides of this edge are equal or differ by 1. Let us start by proving that they cannot be equal.

Since the loop  $l$  is elementary, it traverses each edge exactly once. Hence, the value of the winding number  $n_l$  changes by 1 or  $-1$  when one crosses an edge. The set of vertices of the graph  $\widehat{\mathbb{G}(l)}$  can be partitioned according to the parity of the value of  $n_l$  and we shall speak of even and odd vertices. This partition is a bipartition in the sense that any two neighbours have different parities. The dual root  $\hat{v}_\infty$  is an even vertex. Hence, the parity of any vertex  $\hat{v}$  is that of  $\hat{d}(\hat{v}, \hat{v}_\infty)$ . If two neighbours were to have the same distance to  $\hat{v}_\infty$ , they would also have the same parity and this is impossible. Hence,  $\bar{n}_l$  cannot take the same value on two faces which share a bounding edge.

Let us use this observation to prove the first assertion. Consider  $x \in \mathbb{R}^2 \setminus l$ . Choose a shortest path from  $\hat{v}_\infty$  to  $\hat{v}(x)$  in  $\widehat{\mathbb{G}(l)}$ . At each step of this path,  $n_l$  varies by  $\pm 1$ , and  $\bar{n}_l$  increases by 1. The conclusion follows immediately, as well as the second part of the first assertion.

2. We look for  $\bar{l}$  as a Eulerian circuit in  $\mathbb{G}(l)$ , that is, a cycle which traverses exactly once each edge. Let us start by proving that the direction in which each edge should be traversed by  $\bar{l}$  is determined by  $\bar{n}_l$ . Consider an edge of  $\mathbb{G}(l)$ . We have proved that the values of  $\bar{n}_l$  on both sides of this edge differ by 1. If  $n_{\bar{l}}$  is to be equal to  $\bar{n}_l$ , then  $\bar{l}$  must traverse this edge in such a way that the largest value of  $\bar{n}_l$  is on its left-hand side. Thus, each edge of  $\mathbb{G}(l)$  carries an orientation which is the direction in which  $\bar{l}$  must traverse it in order to have the desired properties. We claim that there are, at each vertex of  $\mathbb{G}(l)$ , as many incoming edges as there are outgoing ones. Indeed, the values of  $\bar{n}_l$  around each vertex read in cyclic order form a sequence of integers which jumps by 1 or  $-1$  and comes back to its initial point. Thus, there must be an equal number of rises and falls, which correspond respectively to incoming and outgoing edges.

We use now the classical fact that a directed graph in which each vertex has equal incoming and outgoing degrees carries a Eulerian circuit, that is, a loop which traverses each edge exactly once, and does so in the direction given by the orientation of the edge. We choose one of these circuits and call it  $\bar{l}$ . It is a loop with the same length as  $l$ . The functions  $n_{\bar{l}}$  and  $\bar{n}_l$  are both integer-valued, locally constant on the complement of  $l$ , equal to 0 at infinity, and both vary by 1 or  $-1$  in the same way across each edge of  $\mathbb{G}(l)$ . Hence, they are equal.  $\square$

We now turn to the main result of this section, which is the following.

**Proposition 5.8.** *Let  $l$  be an elementary loop. Let  $v_0$  be the base point of  $l$ . Let  $F_1, \dots, F_q$  be the bounded faces of  $\mathbb{G}(l)$ . Let  $t = (|F_1|, \dots, |F_q|)$  be the vector of the areas of these faces.*

*It is possible to choose a spanning tree  $\mathbb{T}$  of  $\mathbb{G}(l)$  in such a way that, denoting by  $\{\lambda_F : F \in \mathbb{F}^b(l)\}$  the lasso basis of  $\text{RL}_{v_0}(\mathbb{G}(l))$  determined by the choice of  $\mathbb{T}$ , and by  $w$  the unique element of the free group on  $q$  letters such that  $l = w(\lambda_{F_1}, \dots, \lambda_{F_q})$ , one has the inequality*

$$\bar{A}_t(w) \leq \frac{1}{4\pi} \ell(l)^2.$$

The crucial step in the proof of this result is given by the next proposition. Recall from (47) the definition of the Amperean area of  $w$  relative to  $t$ , denoted by  $\bar{A}_t(w)$ .

**Proposition 5.9.** *With the notation of Proposition 5.8, it is possible to choose the spanning tree  $\mathbb{T}$  of  $\mathbb{G}(l)$  in such a way that the inequality*

$$\bar{A}_t(w) \leq \bar{A}(l)$$

*holds.*

Let us for one minute take this assertion for granted and see how it implies Proposition 5.8.

*Proof of Proposition 5.8.* By Proposition 5.9, one can choose the basis of  $\text{RL}_{v_0}(\mathbb{G}(l))$  in such a way that  $\bar{A}_t(w) \leq \bar{A}(l)$ . On the other hand, by Proposition 5.6,  $\bar{A}(l) \leq \frac{1}{4\pi} \ell(l)^2$ .  $\square$

It remains to prove Proposition 5.9. Rather than choosing a spanning tree of  $\mathbb{G}(l)$ , we will in fact choose a spanning tree  $\widehat{\mathbb{T}}$  of the dual graph  $\widehat{\mathbb{G}(l)}$ . Given such a spanning tree  $\widehat{\mathbb{T}}$ , define, for each dual vertex  $\hat{v}$ , the integer  $\hat{d}_{\widehat{\mathbb{T}}}(\hat{v}, \hat{v}_\infty)$  as the graph distance between  $\hat{v}$  and  $\hat{v}_\infty$  in  $\widehat{\mathbb{T}}$ , also called the height of  $\hat{v}$  in  $\widehat{\mathbb{T}}$ . The inequality  $\hat{d}(\hat{v}, \hat{v}_\infty) \leq \hat{d}_{\widehat{\mathbb{T}}}(\hat{v}, \hat{v}_\infty)$  holds for all  $\hat{v}$ . We claim that  $\widehat{\mathbb{T}}$  can be chosen in such a way that it is an equality. This is in fact a perfectly general property of any finite graph.

**Lemma 5.10.** *There exists a spanning tree  $\widehat{\mathbb{T}}$  such that, for all dual vertex  $\hat{v}$  of  $\widehat{\mathbb{G}(l)}$ , the equality  $\hat{d}(\hat{v}, \hat{v}_\infty) = \hat{d}_{\widehat{\mathbb{T}}}(\hat{v}, \hat{v}_\infty)$  holds.*

*Proof.* Construct  $\widehat{\mathbb{T}}$  by choosing, for each dual vertex different from  $\hat{v}_\infty$ , one edge which joins this vertex to a vertex which is strictly closer from  $\hat{v}_\infty$ . The subgraph thus obtained is connected, for each vertex is joined to the dual root. It has one vertex more than it has edges, it is thus a tree. It is a spanning tree by construction.  $\square$

We can now finish the proof of Proposition 5.9.

*Proof of Proposition 5.9.* Let  $\widehat{\mathbb{T}}$  be a spanning tree of  $\widehat{\mathbb{G}(l)}$  such that  $\hat{d} = \hat{d}_{\widehat{\mathbb{T}}}$ . Such a spanning tree exists by Lemma 5.10. Let  $v_0$  be the base point of  $l$ . Let  $\{\lambda_F : F \in \mathbb{F}^b(l)\}$  be the basis of  $\text{RL}_{v_0}(\mathbb{G}(l))$  determined by  $\widehat{\mathbb{T}}$ , according to Proposition 4.2. Assume that the bounded faces of  $\mathbb{G}(l)$  are labelled  $\{F_1, \dots, F_q\}$ . Let  $w$  be the element of  $F_q$  such that  $l = w(\lambda_{F_1}, \dots, \lambda_{F_q})$ .

In order to bound the Amperean area of  $w$ , we need to understand as concretely as possible how the loop  $l$  is decomposed as a word in the lassos  $\lambda_{F_1}, \dots, \lambda_{F_q}$ . This decomposition can be computed in two steps. For the first step,  $\mathbb{T} = \{e : \hat{e} \notin \widehat{\mathbb{T}}\}$  be the spanning tree of  $\mathbb{G}(l)$  determined by  $\widehat{\mathbb{T}}$ . Let us record the ordered list of edges which are traversed by  $l$  and which do not belong to  $\mathbb{T}$ . We find a finite list  $e_1, \dots, e_n$  of edges, which contains neither repetitions nor two edges which are each other's inverse, because the loop  $l$  is elementary. The loop  $l$  is equivalent, in the sense of Section 4.3, to the loop

$$([v_0, \underline{e}_1]_{\mathbb{T}} e_1 [\overline{e}_1, v_0]_{\mathbb{T}}) \dots ([v_0, \underline{e}_n]_{\mathbb{T}} e_n [\overline{e}_n, v_0]_{\mathbb{T}}).$$

Although in some particular cases this may happen to be true, this decomposition is not in general the decomposition of  $l$  in product of facial lassos. Rather, each loop  $[v_0, \underline{e}_m]_{\mathbb{T}} e_m [\overline{e}_m, v_0]_{\mathbb{T}}$  is a product of some of the lassos  $\lambda_{F_1}, \dots, \lambda_{F_q}$ , which fortunately can be described easily. Each lasso appears at most once in the decomposition of  $[v_0, \underline{e}_m]_{\mathbb{T}} e_m [\overline{e}_m, v_0]_{\mathbb{T}}$ , possibly with an exponent  $-1$ , and the lassos which appear are exactly those which correspond to the faces of  $\mathbb{G}(l)$  which are separated in  $\widehat{\mathbb{T}}$  from the unbounded face by  $\hat{e}_m$ . These faces appear in an order which does not really matter to us, but which is the order of discovery for a certain exploration of the subtree of  $\widehat{\mathbb{T}}$  located above  $\hat{e}_m$ . This is explained in greater detail in [26, Sec. 2.4].

The essential fact is, again, that a lasso  $\lambda_F$  appears, with exponent 1 or  $-1$ , in the decomposition of the loop  $[v_0, \underline{e}_m]_{\mathbb{T}} e_m [\overline{e}_m, v_0]_{\mathbb{T}}$  if and only if the edge  $\hat{e}_m$  separates the faces  $F$  and  $F_\infty$  in the tree  $\widehat{\mathbb{T}}$ . Hence, in the decomposition of  $l$ , and taking possible cancellations into account, a given lasso  $\lambda_F$  appears at most as many times as there are edges separating  $F$  from  $F_\infty$  in the tree  $\widehat{\mathbb{T}}$ , that is, by definition,  $\hat{d}_{\widehat{\mathbb{T}}}(F, F_\infty)$  times. By the choice of  $\widehat{\mathbb{T}}$ , this number is equal to

$\hat{d}(F, F_\infty)$ , that is, by definition, the value of  $\bar{n}_l$  on  $F$ . Hence, the Amperian area of  $w$  satisfies  $\bar{A}_t(w) \leq \sum_{i=1}^q |F_i| \bar{n}_l(F_i)^2 = \bar{A}(l)$ .  $\square$

**5.5. Uniformity of the convergence towards the master field (proof).** We can finally prove the result of uniform convergence of the expected trace of the Yang-Mills field towards its limit on sets of elementary loops of bounded length.

*Proof of Theorem 5.5.* Let  $l$  be an elementary loop. We use the notation of Proposition 5.8. The law of the random variable  $H_{N,l}^{\mathbb{K}}$  does not depend on the graph in which it is computed. We can thus choose the graph  $\mathbb{G}(l)$  which the loop  $l$  itself forms. We choose a basis of  $\text{RL}_{v_0}(\mathbb{G}(l))$  in which the conclusion of Proposition 5.8 holds.

On one hand, thanks to Proposition 4.3, we have

$$\left| \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})] - \Phi^{\text{Aff}}(l) \right| = |\tau_t^{\mathbb{K},N}(w) - \tau_t(w)|.$$

It is understood, as usual, that in the quaternionic case,  $\text{tr}$  has to be replaced by  $\Re\text{tr}$ . On the other hand, by Proposition 3.3,

$$|\tau_t^{\mathbb{K},N}(w) - \tau_t(w)| \leq \frac{1}{N} \bar{A}_t(w) e^{\bar{A}_t(w)},$$

where  $\frac{1}{N}$  can be replaced by  $\frac{1}{N^2}$  if  $\mathbb{K} = \mathbb{C}$ .

The basis of  $\text{RL}_{v_0}(\mathbb{G}(l))$  has been chosen in such a way that  $\bar{A}_t(w) \leq \frac{1}{4\pi} \ell(l)^2$ . Thus, we find

$$\left| \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})] - \Phi^{\text{Aff}}(l) \right| \leq \frac{1}{4\pi N} \ell(l)^2 e^{\frac{1}{4\pi} \ell(l)^2},$$

which is slightly better than the expected inequality.

Let us turn to the second inequality. We are going to apply Proposition 3.4 with a permutation which is not a single cycle. Let  $r$  denote the length of the word  $w$ . Let us apply Proposition 3.4 to the word  $w^2$  and to the permutation  $\sigma = (1 \dots r)(r+1 \dots 2r)$ . With this notation, we have  $p_t^{\mathbb{K},N}(w^2, \sigma) = \mathbb{E}[\text{tr}(H_N(l))^2]$ , or  $\mathbb{E}[\Re\text{tr}(H_N(l))^2]$  if  $\mathbb{K} = \mathbb{H}$ . We also have  $p_t(w^2, \sigma) = \tau(h_l)^2$ . The Amperian area of  $w^2$  satisfies  $\bar{A}_t(w^2) = 4\bar{A}_t(w)$ . Hence,

$$\left| \mathbb{E} \left[ \text{tr}(H_{N,l}^{\mathbb{K}})^2 \right] - \Phi^{\text{Aff}}(l)^2 \right| \leq \frac{4}{N} \bar{A}_t(w) e^{4\bar{A}_t(w)} \leq \frac{1}{\pi N} \ell(l)^2 e^{\ell(l)^2}.$$

Since  $|\mathbb{E}[\text{tr}(H_{N,l})]| \leq 1$  and  $|\Phi^{\text{Aff}}(l)| \leq 1$ , we deduce from this inequality that

$$\begin{aligned} \left| \mathbb{E} \left[ \text{tr}(H_{N,l}^{\mathbb{K}})^2 \right] - \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})]^2 \right| &\leq \left| \mathbb{E} \left[ \text{tr}(H_{N,l}^{\mathbb{K}})^2 \right] - \Phi^{\text{Aff}}(l)^2 \right| + 2|\mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})] - \Phi^{\text{Aff}}(l)| \\ &\leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2} \left( \frac{1}{\pi} + \frac{1}{2\pi} \right) \\ &\leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2}, \end{aligned}$$

as desired. Here as in the first inequality, the factor  $\frac{1}{N}$  can be replaced by  $\frac{1}{N^2}$  when  $\mathbb{K} = \mathbb{C}$ .  $\square$

**5.6. The distribution of the master field.** The situation so far is the following. We have three classes of loops, each containing the next : loops, piecewise affine loops and elementary loops, so that  $\text{EL}(\mathbb{R}^2) \subset \text{Aff}(\mathbb{R}^2) \subset \text{L}(\mathbb{R}^2)$ . A division algebra  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  being fixed, we have for all  $N \geq 1$  a function  $\Phi_N^{\mathbb{K}} : \text{L}(\mathbb{R}^2) \rightarrow \mathbb{C}$  defined by

$$\forall l \in \text{L}(\mathbb{R}^2), \quad \Phi_N^{\mathbb{K}}(l) = \mathbb{E}[\text{tr}(H_{N,l}^{\mathbb{K}})],$$

where, as always,  $\text{tr}$  must be replaced by  $\Re\text{tr}$  if  $\mathbb{K} = \mathbb{H}$ . Note that, by Proposition 3.5, these functions are real-valued. By construction, they are bounded by 1.



We proved that, on  $\text{Aff}(\mathbb{R}^2)$ , the sequence  $(\Phi_N^{\mathbb{K}})_{N \geq 1}$  converges pointwise towards a function  $\Phi^{\text{Aff}}$ , which does not depend on  $\mathbb{K}$ . This is the convergence expressed by Proposition 5.4. Moreover, we have proved that for each positive  $L \geq 0$ , this convergence is uniform on the set of elementary loops whose length is smaller than or equal to  $L$ . This is a consequence of Theorem 5.5. From this and a straightforward observation, we will deduce that the convergence holds and is uniform on the whole space  $\mathbf{L}(\mathbb{R}^2)$ .

For all  $L \geq 0$ , set  $\mathbf{L}_L(\mathbb{R}^2) = \{l \in \mathbf{L}(\mathbb{R}^2) : \ell(l) \leq L\}$  and  $\mathbf{EL}_L(\mathbb{R}^2) = \mathbf{EL}(\mathbb{R}^2) \cap \mathbf{L}_L(\mathbb{R}^2)$ . We also set  $\mathbf{L}_{L-}(\mathbb{R}^2) = \{l \in \mathbf{L}(\mathbb{R}^2) : \ell(l) < L\}$ . In the following lemma, we consider, as always,  $\mathbf{L}(\mathbb{R}^2)$  endowed with the topology of the convergence in 1-variation with fixed endpoints.

**Lemma 5.11.** *1. For all  $L \geq 0$ , the closure of  $\mathbf{EL}_L(\mathbb{R}^2)$  is  $\mathbf{L}_L(\mathbb{R}^2)$ .*

*2. For all  $L \geq 0$ , the interior of  $\mathbf{L}_L(\mathbb{R}^2)$  is  $\mathbf{L}_{L-}(\mathbb{R}^2)$ .*

*3. The union of the sets  $\mathbf{L}_{L-}(\mathbb{R}^2)$  for  $L \geq 0$  is  $\mathbf{L}(\mathbb{R}^2)$ .*

*Proof.* 1. The piecewise affine interpolations of a given loop parametrized at constant speed converge with fixed endpoints, as the mesh of the interpolation tends to 0, to the loop itself. These piecewise affine interpolations have moreover a shorter length than the original loop. By contracting slightly the plane around the base point of the loop by an affine homothety, one makes sure that the approximations are strictly shorter than the original loop. Finally, any piecewise affine loop can be turned into an elementary loop by an arbitrarily small modification of its vertices, by making sure that they are all different.

2. For all  $L \geq 0$ , the set of loops whose length is strictly smaller than  $L$  is open, for the length is a continuous function on  $\mathbf{L}(\mathbb{R}^2)$ . Hence, the interior of  $\mathbf{L}_L(\mathbb{R}^2)$  contains  $\mathbf{L}_{L-}(\mathbb{R}^2)$ . To prove the other inclusion, consider a loop of length greater than or equal to  $L$ . Any neighbourhood of this loop contains its images by sufficiently small affine dilations around its basepoint, and these images have length strictly larger than  $L$ . Hence, no loop of length  $L$  or more belongs to the interior of  $\mathbf{L}_L(\mathbb{R}^2)$ .

3. This assertion barely deserves a proof.  $\square$

The next result summarises the application to the Yang-Mills field of the results which we obtained in the first part of this work.

**Theorem 5.12.** *The function  $\Phi^{\text{Aff}} : \text{Aff}(\mathbb{R}^2) \rightarrow \mathbb{R}$  can be extended in a unique way to a continuous function  $\Phi : \mathbf{L}(\mathbb{R}^2) \rightarrow \mathbb{R}$ , which for all  $L \geq 0$  is the uniform limit of the sequence of functions  $(\Phi_N^{\mathbb{K}})_{N \geq 1}$  on  $\mathbf{L}_L(\mathbb{R}^2)$ .*

*More precisely, for all loop  $l \in \mathbf{L}(\mathbb{R}^2)$  and all  $N \geq 1$ , the following inequalities hold:*

$$\begin{aligned} \left| \mathbb{E} \left[ \text{tr}(H_{N,l}^{\mathbb{K}}) \right] - \Phi(l) \right| &\leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2} \\ \text{Var} \left( \text{tr}(H_{N,l}^{\mathbb{K}}) \right) &\leq \frac{1}{N} \ell(l)^2 e^{\ell(l)^2}, \end{aligned}$$

where  $\text{tr}$  must be replaced by  $\Re \text{tr}$  if  $\mathbb{K} = \mathbb{H}$  and the factor  $\frac{1}{N}$  can be replaced by  $\frac{1}{N^2}$  if  $\mathbb{K} = \mathbb{C}$ .

*In particular the following convergence in probability holds:*

$$\text{tr}(H_{N,l}^{\mathbb{K}}) \xrightarrow[N \rightarrow \infty]{P} \Phi(l),$$

and in the case where  $\mathbb{K} = \mathbb{C}$ , this convergence is fast in the sense that the series

$$\sum_{N \geq 1} \mathbb{P}(|\text{tr}(H_{N,l}^{\mathbb{C}}) - \Phi(l)| > \varepsilon)$$

converges for all  $\varepsilon > 0$ .

*Proof.* For each  $L \geq 0$ , it follows from Theorem 5.5 that the sequence  $(\Phi_N^{\mathbb{K}})_{N \geq 1}$  of continuous functions on  $\mathsf{L}_L(\mathbb{R}^2)$  converges uniformly to  $\Phi^{\text{Aff}}$  on the subset  $\text{EL}_L(\mathbb{R}^2)$  of  $\mathsf{L}_L(\mathbb{R}^2)$ , which is dense by Lemma 5.11. Hence, the sequence  $(\Phi_N^{\mathbb{K}})_{N \geq 1}$  converges uniformly on  $\mathsf{L}_L(\mathbb{R}^2)$  to the unique continuous extension of  $\Phi^{\text{Aff}}$ . Since, by Lemma 5.11 again, the interiors of the subspaces  $\mathsf{L}_L(\mathbb{R}^2)$  with  $L \geq 0$  cover  $\mathsf{L}(\mathbb{R}^2)$ , the convergence holds pointwise on the whole space  $\mathsf{L}(\mathbb{R}^2)$  and the limiting function is continuous.

Let us turn to the second part of the theorem. Consider a loop  $l \in \mathsf{L}(\mathbb{R}^2)$ . Let  $(l_n)_{n \geq 1}$  be a sequence of elementary loops converging to  $l$  with fixed endpoints. By the second assertion of Theorem 4.1, and for all  $N \geq 1$ , the sequence  $(H_{N,l_n}^{\mathbb{K}})_{n \geq 1}$  converges in probability to  $H_{N,l}^{\mathbb{K}}$ . Hence, the same convergence holds for the traces and, since those are bounded, the convergence holds in  $L^2$ . Thus, we have

$$\text{Var}(\text{tr}(H_{N,l}^{\mathbb{K}})) = \lim_{n \rightarrow \infty} \text{Var}(\text{tr}(H_{N,l_n}^{\mathbb{K}})).$$

By the second assertion of Theorem 5.5, we have thus the inequality

$$\text{Var}(\text{tr}(H_{N,l}^{\mathbb{K}})) \leq \limsup_{n \rightarrow \infty} \frac{1}{N} \ell(l_n)^2 e^{\ell(l_n)^2} = \frac{1}{N} \ell(l)^2 e^{\ell(l)^2},$$

with  $\frac{1}{N}$  replaced by  $\frac{1}{N^2}$  if  $\mathbb{K} = \mathbb{C}$ . The rest of the theorem follows immediately.  $\square$

As a consequence of this theorem, we have the following result of convergence in non-commutative distribution, which extends Theorem 5.1 to the set of rectifiable loops based at the origin.

**Theorem 5.13.** *For all  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ , the collection  $(H_{N,l}^{\mathbb{K}})_{l \in \mathsf{L}_0(\mathbb{R}^2)}$  has a limit in non-commutative distribution as  $N$  tends to infinity, which does not depend on  $\mathbb{K}$ . This limiting distribution is that of the family  $(X_l)_{l \in \mathsf{L}_0(\mathbb{R}^2)}$  in the algebra  $\mathbb{C}\langle X_l, X_l^{-1} : l \in \mathsf{L}_0(\mathbb{R}^2) \rangle$  endowed with the involution  $X_l^* = X_l^{-1}$  and with the linear form  $\tau$  characterised by the equality*

$$\tau(X_{l_1}^{\varepsilon_1} \dots X_{l_n}^{\varepsilon_n}) = \Phi(l_n^{\varepsilon_n} \dots l_1^{\varepsilon_1})$$

for all  $l_1, \dots, l_n \in \mathsf{L}_0(\mathbb{R}^2)$  and all  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ .

*Proof.* By the multiplicativity of the Yang-Mills field and Theorem 5.12,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \text{tr}((H_{N,l_1}^{\mathbb{K}})^{\varepsilon_1} \dots (H_{N,l_n}^{\mathbb{K}})^{\varepsilon_n}) \right] = \lim_{N \rightarrow \infty} \mathbb{E} \left[ \text{tr}(H_{N,l_1^{\varepsilon_1} \dots l_n^{\varepsilon_n}}^{\mathbb{K}}) \right] = \Phi(l_n^{\varepsilon_n} \dots l_1^{\varepsilon_1}).$$

This is exactly the desired convergence.  $\square$

Our description of the limiting process does not incorporate the multiplicativity of the Yang-Mills field. Along the lines of the discussion of Section 5.1, we will give a better alternative description of the limiting process. The first and main problem which we encounter in trying to do this is to define the continuous analogue of the group  $\text{RL}_{v_0}(\mathbb{G})$  of reduced loops traced in a graph. This is a deep problem, and a fascinating one in its own right, which, fortunately for us, has already been solved for rectifiable paths by B. Hambly and T. Lyons, in a way which we briefly review in the next section.

**5.7. The group of rectifiable loops.** A beautiful result of B. Hambly and T. Lyons on rectifiable paths [18] allows one, among other things, to make sense on  $\mathsf{L}_0(\mathbb{R}^2)$  of an equivalence relation analogous to the one which we used on the set of loops traced in a graph. The central notion in their approach is that of tree-like loop, which is the continuous analogue of a loop equivalent to the constant loop. In order to define a tree-like loop, one needs to use a certain notion of continuous tree of which we start by recalling the definition.

A *compact  $\mathbb{R}$ -tree* is an arcwise connected compact metric space in which any two distinct points are joined by a unique subset homeomorphic to a segment, and such that the unique such subset which joins two distinct points is not only homeomorphic, but isometric to a segment<sup>2</sup>.

The next theorem gives five equivalent properties of a Lipschitz continuous loop, which all characterise tree-like loops. In this theorem,  $(E, \|\cdot\|)$  denotes a Banach space and we think of the circle  $S^1$  as  $\mathbb{R}/\mathbb{Z}$ .

**Theorem 5.14.** *Let  $l : S^1 \rightarrow E$  be a Lipschitz continuous loop. The following assertions are equivalent.*

1. *There exists a compact  $\mathbb{R}$ -tree  $T$  and two Lipschitz continuous mappings  $f : S^1 \rightarrow T$  and  $g : T \rightarrow E$  such that  $l = g \circ f$ .*

1'. *There exists a compact  $\mathbb{R}$ -tree  $T$  and two continuous mappings  $f : S^1 \rightarrow T$  and  $g : T \rightarrow E$  such that  $l = g \circ f$ .*

2. *There exists a Lipschitz continuous function  $h : [0, 1] \rightarrow \mathbb{R}_+$  such that  $h(0) = h(1) = 0$  and, for all  $s, t \in [0, 1]$ , the following inequality holds:*

$$\|l(t) - l(s)\| \leq h(s) + h(t) - 2 \inf\{h(u) : u \in [s, t]\}.$$

3. *The loop  $l$  is homotopic to a constant loop within its own range, that is, the mapping  $l : S^1 \rightarrow l(S^1)$  is homotopic to a constant map.*

3'. *The loop  $l$  is homotopic to a constant loop within the union of the ranges of finitely many loops, that is, there exist some loops  $l_1, \dots, l_n$  such that mapping  $l : S^1 \rightarrow l(S^1) \cup l_1(S^1) \cup \dots \cup l_n(S^1)$  is homotopic to a constant map.*

If any of these equivalent properties is satisfied, the loop  $l$  is said to be *tree-like*. A loop which satisfies property 3 is also called a *thin loop* by some authors (see [11]).

Not all these characterisations appear in the work of Hambly and Lyons, in particular the last ones, which are slightly remote from their point of view. We thus offer a proof of their equivalence.

*Proof.* 1'  $\Rightarrow$  2. Set  $\rho = f(0)$ , of which we think as the root of the tree. For all  $x, y \in T$ , let us denote by  $V_g(x, y)$  the total variation of  $g$  along the unique segment which joins  $x$  to  $y$ , that is, the total variation of the function  $g \circ \gamma_{x,y}$ , where  $\gamma_{x,y} : [0, 1] \rightarrow T$  is an injective continuous path from  $x$  to  $y$ . We claim that the function  $h : [0, 1] \rightarrow \mathbb{R}_+$  defined by

$$h(t) = V_g(\rho, f(t))$$

satisfies the second property.

Let us prove that  $h$  is finite and Lipschitz continuous. Let  $K$  denote the Lipschitz norm of  $l$ . For all  $x, y \in T$ , let us denote by  $x \wedge y$  the midpoint of  $\rho$ ,  $x$  and  $y$ , that is, the unique point located simultaneously on the three geodesics from  $\rho$  to  $x$ , from  $x$  to  $y$  and from  $y$  to  $\rho$ . Firstly, we have, for all  $s, t \in [0, 1]$ ,

$$|h(t) - h(s)| = |V_g(f(s) \wedge f(t), f(t)) - V_g(f(s) \wedge f(t), f(s))| \leq V_g(f(t), f(s)) \leq K|t - s|.$$

---

<sup>2</sup>It is not contained in the definition of an arcwise connected space, at least not the one which we use nowadays, that any two distinct points of such a space are the endpoints of a subspace homeomorphic to a segment. We merely insist that they be joined by a curve, which may have self-intersection. The fact that, in an arcwise connected metric space, any two distinct points are indeed joined by an injective curve is a consequence of various substantial theorems due to Hahn, Mazurkiewicz, Moore, Menger, Serpienski and which are summarised in the treatise of Kuratowski [22], §45.

Now, for all  $s, t \in [0, 1]$ ,  $l(s)$  is joined to  $l(t)$  by the image by  $g$  of the geodesic from  $f(s)$  to  $f(t)$ . Hence, if  $v \in [s, t]$  is such that  $f(v) = f(s) \wedge f(t)$ , then

$$\begin{aligned} \|l(t) - l(s)\| &\leq V_g(f(s), f(t)) = V_g(f(s), f(v)) + V_g(f(v), f(t)) \\ &= V_g(\rho, f(t)) - V_g(\rho, f(v)) + V_g(\rho, f(s)) - V_g(\rho, f(v)) \\ &= h(t) + h(s) - 2h(v) \\ &\leq h(t) + h(s) - 2 \inf\{h(u) : u \in [s, t]\}. \end{aligned}$$

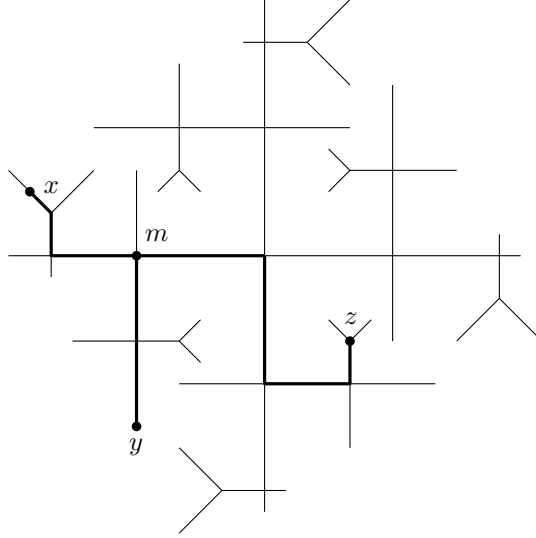


FIGURE 12. A compact  $\mathbb{R}$ -tree on which three points  $x, y, z$  have been chosen, and the geodesics which join them. The intersection of the three geodesics, denoted by  $m$ , is the midpoint of  $x, y$  and  $z$ .

$2 \Rightarrow 1$ . It is a classical fact that the function  $d(s, t) = h(s) + h(t) - 2 \inf\{h(u) : u \in [s, t]\}$  is a pseudo-distance on  $[0, 1]$  and that the quotient by the relation which identifies  $s$  and  $t$  whenever  $d(s, t) = 0$  is a compact  $\mathbb{R}$ -tree, which we denote by  $T$ . Moreover, the canonical projection  $p : [0, 1] \rightarrow T$  is Lipschitz continuous with a Lipschitz constant bounded by that of  $h$ . By assumption,  $l$  descends to a 1-Lipschitz continuous function  $\tilde{l} : T \rightarrow E$ . With this notation,  $l = \tilde{l} \circ p$  is the sought decomposition.

Since 1 tautologically implies  $1'$ , this proves the equivalence of the first three assertions. Let us turn to the proof of the equivalence of  $1'$  and 3. Our argument relies on a result due to Fort [10].

$1' \Rightarrow 3$ . As a compact  $\mathbb{R}$ -tree,  $T$  is contractible. Let  $\eta : [0, 1] \times T \rightarrow T$  be a homotopy from the identity to the constant map equal to  $\rho = f(0)$ . Then  $(s, t) \mapsto g(\eta(s, f(t)))$  is a homotopy between the loop  $l(S^1)$  and the constant loop equal to  $l(0)$ , within  $l(S^1)$ .

$3 \Rightarrow 1'$ . This is the most difficult part. The crucial fact is that the range of  $l$ , being the Lipschitz continuous image of an interval, has covering topological dimension equal to 1 (see [19] for an account of the theory of this dimension). Hence, by a theorem of Fort [10], the fact that  $l$  is homotopic to a constant loop within its own range implies that  $l$  factorises, not through a  $\mathbb{R}$ -tree, for that notion did not exist at the time where Fort wrote this paper, but through a dendrite, which we may of course assume to be compact. A dendrite is a connected

locally connected separable Hausdorff topological space which contains no closed simple curve. Fortunately, compact dendrites are exactly the topological spaces which underlie compact  $\mathbb{R}$ -trees, in the sense that a compact  $\mathbb{R}$ -tree is a compact dendrite and a compact dendrite can be metrised as an  $\mathbb{R}$ -tree. Thus, property 1' holds. It can even be assumed that the mapping  $g$  is light, which is to say, with the charm of a slightly old-fashioned terminology, that its fibres are totally disconnected subsets of  $T$ .

The assertion 3 certainly implies 3'. We finish by proving that 3' implies 1. This is the same argument as the proof that 3 implies 1, with the following modification. The assumptions of Fort's theorem are in fact that the loop is homotopic to a constant loop within a space of topological dimension 1. A classical theorem of addition (see [19]) asserts that a countable union of subspaces of dimension 1 of a topological space is still of dimension 1. Hence, the union of the ranges of finitely many loops has dimension 1.  $\square$

The result of B. Hambly and T. Lyons which matters most for our purposes, and which actually is a corollary of their main result, is the following.

**Theorem 5.15.** *The relation  $\sim$  defined on  $L_0(\mathbb{R}^2)$  by declaring  $l_1 \sim l_2$  if and only if  $l_1 l_2^{-1}$  is tree-like is an equivalence relation. Moreover, each equivalence class contains a unique loop of shortest length, which is characterised by the fact that no restriction of this loop is a tree-like loop.*

A loop which is the shortest in its equivalence class is said to be reduced.

In [18], this theorem is inferred from considerations on an algebraic object associated to a path which the authors call its signature. It turns out that the fact that the relation  $\sim$  is an equivalence relation can be deduced in a slightly more elementary way from the definition 3' of a tree-like loop and, since this definition was not considered in [18], we take a moment to give the argument. The point is of course that 3' allows one to see  $\sim$  as a relation of homotopy, of which we know that it is an equivalence relation.

*Proof of the fact that  $\sim$  is an equivalence relation.* That the relation  $\sim$  is reflexive and symmetric is straightforward. The problem is to prove that it is transitive. Let us assume that three loops  $l_1$ ,  $l_2$  and  $l_3$  are such that  $l_1 \sim l_2$  and  $l_2 \sim l_3$ . Then, in the union of the ranges of  $l_1$ ,  $l_2$  and  $l_3$ , the loops  $l_1$  and  $l_2$  are homotopic, as well as  $l_2$  and  $l_3$ . Hence,  $l_1$  and  $l_3$  are homotopic, so that  $l_1 \sim l_3$ .  $\square$

**Remark 5.16.** *Note that the equivalence between the characterisations 1' and 3 of tree-like loops holds for all loops whose range has topological dimension 1. Since the topological dimension is smaller than the Hausdorff dimension, this holds for loops whose range has Hausdorff dimension strictly smaller than 2, in particular for loops with finite  $p$ -variation for  $p \in [1, 2)$ . It even holds for loops whose range has Hausdorff dimension 2, provided the measure of their range is zero (see [19]). It follows in particular that  $\sim$  is an equivalence relation on the set of loops with finite  $p$ -variation for some  $p < 2$ .*

With the help of this very natural equivalence relation, we may proceed in the same way as in the discrete setting and consider the quotient of  $L_0(\mathbb{R}^2)$  equipped with the operation of concatenation. Equivalently, we may consider the group  $RL_0(\mathbb{R}^2)$  of reduced loops with the operation of concatenation and reduction. Note that, in contrast to the discrete case, this group is not a free group. Indeed, it contains a subgroup isomorphic to the fundamental group of a topological space called the Hawaiian earring [8], which is known not to be free. On the other hand, by a classical theorem of Nielsen and Schreier, any subgroup of a free group is free.

It would be very desirable at this point to possess a nice structure of topological group on  $\text{RL}_0(\mathbb{R}^2)$ . Unfortunately, we do not know how to define such a structure. We shall therefore content ourselves with the bare algebraic structure.

**5.8. The master field as a free process.** The discussion of the previous section provides us with a natural algebra on which the master field is defined, namely the algebra  $\mathbb{C}[\text{RL}_0(\mathbb{R}^2)]$  of the group of reduced rectifiable loops based at the origin.

Not only can we restrict the function  $\Phi$  defined by Theorem 5.12 to the set of reduced loops, but it is in fact compatible with the equivalence of paths.

**Lemma 5.17.** *Let  $l_1, l_2 \in \text{L}_0(\mathbb{R}^2)$  be two loops based at the origin. Assume that  $l_1 \sim l_2$ . Then for all  $N \geq 1$ , the equality  $H_{N,l_1}^{\mathbb{K}} = H_{N,l_2}^{\mathbb{K}}$  holds almost surely. In particular, for all  $\mathbb{K}$  and all  $N \geq 1$ ,  $\Phi_N^{\mathbb{K}}(l_1) = \Phi_N^{\mathbb{K}}(l_2)$ , and  $\Phi(l_1) = \Phi(l_2)$ .*

*Proof.* The second assertion follows immediately from the first. By the multiplicativity of the Yang-Mills field, the first assertion will be proved if we show that for all tree-like loop  $l$ , we have  $H_{N,l}^{\mathbb{K}}$  is almost surely equal to the identity matrix for all  $N \geq 1$ .

In a first step, let us consider a tree-like loop  $l$  traced in a graph  $\mathbb{G}$ , based at some vertex  $v_0$ . We need to show that  $l$  is combinatorially equivalent to a constant loop. We know that  $l$  is homotopic to a constant loop within its own range, hence within the skeleton of  $\mathbb{G}$ . The description of the group  $\text{RL}_{v_0}(\mathbb{G})$  as the free group over a set of facial lassos shows that this group is isomorphic to the fundamental group of  $\text{Sk}(\mathbb{G})$ . Hence,  $l$  is equal to 1 in this group, which means that it is combinatorially equivalent to a constant loop. Then, the multiplicativity of the Yang-Mills field entails that  $H_{N,l}^{\mathbb{K}} = I_N$  almost surely for all  $N \geq 1$ . The conclusion of this paragraph applies in particular to any piecewise affine tree-like loop.

In a second step, let us consider a tree-like loop  $l$ , without any further assumption. We claim that  $l$  is the limit of a sequence of piecewise affine tree-like loops. In order to prove this, let us consider a factorisation  $l = g \circ f$  through an  $\mathbb{R}$ -tree  $T$ . For each  $n \geq 1$ , consider a finite subset of  $T$  whose  $2^{-n}$ -neighbourhood covers  $T$  and let  $T_n$  be the convex hull of this subset, which is a finite sub-tree of  $T$ . Construct  $\tilde{g}_n : T_n \rightarrow \mathbb{R}^2$  as the unique mapping which coincides with  $g$  on the vertices of  $T_n$  and is affine on each edge of  $T_n$ . Finally, let  $p_n : T \rightarrow T_n$  denote the projection. Let us define, for all  $x \in T$ ,  $g_n(x) = \tilde{g}_n(p_n(x))$ . Then it is not difficult to check that  $g_n \circ f$  is piecewise affine and converges to  $l$  as  $n$  tends to infinity. Hence, by continuity of the Yang-Mills field for fixed  $N \geq 1$ , we have  $H_{N,l}^{\mathbb{K}} = I_N$  almost surely for all  $N \geq 1$ . This concludes the proof.  $\square$

According to this lemma, the functions  $\Phi_N^{\mathbb{K}}$  and  $\Phi$  descend to functions on the quotient  $\text{L}_0(\mathbb{R}^2)/\sim$ , or on  $\text{RL}_0(\mathbb{R}^2)$ . We still denote these functions by  $\Phi_N^{\mathbb{K}}$  and  $\Phi$ . It follows from Theorem 5.12 that, on the complex involutive unital algebra  $\mathbb{C}[\text{RL}_0(\mathbb{R}^2)]$ , the sequence of states  $(\Phi^{\mathbb{K},N})_{N \geq 1}$  converges pointwise to  $\Phi$ , which is also a state.

We can thus define the master field as a free process.

**Definition 5.18.** *Let  $\mathbb{C}[\text{RL}_0(\mathbb{R}^2)]$  be the complex group algebra of the group of reduced rectifiable loops on  $\mathbb{R}^2$  endowed with the operation of concatenation-reduction. Let  $\Phi$  be the linear form on this algebra characterised by the equality*

$$(78) \quad \forall l \in \text{RL}_0(\mathbb{R}^2), \quad \Phi(l) = \lim_{N \rightarrow \infty} \mathbb{E}_{\text{YM}_{\text{U}(N)}} \left[ \text{tr}(H_{N,l}^{\mathbb{C}}) \right].$$

*On the non-commutative space  $(\mathbb{C}[\text{RL}_0(\mathbb{R}^2)], \Phi)$ , define the process  $\{h_l : l \in \text{L}_0(\mathbb{R}^2)\}$  by letting, for all  $l \in \text{L}_0$ , the non-commutative random variable  $h_l$  be the image of  $l$  by the composed mapping*

$L_0(\mathbb{R}^2) \rightarrow \mathbf{RL}_0(\mathbb{R}^2) \rightarrow \mathbb{C}[\mathbf{RL}_0(\mathbb{R}^2)]$ . In other words,  $h_l$  is the unique reduced loop equivalent to  $l$ , seen as an element of the group algebra of the group of reduced loops.

The process  $(h_l)_{l \in L_0(\mathbb{R}^2)}$  is called the master field on the plane.

We can state the main theorem of the present work in its final form. We call *lasso* a loop of the form  $sbs^{-1}$ , where  $s$  is an arbitrary path and  $b$  is a simple loop based at the endpoint of  $s$ .

**Theorem 5.19.** *Choose  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ . For each  $N \geq 1$ , consider the Yang-Mills field on the plane  $\mathbb{R}^2$  with structure group  $U(N, \mathbb{K})$ , associated to the Lebesgue measure on  $\mathbb{R}^2$  and the scalar product on  $\mathfrak{u}(N, \mathbb{K})$  given by 7. This is a process  $(H_{N,l}^{\mathbb{K}})_{l \in L_0(\mathbb{R}^2)}$  with values in  $U(N, \mathbb{K})$ . Consider this process as a non-commutative process with respect to the state  $\mathbb{E} \otimes \text{tr}$  if  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  and  $\mathbb{E} \otimes \Re \text{tr}$  if  $\mathbb{K} = \mathbb{H}$ .*

0. As  $N$  tends to infinity, the Yang-Mills field converges in non-commutative distribution towards the master field on the plane.

1. If  $l$  is a lasso enclosing an open subset of  $\mathbb{R}^2$  of area  $t$ , then  $h_l$  has the distribution of a free multiplicative Brownian motion at time  $t$ .

2. If  $l_1, \dots, l_n$  are  $n$  lassos enclosing pairwise disjoint open subsets of  $\mathbb{R}^2$ , then the non-commutative random variables  $h_{l_1}, \dots, h_{l_n}$  are free.

3. The process  $\{h_l : l \in L_0(\mathbb{R}^2)\}$  is continuous in the  $L^2$  topology. This means that if a sequence of loops  $(l_n)_{n \geq 0}$  converges to a loop  $l$ , then  $\Phi((h_{l_n} - h_l)(h_{l_n} - h_l)^*)$  tends to 0 as  $n$  tends to infinity. More generally, in this case, for all integer  $q \geq 1$ , all loops  $m_2, \dots, m_q$  and all word  $w \in \mathbb{F}_q$  in  $q$  letters, the following convergence holds:

$$\lim_{n \rightarrow \infty} \Phi(w(l_n, l_n^{-1}, m_2, \dots, m_q)) = \Phi(w(l, l^{-1}, m_2, \dots, m_q)).$$

4. The properties 1, 2 and 3 characterise the distribution of the master field.

5. The function  $\Phi : L_0(\mathbb{R}^2) \rightarrow \mathbb{C}$  determined by  $\Phi(l) = \Phi(h_l)$  takes its values in the real segment  $[-1, 1]$ , is continuous with respect to the convergence of loops, and satisfies  $\Phi(l^{-1}) = \Phi(l)$  for all loop  $l$ .

*Proof.* 0. This is the content of Theorem 5.12.

1. If  $l$  is a lasso enclosing a domain of area  $t$ , then for all  $N \geq 1$ , the distribution of  $H_{N,l}^{\mathbb{C}}$  is that of  $U_t$ . The claim is thus a consequence of Theorem 2.1.

We prove 3 before proving 2.

3. Assume that  $(l_n)_{n \geq 1}$  converges to  $l$ . For each  $N \geq 1$ ,  $H_{N,l_n}^{\mathbb{C}}$  converges in probability to  $H_{N,l}^{\mathbb{C}}$ , so that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Phi^{\mathbb{C}, N}((h_{l_n} - h_l)(h_{l_n} - h_l)^*) &= 2 - 2 \lim_{n \rightarrow \infty} \Re \Phi^{\mathbb{C}, N}(h_{l_n l^{-1}}) \\ &= 2 - 2 \lim_{n \rightarrow \infty} \Re \mathbb{E}[\text{tr}(H_{N,l_n}^{\mathbb{C}} (H_{N,l}^{\mathbb{C}})^{-1})] = 0. \end{aligned}$$

Since the convergence of  $\Phi^{\mathbb{C}, N}$  towards  $\Phi$  is uniform on the set  $\{l_n : n \geq 1\} \cup \{l\}$ , for the length function is bounded on this set, the convergence holds at the limit when  $N$  tends to infinity. The last assertion follows from the same argument applied to  $w(H_{N,l_n}^{\mathbb{C}}, H_{N,l_n^{-1}}^{\mathbb{C}} l_n^{-1}, H_{N,m_2}^{\mathbb{C}}, \dots, H_{N,m_q}^{\mathbb{C}})$ .

2. By deforming slightly the lassos  $l_1 = s_1 b_1 s_1^{-1}, \dots, l_n s_n b_n s_n^{-1}$ , we may assume that each path  $s_1, \dots, s_n$  contains a subpath which it traverses exactly once, and that none of the other paths crosses. In this case, the random variables  $H_{N,l_1}^{\mathbb{C}}, \dots, H_{N,l_n}^{\mathbb{C}}$  are independent. The claim is thus a consequence of the theorem of Voiculescu [37] which asserts asymptotic freeness for large independent random matrices invariant in distribution by unitary conjugation.

4. Since the group of loops traced in a graph admits a basis formed by lassos enclosing the faces of the graph, which are pairwise disjoint, the properties 1 and 2 characterise the distribution

of the master field on the set of loops traced in a graph, hence on the set of piecewise affine loops. The property 3 guarantees that the distribution on  $L_0(\mathbb{R}^2)$  is given by the unique extension by continuity of that on  $\text{Aff}_0(\mathbb{R}^2)$ .

5. By theorem 3.5, the function  $\Phi$  is real on  $\text{Aff}_0(\mathbb{R}^2)$ . It is continuous on  $L(\mathbb{R}^2)$  by the third assertion of the present theorem. It is thus real-valued on  $L_0(\mathbb{R}^2)$ . The definition (78) shows that it is bounded by 1. Since  $\Phi(l^{-1}) = \Phi(l)^*$ , it is also equal to  $\Phi(l)$ .  $\square$

## 6. COMPUTING WITH THE MASTER FIELD

In this section, we address the following question : given a loop  $l$  on the plane, how can we actually compute  $\Phi_N^{\mathbb{K}}(l)$  and  $\Phi(l)$  ?

We are going to provide several more or less explicit pieces of answer to this question. They all rely on the fundamental principle that one should see  $\Phi(l)$  and  $\Phi_N^{\mathbb{K}}(l)$  as functions of the areas of the faces delimited by  $l$  and that the information we are looking for can be obtained by differentiating this function.

It is clear from this general description that this approach works only for loops which delimit a finite number of faces. Accordingly, the level at which we attack the problem is that of the discrete theory. The answer which we are seeking is combinatorial in nature.

The content of the present section is in part guided by the desire to understand and elaborate on previous work of Makeenko and Migdal [28], Kazakov [20], and Kostov [21] on this question.

In a first step, we shall compute the derivative of the Yang-Mills measure on a graph with respect to the areas of the faces. Our expressions will involve differential operators on the configuration space which we will, in a second time, interpret in a combinatorial language. This second step will be meaningful only for a special class of observables known as the Wilson loops, which are both very natural, and general enough to generate the algebra of all invariant observables.

**6.1. Differential operators on the configuration space.** To start with, we introduce some differential operators on the configuration space of the discrete Yang-Mills theory. The computations which we are going to do in the first sections are valid for any structure group. We thus choose a connected compact Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ .

Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. The configuration space  $\mathcal{C}^{\mathbb{G}} = \mathcal{M}(\mathbb{P}(\mathbb{G}), G)$  is in a canonical way a smooth manifold through the identification  $\mathcal{C}^{\mathbb{G}} \simeq G^{\mathbb{E}^+}$ , regardless of the orientation  $\mathbb{E}^+$  of  $\mathbb{G}$ . We are going to define certain vector fields on this manifold. It is tempting to this end to use the Lie group structure inherited from  $G^{\mathbb{E}^+}$ , but this structure depends on the orientation. In a first time, it is more convenient to use the following description of the configuration space :

$$\mathcal{C}^{\mathbb{G}} = \mathcal{M}(\mathbb{E}, G) = \{(h(e))_{e \in \mathbb{E}} : \forall e \in \mathbb{E}, h(e^{-1}) = h(e)^{-1}\},$$

thus defined as a submanifold of  $G^{\mathbb{E}}$ .

Let  $e \in \mathbb{E}$  be an edge. Let  $X$  be an element of the Lie algebra  $\mathfrak{g}$ . We define the vector field  $\mathbf{D}_X^{(e)}$  on  $\mathcal{C}^{\mathbb{G}}$  by setting, for all  $h \in \mathcal{C}^{\mathbb{G}}$ ,

$$\left(\mathbf{D}_X^{(e)}\right)(h) = \frac{d}{dt}\Big|_{t=0} (h_t(f))_{f \in \mathbb{E}},$$

where, for all  $t \in \mathbb{R}$  and all  $f \in \mathbb{E}$ ,

$$h_t(f) = \begin{cases} h(f) & \text{if } f \notin \{e, e^{-1}\}, \\ h(e)e^{tX} & \text{if } f = e, \\ e^{-tX}h(e^{-1}) & \text{if } f = e^{-1}. \end{cases}$$



Let us generalise slightly this definition. Let  $c \in \mathcal{P}(\mathbb{G})$  be a path which ends at the starting point of  $e$ . We define the vector field  $\mathbf{D}_X^{(c,e)}$  by setting, for all  $h \in \mathcal{C}^{\mathbb{G}}$ ,

$$\left(\mathbf{D}_X^{(c,e)}\right)(h) = \left(\mathbf{D}_{\text{Ad}(h(c))X}^{(e)}\right)(h).$$

In vague but perhaps more intuitive terms, the field  $\mathbf{D}_X^{(e)}$  corresponds to the adjunction of an infinitesimal loop with holonomy  $X$  at the starting point of the edge  $e$ . This starting point should however not be understood as the vertex  $\underline{e}$ , since there may be edges other than  $e$  which are issued from  $\underline{e}$ , but the field  $\mathbf{D}_X^{(e)}$  does not affect the configuration on these other edges. Let us say that an infinitesimal loop with holonomy  $X$  is inserted *at the very beginning* of the edge  $e$  (see Figure 13). Similarly, the field  $\mathbf{D}_X^{(c,e)}$  varies the current configuration by inserting, at the very beginning of  $e$ , a loop formed by following the path  $c^{-1}$ , then going around an infinitesimal loop with holonomy  $X$  and then coming back along  $c$ .

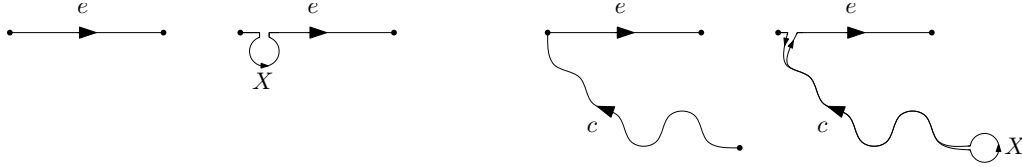


FIGURE 13. The action of the vector fields  $\mathbf{D}_X^{(e)}$  (on the left) and  $\mathbf{D}_X^{(c,e)}$  (on the right).

Note that for all  $e \in \mathbb{E}$ , the mapping  $\mathfrak{g} \rightarrow \mathcal{X}(\mathcal{C}^{\mathbb{G}})$  into the Lie algebra of smooth vector fields on  $\mathcal{C}^{\mathbb{G}}$  which sends  $X$  to  $\mathbf{D}_X^{(e)}$  is linear and a homomorphism of Lie algebras.

Let us assume that  $\mathfrak{g}$  is endowed with an invariant scalar product, which we denote by  $\langle \cdot, \cdot \rangle$ . Let  $f : \mathcal{C}^{\mathbb{G}} \rightarrow \mathbb{C}$  be a smooth function. We define the gradient at  $e$  of  $f$  by choosing an orthonormal basis  $(X_1, \dots, X_d)$  of  $\mathfrak{g}$  and setting, for all  $h \in \mathcal{C}^{\mathbb{G}}$ ,

$$\left(\nabla^{(e)} f\right)(h) = \sum_{k=1}^d \left(\left(\mathbf{D}_{X_k}^{(e)} f\right)(h)\right) X_k.$$

The gradient thus defined does not depend on the choice of the orthonormal basis of  $\mathfrak{g}$ , since it is a linear function of the Casimir element of  $\mathfrak{g}$  (see Section 1.3). Similarly, we define

$$\left(\nabla^{(c,e)} f\right)(h) = \sum_{k=1}^d \left(\left(\mathbf{D}_{X_k}^{(c,e)} f\right)(h)\right) X_k.$$

Both  $\nabla^{(e)} f$  and  $\nabla^{(c,e)} f$  are smooth  $\mathfrak{g}$ -valued functions on  $\mathcal{C}^{\mathbb{G}}$ . They are related by

$$\left(\nabla^{(c,e)} f\right)(h) = \text{Ad}(h(c))^{-1} \left(\nabla^{(e)} f\right)(h).$$

In particular, if  $f_1$  and  $f_2$  are smooth functions on  $\mathcal{C}^{\mathbb{G}}$  and  $c$  is a path which joins the starting point of the edge  $e_2$  to the starting point of the edge  $e_1$ , then the equality

$$(79) \quad \left\langle \nabla^{(c,e_1)} f_1, \nabla^{(e_2)} f_2 \right\rangle = \left\langle \nabla^{(e_1)} f_1, \nabla^{(c^{-1},e_2)} f_2 \right\rangle$$

holds pointwise on  $\mathcal{C}^{\mathbb{G}}$ .

Let us now define second-order differential operators. Let  $e_1, e_2$  be two edges of  $\mathbb{G}$ . Let  $c \in \mathbf{P}(\mathbb{G})$  be a path which joins the starting point of  $e_2$  to the starting point of  $e_1$ . Let  $(X_1, \dots, X_d)$  be an orthonormal basis of  $\mathfrak{g}$ . We define

$$\Delta^{(e_2)(c,e_1)} = \sum_{k=1}^d \mathbf{D}_{X_k}^{(e_2)} \mathbf{D}_{X_k}^{(c,e_1)}.$$

If  $e_1 = e_2 = e$  and  $c$  is the constant path at the starting point of  $e$ , we write

$$\Delta^{(e)} = \Delta^{(e_2)(c,e_1)} = \sum_{k=1}^d \left( \mathbf{D}_{X_k}^{(e)} \right)^2.$$

As before, none of these definitions depend on the choice of the orthonormal basis of  $\mathfrak{g}$ . Let us however emphasise that the order of the derivatives in the definition of  $\Delta^{(e_2)(c,e_1)}$  matters, since in general,

$$\sum_{k=1}^d \mathbf{D}_{X_k}^{(e_2)} \mathbf{D}_{X_k}^{(c,e_1)} \neq \sum_{k=1}^d \mathbf{D}_{X_k}^{(c,e_1)} \mathbf{D}_{X_k}^{(e_2)},$$

unless  $e_1 \neq e_2$  and the path  $c$  does not traverse the edge  $e_2$ .

We have defined the differential operators  $\mathbf{D}_X^{(c,e)}$ ,  $\nabla^{(c,e)}$  and  $\Delta^{(e_2)(c,e_1)}$  on the configuration space  $\mathcal{C}^{\mathbb{G}}$  seen as a submanifold of  $G^{\mathbb{E}}$ . It is however usually simpler, when computing on the configuration space, to choose an orientation  $\mathbb{E}^+$  of  $\mathbb{G}$  and to use the identification  $\mathcal{C}^{\mathbb{G}} \simeq G^{\mathbb{E}^+}$ . Let us write down the definition of our differential operators in this language. It is enough to write the definition of  $\mathbf{D}_X^{(e)}$ , since all others are built from this one.

Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . For each  $e \in \mathbb{E}^+$  and all  $X \in \mathfrak{g}$ , let us denote by  $X^{(e)}$  the element  $(0, \dots, 0, X, 0, \dots, 0)$  of  $\mathfrak{g}^{\mathbb{E}^+}$  whose only possibly non-zero component is that corresponding to the edge  $e$  and is equal to  $X$ . Let  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{C}$  be a smooth observable. Let  $h \in G^{\mathbb{E}^+}$  be a configuration. Let  $e \in \mathbb{E}$  be an edge, and  $X$  an element of  $\mathfrak{g}$ . If  $e$  belongs to  $\mathbb{E}^+$ , we have

$$\left( \mathbf{D}_X^{(e)} f \right) (h) = \frac{d}{dt} \Big|_{t=0} f(h e^{tX^{(e)}}),$$

and if  $e^{-1}$  belongs to  $\mathbb{E}^+$ , then

$$(80) \quad \left( \mathbf{D}_X^{(e)} f \right) (h) = \frac{d}{dt} \Big|_{t=0} f(e^{-tX^{(e^{-1})}} h).$$

Let us collect some properties of the differential operators which we have just defined which we will need in the proof of Proposition 6.2. We denote, as usual, by  $\Delta$  the Laplace operator on  $G$ , and by  $(Q_t)_{t>0}$  the associated heat kernel.

**Lemma 6.1.** *1. Let  $f_1, f_2 : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$  be two smooth functions. Let  $e$  be an edge of  $\mathbb{G}$ . We have, for all  $X \in \mathfrak{g}$ ,*

$$\int_{G^{\mathbb{E}^+}} f_1(h) (\mathbf{D}_X^{(e)} f_2)(h) dh = - \int_{G^{\mathbb{E}^+}} (\mathbf{D}_X^{(e)} f_1)(h) f_2(h) dh.$$

*In particular,*

$$\int_{G^{\mathbb{E}^+}} f_1(h) (\Delta^{(e)} f_2)(h) dh = \int_{G^{\mathbb{E}^+}} (\Delta^{(e)} f_1)(h) f_2(h) dh.$$

*2. Let  $e$  be an edge of  $\mathbb{G}$ . Let  $c \in \mathbf{P}(\mathbb{G})$  be a path in  $\mathbb{G}$  from the finishing point of  $e$  to its starting point, such that  $c$  does not traverse  $e$  nor  $e^{-1}$ . Let  $l$  be the loop  $ec$ . Let  $q : G \rightarrow \mathbb{R}$  be a smooth function invariant by conjugation.*

The two functions  $h \mapsto (\Delta q)(h(l))$  and  $\Delta^{(e)}(h \mapsto q(h(l)))$  are equal and the two functions  $h \mapsto (\Delta q)(h(l^{-1}))$  and  $\Delta^{(e)}(h \mapsto q(h(l^{-1})))$  are also equal.

In particular, if  $e$  is an edge which bounds a face  $F$ , whether positively or negatively, then for all  $t > 0$ ,

$$(\Delta Q_{|F|})(h(\partial F)) = \Delta^{(e)}(h \mapsto Q_{|F|}(h(\partial F))).$$

3. Let  $F$  be a face of  $\mathbb{G}$ . Let  $e$  and  $e'$  be two edges which bound  $F$ , respectively negatively and positively. Let  $c$  be the portion of the boundary of  $F$  which joins the starting point of  $e'$  to the starting point of  $e$ , turning clockwise around  $F$  (see Figure 14 below). Let  $X$  be an element of  $\mathfrak{g}$ . Then

$$\mathbf{D}_X^{(e)}(h \mapsto Q_{|F|}(h(\partial F))) = -\mathbf{D}_X^{(c^{-1}, e')}(h \mapsto Q_{|F|}(h(\partial F))).$$

More generally, if  $d$  is a path which starts from the starting point of  $e$ , then

$$\mathbf{D}_X^{(d^{-1}, e)}(Q_{|F|}(h(\partial F))) = -\mathbf{D}_X^{(d^{-1}c^{-1}, e')}(Q_{|F|}(h(\partial F))).$$

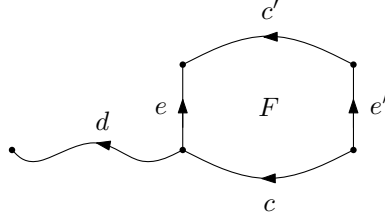


FIGURE 14. The paths involved in the third assertion of Lemma 6.1.

*Proof.* 1. The operator  $\mathbf{D}_X^{(e)}$  satisfies the Leibniz rule. Hence, the first set of assertions is a consequence of the fact that for all smooth function  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$ , one has

$$\int_{G^{\mathbb{E}^+}} (\mathbf{D}_X^{(e)} f)(h) dh = 0.$$

This equality in turn follows from the fact that the Haar measure on  $G^{\mathbb{E}^+}$  is invariant by the flow of  $\mathbf{D}_X^{(e)}$ , which is a flow of translations, on the right if  $e \in \mathbb{E}^+$  and on the left if  $e^{-1} \in \mathbb{E}^+$ .

2. Since  $q$  is invariant by conjugation, so is  $\Delta q$ . We thus have

$$\begin{aligned} (\Delta q)(h(l)) &= (\Delta q)(h(ce)) = (\Delta q)(h(e)h(c)) = (\Delta q)(h(c)h(e)) \\ &= \sum_{k=1}^d \frac{d^2}{dt^2} \Big|_{t=0} q(h(c)h(e)e^{tX_k}) = \Delta^{(e)}(q(h(ce))). \end{aligned}$$

Similarly,

$$\begin{aligned} (\Delta q)(h(l^{-1})) &= (\Delta q)(h(e)^{-1}h(c)^{-1}) = \sum_{k=1}^d \frac{d^2}{dt^2} \Big|_{t=0} q(e^{tX_k}h(e)^{-1}h(c)^{-1}) \\ &= \sum_{k=1}^d \frac{d^2}{dt^2} \Big|_{t=0} q(e^{-tX_k}h(e)^{-1}h(c)^{-1}) = \Delta^{(e)}(q(h(l^{-1}))). \end{aligned}$$

3. The first assertion is a particular case of the second, by taking  $d$  to be the constant path at the starting point of  $e$ . Let us write  $\partial F = e^{-1}c^{-1}e'c'$ , with  $c'$  the appropriate path (see Figure 14 above). We have

$$\begin{aligned} \mathbf{D}_X^{(d^{-1},e)}(Q_{|F|}(h(\partial F))) &= \frac{d}{dt}\Big|_{t=0} Q_{|F|}\left(h(c')h(e')h(c)^{-1}e^{-t\text{Ad}(h(d)^{-1})X}h(e)^{-1}\right) \\ &= \frac{d}{dt}\Big|_{t=0} Q_{|F|}\left(h(c')h(e')e^{-t\text{Ad}(h(c)^{-1}h(d)^{-1})X}h(c)^{-1}h(e)^{-1}\right) \\ &= \mathbf{D}_{-\text{Ad}(h(d^{-1}c^{-1}))X}^{(e')} (Q_{|F|}(h(\partial F))) \\ &= -\mathbf{D}_X^{(d^{-1}c^{-1},e')} (Q_{|F|}(h(\partial F))), \end{aligned}$$

as expected.  $\square$

Let us emphasise that the operator  $\mathbf{D}_X^{(c,e)}$  does not satisfy in general a formula of integration by parts analogous to the one satisfied by  $\mathbf{D}_X^{(e)}$ . We shall make further comments on this point in Section 6.6.

**6.2. Variation of the area in the abstract.** The main result of this section provides us with an expression of the derivative  $\frac{d}{d|F_1|}\mathbb{E}_{\text{YM}^\mathbb{G}}[f]$  in terms of the differential operators which we introduced in the previous section, and without any assumption on the observable  $f$ .

Before we state the result, let us give a more formal description of what we mean by the derivative  $\frac{d}{d|F_1|}\mathbb{E}_{\text{YM}^\mathbb{G}}[f]$ . Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $\mathbb{F}^b = \{F_1, \dots, F_q\}$  be the set of bounded faces. For all  $t = (t_1, \dots, t_q) \in (\mathbb{R}_+^*)^q$ , we define the Yang-Mills measure with areas  $t$  on the configuration space  $\mathcal{C}_\mathbb{G}^t$  by the following formula, analogous to (74):

$$(81) \quad \text{YM}_t^\mathbb{G}(dh) = \prod_{i=1}^q Q_{t_i}(h(\partial F_i)) dh.$$

Then we define, for all  $i \in \{1, \dots, q\}$ ,

$$\frac{d}{d|F_i|}\mathbb{E}_{\text{YM}^\mathbb{G}}[f] = \frac{\partial}{\partial t_i}\Big|_{t=(|F_1|, \dots, |F_q|)} \mathbb{E}_{\text{YM}_t^\mathbb{G}}[f].$$

**Proposition 6.2.** *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $n \geq 1$  be an integer. Let  $F_1, \dots, F_{n+1}$  be a sequence of faces of  $\mathbb{G}$ . Assume that each face of this sequence is different from the next, and only the last face  $F_{n+1}$  may be the unbounded face of  $\mathbb{G}$ . Assume also that for all  $r \in \{1, \dots, n\}$ , the faces  $F_r$  and  $F_{r+1}$  share a common edge, which we denote by  $e_r$ , with the convention that  $e_r$  bounds  $F_r$  positively. For each  $r \in \{2, \dots, n\}$ , denote by  $c_r^{-1}$  the portion of the boundary of  $F_r$  which joins the starting point of  $e_{r-1}$  to the starting point of  $e_r$ . Finally, let  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$  be a smooth function. Then, if  $n \geq 2$ , we have the following formula :*

$$(82) \quad \begin{aligned} \left(\frac{d}{d|F_1|} - \frac{d}{d|F_2|}\right) \mathbb{E}_{\text{YM}^\mathbb{G}}[f] &= \mathbb{E}_{\text{YM}^\mathbb{G}} \left[ \frac{1}{2} \Delta^{(e_1)} f + \sum_{i=2}^n \Delta^{(e_i)(c_i \dots c_2, e_1)} f \right] \\ &\quad + \mathbb{E}_{\text{YM}^\mathbb{G}} \left[ \left\langle \nabla^{(e_n)} (h \mapsto \log Q_{|F_{n+1}|}(h(\partial F_{n+1}))), \nabla^{(c_n \dots c_2, e_1)} f \right\rangle \right] \end{aligned}$$

in which the last term of the right-hand side must be replaced by 0 in the case where  $F_{n+1} = F_\infty$ . If  $n = 1$  and  $F_2$  is not the unbounded face, then the same formula holds after dropping the sum

over  $i$  in the first expectation and replacing  $\nabla^{(c_n \dots c_2, e_1)}$  by  $\nabla^{(e_1)}$ . Finally, if  $n = 1$  and  $F_2$  is the unbounded face of  $\mathbb{G}$ , the formula simply reads

$$(83) \quad \frac{d}{d|F_1|} \mathbb{E}_{\text{YM}^{\mathbb{G}}} [f] = \mathbb{E}_{\text{YM}^{\mathbb{G}}} \left[ \frac{1}{2} \Delta^{(e_1)} f \right].$$

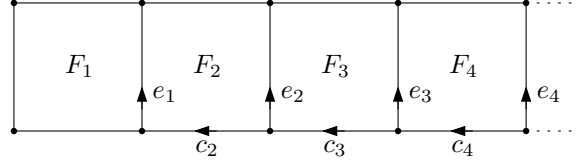


FIGURE 15. The paths involved in the differentiation with respect to the area of the face  $F_1$ .

*Proof.* The proof is but a straightforward computation. Let us go through it step by step.

The first step is of course to use the heat equation satisfied by the heat kernel  $(Q_t)_{t>0}$ , together with the second assertion of Lemma 6.1. We find

$$\begin{aligned} \frac{d}{d|F_1|} \mathbb{E}_{\text{YM}^{\mathbb{G}}} [f] &= \frac{d}{d|F_1|} \int_{G^{\mathbb{E}^+}} f(h) \prod_{F \in \mathbb{F}^b} Q_{|F|}(h(\partial F)) dh \\ &= \frac{1}{2} \int_{G^{\mathbb{E}^+}} f(h) (\Delta Q_{|F_1|})(h(\partial F_1)) \prod_{F \in \mathbb{F}^b \setminus \{F_1\}} Q_{|F|}(h(\partial F)) dh \\ &= \frac{1}{2} \int_{G^{\mathbb{E}^+}} f(h) \Delta^{(e_1)} (Q_{|F_1|}(h(\partial F_1))) \prod_{F \in \mathbb{F}^b \setminus \{F_1\}} Q_{|F|}(h(\partial F)) dh. \end{aligned}$$

Among all edges bounding  $F_1$  positively, we have chosen the edge  $e_1$ . Now, we use integration by parts, or equivalently the fact that  $\Delta^{(e_1)}$  is self-adjoint (see Lemma 6.1). If  $n = 1$  and  $F_2$  is the unbounded face, then  $f(h)$  and  $Q_{|F_1|}(h(\partial F_1))$  are the only edges in the integrand which depend on the edge  $e$  and we find

$$\begin{aligned} \frac{d}{d|F_1|} \mathbb{E}_{\text{YM}^{\mathbb{G}}} [f] &= \frac{1}{2} \int_{G^{\mathbb{E}^+}} Q_{|F_1|}(h(\partial F_1)) \Delta^{(e_1)} \left( f(h) \prod_{F \in \mathbb{F}^b \setminus \{F_1\}} Q_{|F|}(h(\partial F)) \right) dh \\ &= \frac{1}{2} \int_{G^{\mathbb{E}^+}} (\Delta^{(e_1)} f)(h) Q_{|F_1|}(h(\partial F_1)) \prod_{F \in \mathbb{F}^b \setminus \{F_1\}} Q_{|F|}(h(\partial F)) dh \\ &= \mathbb{E}_{\text{YM}^{\mathbb{G}}} \left[ \frac{1}{2} \Delta^{(e_1)} f \right], \end{aligned}$$

which proves the result in this case. If  $F_2$  is not the unbounded face, then  $Q_{|F_2|}(h(\partial F_2))$  also depends on the edge  $e$ , and no other term does. We find, applying the Leibniz rule,

$$\begin{aligned}
\frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^G}[f] &= \frac{1}{2} \int_{G^{\mathbb{E}^+}} (\Delta^{(e_1)} f)(h) Q_{|F_1|}(h(\partial F_1)) \prod_{F \in \mathbb{F}^b \setminus \{F_1\}} Q_{|F|}(h(\partial F)) dh \\
&+ \frac{1}{2} \int_{G^{\mathbb{E}^+}} f(h) \Delta^{(e_1)} (Q_{|F_2|}(h(\partial F_2))) \prod_{F \in \mathbb{F}^b \setminus \{F_2\}} Q_{|F|}(h(\partial F)) dh \\
(84) \quad &+ \sum_{k=1}^d \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(e_1)} f \right) (h) \mathbf{D}_{X_k}^{(e_1)} (Q_{|F_2|}(h(\partial F_2))) \prod_{F \in \mathbb{F}^b \setminus \{F_2\}} Q_{|F|}(h(\partial F)) dh.
\end{aligned}$$

We have already recognised the first term of this sum as the first term of the right-hand side of (82). In the second term we recognise, using backwards the second assertion of Lemma 6.1, the derivative of the integral of  $f$  with respect to the area of  $F_2$ . If  $n = 1$ , we thus find

$$\begin{aligned}
\left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} \right) \mathbb{E}_{\mathcal{YM}^G}[f] &= \mathbb{E}_{\mathcal{YM}^G} \left[ \frac{1}{2} \Delta^{(e_1)} f \right] \\
&+ \mathbb{E}_{\mathcal{YM}^G} \left[ \left\langle \nabla^{(e_1)} (h \mapsto \log Q_{|F_{n+1}|}(h(\partial F_{n+1}))), \nabla^{(e_1)} f \right\rangle \right],
\end{aligned}$$

proving the result in this case.

In order to treat the case where  $n > 1$ , we need to transform the last term of (84), and we do so by using the third assertion of Lemma 6.1. We find

$$\begin{aligned}
\frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^G}[f] &= \frac{1}{2} \mathbb{E}_{\mathcal{YM}^G}[\Delta^{(e_1)} f] + \frac{d}{d|F_2|} \mathbb{E}_{\mathcal{YM}^G}[f] \\
(85) \quad &- \sum_{k=1}^d \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(e_1)} f \right) (h) \mathbf{D}_{X_k}^{(e_2^{-1}, e_2)} (Q_{|F_2|}(h(\partial F_2))) \prod_{F \in \mathbb{F}^b \setminus \{F_2\}} Q_{|F|}(h(\partial F)) dh.
\end{aligned}$$

We have of course chosen the edge  $e_2$  as bounding  $F_2$  positively. This allows us to move one step forward along the sequence of faces which we are given. For this, we proceed to an integration by parts with respect to the edge  $e_2$ , which brings in the term  $Q_{|F_3|}(h(\partial F_3))$ . In order to prepare for this, we use (79) :

$$\begin{aligned}
\frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^G}[f] &= \frac{1}{2} \mathbb{E}_{\mathcal{YM}^G}[\Delta^{(e_1)} f] + \frac{d}{d|F_2|} \mathbb{E}_{\mathcal{YM}^G}[f] \\
(86) \quad &- \sum_{k=1}^{N^2} \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(e_2, e_1)} f \right) (h) \mathbf{D}_{X_k}^{(e_2)} (Q_{|F_2|}(h(\partial F_2))) \prod_{F \in \mathbb{F}^b \setminus \{F_2\}} Q_{|F|}(h(\partial F)) dh.
\end{aligned}$$

The integration by parts is now possible and yields

$$\begin{aligned}
\frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] &= \frac{d}{d|F_2|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] + \frac{1}{2} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[\Delta^{(e_1)} f] \\
&+ \sum_{k=1}^d \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(e_2)} \mathbf{D}_{X_k}^{(c_2, e_1)} f \right) (h) \prod_{F \in \mathbb{F}} Q_{|F|}(h(\partial F)) dh \\
&+ \sum_{k=1}^d \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(c_2, e_1)} f \right) (h) \mathbf{D}_{X_k}^{(e_2)} (Q_{|F_3|}(h(\partial F_3))) \prod_{F \in \mathbb{F}^b \setminus \{F_3\}} Q_{|F|}(h(\partial F)) dh \\
&= \frac{d}{d|F_2|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] + \frac{1}{2} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[\Delta^{(e_1)} f] + \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[\Delta^{(e_2)(e_2, c_1)} f] \\
&- \sum_{k=1}^d \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{X_k}^{(e_1)} f \right) (h) \mathbf{D}_{X_k}^{(c_3^{-1} c_2^{-1}, e_3)} (Q_{|F_3|}(h(\partial F_3))) \prod_{F \in \mathbb{F}^b \setminus \{F_3\}} Q_{|F|}(h(\partial F)) dh.
\end{aligned}$$

The last term is similar to the last term of the right-hand side of (85), just one step further in the sequence of faces  $F_1, \dots, F_{n+1}$ . We can continue this until we reach the end of this sequence: if  $F_{n+1}$  is not the unbounded face, a straightforward induction argument finishes the proof. If on the other hand  $F_{n+1} = F_{\infty}$ , we still need to observe, as we did in the case where  $n = 1$ , that there are only two terms in the integrand which depend on the edge  $e_n$ , namely  $f(h)$  and  $Q_{|F_n|}(h(\partial F_n))$ . Hence, in the last integration by parts, with respect to this edge  $e_n$ , only one term is produced, which is the integral of  $\Delta^{(e_n)(c_n \dots c_2, e_1)} f$ . This concludes the proof also in this case.  $\square$

In the process of computing the derivative with respect to the area of  $F_1$  of the integral of  $f$ , the derivative with respect to the area of  $F_2$  of the same integral has unexpectedly appeared. We can easily correct this as follows, thus generalising (83).

**Corollary 6.3.** *Recall the notation of Proposition 6.2. Let us assume that the face  $F_{n+1}$  is the unbounded face of  $\mathbb{G}$ . Then the following equality holds :*

$$(87) \quad \frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] = \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}} \left[ \frac{1}{2} \sum_{i=1}^n \Delta^{(e_i)} f + \sum_{1 \leq i < j \leq n} \Delta^{(e_j)(c_j \dots c_{i+1}, e_i)} f \right].$$

*Proof.* Simply write

$$\frac{d}{d|F_1|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] = \sum_{i=1}^{n-1} \left( \frac{d}{d|F_i|} - \frac{d}{d|F_{i+1}|} \right) \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f] + \frac{d}{d|F_n|} \mathbb{E}_{\mathcal{YM}^{\mathbb{G}}}[f]$$

and apply Proposition 6.2 to each term.  $\square$

Let us emphasise that in Proposition 6.2 and its Corollary 6.3, the sequence of faces  $F_1, \dots, F_n$  may contain repetitions. We only assumed that each face is different from the next.

**6.3. Derivatives of spin networks.** The results of Section 6.2 are very general in the sense that they hold for an arbitrary observable  $f$ . Our original motivation is to compute the functions  $\Phi_N^{\mathbb{K}}$  and  $\Phi$ , and in order to achieve this, we are going to apply (87) to observables of the form  $f : h \mapsto \text{tr}(h(l))$ , which are called Wilson loops. The point is that when  $f$  is a Wilson loop, the

right-hand side of (87) can be computed fairly explicitly, and turns out to be a polynomial of Wilson loops. This is what we explain in the present section and in the next.

Although our short term interest is in Wilson loops, we start by discussing spin networks, which are slightly more general and, in a sense, more intrinsic observables. We believe that this discussion may be relevant to an interpretation of Proposition 6.2 as giving some insight into the structure of the infinitesimal generator of the Yang-Mills measure, with the role of time played by the area. This interpretation has yet to be given a consistent and substantial form.

Let us recall how spin networks are defined. Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. For each vertex  $v$ , we define the set  $\text{Out}(v)$  as the set of edges issued from  $v$ :

$$\text{Out}(v) = \{e \in \mathbb{E} : \underline{e} = v\}.$$

In order to define a spin network, we need first to choose a collection  $\alpha = (\alpha_e)_{e \in \mathbb{E}}$  of representations of  $G$ , acting respectively on the real or complex linear spaces  $(V_e)_{e \in \mathbb{E}}$ , and such that for all  $e \in \mathbb{E}$ , we have  $V_{e^{-1}} = V_e^*$  and  $\alpha_{e^{-1}} = \alpha_e^\vee$ , the contragredient representation of  $\alpha_e$ . We also need to choose a collection  $I = (I_v)_{v \in \mathbb{V}}$  of tensors such that for all  $v \in \mathbb{V}$ , the tensor  $I_v$  belongs to  $\bigotimes_{e \in \text{Out}(v)} V_e$ .

From the data of  $\alpha$  and  $I$ , we build a function  $\psi_{\alpha, I} : \mathcal{C}_G^{\mathbb{G}} \rightarrow \mathbb{C}$  as follows. Let us choose an orientation  $\mathbb{E}^+ \subset \mathbb{E}$  of  $\mathbb{G}$ . Let  $h \in \mathcal{C}_G^{\mathbb{G}} = \mathcal{M}(\mathbb{P}(\mathbb{G}), G)$  be an element of the configuration space. On one hand, the tensor  $\bigotimes_{v \in \mathbb{V}} I_v$  belongs to  $\bigotimes_{v \in \mathbb{V}} \bigotimes_{e \in \text{Out}(v)} V_e$ . On the other hand, through the natural identification  $\text{End}(V_e) \simeq V_e^* \otimes V_e$ , the tensor  $\bigotimes_{e \in \mathbb{E}^+} \alpha_e(h(e))$  belongs to  $\bigotimes_{e \in \mathbb{E}^+} V_e^* \otimes V_e \simeq \bigotimes_{e \in \mathbb{E}} V_e^* \simeq \bigotimes_{v \in \mathbb{V}} \bigotimes_{e \in \text{Out}(v)} V_e^*$ . We define, according to these identifications,

$$(88) \quad \psi_{\alpha, I}(h) = \left\langle \bigotimes_{e \in \mathbb{E}^+} \alpha_e(h(e)), \bigotimes_{v \in \mathbb{V}} I_v \right\rangle.$$

If the structure group is a compact matrix group, then spin networks are exactly the polynomial functions on  $\mathcal{C}_G^{\mathbb{G}}$ , that is, the functions which map a configuration  $h$  to a polynomial in the entries of the matrices  $\{h(e) : e \in \mathbb{E}\}$ . Using the classical theory of characters, in particular their orthogonality properties, it can easily be shown that the spin networks  $\psi_{\alpha, I}$  and  $\psi_{\alpha', I'}$  are equal only if  $\alpha = \alpha'$  and  $I = I'$ .

We will need a variant of the definition of a spin network in which not all the pairs  $V_e^* \otimes V_e$  which appear in (88) are contracted. Instead, we contract all the pairs but one, which corresponds to a certain edge  $e$ . In order to define this properly, let us denote, for each pair  $(v, e) \in \mathbb{V} \times \mathbb{E}$  such that  $e \in \text{Out}(v)$ , by  $\text{Tr}_{(v, e)} : V_e^* \otimes V_e \rightarrow \mathbb{C}$  the natural contraction. We thus have

$$\psi_{\alpha, I}(h) = \left( \bigotimes_{(v, e)} \text{Tr}_{(v, e)} \right) \left( \bigotimes_{e \in \mathbb{E}^+} \alpha_e(h(e)) \otimes \bigotimes_{v \in \mathbb{V}} I_v \right),$$

where the first tensor product is taken over all pairs  $(v, e)$  with  $e \in \text{Out}(v)$ . Given an edge  $e \in \mathbb{E}$ , we now define

$$\psi_{\alpha, I}^{(e)} : \mathcal{C}_G^{\mathbb{G}} \rightarrow \text{End}(V_e)$$

by setting

$$\psi_{\alpha, I}^{(e)}(h) = \left( \bigotimes_{(w, f) \neq (e, e)} \text{Tr}_{(w, f)} \right) \left( \bigotimes_{e \in \mathbb{E}^+} \alpha_e(h(e)) \otimes \bigotimes_{v \in \mathbb{V}} I_v \right).$$

Similarly, if  $e_1$  and  $e_2$  are two distinct edges, we define

$$\psi_{\alpha, I}^{(e_2)(e_1)} : \mathcal{C}_G^{\mathbb{G}} \rightarrow \text{End}(V_{e_2}) \otimes \text{End}(V_{e_1})$$

by not contracting the pairs  $V_{e_2}^* \otimes V_{e_2}$  and  $V_{e_1}^* \otimes V_{e_1}$ .



We shall also need the following notation. Recall that  $C_{\mathfrak{g}}$  denotes the Casimir element of  $\mathfrak{g}$ , equal to  $\sum_{k=1}^d X_k \otimes X_k$  for any choice of an orthonormal basis  $\{X_1, \dots, X_d\}$  of  $\mathfrak{g}$ . If  $\alpha$  is a representation of  $G$  on a vector space  $V$ , then we denote by  $\alpha(C_{\mathfrak{g}})$  the endomorphism  $\sum_{k=1}^d \alpha(X_k)^2$  of  $V$ .

We can now compute the effect of some of our differential operators on spin networks. We shall use the notation  $\iota_e : \text{End}(V_e) \rightarrow \bigotimes_{f \in \text{Out}(v)} \text{End}(V_f)$  and  $\iota_{e_1, e_2} : \text{End}(V_{e_1}) \otimes \text{End}(V_{e_2}) \rightarrow \bigotimes_{e \in \text{Out}(v)} \text{End}(V_e)$  for the natural operators analogous to  $\iota_{i,j}$  defined by (18).

**Proposition 6.4.** *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $\psi_{\alpha, I} : \mathcal{C}^{\mathbb{G}} \rightarrow \mathbb{C}$  be a spin network. Let  $e, e_1, e_2 \in \mathbb{E}$  be three edges issued from the same vertex  $v$ . Assume that  $e_1 \neq e_2$ . Choose  $X \in \mathfrak{g}$ . The following equalities hold. Firstly,*

$$\mathbf{D}_X^{(e)} \psi_{\alpha, I} = \text{Tr}_{V_e} \left( \psi_{\alpha, I}^{(e)} \circ \alpha_e(X) \right) = \psi_{\alpha, I'},$$

where  $I'_w = I_w$  for all  $w \neq v$ , and  $I'_v = \iota_e(\alpha_e(X))(I_v)$ . Secondly,

$$\Delta_X^{(e)} \psi_{\alpha, I} = \text{Tr}_{V_e} \left( \psi_{\alpha, I}^{(e)} \circ \alpha_e(C_{\mathfrak{g}}) \right) = \psi_{\alpha, I''},$$

where  $I''_w = I_w$  for all  $w \neq v$ , and  $I''_v = \iota_e(\alpha_e(C_{\mathfrak{g}}))(I_v)$ . Finally,

$$(89) \quad \Delta^{(e_2)(e_1)} \psi_{\alpha, I} = \text{Tr}_{V_{e_2} \otimes V_{e_1}} \left( \psi_{\alpha, I}^{(e_2)(e_1)} \circ (\alpha_{e_2} \otimes \alpha_{e_1})(C_{\mathfrak{g}}) \right) = \psi_{\alpha, I'''},$$

where  $I'''_w = I_w$  for all  $w \neq v$  and  $I'''_v = \iota_{e_1, e_2}((\alpha_{e_1} \otimes \alpha_{e_2})(C_{\mathfrak{g}}))(I_v)$ .

These assertions are illustrated by Figure 16 below.

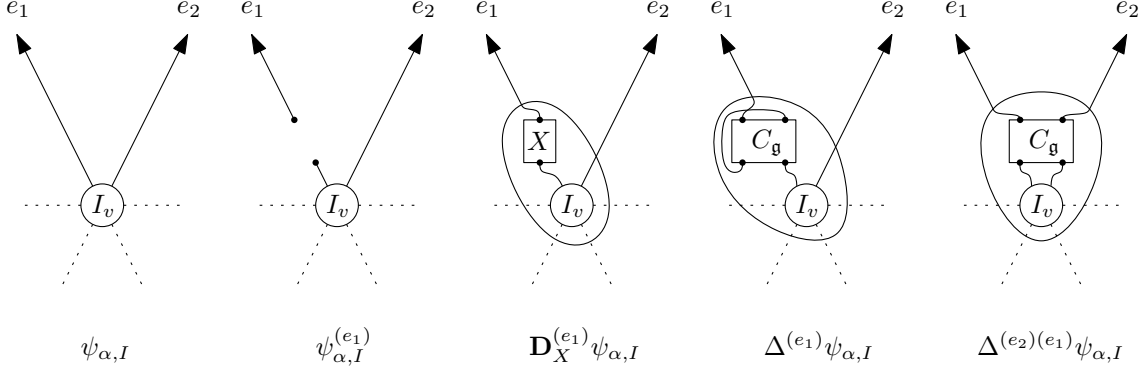


FIGURE 16. The first picture shows a spin network around the vertex  $v$ . The representations associated with the edges are not indicated explicitly. In the second picture, the two dots indicate that  $\psi_{\alpha, I}^{(e_1)}$  takes its values in  $V_{e_1}^* \otimes V_{e_1}$ . In the rightmost picture, the Casimir operator  $C_{\mathfrak{g}}$  acts through the representation  $\alpha_{e_1} \otimes \alpha_{e_2}$ .

*Proof.* Let us assume that  $\mathbb{G}$  is oriented in such a way that  $e \in \mathbb{E}^+$ . From the definition of  $\psi_{\alpha, I}^{(e)}$  we have, for all  $g \in G$  and all  $h \in \mathcal{C}^{\mathbb{G}} \simeq G^{\mathbb{E}^+}$ ,

$$\text{Tr}_{V_e} \left( \psi_{\alpha, I}^{(e)} \circ \alpha_e(g) \right) = \psi_{\alpha, I}(h'),$$

where  $h' \in G^{\mathbb{E}^+}$  has all components equal to those of  $h$ , except for  $h'(e)$  which is given by  $h'(e) = h(e)g$ . Differentiating with respect to  $g$  yields the first equality. Differentiating a second time yields the second.

For all  $X, Y \in \mathfrak{g}$ , one has

$$\mathbf{D}_Y^{(e_2)} \mathbf{D}_X^{(e_1)} \psi_{\alpha, I} = \text{Tr}_{V_{e_2} \otimes V_{e_1}} \left( \psi_{\alpha, I}^{(e_2)(e_1)} \circ (\alpha_{e_2}(Y) \otimes \alpha_{e_1}(X)) \right),$$

from which we deduce the third assertion.  $\square$

Note that a spin network built from representations of  $G$  on real vector spaces is a real-valued observable. This case will be useful to treat the orthogonal and symplectic cases.

Let us generalise our formulas to the case of  $\mathbf{D}_X^{(c, e)} \psi_{\alpha, I}$  and  $\Delta^{(e_2)(c, e_1)} \psi_{\alpha, I}$ . This requires a construction which is slightly unpleasant to describe verbally, but much more easily explained by a picture: see Figure 17 below.

Let  $\psi_{\alpha, I} : \mathcal{C}^{\mathbb{G}} \rightarrow \mathbb{C}$  be a spin network. Let  $e_1, e_2 \in \mathbb{E}$  be two edges, and  $c \in \mathbf{P}(\mathbb{G})$  be such that  $c$  joins the starting point of  $e_2$  to the starting point of  $e_1$ . Choose  $X \in \mathfrak{g}$ . For each edge  $e \in \mathbb{E}$ , let  $n_e^+$  and  $n_e^-$  be respectively the number of times  $c$  traverses  $e$  and  $e^{-1}$ , and set  $\alpha'_e = \alpha_e \otimes (\alpha_{e_1}^\vee \otimes \alpha_{e_1})^{\otimes n_e^+} \otimes (\alpha_{e_1} \otimes \alpha_{e_1}^\vee)^{\otimes n_e^-}$ . We are adding twice as many new factors as the number of times  $c$  traverses  $e$  or  $e^{-1}$ , because we are, in a sense, inserting both  $c$  and  $c^{-1}$  to the spin network. For each vertex  $v \in \mathbb{V}$  which is not  $\underline{e_1}$  nor  $\underline{e_2}$ , let  $n_v$  be the number of times  $c$  visits  $v$ , and set  $I'_v = I_v \otimes (\text{id}_{V_{e_1}^*} \otimes \text{id}_{V_{e_1}})^{\otimes n_v}$ , seen as an element of  $\bigotimes_{e \in \text{Out}(v)} V_e$  in such a way as to connect, for each visit of  $c$  and  $c^{-1}$ , the incoming edge with the outgoing one. Then, set  $v_1 = \underline{e_1}$  and  $I'_{v_1} = I_{v_1} \otimes (\text{id}_{V_{e_1}^*} \otimes \text{id}_{V_{e_1}})^{\otimes (n_{v_1}-1)} \otimes \text{id}_{V_{e_1}}$ . In this tensor, the component of  $I_{v_1}$  in  $V_{e_1}$  is now seen as a part of the component associated to the last edge of  $c$ , the  $n_{v_1} - 1$  factors  $\text{id}_{V_{e_1}^*} \otimes \text{id}_{V_{e_1}}$  connect the incoming and outgoing strands of  $c$  at each visit except the last, and the last factor  $\text{id}_{V_{e_1}}$  connects the last edge of  $c$  to  $e_1$ . Finally, set  $v_2 = \underline{e_2}$  and  $I'_{v_2} = I_{v_2} \otimes (\text{id}_{V_{e_1}^*} \otimes \text{id}_{V_{e_1}})^{\otimes (n_{v_2}-1)} \otimes \alpha_{e_1}(X)$ , in which the interpretation of  $I_{v_2}$  is unchanged, the middle factor connects the strands of  $c$  at each visit but the first, and  $\alpha_{e_1}(X)$  belongs to the  $V_{e_1}^* \otimes V_{e_1}$  part of the first edge of  $c$ .

Let us also define  $\alpha'' = \alpha'$ , and  $I''_v = I'_v$  for all  $v \neq v_2$ . For  $v = v_2$ , we set  $I''_{v_2} = c_{e_2}(I_{v_2} \otimes (\text{id}_{V_{e_1}^*} \otimes \text{id}_{V_{e_1}})^{\otimes (n_{v_2}-1)} \otimes \alpha_{e_1} \otimes (\alpha_{e_2})(C_{\mathfrak{g}}))$ , where  $c_{e_2}$  is the contraction of the  $V_{e_2}$  factor of  $I_{v_2}$  and the  $V_{e_2}^*$  factor of  $(\alpha_{e_2})(C_{\mathfrak{g}})$ .

We leave the details of the proof of the following proposition to the reader.

**Proposition 6.5.** *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $\psi_{\alpha, I} : \mathcal{C}_{\mathbb{G}} \rightarrow \mathbb{C}$  be a spin network. Let  $e_1, e_2 \in \mathbb{E}$  be two edges, and  $c \in \mathbf{P}(\mathbb{G})$  be such that  $c$  joins the starting point of  $e_2$  to the starting point of  $e_1$ . Choose  $X \in \mathfrak{g}$ . The following equalities hold:*

$$(90) \quad \begin{aligned} \mathbf{D}_X^{(c, e)} \psi_{\alpha, I} &= \text{Tr}_{V_e} \left( \psi_{\alpha, I}^{(e)} \circ \alpha_e(\text{Ad}(h(c))X) \right) = \psi_{\alpha', I'}, \\ \Delta^{(e_2)(c, e_1)} \psi_{\alpha, I} &= \text{Tr}_{V_{e_2} \otimes V_{e_1}} \left( \psi_{\alpha, I}^{(e_2)(e_1)} \circ (\alpha_{e_2} \otimes \alpha_{e_1})([\text{id}_{\mathfrak{g}} \otimes \text{Ad}(h(c))]C_{\mathfrak{g}}) \right) = \psi_{\alpha'', I''}. \end{aligned}$$

**6.4. Derivatives of Wilson loops.** Wilson loops are a particular case of spin networks, so that we already know, in principle, how our differential operators act on them. However, we shall interpret (89) and (90) very concretely when the group  $G$  is one of the groups  $U(N, \mathbb{K})$  which we studied in the first part of this work.

Let us start by giving a formal definition of Wilson loops. Let  $\mathbb{G}$  be a graph,  $l \in \mathbf{L}(\mathbb{G})$  a loop,  $\chi : G \rightarrow \mathbb{C}$  a conjugation-invariant function. The Wilson loop associated to this data is the

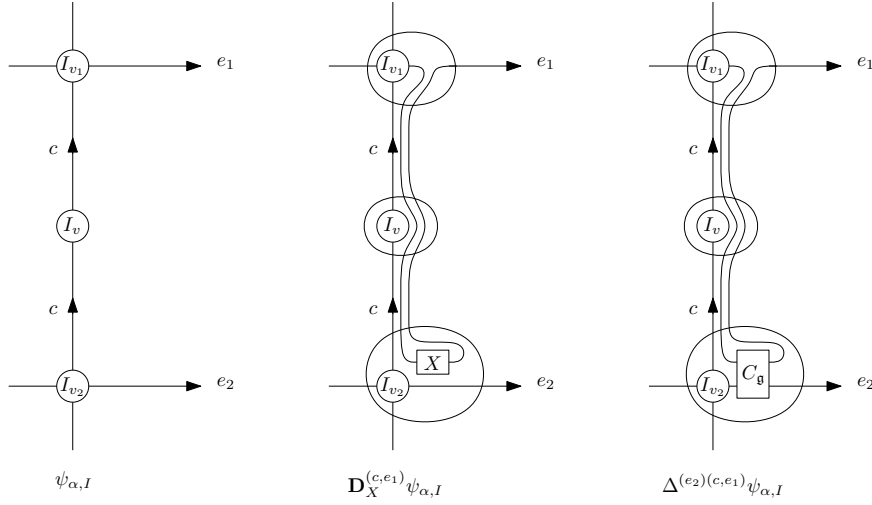


FIGURE 17. As in Figure 16, we do not indicate the representations explicitly. In this example, the path  $c$  is constituted by two edges and crosses a vertex  $v$  between  $v_1$  and  $v_2$ .

invariant observable  $W_{\chi, l} : \mathcal{C}_G^{\mathbb{G}} \rightarrow \mathbb{C}$  defined by

$$W_{\chi, l}(h) = \chi(h(l)).$$

When  $G$  is a matrix group, the function  $\chi$  is often taken to be the normalised trace, or the real part of the normalised trace in the case of a quaternionic group. More precisely, if  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  then we define  $W_l^{\mathbb{K}, N} : \mathcal{C}_{\mathbb{U}(N, \mathbb{K})}^{\mathbb{G}} \rightarrow \mathbb{C}$  by setting

$$W_l^{\mathbb{K}, N}(h) = \text{tr}(h(l)) \text{ if } \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, \text{ and } W_l^{\mathbb{H}, N}(h) = \Re \text{tr}(h(l)).$$

In contrast with spin networks, which form an algebra of observables, Wilson loops do not, for polynomials of Wilson loops are not Wilson loops in general. Let us introduce a convenient notation in order to deal with products of Wilson loops.

Let us call *skein* a finite collection  $\mathcal{S} = \{l_1, \dots, l_r\}$  of elementary loops (see Section 5.3) such that there exists a graph  $\mathbb{G}$  whose skeleton is the union of the ranges of  $l_1, \dots, l_r$  and such that in each pair  $\{e, e^{-1}\}$  of edges, exactly one edge is traversed exactly once by exactly one of the loops  $l_1, \dots, l_r$ . If we insist that each vertex of  $\mathbb{G}$  has degree at least 4, then  $\mathbb{G}$  is completely determined by  $\mathcal{S}$  and we denote it by  $\mathbb{G}(\mathcal{S})$ . In particular, if  $\mathcal{S} = \{l\}$  consists in a single elementary loop, then  $\mathbb{G}(\mathcal{S}) = \mathbb{G}(l)$  as defined in Section 5.4. To a skein  $\mathcal{S} = \{l_1, \dots, l_r\}$  we associate the observable  $W_{N, \mathcal{S}}^{\mathbb{K}} = W_{N, l_1}^{\mathbb{K}} \dots W_{N, l_r}^{\mathbb{K}}$ , or more generally  $W_{\chi, \mathcal{S}} = W_{\chi, l_1} \dots W_{\chi, l_r}$ , which we call a *Wilson skein*.

Let  $\mathcal{S} = \{l_1, \dots, l_r\}$  be a skein and set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . Each edge of  $\mathbb{G}$  is naturally oriented by the unique loop of  $\mathcal{S}$  which traverses it. Let us denote by  $\mathbb{E}^+ = \{e_1, \dots, e_n\}$  the corresponding orientation. The skein  $\mathcal{S}$  determines a permutation of  $\mathbb{E}^+$ , which to each edge  $e$  associates the edge traversed immediately after  $e$  by the unique loop of  $\mathcal{S}$  which traverses  $e$ . We denote this permutation by  $\lambda_{\mathcal{S}}$ . The cycles of  $\lambda_{\mathcal{S}}$  are naturally in bijection with the elements of  $\mathcal{S}$ . In particular,  $\ell(\lambda_{\mathcal{S}}) = r$ . Through the labelling of  $\mathbb{E}^+$  which we have chosen, we identify it with the set  $\{1, \dots, n\}$ , and  $\lambda_{\mathcal{S}}$  with an element of  $\mathfrak{S}_n$ .

Recall the notation of Sections 2.5 and 2.7, in particular the definitions of the morphisms  $\rho_{\mathbb{K}}$  (see (43)).

**Lemma 6.6.** *With the notation above, the Wilson skein  $W_{N,\mathcal{S}}^{\mathbb{K}}$  can be written as follows:*

(91)

$$\forall h \in \mathcal{C}^{\mathbb{G}}, W_{N,\mathcal{S}}^{\mathbb{K}}(h) = \begin{cases} N^{-r} \mathrm{Tr}^{\otimes n}(\rho_{\mathbb{K}}(\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_n)) & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}, \\ (-2N)^{-r} (-2\Re \mathrm{Tr})^{\otimes n}(\rho_{\mathbb{H}}(\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_n)) & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

*Proof.* According to the formula (39), we have, in the real and complex cases,

$$N^{-r} \mathrm{Tr}^{\otimes n}(\rho_{\mathbb{K}}(\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_n)) = \prod_{(i_1 \dots i_s) \preceq \lambda_{\mathcal{S}}} \mathrm{tr}(h(e_{i_s}) \dots h(e_{i_1})).$$

Observe that, since  $\lambda_{\mathcal{S}}$  is a permutation, the signs  $\varepsilon_1, \dots, \varepsilon_n$  are all equal to 1. Now, each cycle  $(i_1 \dots i_s)$  of  $\lambda_{\mathcal{S}}$  corresponds to a loop  $e_{i_1} \dots e_{i_s}$  of  $\mathcal{S}$  and the multiplicativity of  $h$  (recall (72)) reads  $h(e_{i_s}) \dots h(e_{i_1}) = h(e_{i_1} \dots e_{i_s})$ . Thus, the right-hand side of the equality above is exactly  $W_{N,\mathcal{S}}^{\mathbb{K}}(h)$ .

In the quaternionic case, we apply Lemma 2.6 and use the same argument.  $\square$

Let us define two operations on skeins, analogous to the operations  $S_{i,j}$  and  $F_{i,j}$  which we defined on the Brauer algebra in Section 3.4. Let  $\mathcal{S} = \{l_1, \dots, l_r\}$  be a skein. Let  $e_1$  and  $e_2$  be two edges of  $\mathbb{G}(\mathcal{S})$  such that  $\underline{e_1} = \underline{e_2}$ . Let us assume that  $e_1$  and  $e_2$  belong to  $\mathbb{E}^+$ . We can assume that  $e_1$  is traversed by the loop  $l_1$ . Let us first assume that  $e_2$  is also traversed by  $l_1$ . We can write  $l_1 = ae_1be_2c$ , where  $a, b, c$  are paths in  $\mathbb{G}$ . Let us define  $l'_1 = e_1b$ ,  $l''_1 = e_2ca$  and  $\tilde{l}_1 = e_1b(e_2ca)^{-1}$ . These are elementary loops in  $\mathbb{G}(\mathcal{S})$ . We define

$$S^{(e_1)(e_2)}\mathcal{S} = \{l'_1, l''_1, l_2, \dots, l_r\} \text{ and } F^{(e_1)(e_2)}\mathcal{S} = \{\tilde{l}_1, l_2, \dots, l_r\}.$$

In the case where  $e_2$  is not traversed by  $l_1$ , we may assume that it is traversed by  $l_2$ . Let us write the loops as  $l_1 = ae_1b$  and  $l_2 = ce_2d$ , where  $a, b, c, d$  are paths. We define  $l' = e_1bae_2dc$  and  $\tilde{l} = e_1ba(e_2dc)^{-1}$ , and set

$$S^{(e_1)(e_2)}\mathcal{S} = \{l', l_3, \dots, l_r\} \text{ and } F^{(e_1)(e_2)}\mathcal{S} = \{\tilde{l}, l_3, \dots, l_r\}.$$

Note that, in both cases, the graph which underlies  $S^{(e_1)(e_2)}\mathcal{S}$  and  $F^{(e_1)(e_2)}\mathcal{S}$  is the same graph which underlies  $\mathcal{S}$ .

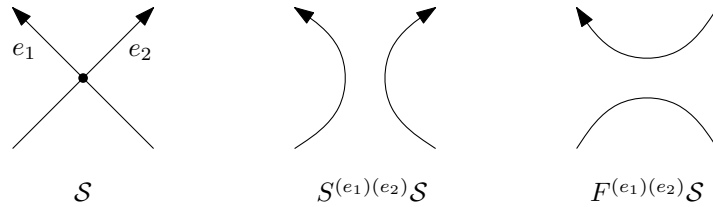


FIGURE 18. The operations  $S^{(e_1)(e_2)}$  and  $F^{(e_1)(e_2)}$  can be understood as acting locally at the common origin of  $e_1$  and  $e_2$ . From this point of view, the fact that  $e_1$  and  $e_2$  are on the same loop or not does not matter. The only difference is, in the case of the operation  $F^{(e_1)(e_2)}$ , the direction in which the lower strand is traversed. In fact, in this case, the orientation of the loop  $\tilde{l}$  itself is arbitrary. We chose to let it traverse  $e_1$  positively, but the other choice would not make any difference, since this operation is used only in the real and quaternionic cases, where a Wilson loop is not altered by changing the loop into its inverse.

The following proposition shows that the vector space of smooth complex-valued functions on  $\mathcal{C}_{U(N,\mathbb{K})}^{\mathbb{G}}$  spanned by Wilson skeins is stable under the action of the operators  $\Delta^{(e_1)}$  and  $\Delta^{(e_2)(e_1)}$ .

**Proposition 6.7.** *Let  $\mathcal{S}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . Let  $\mathbb{E}^+$  be the orientation of  $\mathbb{G}$  induced by  $\mathcal{S}$ . Let  $e_1, e_2$  be two distinct edges of  $\mathbb{G}$  issued from the same vertex  $v$ . The following properties hold.*

1.  $\Delta^{(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = c_{\mathfrak{u}(N,\mathbb{K})} W_{N,\mathcal{S}}^{\mathbb{K}}$ .
2.  $\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = \Delta^{(e_1)(e_2)} W_{N,\mathcal{S}}^{\mathbb{K}}$ .
3. If  $e_1 \notin \mathbb{E}^+$ , then  $\lambda_{\mathcal{S}}(e_1^{-1}) \in \text{Out}(v) \cap \mathbb{E}^+$  and  $\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = -\Delta^{(e_2)(\lambda_{\mathcal{S}}(e_1^{-1}))} W_{N,\mathcal{S}}^{\mathbb{K}}$ .
4. Let us assume that  $e_1$  and  $e_2$  belong to  $\mathbb{E}^+$ . If  $e_1$  and  $e_2$  are traversed by the same loop of  $\mathcal{S}$ , then

$$\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = \begin{cases} -W_{N,\mathcal{S}(e_1)(e_2)} + \frac{1}{N} W_{N,F(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{R}, \\ -W_{N,\mathcal{S}(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{C}, \\ -W_{N,\mathcal{S}(e_1)(e_2)} - \frac{1}{2N} W_{N,F(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

If, on the contrary,  $e_1$  and  $e_2$  are traversed by distinct loops of  $\mathcal{S}$ , then

$$N^2 \Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = \begin{cases} -W_{N,\mathcal{S}(e_1)(e_2)} + W_{N,F(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{R}, \\ -W_{N,\mathcal{S}(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{C}, \\ -\frac{1}{4} W_{N,\mathcal{S}(e_1)(e_2)} + \frac{1}{4} W_{N,F(e_1)(e_2)} & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

The superscripts  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  attached to the Wilson skeins are implicit in the right-hand sides of the last two equations.

*Proof.* 1. The easiest way to derive this relation is to start from (91) and to use the definition of  $c_{\mathfrak{u}(N,\mathbb{K})}$  given by (13). In the orthogonal case for example, we have, for all  $h \in \mathcal{C}_{U(N,\mathbb{R})}^{\mathbb{G}}$ ,

$$\begin{aligned} \Delta^{(e_1)} W_{N,\mathcal{S}}^{\mathbb{R}}(h) &= N^{-r} \sum_{k=1}^{\frac{N(N-1)}{2}} \frac{d^2}{dt^2} \Big|_{t=0} \text{Tr}^{\otimes n}(\rho_{\mathbb{R}}(\lambda_{\mathcal{S}}) \circ h(e_1) e^{tX_k} \otimes \dots \otimes h(e_n)) \\ &= N^{-r} \text{Tr}^{\otimes n}(\rho_{\mathbb{R}}(\lambda_{\mathcal{S}}) \circ c_{\mathfrak{u}(N,\mathbb{R})} h(e_1) \otimes \dots \otimes h(e_n)) \\ &= c_{\mathfrak{u}(N,\mathbb{R})} W_{N,\mathcal{S}}^{\mathbb{R}}(h). \end{aligned}$$

2. For all  $X, Y \in \mathfrak{g}$ , the operators  $\mathbf{D}_X^{(e_1)}$  and  $\mathbf{D}_Y^{(e_2)}$  commute. The equality follows immediately.

3. Let us assume that  $e_1 \notin \mathbb{E}^+$ . Let us consider  $h \in \mathcal{C}_{U(N,\mathbb{K})}^{\mathbb{G}}$ . Then  $W_{N,\mathcal{S}}^{\mathbb{K}}(h)$ , according to its initial definition, is a product of traces, one of which involves  $h(e_1^{-1})$  and  $h(\lambda_{\mathcal{S}}(e_1^{-1}))$ . Thus, for all  $X \in \mathfrak{g}$  and thanks to (80), we can compute

$$\begin{aligned} \mathbf{D}_X^{(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}}(h) &= \dots \frac{d}{dt} \Big|_{t=0} \text{Tr}(h(\lambda_{\mathcal{S}}(e_1^{-1})) e^{-tX} h(e_1^{-1}) \dots) \dots \\ &= \mathbf{D}_{-X}^{(\lambda_{\mathcal{S}}(e_1^{-1}))} W_{N,\mathcal{S}}^{\mathbb{K}}(h). \end{aligned}$$

This implies the desired equality.

4. The enumeration of  $\mathbb{E}^+$  which we have chosen there does not play any particular role, and we may assume that it is compatible with our choice of the edges  $e_1$  and  $e_2$ . Using the definition of the operator  $\Delta^{(e_2)(e_1)}$ , we find, in close analogy with (54),

$$\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} = N^{-r} \text{Tr}^{\otimes n}(\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{K})}) \circ \rho_{\mathbb{K}}(\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_r))$$

if  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and

$$\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{H}} = (-2N)^{-r} (-2\Re \text{Tr})^{\otimes n}(\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{H})}) \circ \rho_{\mathbb{H}}(\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_r))$$

if  $\mathbb{K} = \mathbb{H}$ . Note that the operator  $\mathbf{D}_X^{(e_1)}$  multiplies  $h(e_1)$  on the right by  $X$ , so that  $\rho(\lambda_{\mathcal{S}})$  is multiplied on the left.

Thanks to the expressions (11), (12), (34) and (42) of the Casimir operators, we know that

$$N\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{K})}) = \begin{cases} -(\rho((1\ 2)) - \rho(\langle 1\ 2 \rangle)) & (\mathbb{K} = \mathbb{R}), \\ -\rho((1\ 2)) & (\mathbb{K} = \mathbb{C}), \\ \frac{1}{2}(\rho_{\mathbb{H}}((1\ 2)) - \rho_{\mathbb{H}}(\langle 1\ 2 \rangle)) & (\mathbb{K} = \mathbb{H}). \end{cases}$$

Since the mappings  $\rho_{\mathbb{K}}$  are homomorphisms of algebra, we can perform the computation easily. If  $\mathbb{K} = \mathbb{C}$ , then

$$\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{C})})\rho_{\mathbb{C}}(\lambda_{\mathcal{S}}) = -\frac{1}{N}\rho_{\mathbb{C}}((e_1\ e_2)\lambda_{\mathcal{S}}).$$

A simple verification shows that  $(e_1\ e_2)\lambda_{\mathcal{S}} = \lambda_{\mathcal{S}^{(e_1)(e_2)}}$ . Thus,  $\Delta^{(e_2)(e_1)}W_{N,\mathcal{S}}^{\mathbb{C}}$  is a multiple of  $W_{N,\mathcal{S}^{(e_1)(e_2)}}^{\mathbb{C}}$ , with a coefficient which depends on the number of loops in the skein  $\mathcal{S}^{(e_1)(e_2)}$  and which is easily checked to give the expected equalities.

If  $\mathbb{K} = \mathbb{R}$ , then

$$\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{R})})\rho_{\mathbb{R}}(\lambda_{\mathcal{S}}) = -\frac{1}{N}\rho_{\mathbb{R}}((e_1\ e_2)\lambda_{\mathcal{S}}) + \frac{1}{N}\rho_{\mathbb{R}}(\langle e_1\ e_2 \rangle\lambda_{\mathcal{S}}).$$

Using (39), we find that for all  $h \in \mathcal{C}^{\mathbb{G}}$ ,  $N^{-r}\mathrm{Tr}^{\otimes n}(\rho_{\mathbb{R}}(\langle e_1\ e_2 \rangle\lambda_{\mathcal{S}}) \circ h(e_1) \otimes \dots \otimes h(e_n))$  is equal either to  $W_{N,\mathcal{S}^{(e_1)(e_2)}}^{\mathbb{R}}(h)$ , if  $e_1$  and  $e_2$  are traversed by the same loop of  $\mathcal{S}$ , or to  $\frac{1}{N}W_{N,\mathcal{S}^{(e_1)(e_2)}}^{\mathbb{R}}(h)$  if they are not.

Finally, if  $\mathbb{K} = \mathbb{H}$ , then

$$\iota_{1,2}(C_{\mathfrak{u}(N,\mathbb{H})})\rho_{\mathbb{H}}(\lambda_{\mathcal{S}}) = -\frac{1}{-2N}\rho_{\mathbb{H}}((e_1\ e_2)\lambda_{\mathcal{S}}) + \frac{1}{-2N}\rho_{\mathbb{H}}(\langle e_1\ e_2 \rangle\lambda_{\mathcal{S}}).$$

Lemma 2.6 allows us to conclude the proof as in the orthogonal case.  $\square$

It is necessary for our purposes to study the action of the more general operators  $\Delta^{(e_2)(c,e_1)}$  on Wilson loops. Unfortunately, the linear space generated by the set of Wilson skeins on a given graph is not stable under the action of these operators. We thus need to extend the definition of a skein.

Let  $\mathbb{G}$  be a graph. Let  $T \subset \mathbb{E}$  be a subtree of  $\mathbb{G}$ , that is, a subset of  $\mathbb{E}$  which is stable under the map  $e \mapsto e^{-1}$  and such that any two vertices which are endpoints of edges of  $T$  are joined by a unique reduced path in  $T$ . We say that a collection of loops  $\mathcal{G} = \{l_1, \dots, l_r\} \subset \mathbb{L}(\mathbb{G})$  is a *garland* with respect to  $\mathbb{G}$  and  $T$  if in each pair  $\{e, e^{-1}\}$  of edges contained in  $\mathbb{E} \setminus T$ , exactly one edge is traversed exactly once by exactly one of the loops  $l_1, \dots, l_r$ . The edges of  $T$ , on the other hand, can be traversed many times. A garland determines uniquely the underlying graph, but not necessarily the underlying subtree. Given a garland  $\mathcal{G}$  on  $(\mathbb{G}, T)$ , we define naturally the *Wilson garland*  $W_{N,\mathcal{G}}^{\mathbb{K}} : \mathcal{C}_{\mathfrak{U}(N,\mathbb{K})}^{\mathbb{G}} \rightarrow \mathbb{C}$  by  $W_{N,\mathcal{G}}^{\mathbb{K}} = W_{N,l_1}^{\mathbb{K}} \dots W_{N,l_r}^{\mathbb{K}}$ .

Let us extend the definition of the permutation  $\lambda_{\mathcal{S}}$  to the case of garlands. Let  $\mathcal{G} = \{l_1, \dots, l_r\}$  be a garland on  $(\mathbb{G}, T)$ . Each edge of  $\mathbb{G}$  which is not in  $T$  is given an orientation by the unique loop of  $\mathcal{G}$  which traverses it. This determines a partial orientation  $(\mathbb{E} \setminus T)^+$  of  $\mathbb{G}$ . The order in which the loops of  $\mathcal{G}$  traverse the edges of  $(\mathbb{E} \setminus T)^+$  determines a permutation  $\lambda_{\mathcal{G}}$  of  $(\mathbb{E} \setminus T)^+$  and we claim that this permutation suffices to determine the Wilson garland associated to  $\mathcal{G}$ . Indeed, recall that for all vertices  $v$  and  $w$  adjacent to an edge of  $T$ , we denote by  $[v, w]_T$  the unique reduced path in  $T$  which joins  $v$  to  $w$ . Now if  $(e_1 \dots e_n)$  is the cycle of  $\lambda_{\mathcal{G}}$  corresponding to the loop  $l_1$ , then the loop  $l_1$  is equivalent to the loop

$$l_1^{\mathrm{red}} = e_1[\bar{e}_1, \underline{e}_2]_T e_2 \dots e_{n-1}[\bar{e}_{n-1}, \underline{e}_n]_T e_n[\bar{e}_n, \underline{e}_1]_T,$$

which is reduced (see Section 4.3) and completely determined by  $\mathbb{G}$ ,  $T$  and  $\lambda_{\mathcal{G}}$ . The set of loops  $\mathcal{G}^{\text{red}} = \{l_1^{\text{red}}, \dots, l_r^{\text{red}}\}$  on  $\mathbb{G}$  is a garland with respect to  $T$  which is completely determined by  $\lambda_{\mathcal{G}}$  and satisfies  $W_{N, \mathcal{G}^{\text{red}}}^{\mathbb{K}} = W_{N, \mathcal{G}}^{\mathbb{K}}$ .

Let us extend the operations  $S$  and  $F$  which we defined above to garlands. Let  $\mathcal{G} = \{l_1, \dots, l_r\}$  be a garland on  $(\mathbb{G}, T)$ . Let  $e_1$  and  $e_2$  be two edges of  $\mathbb{G}$  such that  $\underline{e_1}$  and  $\underline{e_2}$  are adjacent to  $T$ . Let  $c$  be a path in  $T$  which joins  $\underline{e_2}$  to  $\underline{e_1}$ . Let us assume that  $e_1$  and  $e_2$  belong to  $(\mathbb{E} \setminus T)^+$ , and that  $e_1$  is traversed by the loop  $\underline{l_1}$ . Let us first treat the case where  $e_2$  is also traversed by  $\underline{l_1}$ . We can write  $\underline{l_1} = ae_1be_2d$ , where  $a, b, d$  are paths in  $\mathbb{G}$ . Let us define  $l'_1 = e_1bc$ ,  $l''_1 = e_2dac^{-1}$  and  $\tilde{l}_1 = e_1bc(e_2da)^{-1}c$ . We define

$$S^{(e_1)(c, e_2)}\mathcal{G} = \{l'_1, l''_1, l_2, \dots, l_r\} \text{ and } F^{(e_1)(c, e_2)}\mathcal{G} = \{\tilde{l}_1, l_2, \dots, l_r\}.$$

In the case where  $e_2$  is not traversed by  $\underline{l_1}$ , we may assume that it is traversed by  $\underline{l_2}$ . Let us write the loops as  $\underline{l_1} = ae_1b$  and  $\underline{l_2} = de_2f$ , where  $a, b, d, f$  are paths. We define  $l' = e_1bac^{-1}e_2fdc$  and  $\tilde{l} = e_1bac^{-1}(e_2fd)^{-1}c$ , and set

$$S^{(e_1)(c, e_2)}\mathcal{G} = \{l', l_3, \dots, l_r\} \text{ and } F^{(e_1)(c, e_2)}\mathcal{G} = \{\tilde{l}, l_3, \dots, l_r\}.$$

One checks easily in both cases that  $S^{(e_1)(c, e_2)}\mathcal{G}$  and  $F^{(e_1)(c, e_2)}\mathcal{G}$  are still garlands on  $(\mathbb{G}, T)$ .

We can formulate a result analogous to Proposition 6.7. This one shows that the linear space of smooth complex-valued functions on  $\mathcal{C}_{\mathbb{U}(N, \mathbb{K})}^{\mathbb{G}}$  spanned by Wilson garlands with respect to  $T$  is stable under the action of the operators  $\Delta^{(e_2)(c, e_1)}$ , where  $e_1$  and  $e_2$  are edges of  $\mathbb{E} \setminus T$  issued from a vertex adjacent to  $T$  and  $c$  is a path in  $T$ .

**Proposition 6.8.** *Let  $\mathbb{G}$  be a graph and  $T$  a subtree of  $\mathbb{G}$ . Let  $\mathcal{G}$  be a garland on  $(\mathbb{G}, T)$ . Let  $(\mathbb{E} \setminus T)^+$  be the partial orientation of  $\mathbb{G}$  induced by  $\mathcal{G}$ . Let  $e_1, e_2$  be two distinct edges of  $\mathbb{E} \setminus T$  issued from two vertices adjacent to  $T$ . Let  $c$  be a path in  $T$  from the starting point of  $e_1$  to the starting point of  $e_2$ . The following properties hold.*

1. *If  $e_1 \notin (\mathbb{E} \setminus T)^+$ , then  $\lambda_{\mathcal{G}}(e_1^{-1}) \in (\mathbb{E} \setminus T)^+$  and is issued from a vertex adjacent to  $T$ . Moreover,  $\Delta^{(e_2)(c, e_1)}W_{N, \mathcal{G}}^{\mathbb{K}} = -\Delta^{(e_2)(c', \lambda_{\mathcal{G}}(e_1^{-1}))}W_{N, \mathcal{G}}^{\mathbb{K}}$ , where  $c' = c[e_1, \underline{\lambda_{\mathcal{G}}(e_1^{-1})}]_T$ .*
2. *If  $e_2 \notin (\mathbb{E} \setminus T)^+$ , then  $\lambda_{\mathcal{G}}(e_2^{-1}) \in (\mathbb{E} \setminus T)^+$  and is issued from a vertex adjacent to  $T$ . Moreover,  $\Delta^{(e_2)(c, e_1)}W_{N, \mathcal{G}}^{\mathbb{K}} = -\Delta^{(\lambda_{\mathcal{G}}(e_2^{-1}))(c'', e_1)}W_{N, \mathcal{G}}^{\mathbb{K}}$ , where  $c'' = [\underline{\lambda_{\mathcal{G}}(e_2^{-1})}, e_2]_T$ .*
3. *Let us assume that  $e_1$  and  $e_2$  belong to  $(\mathbb{E} \setminus T)^+$ . The fourth assertion of Proposition 6.7 holds after substituting everywhere  $W_{N, \mathcal{S}}^{\mathbb{K}}$  by  $W_{N, \mathcal{G}}^{\mathbb{K}}$ ,  $\Delta^{(e_2)(e_1)}$  by  $\Delta^{(e_2)(c, e_1)}$ ,  $S^{(e_2)(e_1)}$  by  $S^{(e_2)(c, e_1)}$  and  $F^{(e_2)(e_1)}$  by  $F^{(e_2)(c, e_1)}$ .*

The most natural proof of this proposition involves the gauge invariance of the Wilson loops, which we have not yet discussed. Let us give a brief and general account of this property.

Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $G$  be a compact connected Lie group. The gauge group is by definition the group  $G^{\mathbb{V}}$  equipped with pointwise multiplication. It acts on  $\mathcal{M}(\mathbb{P}(\mathbb{G}), G)$  according to the following rule : given  $j = (j(v))_{v \in \mathbb{V}} \in G^{\mathbb{V}}$  and a multiplicative function  $h$ , we have for all path  $c$

$$(j \cdot h)(c) = j(\bar{c})^{-1}h(c)j(\underline{c}).$$

One checks easily that this is a right action in the sense that if  $j$  and  $k$  belong to the gauge group and  $h$  is a multiplicative function, then  $(jk) \cdot h = k \cdot (j \cdot h)$ .

The gauge group acts naturally on the space of smooth functions on the configuration space  $G^{\mathbb{E}^+}$  : if  $f$  is such a smooth function,  $j$  a gauge transformation and  $h$  a multiplicative function, then we have, by definition,

$$(j \cdot f)(h) = f(j^{-1} \cdot h),$$

so that again, if  $k$  is another gauge transformation,  $(jk) \cdot f = k \cdot (j \cdot f)$ . We say that a function on the configuration space  $G^{\mathbb{E}^+}$  is *invariant* if it is invariant under the action of the gauge group.

It is straightforward to check that Wilson loops, hence Wilson skeins and Wilson garlands, are invariant functions on  $\mathcal{C}_G^{\mathbb{G}}$ . Not all the differential operators which we have defined preserve the subspace of smooth invariant functions. The following lemma clarifies this point.

**Lemma 6.9.** *Let  $f : \mathcal{C}_G^{\mathbb{G}} \rightarrow \mathbb{C}$  be a smooth function. Let  $e_1$  and  $e_2$  be two edges of  $\mathbb{G}$ . Let  $c$  be a path joining the starting point of  $e_2$  to the starting point of  $e_1$ .*

1. *For all  $j \in G^V$  and all  $X \in \mathfrak{g}$ , the following equality holds:*

$$(92) \quad j \cdot \left( \mathbf{D}_X^{(c, e_1)}(j^{-1} \cdot f) \right) = \mathbf{D}_{\text{Ad}(j(c)^{-1})X}^{(c, e_1)} f.$$

2. *The operator  $\Delta^{(e_2)(c, e_1)}$  is invariant. In other words, for all  $j \in G^V$ , the following equality holds:*

$$j \cdot \left( \Delta^{(e_2)(c, e_1)}(j^{-1} \cdot f) \right) = \Delta^{(e_2)(c, e_1)} f.$$

The proof of this lemma is a straightforward application of the definitions and we leave it to the reader.

We need another piece of information. Let  $T$  be a subtree of  $\mathbb{G}$ . Let  $v_0$  be a vertex of  $T$ . For all configuration  $h \in \mathcal{C}_G^{\mathbb{G}}$ , let us consider the element  $j_{h, T}$  of the gauge group defined by

$$j_{h, T}(v) = \begin{cases} h([v_0, v]_T) & \text{if } v \in T \\ 1 & \text{otherwise.} \end{cases}$$

Then  $j_{h, T} \cdot h$  is identically equal to 1 on the edges of  $T$ .

We have now gathered the tools to prove Proposition 6.8.

*Proof of Proposition 6.8.* The equalities which we need to prove are pointwise equalities of functions on  $\mathcal{C}_{\mathbb{U}(N, \mathbb{K})}^{\mathbb{G}}$ . By Lemma 6.9, all the functions which we consider are invariant. Let  $h$  be an element of the configuration space  $\mathcal{C}_{\mathbb{U}(N, \mathbb{K})}^{\mathbb{G}}$ . Two invariant functions agree at  $h$  if and only if they agree at  $j_{h, T} \cdot h$ .

Now, if we write  $(\mathbb{E} \setminus T)^+ = \{e_1, \dots, e_n\}$ , then Lemma 6.6 holds if  $W_{N, \mathcal{S}}^{\mathbb{K}}$  is replaced by  $W_{N, \mathcal{G}}^{\mathbb{K}}$ ,  $\lambda_{\mathcal{S}}$  by  $\lambda_{\mathcal{G}}$  and  $h$  by  $j_{h, T} \cdot h$ . Let us use this to prove the first assertion. The proof of the second assertion is very similar.

Let us assume that  $e_1 \notin (\mathbb{E} \setminus T)^+$ . As in the proof of the third assertion of Proposition 6.7, we have

$$\begin{aligned} \mathbf{D}_X^{(c, e_1)} W_{N, \mathcal{G}}^{\mathbb{K}}(j_{h, T} \cdot h) &= \dots \frac{d}{dt} \Big|_{t=0} \text{Tr}((j_{h, T} \cdot h)(\lambda_{\mathcal{G}}(e_1^{-1})) e^{-t \text{Ad}((j_{h, T} \cdot h)(c))X} (j_{h, T} \cdot h)(e_1^{-1}) \dots) \dots \\ &= \mathbf{D}_{-\text{Ad}((j_{h, T} \cdot h)(c'))X}^{(\lambda_{\mathcal{G}}(e_1^{-1}))} W_{N, \mathcal{G}}^{\mathbb{K}}(j_{h, T} \cdot h) \\ &= -\mathbf{D}_X^{(c', \lambda_{\mathcal{G}}(e_1^{-1}))} W_{N, \mathcal{G}}^{\mathbb{K}}(j_{h, T} \cdot h), \end{aligned}$$

the point being that  $(j_{h, T} \cdot h)(c) = (j_{h, T} \cdot h)(c') = 1$ .

The proof of the third assertion is a mild adaptation of the proof of the fourth assertion of Proposition 6.7, of which we leave the detail to the reader. Note that Figure 17 is relevant to the situation which is being analysed here.  $\square$

Let us conclude this section by a formula of integration by parts related to gauge invariance and which completes the remark which we made just before stating Proposition 6.2. This is also a general statement in the sense that it does not depend on the structure group with which we are working.



**Proposition 6.10.** *Let  $f : \mathcal{C}_G^{\mathbb{G}} \rightarrow \mathbb{C}$  be a smooth invariant function. Let  $e$  be an edge of  $\mathbb{G}$ . Let  $c$  be a path finishing at the starting point of  $e$ . Let  $X$  be an element of  $\mathfrak{g}$ . The following equality holds:*

$$\int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_X^{(c,e)} f \right) (h) dh = 0.$$

*Proof.* In a word, we are going to average the equality (92) over the gauge group, which is a compact Lie group.

The invariance of  $f$  and (92) imply that for all  $j \in G^{\mathbb{V}}$ , we have

$$\int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_X^{(c,e)} f \right) (h) dh = \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_{\text{Ad}(j(\underline{c})^{-1})X}^{(c,e)} f \right) (j^{-1} \cdot h) dh.$$

Since the Haar measure on  $G^{\mathbb{E}^+}$  is invariant under the action of  $G^{\mathbb{V}}$ , we can replace  $j^{-1} \cdot h$  by  $h$  in the right-hand side and, averaging over  $j$ , we find

$$\int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_X^{(c,e)} f \right) (h) dh = \int_{G^{\mathbb{E}^+} \times G^{\mathbb{V}}} \left( \mathbf{D}_{\text{Ad}(j(\underline{c})^{-1})X}^{(c,e)} f \right) (h) dj dh,$$

which by linearity of the map  $X \mapsto \mathbf{D}_X^{(c,e)}$ , is equal to

$$\int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_Z^{(c,e)} f \right) (h) dh,$$

where we have set  $Z = \int_{G^{\mathbb{V}}} \text{Ad}(j(\underline{c})^{-1})X dj$ . We compute  $Z = \int_G \text{Ad}(x)X dx$ , which belongs to the centre of the Lie algebra  $\mathfrak{g}$ . Hence, for all  $h \in G^{\mathbb{E}^+}$ ,  $\mathbf{D}_Z^{(c,e)} f(h) = \mathbf{D}_{\text{Ad}(h(\underline{c}))Z}^{(e)} f(h) = \mathbf{D}_Z^{(e)} f(h)$ , and we finally find

$$\int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_X^{(c,e)} f \right) (h) dh = \int_{G^{\mathbb{E}^+}} \left( \mathbf{D}_Z^{(e)} f \right) (h) dh,$$

which is equal to zero by the first assertion of Lemma 6.1.  $\square$

**6.5. Expectation of products of Wilson loops.** In this section, we prove that the results which we obtained so far allow us, at least in principle, to compute the expectation of any Wilson skein, for any  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  and any integer  $N$ .

The strategy is the following. Let us consider a Wilson skein  $\mathcal{S} = \{l_1, \dots, l_r\}$ . In order to compute the expectation  $\mathbb{E}_{\text{YM}^{\mathbb{G}}}[W_{N,\mathcal{S}}^{\mathbb{K}}]$ , we would like to use Corollary 6.3 to differentiate this expectation with respect to the areas of the faces of the graph  $\mathbb{G}$  which underlies  $\mathcal{S}$ . Unfortunately, (87) involves differential operators which do not preserve the space of functions on  $\mathcal{C}_{\text{U}(N,\mathbb{K})}^{\mathbb{G}}$  spanned by Wilson skeins on  $\mathbb{G}$ . On the other hand, Wilson skeins are particular cases of Wilson garlands, and we now by Proposition 6.8 that the space of functions on  $\mathcal{C}_{\text{U}(N,\mathbb{K})}^{\mathbb{G}}$  linearly spanned by Wilson garlands is stable by the differential operators which appear in (87), provided the paths which are denoted as  $c_2, \dots, c_r$  belong to the tree with respect to which we consider the garlands. Moreover, it turns out that if we choose a spanning tree  $T$  of  $\mathbb{G}$ , we can apply (87) in such a way that the paths  $c$  which appear in the operators  $\Delta^{(e_2)(c,e_1)}$  stay confined in  $T$ . Since there is a finite number of garlands with respect to a pair  $(\mathbb{G}, T)$ , we are thus able to write down a finite-dimensional differential equation, the solution of which has one particular component which is the expected value of the Wilson skein in which we were originally interested.

Given a graph  $\mathbb{G}$  and a subtree  $T$  of  $\mathbb{G}$ , let us denote by  $\mathbf{G}(\mathbb{G}, T)$  the set of garlands on  $\mathbb{G}$  with respect to  $T$ . It is a finite set. The following proposition is a rigorous formulation of the discussion of the previous paragraph.

**Proposition 6.11.** *Choose  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  and an integer  $N \geq 1$ . Let  $\mathbb{G}$  be a graph. Let  $T$  be a spanning tree of  $\mathbb{G}$ . For each face  $F$  of  $\mathbb{G}$ , there exists a  $\mathbb{G}(\mathbb{G}, T) \times \mathbb{G}(\mathbb{G}, T)$  matrix, which depends on  $\mathbb{K}, N, \mathbb{G}, T, F$ , and which we simply denote by  $M_F^{\mathbb{K}, N}$ , such that*

$$\left( \frac{d}{d|F|} - M_F^{\mathbb{K}, N} \right) \left( \mathbb{E}_{\mathbb{Y}\mathbb{M}\mathbb{G}} \left[ W_{\mathbb{G}}^{\mathbb{K}, N} \right] : \mathcal{G} \in \mathbb{G}(\mathbb{G}, T) \right) = 0.$$

*Proof.* Recall from Section 4.3 (see in particular Figure 9) the way in which the spanning tree  $T$ , once we have chosen a neighbour of the dual root, determines a labelling of the set  $\mathbb{F}$  of faces by words of integers. We use this structure on  $\mathbb{F}$  to determine a specific sequence of faces to which we shall apply Corollary 6.3.

If  $k_1 \dots k_p$  is a word of integers corresponding to a face of  $\mathbb{G}$ , we denote by  $c(k_1 \dots k_p)$  the number of children of  $k_1 \dots k_p$ , that is, the largest integer  $l$  such that  $k_1 \dots k_p l$  corresponds to a face of  $\mathbb{G}$ . If  $l \in \{0, \dots, c(k_1 \dots k_p)\}$ , we define

$$s(k_1 \dots k_p, l) = \begin{cases} (k_1 \dots k_{p-1}, k_p) & \text{if } l = c(k_1 \dots k_p), \\ (k_1 \dots k_p(l+1), 0) & \text{if } l < c(k_1 \dots k_p). \end{cases}$$

Starting from  $(\emptyset, 0)$  and iterating  $s$  until one reaches  $(\emptyset, c(\emptyset))$ , whose image by  $s$  is not defined, corresponds to the exploration of the dual tree by a person who keeps it on her left-hand side.

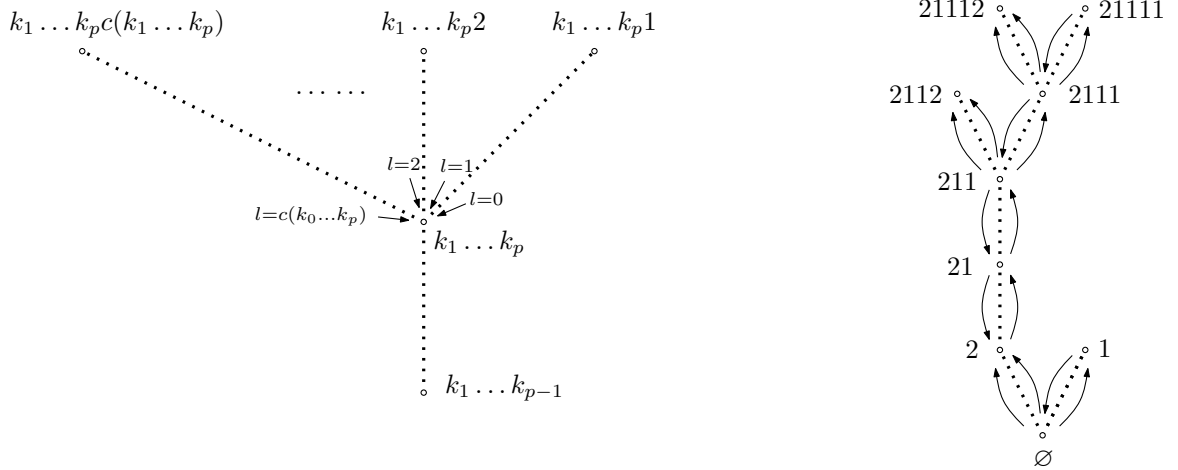


FIGURE 19. The left-hand side explains the meaning of the integer  $l$ , namely the location of the explorer around the vertex which is currently visited. The right-hand side shows the trajectory of the left-handed exploration of the dual tree of the example depicted in Figure 9.

Let us consider a bounded face  $F$  and its label  $k_1 \dots k_p$ . Let us construct a sequence of faces by starting from  $(k_1 \dots k_p, c(k_1 \dots k_p))$ , iterating  $s$  until we reach the unbounded face for the first time, and forgetting the values of  $l$  in each term of the sequence obtained. We find a sequence  $F = F_1, F_2, \dots, F_n, F_{n+1} = F_\infty$ . For example, if we use the graph depicted in Figure 9 and start from the face 21111, we find the sequence  $(21111, 2111, 21112, 2111, 211, 2112, 211, 21, 2, \emptyset)$ .

Each face of the sequence  $(F_1, \dots, F_{n+1})$  is adjacent to the next, for they correspond to adjacent vertices in the dual spanning tree. For each  $r \in \{1, \dots, n\}$ , let  $e_r$  be the unique edge of  $\mathbb{G}$  such that  $(F_r, e_r, F_{r+1})$  is a dual edge.

The fact that we chose the sequence of faces by left-handed exploration of the tree  $T$  implies that the paths  $c_2, \dots, c_r$  defined in the statement of Proposition 6.2 are paths in  $T$ .

We now apply Corollary 6.3 to the sequence  $F_1, \dots, F_{n+1}$ , the edges  $e_1, \dots, e_r$ , once for each observable of the form  $W_{N, \mathcal{G}}^{\mathbb{K}}$ , where  $\mathcal{G}$  spans  $\mathbb{G}(\mathbb{G}, T)$ . By Proposition 6.8, each derivative with respect to the area of  $F$  is expressed as a linear combination of Wilson garlands belonging to  $\mathbb{G}(\mathbb{G}, T)$ .  $\square$

Recall from (81) the definition of the Yang-Mills measure  $\mathbb{Y}\mathbb{M}_t$ . When the components of  $t$  tend to 0, for all lasso  $\lambda$  on  $\mathbb{G}$  (see Proposition 4.2), the distribution of  $h(\lambda)$  under  $\mathbb{Y}\mathbb{M}_t$  converges weakly to the Dirac mass at the identity matrix  $I_N$ . Hence, all the components of the vector  $\left(\mathbb{E}_{\mathbb{Y}\mathbb{M}_t^{\mathbb{G}}} \left[ W_{\mathcal{G}}^{\mathbb{K}, N} \right] : \mathcal{G} \in \mathbb{G}(\mathbb{G}, T) \right)$  tend to 1. Let us denote by  $\mathbb{1}$  the vector of size  $\mathbb{G}(\mathbb{G}, T)$  with all its components equal to 1. In analogy with (61), we have

$$(93) \quad \left(\mathbb{E}_{\mathbb{Y}\mathbb{M}_t^{\mathbb{G}}} \left[ W_{\mathcal{G}}^{\mathbb{K}, N} \right] : \mathcal{G} \in \mathbb{G}(\mathbb{G}, T) \right) = \left( \prod_{k=1}^q e^{t_k M_{F_k}^{\mathbb{K}, N}} \right) \mathbb{1}.$$

This is a formula of the sort we were aiming at: it provides us with a graphical procedure to compute the expectation of products of Wilson loops. It is however rather impractical. The number of garlands on a given pair  $(\mathbb{G}, T)$  is large: if all the vertices of  $\mathbb{G}$  have degree 4, which is the generic case, there are  $2^{v+1}(v+1)!$  garlands, where  $v$  is the number of vertices of  $\mathbb{G}$ . Moreover, solving the differential system (93) needs not be particularly simple.

It follows immediately from the expressions given in Proposition 6.7 that the matrices  $M_F^{\mathbb{K}, N}$  have a limit as  $N$  tends to infinity, and that this limit does not depend on  $K$ . We denote this limit simply by  $M_F$ . Since Wilson garlands are bounded by 1 by construction, we can let  $N$  tend to infinity in the equality (93), and find an algorithm to compute the master field. We shall see in the forthcoming sections that this algorithm can be significantly improved.

**6.6. Extended gauge invariance and local graphical operations.** The formulas (82) and (87) share the unpleasant feature that, in order to express the derivative of the expectation of an observable with respect to the area of a single face, they involve derivatives of the observable with respect to edges which may be very far from this face, indeed all the edges located on a path from this face to the unbounded face. We suffered from this problem in our study of the expectations of products of Wilson loops, and this led us to the introduction of Wilson garlands.

As far as concrete computations in the  $N < \infty$  case are concerned, we shall not be able to go much further than (93). However, with the limiting case  $N \rightarrow \infty$  in view, that is, for the master field itself, we shall elaborate on several works of Makeenko and Migdal [28], Kazakov [20], Kazakov and Kostov [21], and describe a much more appealing way of computing.

The main discovery of Makeenko and Migdal is that a certain linear combination of the derivatives of the expectation of a Wilson loop with respect to the area of the faces which surround a given vertex involves only local terms, in the sense that the differential operators which involve edges located far away cancel out. The original statement of the Makeenko-Migdal relation was essentially pictorial, and its proof was based on an ill-defined path integral with respect to the continuous Yang-Mills measure over the space of gauge fields. In this section, we propose a general framework in which cancellations of this sort happen. This turns out to be related with properties of invariance of the observable under consideration with respect to the action of a group larger than the gauge group. The results of this section are valid for an arbitrary gauge group.

Before we introduce the notion of invariance which will be useful, let us recall that the usual gauge symmetry of the theory is that of the configuration space of the discrete Yang-Mills theory being acted on by the gauge group, which we described after the statement of Proposition 6.8.

Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. The gauge group is the group  $G^{\mathbb{V}}$  acting on  $\mathcal{M}(\mathbb{P}(\mathbb{G}), G)$  according to the following rule : given  $j = (j(v))_{v \in \mathbb{V}} \in G^{\mathbb{V}}$  and a multiplicative function  $h$ , we have for all path  $c$

$$(j \cdot h)(c) = j(\bar{c})^{-1} h(c) j(c).$$

Let us give an infinitesimal version of the gauge invariance of a function. For each vertex  $v$ , recall that we defined the set  $\text{Out}(v)$  as the set of edges issued from  $v$ :  $\text{Out}(v) = \{e \in \mathbb{E} : \underline{e} = v\}$ .

**Lemma 6.12.** *Let  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$  be a smooth invariant function. For all vertex  $v$  and all  $X \in \mathfrak{g}$ , we have*

$$\sum_{e \in \text{Out}(v)} \mathbf{D}_X^{(e)} f = 0.$$

*Proof.* Let  $v \in \mathbb{V}$  be a vertex. We may assume that all edges adjacent to  $v$  are oriented in such a way that  $v$  is their starting point, so that  $\text{Out}(v) \subset \mathbb{E}^+$ . Choose  $X \in \mathfrak{g}$  and consider the one-parameter subgroup of gauge transformations  $j_t(v) = e^{tX}$  and  $j_t(w) = 1$  for all vertex  $w \neq v$ . Differentiating the equality  $j_t \cdot f = f$  with respect to  $t$  and evaluating at  $t = 0$  yields the desired equality.  $\square$

We want to consider invariant functions which are invariant by more transformations than just the gauge transformations. We shall give natural examples of such functions in a moment.

**Definition 6.13.** *Let  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$  be a smooth invariant function. Let  $v \in \mathbb{V}$  be a vertex. Let  $I$  be a subset of  $\text{Out}(v)$ . We say that  $f$  is  $I$ -invariant at  $v$  if for all  $X \in \mathfrak{g}$  we have the equality*

$$\sum_{e \in I} \mathbf{D}_X^{(e)} f = 0.$$

We have seen that any invariant function is  $\text{Out}(v)$ -invariant at each vertex  $v$ . It follows for instance that a smooth invariant function which is  $I$ -invariant at  $v$  is also  $(\text{Out}(v) \setminus I)$ -invariant.

The simplest examples of functions which are  $I$ -invariant at some vertex  $v$  for some proper subset  $I$  of  $\text{Out}(v)$  are provided by Wilson loops. For instance, let  $l$  be a loop in  $\mathbb{G}$  which visits exactly once the vertex  $v$ . Assume that  $l$  arrives at  $v$  through the edge  $e_1^{-1}$  and leaves  $v$  through the edge  $e_3$  (see the left-hand side of Figure 20). Then the Wilson loop  $W_{X,l}$  is invariant and  $\{e_1, e_3\}$ -invariant at  $v$ .

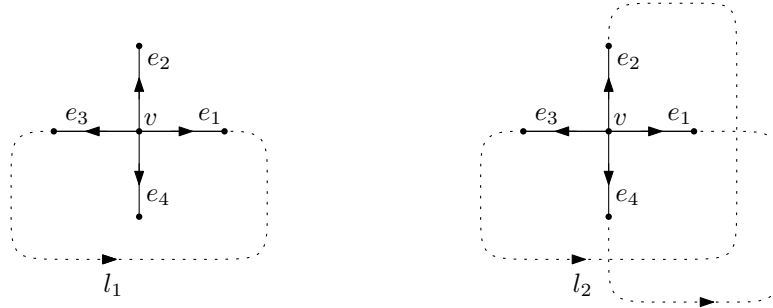


FIGURE 20. The Wilson loops associated to the loops  $l_1$  and  $l_2$  are both  $\{e_1, e_3\}$ -invariant, and also  $\{e_2, e_4\}$ -invariant, at  $v$ .

This example is however in a sense trivial, for the Wilson loop we chose does not depend at all on  $e_2$  nor  $e_4$ . The next simplest example is also the fundamental one with the Makeenko-Migdal equations in mind. It is that of a loop which visits exactly twice the vertex  $v$ , once arriving through  $e_1^{-1}$  and leaving through  $e_3$ , and once arriving through  $e_2^{-1}$  and leaving through  $e_4$  (see the right-hand side of Figure 20).

The next result shows that an observable which enjoys a property of local invariance as we have defined it also satisfies a local differential relation with respect to the areas of the faces of the graph.

**Proposition 6.14.** *Let  $f : G^{\mathbb{E}^+} \rightarrow \mathbb{R}$  be a smooth invariant function. Let  $v \in \mathbb{V}$  be a vertex. Let  $I$  be a proper non-empty subset of  $\text{Out}(v)$ . Enumerate the edges of  $\text{Out}(v)$  in cyclic clockwise order around  $v$  as  $\{e_1, \dots, e_p\}$  in such a way that  $I$  contains  $e_1$  but not  $e_p$ . Set  $r = \max\{i : e_i \in I\}$ . For each  $i \in \{1, \dots, p\}$ , let  $F_i$  denote the face adjacent to  $v$  which is bounded positively by  $e_i$ .*

*Assume that  $f$  is  $I$ -invariant at  $v$ . The following equality holds:*

$$(94) \quad \sum_{i \in I} \left( \frac{d}{d|F_i|} - \frac{d}{d|F_{i+1}|} \right) \mathbb{E}_{\text{YM}^G}[f] = \sum_{\substack{1 \leq i < j \leq r \\ i \in I, j \notin I}} \mathbb{E}_{\text{YM}^G} \left[ \Delta^{(e_i)(e_j)} f \right].$$

Since the problem is cyclically invariant around  $v$  and  $I$  is a proper subset of  $\text{Out}(v)$ , the assumption that  $I$  contains  $e_1$  and not  $e_p$  does not restrict the generality. The case where  $I$  is empty or equal to  $\text{Out}(v)$  has little interest and the result is anyway trivially true in this case.

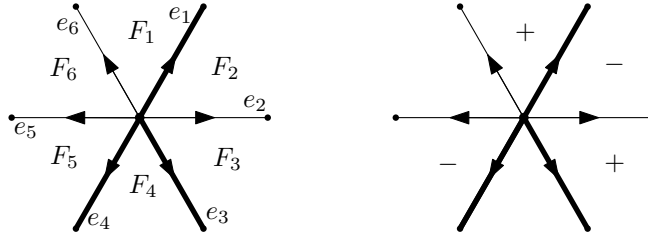


FIGURE 21. In this example, we take  $I = \{e_1, e_3, e_4\}$ . The relation is then

$$\left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_5|} \right) \mathbb{E}_{\text{YM}_N^G}[f] = \mathbb{E}_{\text{YM}_N^G} \left[ \Delta^{(e_1)(e_2)} f \right].$$

*Proof.* The proof consists in applying Proposition 6.2 as many times as there are edges in  $I$ . For this, we need to choose an appropriate path in the dual graph from  $F_1$  to the unbounded face. We do this by considering the sequence  $F_1, \dots, F_{r+1}$  (observe that  $r < p$ ), which we complete by an arbitrary sequence  $\tilde{F}_{r+2}, \dots, \tilde{F}_{n+1}$ , where  $\tilde{F}_{n+1}$  is the unbounded face. For each  $i \in I$ , we apply Proposition 6.2 with the sequence  $F_i, \dots, F_{r+1}, \tilde{F}_{r+2}, \dots, \tilde{F}_{n+1}$ . Since the edges  $e_i, \dots, e_r$  are all issued from  $v$ , the paths  $c_{i+1}, \dots, c_r$  are constant, and the paths  $c_{r+1}, \dots, c_n$  do not depend on  $i$ . For each  $i$ , we find

$$\left( \frac{d}{d|F_i|} - \frac{d}{d|F_{i+1}|} \right) \mathbb{E}_{\text{YM}^G}[f] = \frac{1}{2} \mathbb{E}_{\text{YM}^G} [\Delta^{(e_i)} f] + \sum_{j=i+1}^n \mathbb{E}_{\text{YM}^G} [\Delta^{(e_j)(c_j \dots c_{i+1}, e_i)} f].$$

Summing the right-hand side over  $i \in I$  and splitting according to the values of  $j$  gives

$$(95) \quad \mathbb{E}_{\text{YMG}} \left[ \frac{1}{2} \sum_{i \in I} \Delta^{(e_i)} f + \sum_{i \in I, i < j \leq r} \Delta^{(e_j)(e_i)} f \right] + \sum_{k=1}^d \sum_{i \in I} \sum_{j=r+1}^n \mathbb{E}_{\text{YMG}} \left[ \frac{1}{2} \mathbf{D}_{X_k}^{(e_j)} \mathbf{D}_{X_k}^{(c_j \dots c_{i+1}, e_i)} f \right].$$

The last term of (95) can be rewritten as

$$\sum_{k=1}^d \sum_{j=r+1}^n \mathbb{E}_{\text{YMG}} \left[ \frac{1}{2} \mathbf{D}_{X_k}^{(e_j)} \sum_{i \in I} \mathbf{D}_{X_k}^{(c_j \dots c_{i+1}, e_i)} f \right],$$

and for all  $h \in G^{\mathbb{E}^+}$ , all  $k \in \{1, \dots, d\}$  and all  $j \in \{r+1, \dots, n\}$ , we have

$$\left( \sum_{i \in I} \mathbf{D}_{X_k}^{(c_j \dots c_{i+1}, e_i)} f \right) (h) = \sum_{i \in I} \left( \mathbf{D}_{\text{Ad}(h(c_j \dots c_{i+1})) X_k}^{(e_i)} f \right) (h) = 0,$$

thanks to the  $I$ -invariance of  $f$ .

Another consequence of the  $I$ -invariance of  $f$  is

$$0 = \sum_{k=1}^d \left( \sum_{i \in I} \mathbf{D}_{X_k}^{(e_i)} \right)^2 f = \sum_{i \in I} \Delta^{(e_i)} f + 2 \sum_{\substack{i, j \in I \\ i < j}} \Delta^{(e_i)(e_j)} f.$$

It follows that the first term of (95) is equal to

$$\mathbb{E}_{\text{YMG}} \left[ \sum_{\substack{i < j \leq r \\ i \in I, j \notin I}} \Delta^{(e_j)(e_i)} f \right],$$

as expected.  $\square$

We mentioned before stating Proposition 6.14 that the main situation where we intended to apply it is at a point of self-intersection of a Wilson loop, or at the intersection point of two Wilson loops. First of all, let us state and prove the extended gauge-invariance properties of Wilson loops, indeed of Wilson skeins.

**Lemma 6.15.** *Let  $\mathcal{S} = \{l_1, \dots, l_r\}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . For each  $i \in \{1, \dots, r\}$ , let  $l_i = e_{i,1} \dots e_{i,n_i}$  be the decomposition of  $l_i$  as a product of edges, and set  $e_{i,n_i+1} = e_{i,1}$ . Let  $\chi : G \rightarrow \mathbb{C}$  be a central function. For each  $i \in \{1, \dots, r\}$  and each  $j \in \{1, \dots, n_i\}$ , the Wilson skein  $W_{\chi, \mathcal{S}} = W_{\chi, l_1} \dots W_{\chi, l_r}$  is  $\{e_{i,j}^{-1}, e_{i,j+1}\}$ -invariant at  $\bar{e}_{i,j}$ .*

*Proof.* Without loss of generality, we may assume that  $i = 1$ . For all  $h \in \mathcal{C}^{\mathbb{G}}$ , we have  $W_{\chi, l_1}(h) = \chi(h(e_{1,n_1}) \dots h(e_{1,1}))$ . Since  $l_1$  is an elementary loop, the edges  $e_{1,1}, \dots, e_{1,n_1}$  are pairwise distinct, even up to inversion. Thus, for each  $j \in \{1, \dots, n_1\}$ , and all  $X \in \mathfrak{g}$ , we have

$$\begin{aligned} \left( \left( \mathbf{D}_X^{(e_{1,i}^{-1})} + \mathbf{D}_X^{(e_{1,i+1})} \right) W_{\chi, l_1} \right) (h) &= \frac{d}{dt} \Big|_{t=0} \chi(h(e_{1,n_1}) \dots e^{-tX} h(e_{1,i}) \dots h(e_{1,1})) \\ &\quad + \frac{d}{dt} \Big|_{t=0} \chi(h(e_{1,n_1}) \dots h(e_{1,i+1}) e^{tX} \dots h(e_{1,1})) \\ &= 0. \end{aligned}$$

On the other hand, the product  $W_{\chi,l_2} \dots W_{\chi,l_r}$  does not depend on  $h(e_{1,i})$  nor  $h(e_{1,i+1})$ , so that for all  $X \in \mathfrak{g}$ ,

$$\mathbf{D}_X^{(e_{1,i}^{-1})}(W_{\chi,l_2} \dots W_{\chi,l_r}) = \mathbf{D}_X^{(e_{1,i+1})}(W_{\chi,l_2} \dots W_{\chi,l_r}) = 0.$$

An application of the Leibniz rule completes the proof.  $\square$

Combining the extended invariance properties of Wilson skeins (Lemma 6.15), the local differential relation which this entails for their expectation (Proposition 6.14) and the result of the action on Wilson skeins of the differential operators which appear in (94) (Proposition 6.7), we find the following result.

**Proposition 6.16** (The Makeenko-Migdal equations for Wilson loop expectations). *Let  $\mathcal{S} = \{l_1, \dots, l_r\}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . Let  $\mathbb{E}^+$  be the orientation of  $\mathbb{G}$  induced by  $\mathcal{S}$ . Let  $v$  be a vertex of  $\mathbb{G}$  of degree 4. Let  $e_1$  and  $e_2$  be the two edges of  $\mathbb{E}^+$  which start at  $v$ . Assume that  $e_1$  precedes immediately  $e_2$  in the clockwise cyclic order of the edges outgoing at  $v$ . Let  $F_1, F_2, F_3, F_4$  be the faces adjacent to  $v$ , listed in clockwise order and starting by the face located between  $e_1$  and  $e_2$ . The faces  $F_1, F_2, F_3, F_4$  need not be pairwise distinct. Then*

$$(96) \quad \left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) \mathbb{E}_{\mathbb{YMG}} \left[ W_{N,\mathcal{S}}^{\mathbb{K}} \right] = \mathbb{E}_{\mathbb{YMG}} \left[ -\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}} \right],$$

where  $\Delta^{(e_2)(e_1)} W_{N,\mathcal{S}}^{\mathbb{K}}$  is given by Proposition 6.7. If a face is the unbounded face, then the corresponding derivative must be removed from the left-hand side of (96).

In particular, if  $\mathbb{K} = \mathbb{C}$  and if  $e_1$  and  $e_2$  are traversed by the same loop,

$$(97) \quad \left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) \mathbb{E}_{\mathbb{YMG}} \left[ W_{N,\mathcal{S}}^{\mathbb{C}} \right] = \mathbb{E}_{\mathbb{YMG}} \left[ W_{N,\mathcal{S}^{(e_2)(e_1)}}^{\mathbb{C}} \right].$$

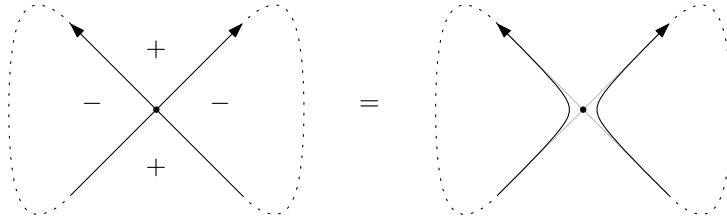


FIGURE 22. This picture illustrates the original instance of the Makeenko-Migdal equations, corresponding to (97).

*Proof.* Let  $e_3$  and  $e_4$  denote the other two outgoing edges at  $v$ , in such a way that  $e_1, e_2, e_3, e_4$  are cyclically ordered in this way around  $v$ . The function  $W_{N,\mathcal{S}}^{\mathbb{K}}$  is  $\{e_1, e_3\}$ -invariant at  $v$ . Hence, Proposition 6.14 implies

$$\left( \frac{d}{d|F_4|} - \frac{d}{d|F_1|} + \frac{d}{d|F_2|} - \frac{d}{d|F_3|} \right) \mathbb{E}_{\mathbb{YMG}} \left[ W_{N,\mathcal{S}}^{\mathbb{K}} \right] = \mathbb{E}_{\mathbb{YMG}} \left[ \Delta^{(e_1)(e_2)} W_{N,\mathcal{S}}^{\mathbb{K}} \right],$$

which is the expected equality.  $\square$

In the equality (97), we can let  $N$  tend to infinity. Recall that we denote by  $\Phi : \mathbf{L}(\mathbb{R}^2) \rightarrow \mathbb{R}$  the master field. For each skein  $\mathcal{S} = \{l_1, \dots, l_r\}$ , we set  $\Phi(\mathcal{S}) = \Phi(l_1) \dots \Phi(l_r)$ , which is the limit of  $\mathbb{E}_{\mathbb{YMG}}[W_{N,\mathcal{S}}^{\mathbb{K}}]$  as  $N$  tends to infinity. We have the following result.

**Proposition 6.17** (The Makeenko-Migdal equation for the master field). *With the notation of Proposition 6.16, we have*

$$(98) \quad \left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) \Phi(\mathcal{S}) = \Phi(S^{(e_2)(e_1)}\mathcal{S}).$$

*Proof.* It suffices to write (97) in integral form and to use the dominated convergence theorem to take the limit where  $N$  tends to infinity.  $\square$

**6.7. Underdetermination of local relations.** The Makeenko-Migdal equations convey in an extremely pleasant and practical way a lot of information on Wilson loop and Wilson skein expectations. It is clear however that these equations alone do not suffice to determine the Wilson skein expectations, if only because the derivation which appears in the left-hand side of (96) leaves invariant the sum of the areas of the bounded faces of the graph. It only redistributes it around the vertex at which the equations are applied. Another piece of information is provided by the first assertion of Proposition 6.7, which allows us to determine how the expectation changes when one increases the area of a face which is adjacent to the unbounded face.

In this section, we show that these two relations, namely the first assertion of Proposition 6.7 and the Makeenko-Migdal equation (96) do not suffice to solve the problem of computing the Wilson loop expectations for finite  $N$ . However, as we shall explain in the next section, they do suffice in the large  $N$  limit, thanks to the factorisation property of the master field.

Let us define a linear operator  $\Omega : C^\infty(\mathcal{C}^{\mathbb{G}}) \rightarrow C^\infty((\mathbb{R}_+^*)^{\mathbb{F}^b}, \mathbb{C}^{\mathbb{F}})$  by setting, for all smooth observable  $f : \mathcal{C}^{\mathbb{G}} \rightarrow \mathbb{C}$ ,

$$(99) \quad \Omega f = \left( \frac{d}{d|F|} \mathbb{E}_{\mathbb{Y}\mathbb{M}_{\mathbb{G}}} [f] : t \in (\mathbb{R}_+^*)^{\mathbb{F}^b}, F \in \mathbb{F} \right),$$

where it is understood the the derivative with respect to the unbounded face is 0. We call  $\Omega f$  the *area-derivative* of the expectation of  $f$ . We shall often see it as a vector of  $\mathbb{C}^{\mathbb{F}}$ , keeping implicit the dependence with respect to  $t$ .

Let  $\mathcal{S} = \{l_1, \dots, l_r\}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . For each  $i \in \{1, \dots, r\}$ , let  $l_i = e_{i,1} \dots e_{i,n_i}$  be the decomposition of  $l_i$  as a product of edges, and set  $e_{i,n_i+1} = e_{i,1}$ . The set  $\mathbb{E}^+ = \{e_{i,j} : i \in \{1, \dots, r\}, j \in \{1, \dots, n_i\}\}$  is the orientation of  $\mathbb{G}$  determined by  $\mathcal{S}$ . For each  $i \in \{1, \dots, r\}$  and each  $j \in \{1, \dots, n_i\}$ , let  $F_{i,j}^+$  and  $F_{i,j}^-$  denote the faces of  $\mathbb{G}$  which are respectively bounded positively and negatively by  $e_{i,j}$ . For each such  $(i, j)$ , (94) allows us to compute  $(\Omega W_{\chi, \mathcal{S}})_{F_{i,j}^-} - (\Omega W_{\chi, \mathcal{S}})_{F_{i,j}^+} + (\Omega W_{\chi, \mathcal{S}})_{F_{i,j+1}^+} - (\Omega W_{\chi, \mathcal{S}})_{F_{i,j+1}^-}$ . Thus, letting  $n = n_1 + \dots + n_r$  be the number of unoriented edges of  $\mathbb{G}$ , (94) enables us to compute the image of  $\Omega W_{\chi, \mathcal{S}}$  by the linear mapping

$$\begin{aligned} \mu_{\mathcal{S}} : \mathbb{R}^{\mathbb{F}} &\longrightarrow \mathbb{R}^{\mathbb{E}^+} \\ u &= (u_F : F \in \mathbb{F}) \longmapsto \left( (u_{F_{i,j+1}^+} - u_{F_{i,j+1}^-}) - (u_{F_{i,j}^+} - u_{F_{i,j}^-}) \right)_{\substack{i=1 \dots r \\ j=1, \dots, n_i}}. \end{aligned}$$

We must determine whether this mapping is injective or not. It is clearly not, for the vector all of whose components are equal to 1, which we denote by  $\mathbb{1}^{\mathbb{F}}$ , lies in its kernel. Perhaps less obviously, the winding number of each loop  $l_i$ , which we denote by  $n_i$ , and seen in the most natural way as an element of  $\mathbb{R}^{\mathbb{F}}$ , also belongs to the kernel of  $\mu_{\mathcal{S}}$ .

We will also analyse the image of  $\mu_{\mathcal{S}}$ . For this, let us introduce the following notation. For each edge  $e_{i,j} \in \mathbb{E}^+$ , let  $\delta_{e_{i,j}}$  denote the vector of  $\mathbb{R}^{\mathbb{E}^+}$  whose components are all equal to 0 except the  $e_{i,j}$  component, which is equal to 1. For each  $i \in \{1, \dots, r\}$ , let us write  $\delta_{l_i} = \sum_{j=1}^{n_i} \delta_{e_{i,j}}$ . For each vertex  $v$  of  $\mathbb{G}$ , let us define  $\star_v = \sum_{\overline{e_{i,j}}=v} \delta_{e_{i,j}}$ . Finally, let  $\mathbb{1}^{\mathbb{E}^+}$  be the vector of  $\mathbb{R}^{\mathbb{E}^+}$  whose



components are all equal to 1. Note that  $\sum_{i=1}^r \delta_{l_i} = \sum_{v \in \mathbb{V}} \star_v = \mathbb{1}^{\mathbb{E}^+}$ . Let us endow  $\mathbb{R}^{\mathbb{E}^+}$  with the scalar product for which  $(\delta_{e_{i,j}})_{i,j}$  is an orthonormal basis.

**Lemma 6.18.** *Assume that none of the loops  $l_1, \dots, l_r$  is constant.*

1. *The kernel of  $\mu_{\mathcal{S}}$  is spanned by the linearly independent vectors  $\mathbb{1}, \mathbf{n}_{l_1}, \dots, \mathbf{n}_{l_r}$ .*
2. *The intersection of the subspaces of  $\mathbb{R}^{\mathbb{E}^+}$  spanned respectively by  $\delta_{l_1}, \dots, \delta_{l_r}$  and  $\{\star_v : v \in \mathbb{V}\}$  is equal to the line spanned by  $\mathbb{1}^{\mathbb{E}^+}$ .*
3. *The image of  $\mu_{\mathcal{S}}$  is the subspace orthogonal to the sum of the subspaces of  $\mathbb{R}^{\mathbb{E}^+}$  spanned respectively by  $\delta_{l_1}, \dots, \delta_{l_r}$  and  $\{\star_v : v \in \mathbb{V}\}$ .*

*Proof.* In this proof, we think of  $\mathbb{R}^{\mathbb{F}}$  as the vector space of functions on  $\mathbb{R}^2$  which are locally constant on the complement of the skeleton of  $\mathbb{G}$ .

1. Let  $u$  be an element of the kernel of  $\mu_{\mathcal{S}}$ . Set  $\alpha = u_{F_\infty}$ , so that  $u - \alpha \mathbb{1}^{\mathbb{F}}$  vanishes on the unbounded face  $F_\infty$ . The equality  $\mu_{\mathcal{S}}(u - \alpha \mathbb{1}^{\mathbb{F}}) = 0$  means that the jump of  $u$  across any two consecutive edges of  $l_i$  are equal, so that  $u$  varies by a certain fixed quantity  $\beta_i$  when one crosses an edge of  $l_i$ .

The function  $u - \alpha \mathbb{1}^{\mathbb{F}} - \sum_{i=1}^r \beta_i \mathbf{n}_{l_i}$  vanishes on the unbounded face of  $\mathbb{G}$ , and does not vary when one crosses one of the edges of  $\mathbb{G}$ . It is thus identically equal to zero.

The fact that  $\mathbb{1}, \mathbf{n}_{l_1}, \dots, \mathbf{n}_{l_r}$  are linearly independent follows from the fact that given a function of the form  $\alpha \mathbb{1} + \sum_{i=1}^r \beta_i \mathbf{n}_{l_i}$ ,  $\alpha$  can be recovered from its value on the unbounded face, and  $\beta_1, \dots, \beta_r$  from its jumps across the edges of  $l_1, \dots, l_r$ .

2. Consider an equality  $w = \sum_{i=1}^r \alpha_i \delta_{l_i} = \sum_{v \in \mathbb{V}} \beta_v \star_v$  in  $\mathbb{R}^{\mathbb{E}^+}$ . The first expression of  $w$  shows that it has the same value on any two edges of a same loop of  $\mathcal{S}$ . The second expression shows that it has the same value on the edges of any two loops which visit a common vertex. Since  $\mathbb{G}$  is connected,  $w$  has the same value on each edge, that is,  $w$  is a multiple of  $\mathbb{1}^{\mathbb{E}^+}$ .

3. It is a simple verification to check that the range of  $\mu_{\mathcal{S}}$  is orthogonal to each  $\delta_{l_i}, i \in \{1, \dots, r\}$  and to each  $\star_v, v \in \mathbb{V}$ . On the other hand, by the first assertion, the range of  $\mu_{\mathcal{S}}$  has dimension  $|\mathbb{F}| - r - 1$ . We conclude the proof by observing, thanks to Euler's relation, that  $|\mathbb{F}| - r - 1 = |\mathbb{E}^+| - (|\mathbb{V}| + r - 1)$ .  $\square$

This result shows that in general, given a skein consisting of  $r$  loops, the Makeenko-Migdal equations (96) need to be completed by  $r + 1$  independent relations in order to allow one to determine  $\Omega W_{N, \mathcal{S}}^{\mathbb{K}}$ .

It turns out that two such relations are always available. The first is given by the fact that the component of  $\Omega W_{N, \mathcal{S}}^{\mathbb{K}}$  corresponding to the unbounded face is 0. The second is given by the first assertion of Proposition 6.7 applied to a face adjacent to the unbounded face. In general, no other obvious relations seem to be available. In particular, there may be only one face adjacent to the unbounded face (see Figure 23 below).

In the case of a single loop however, these relations suffice to determine  $\Omega W_{N, \mathcal{S}}^{\mathbb{K}}$  and, together with the factorisation property of the master field, this observation will allow us to design an algorithm to compute it. Let us state and prove the observation.

**Corollary 6.19.** *Let  $l$  be a non-constant elementary loop. Let  $\mathbb{G} = \mathbb{G}(l)$  be the underlying graph. Let  $F$  be a face of  $\mathbb{G}$  which shares an edge with the unbounded face. Then the mapping*

$$\begin{aligned} \tilde{\mu}_{\{l\}} : \mathbb{R}^{\mathbb{F}} &\longrightarrow \mathbb{R}^{\mathbb{E}^+} \times \mathbb{R}^2 \\ u &\longmapsto (\mu_{\{l\}}(u), u_{F_\infty}, u_F) \end{aligned}$$

*is injective.*

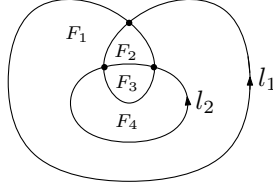


FIGURE 23. In this example, the area derivatives available through the Makeenko-Migdal equations are  $2\partial_{|F_1|} - \partial_{|F_2|}$  and  $\partial_{|F_1|} - \partial_{|F_2|} + \partial_{|F_3|} - \partial_{|F_4|}$ . Moreover, the derivative  $\partial_{|F_1|}$  is given by Proposition 6.7. This does however not suffice to determine the area-derivative of the expectation of this Wilson skein.

*Proof.* Assume that  $\tilde{\mu}_{\{l\}}(u) = 0$ . Then  $\mu_{\{l\}}(u) = 0$  and, by Lemma 6.18,  $u$  is a linear combination of  $\mathbb{1}$  and  $n_l$ . Since  $u_{F_\infty} = u_F = 0$ ,  $u$  must be equal to 0.  $\square$

**6.8. A recursive algorithm to compute the master field.** Recall that if  $\mathcal{S} = \{l_1, \dots, l_r\}$  is a skein, we denote by  $\Phi(\mathcal{S})$  the product  $\Phi(l_1) \dots \Phi(l_r)$ . We are going to prove that the value of the master field on a skein can be computed recursively from the values of the master field on skeins with less self-intersections.

Let us define what we mean by the number of true self-intersections of a skein. For this, we shall use the cyclic order induced by the orientation of  $\mathbb{R}^2$  on the edges issued from each vertex of a graph. In a cyclically ordered set  $\{x_1 < \dots < x_n < x_1\}$ , we shall say that two pairs  $\{x, y\}$  and  $\{z, t\}$  are crossing if  $x < z < y < t$  or  $x < t < y < z$ .

Let  $\mathcal{S}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . For each vertex  $v$  of  $\mathbb{G}$ , let us count how many pairs  $\{\{e_1, e_2\}, \{e_3, e_4\}\}$  of edges of  $\text{Out}(v)$  there exist which are crossing in the cyclic order of  $\text{Out}(v)$ , and such that all of these edges are traversed by the same loop of  $\mathcal{S}$ , with  $e_1$  and  $e_2$  on one hand, and  $e_3$  and  $e_4$  on the other hand, traversed consecutively. We define the number of crossings of  $\mathcal{S}$  as the sum over the vertices of  $\mathbb{G}$  of these numbers of crossing pairs. We denote it by  $\text{cn}(\mathcal{S})$ .

In the case where all vertices of  $\mathbb{G}$  have degree 4, which is the generic situation,  $\text{cn}(\mathcal{S})$  is simply the number of vertices of  $\mathbb{G}$  which are visited twice by the same loop, and at which this loop has a stable crossing with itself. By a stable crossing, we mean a crossing which cannot be removed by slightly deforming the loop.

**Lemma 6.20.** *Let  $\mathcal{S}$  be a skein. Set  $\mathbb{G} = \mathbb{G}(\mathcal{S})$ . Let  $e_1, e_2, e_3, e_4$  be four edges issued from the same vertex of  $\mathbb{G}$  and which are traversed by the same loop of  $\mathcal{S}$ . Assume that  $e_1$  and  $e_2$  on one hand, and  $e_3$  and  $e_4$  on the other hand, are traversed consecutively by this loop.*

1. *The equality  $(\Delta^{(e_1)(e_3)} + \Delta^{(e_1)(e_4)}) W_{\mathcal{S}}^{\mathbb{K}, N} = 0$  holds.*
2. *If the pairs  $\{e_1, e_2\}$  and  $\{e_3, e_4\}$  cross, then  $\text{cn}(\mathcal{S}^{(e_1)(e_3)} \mathcal{S}) < \text{cn}(\mathcal{S})$ .*

*Proof.* 1. The edges  $e_3$  and  $e_4$  are traversed consecutively by one of the loops of  $\mathcal{S}$  and, since  $e_3$  and  $e_4$  are outgoing at the same vertex, one of them is traversed positively and the other negatively. Hence, for all  $X \in \mathfrak{u}(N, \mathbb{K})$ , we have  $\mathbf{D}_X^{(e_3)} W_{\mathcal{S}}^{\mathbb{K}, N} = -\mathbf{D}_X^{(e_4)} W_{\mathcal{S}}^{\mathbb{K}, N}$ . The first assertion follows immediately.

2. Let us assume that  $e_1 < e_3 < e_2 < e_4 < e_1$  in the cyclic order. The first thing to observe is that the two pairs  $\{e_1, e_2\}$  and  $\{e_3, e_4\}$ , which are crossing in  $\mathcal{S}$ , are not crossing anymore in  $\mathcal{S}^{(e_1)(e_3)} \mathcal{S}$ .

Moreover, let us consider a third pair  $\{e_5, e_6\}$  of consecutive edges of the same loop which traverses  $e_1, e_2, e_3, e_4$ . There are, up to obvious symmetries, three ways in which these two edges can fit into the cyclic order  $e_1 < e_3 < e_2 < e_4 < e_1$ . One checks in each case that the number of

crossings involving  $\{e_5, e_6\}$  in  $S_{e_1, e_3}\mathcal{S}$  is at most equal to the same number in  $\mathcal{S}$ . (see Figure 24 below).  $\square$

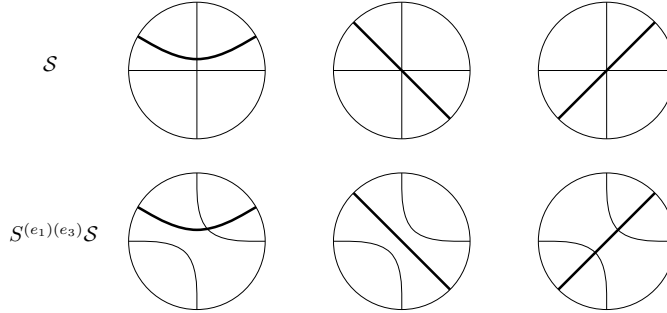


FIGURE 24. The strand corresponding to the pair of edges  $\{e_5, e_6\}$  is heavier.

The algorithm, in its principle, is described by the following proposition.

**Proposition 6.21.** *Let  $l$  be an elementary loop. Consider  $\Phi(l)$  as a function of the areas of the faces of the graph  $\mathbb{G}(l)$ . Each component of the area-derivative of  $\Phi(l)$  can be expressed as a linear combination of  $\Phi(l)$  and a quadratic polynomial in the values of the master field on a set of elementary loops on  $\mathbb{G}$  which all have a crossing number strictly smaller than that of  $l$ .*

*Proof.* Let  $l$  be an elementary loop. By Corollary 6.19, we know that the area derivative of  $\Phi(l)$  is a linear combination of the derivatives which (94) allows us to compute, and the derivative with respect to the area of a face adjacent to the unbounded face. We know that if  $F$  is such a face, then  $\frac{d}{d|F|}\Phi(l) = -\frac{1}{2}\Phi(l)$ .

Lemma 6.20 allows us to bound the number of crossings of the skeins which appear on the right-hand side of (94). Indeed, we apply this equation with  $I$  a pair of consecutive edges of the loop  $l$ . By the first assertion of Lemma 6.20, only those index  $j$  contribute to the sum such that  $e_i$  and  $e_j$  belong to strands of  $l$  which form a crossing. For those, Proposition 6.7 indicates that we are applying the operator  $S^{(e_i)(e_j)}$  to the skein  $\{l\}$ , thus producing a skein which contains two loops, for which the second assertion of Lemma 6.20 assures us that the crossing number is strictly smaller than that of  $l$ .

In particular, any loop which forms such a skein has a crossing number strictly smaller than that of  $l$ .  $\square$

This proposition leads to a recursive algorithm, provided we have a finiteness result. Indeed, on a graph  $\mathbb{G}$ , there are a finite number of skeins. In fact, there are, up to homeomorphism, only a finite number of skeins in generic position whose crossing number is smaller than a certain value.

We can now describe a set of rules which allows one to recursively compute the values of the master field on an arbitrary elementary loop. Consider a skein  $\mathcal{S} = \{l_1, \dots, l_r\}$ .

1. If  $\text{cn}(\mathcal{S}) = 0$ , then each loop  $l_i$  of  $\mathcal{S}$  bounds a disk of area  $t_i$  and

$$\Phi(\mathcal{S}) = e^{-\frac{t_1 + \dots + t_r}{2}}.$$

2. If  $\text{cn}(\mathcal{S}) > 0$ , then the area derivative of  $\Phi(\mathcal{S})$  is a linear combination of  $\Phi(\mathcal{S}')$  with  $\text{cn}(\mathcal{S}') < \text{cn}(\mathcal{S})$ .

3. When all faces have area equal to 0, we have  $\Phi(\mathcal{S}) = 1$ .

Using these rules, it is fairly easy to set up a table of the values of  $\Phi$  on the elementary loops which have few self-intersections. Incidentally, the enumeration of these loops, up to isotopy, is a difficult problem (see [2]).

**6.9. The Kazakov basis.** In [20], V. Kazakov gave without proof a beautiful description of the way in which one can invert the mapping  $\tilde{\mu}_{\{l\}}$  which we introduced in Corollary 6.19. We devote this last section to a proof of his main statements.

Let  $l$  be an elementary loop which is generic in the sense that each vertex of the graph  $\mathbb{G} = \mathbb{G}(l)$  has degree 4 and is a crossing in the sense of the previous section, except for the vertex  $l(0)$  which is of degree 2. The problem which Kazakov studied was to determine as explicitly as possible how to recover the derivative of  $\mathbb{E}[W_{N,l}^{\mathbb{C}}]$  with respect to the area of each face of  $\mathbb{G}$  from the data of the alternated sum of these derivatives with respect to the faces located around each vertex, and the derivative with respect to the area of a face adjacent to the unbounded face.

In this context, Kazakov identified a relevant and non-trivial basis of the module  $\mathbb{Z}^{\mathbb{F}^b}$ , of which we shall think as the space of locally constant integer-valued functions on the complement of the range of  $l$  which vanish at infinity.

Let us write  $\mathbb{F}^b = \{F_1, \dots, F_q\}$ . To start with, the module  $\mathbb{Z}^{\mathbb{F}^b}$  admits the trivial basis indexed by  $\mathbb{F}^b$ , which we simply denote by  $\{F_1, \dots, F_q\}$ . In order to define the second basis, let us make the assumption that the vertex  $l(0)$  is located on the boundary of the unbounded face. The orientation of  $l$  determines an orientation  $\mathbb{E}^+$  of  $\mathbb{G}$ , and it determines an order on  $\mathbb{V}$ , which is the order of first visit starting from  $l(0)$ , which we call  $v_0$ . Thus,  $\mathbb{V} = \{v_0 = l(0), v_1, \dots, v_{q-1}\}$ . Indeed, it follows from Euler's relation that the number of vertices of  $\mathbb{G}$  is the same as the number of faces.

For each  $i \in \{1, \dots, q-1\}$ , let  $l_i$  denote the subloop of  $l$  which is delimited by the two visits at the vertex  $v_i$ . Set also  $l_0 = l$ . For each  $i \in \{0, \dots, q-1\}$ , the winding number  $n_{l_i}$  (see Section 5.4) belongs to  $\mathbb{Z}^{\mathbb{F}^b}$ .

The main statement of Kazakov is a consequence of the fact that  $\{n_{l_0}, \dots, n_{l_{q-1}}\}$  is a basis of  $\mathbb{Z}^{\mathbb{F}^b}$ . We propose to understand this fact in terms of a certain positive quadratic form on the module  $\mathbb{Z}^{\mathbb{F}^b}$ . Consider the space  $\mathbb{Z}^{\mathbb{E}^+}$  of formal linear combinations of edges of  $\mathbb{E}^+$  and, in this space, the subspace formed by linear combinations such that at each vertex, the sum of the coefficients of the incoming edges equals the sum of the coefficients of the outgoing edges. This subspace is the space of algebraic cycles of  $\mathbb{G}$ , which we denote by  $H^1(\mathbb{G}; \mathbb{Z})$ . It is well known that the mapping  $\mathbb{Z}^{\mathbb{F}^b} \rightarrow H^1(\mathbb{G}; \mathbb{Z})$  which sends  $F_i$  to the cycle  $\partial F_i$  is an isomorphism.

We start by defining a bilinear form on  $\mathbb{Z}^{\mathbb{E}^+}$ . For this, consider a vertex  $v \neq v_0$ . Let  $e_1, e_2, e_3, e_4$  the elements of  $\mathbb{E}^+$  such that  $l$  traverses  $e_1, e_2, e_3, e_4$  in this order,  $e_1$  and  $e_3$  are incoming at  $v$  and  $e_2$  and  $e_4$  are outgoing at  $v$ . Observe that it is possible that  $e_2 = e_3$ , but the five other pairs of edges are distinct. If  $e_2 \neq e_3$ , we set  $\langle e_i, e_i \rangle_v = 0$  for each  $i \in \{1, 2, 3, 4\}$  and define

$$\langle e_1, e_2 \rangle_v = \langle e_3, e_4 \rangle_v = 0, \quad \langle e_1, e_3 \rangle_v = \langle e_2, e_4 \rangle_v = -\frac{1}{2} \quad \text{and} \quad \langle e_1, e_4 \rangle_v = \langle e_2, e_3 \rangle_v = \frac{1}{2}.$$

If  $e_2 = e_3$ , we set  $\langle e_2, e_3 \rangle_v = 1$  and keep the other values unchanged.

At the origin  $v_0$ , we let  $e_1$  be the incoming edge and  $e_2$  be the outgoing edge. If  $e_1 = e_2$ , we set  $\langle e_1, e_2 \rangle_{v_0} = 1$ , otherwise we set  $\langle e_1, e_1 \rangle_{v_0} = \langle e_2, e_2 \rangle_{v_0} = 0$  and  $\langle e_1, e_2 \rangle_{v_0} = \frac{1}{2}$ .

We now define a bilinear form on  $\mathbb{Z}^{\mathbb{E}^+}$  by setting

$$\langle \cdot, \cdot \rangle = \sum_{i=0}^{q-1} \langle \cdot, \cdot \rangle_{v_i}.$$

By restriction, this defines a bilinear form on  $H^1(\mathbb{G}; \mathbb{Z}) \simeq \mathbb{Z}^{\mathbb{F}^b}$ .

**Proposition 6.22.** *For all  $i, j \in \{0, \dots, q-1\}$ , the equality*

$$\langle \mathbf{n}_{l_i}, \mathbf{n}_{l_j} \rangle = \delta_{i,j}$$

holds.

*Proof.* Let us describe the bilinear form on  $H^1(\mathbb{G}; \mathbb{Z})$  in words. Since each element of  $H^1(\mathbb{G}; \mathbb{Z})$  can be written, although non uniquely, as a linear combination of loops in  $\mathbb{G}$ , it suffices to understand the bilinear form evaluated on two loops. Let  $m_1$  and  $m_2$  be two loops on  $\mathbb{G}$ . The number  $\langle m_1, m_2 \rangle$  is the sum of local contributions, one for each pair formed by a visit of  $m_1$  at a vertex of  $\mathbb{G}$  and a visit of  $m_2$  at the same vertex. At each vertex of  $\mathbb{G}$ , two strands of  $l$  cross each other and we say that a loop which visits this vertex *turns* during this visit if it arrives along one strand and leaves it along the other. The number  $\langle m_1, m_2 \rangle$  is the sum of the following numbers:

- +1 for each pair of visits of  $m_1$  and  $m_2$  in the same direction at the vertex  $v_0$ ,
- -1 for each pair of visits of  $m_1$  and  $m_2$  in opposite directions at the vertex  $v_0$ ,
- +1 for each pair of visits of  $m_1$  and  $m_2$  at a vertex distinct from  $v_0$ , such that both  $m_1$  and  $m_2$  turn during this visit, and such that  $m_1$  and  $m_2$  arrive along the same strand of  $l$ , regardless of the orientation,
- -1 for each pair of visits of  $m_1$  and  $m_2$  at a vertex distinct from  $v_0$ , such that both  $m_1$  and  $m_2$  turn during this visit, and such that  $m_1$  and  $m_2$  arrive along distinct strands of  $l$ , regardless of the orientation.

It suffices now to observe that the loop  $l_0$  turns nowhere and that for all  $i \in \{1, \dots, q-1\}$ , the loop  $l_i$  turns exactly once at  $v_i$  and does not visit  $v_0$ .  $\square$

By elementary linear algebra, we obtain the following corollary.

**Corollary 6.23.** *The bilinear form  $\langle \cdot, \cdot \rangle$  is symmetric, positive and  $\mathbb{Z}$ -valued on  $\mathbb{Z}^{\mathbb{F}^b}$  and the family  $\{\mathbf{n}_{l_0}, \dots, \mathbf{n}_{l_{q-1}}\}$  is an orthonormal basis of  $\mathbb{Z}^{\mathbb{F}^b}$ .*

It is graphically easy to express the vectors  $\mathbf{n}_{l_i}$  in the canonical basis of  $\mathbb{Z}^{\mathbb{F}^b}$ . Thanks to this corollary, it is easy to express the vectors  $\partial F_i$  in the basis  $\{\mathbf{n}_{l_0}, \dots, \mathbf{n}_{l_{q-1}}\}$ .

**Corollary 6.24.** *Let  $F$  be a bounded face of  $\mathbb{G}$ . Let  $v$  be a vertex located on the boundary of  $F$ . Let  $e_1$  and  $e_2$  be the outgoing edges at  $v$  which bound  $F$ , such that  $e_1, F, e_2$  are located in this counterclockwise order around  $v$ . If  $e_1 \neq e_2$ , set  $\varepsilon(F, v) = 1$  if  $l$  traverses  $e_1$  before  $e_2$ , regardless of the direction in which it traverses them, and  $\varepsilon(F, v) = -1$  otherwise. If  $e_1 = e_2$ , set  $\varepsilon(F, v) = 1$  if  $l$  traverses  $e_1$  and  $\varepsilon(F, v) = -1$  if  $l$  traverses  $e_1^{-1}$ .*

For all bounded face  $F$ , the following equality holds in  $\mathbb{Z}^{\mathbb{F}^b} \simeq H^1(\mathbb{G}; \mathbb{Z})$ :

$$(100) \quad \partial F = \sum_{v_i \sim F} \varepsilon(F, v_i) \mathbf{n}_{l_i},$$

where the sum is taken over the vertices adjacent to  $F$ .

*Proof.* It suffices to check that for all  $i \in \{0, \dots, q-1\}$ , one has  $\langle \partial F, \mathbf{n}_{l_i} \rangle = \varepsilon(F, v_i)$ . This is a verification which we leave to the reader.  $\square$

It follows from (93) that the expectation of a Wilson loop, seen as a function of the areas of the faces of the underlying graph, is a linear combination of functions of the form  $(t_1, \dots, t_q) \mapsto t_i^m e^{\alpha t_j}$ , where  $m$  is a non-negative integer and  $\alpha$  a real. It is in particular a smooth function on  $\mathbb{R}^{\mathbb{F}^b}$ .

Kazakov proposes to use on  $\mathbb{R}^{\mathbb{F}^b}$ , rather than the usual coordinates, the coordinates  $a = (a_0, \dots, a_{q-1})$  given by

$$\forall i \in \{0, \dots, q-1\}, a_i = \int_{\mathbb{R}^2} n_{l_i}(x) dx,$$

that is,  $a_i$  is the algebraic area enclosed by the loop  $l_i$ . Kazakov's main claims are that these are indeed coordinates on  $\mathbb{R}^{\mathbb{F}^b}$  and that the alternated sum of derivatives with respect to the areas of faces around the vertex  $v_i$ , which appears in the Makeenko-Migdal equations, is the derivative with respect to  $a_i$ . If we denote by  $t = (t_1, \dots, t_q)$  the canonical coordinates on  $\mathbb{R}^{\mathbb{F}^b}$ , we have, thanks to (100),

$$\forall j \in \{1, \dots, q\}, t_j = \sum_{v_i \sim F_j} \varepsilon(F_j, v_i) a_i,$$

and we know that the matrix which appears here is invertible. Moreover, if  $u : \mathbb{R}^{\mathbb{F}^b} \rightarrow \mathbb{C}$  is a smooth function, we find

$$(101) \quad \forall i \in \{0, \dots, q-1\}, \frac{\partial u}{\partial a_i}(t_1, \dots, t_q) = \sum_{F_j \sim v_i} \varepsilon(F_j, v_i) \frac{\partial u}{\partial t_j}(t_1, \dots, t_q).$$

The main consequence of this is the following. For all  $i \in \{1, \dots, q-1\}$ , let  $\tilde{l}_i$  denote the loop obtained from  $l$  by erasing the subloop  $l_i$ .

**Proposition 6.25.** *For all  $i \in \{0, \dots, q-1\}$ , the following equality holds:*

$$\frac{\partial}{\partial a_i} \Phi(l) = \begin{cases} -\frac{1}{2} \Phi(l) & \text{if } i = 0, \\ \Phi(l_i) \Phi(\tilde{l}_i) & \text{if } i > 0. \end{cases}$$

*Proof.* It suffices to check that the right-hand side of (101) is exactly the left-hand side of (98).  $\square$

We recover here, in a much more explicit form, the possibility of computing the master field by a recursive algorithm. In fact, Kazakov proposes, for loops of a special kind which he calls *planar*, a fairly complicated but explicit formula for  $\Phi(l)$ . Planar loops can be characterised recursively in a way which is reminiscent of the definition of non-crossing partitions. With the notation of this section, a generic elementary loop  $l$  is planar if it is a simple loop or if there exists  $i \in \{1, \dots, q-1\}$  such that  $l_i$  is a simple loop and  $\tilde{l}_i$  is planar. However, we were not yet able to analyse deeply enough Kazakov's formula to improve it or let it fit into the present work.

## Appendix : Asymptotic freeness results

We found it difficult to collect all the arguments needed to prove the results of asymptotic freeness that we use in this work. Most of these arguments are of course present in the work of Collins and Śniady, but we felt that perhaps the symplectic case deserved a more detailed treatment. We give here a brief account of the proofs in the three cases, with the references needed to fill the gaps that we leave.

**A.1. Unitary case.** Let us start by the unitary case. Let  $N$  and  $n$  be positive integers. Recall from the proof of Theorem 2.1 the definition of the representation  $\rho_{\mathbb{C}} : \mathbb{C}[\mathfrak{S}_n] \rightarrow \text{End}((\mathbb{C}^N)^{\otimes n})$ , which we then still called  $\rho$ . Consider the endomorphism  $P$  acting on the vector space  $\text{End}((\mathbb{C}^N)^{\otimes n})$  according to

$$\forall A \in \text{End}((\mathbb{C}^N)^{\otimes n}), P(A) = \int_{U(N)} U^{\otimes n} \circ A \circ (U^{-1})^{\otimes n} dU.$$

The invariance by translation of the Haar measure implies that  $P$  is a projection on a subspace of  $\text{End}((\mathbb{C}^N)^{\otimes n})$  which is contained in the commutant of the action of  $U(N)$  on  $(\mathbb{C}^N)^{\otimes n}$ . The main assertion of Schur-Weyl duality in this context is that the range of  $P$  is thus contained in the range of  $\rho_{\mathbb{C}}$  (for the three instances of the Schur-Weyl duality which we shall use, see [12]).

Collins and Śniady gave in [7] an expression of  $P(A)$  which makes this inclusion manifest. The element  $\sum_{\sigma \in \mathfrak{S}_n} N^{\ell(\sigma)} \sigma$ , which is equal to  $N^n(\text{id} + O(N^{-1}))$ , is invertible in  $\mathbb{C}[\mathfrak{S}_n]$  for  $N$  large enough, indeed for  $N \geq n$ . Its inverse is called the Weingarten function, and it is denoted by  $\text{Wg}$ . If we denote by  $X_1, \dots, X_n$  the Jucys-Murphy elements of  $\mathfrak{S}_n$ , and by  $h_k$  the classical complete symmetric functions, then we have the equalities

$$(102) \quad \text{Wg} = \prod_{i=1}^n (N + X_i)^{-1} = N^{-n} \sum_{k \geq 0} \frac{(-1)^k}{N^k} h_k(X_1, \dots, X_n).$$

Consider now the endomorphism  $Q$  of  $\text{End}((\mathbb{C}^N)^{\otimes n})$  defined by

$$\forall A \in \text{End}((\mathbb{C}^N)^{\otimes n}), \quad Q(A) = \rho_{\mathbb{C}}(\text{Wg}) \sum_{\sigma \in \mathfrak{S}_n} \text{Tr}(A \circ \rho_{\mathbb{C}}(\sigma^{-1})) \rho_{\mathbb{C}}(\sigma).$$

It satisfies  $Q(\text{id}) = \text{id}$  and  $Q(A \circ \rho_{\mathbb{C}}(\sigma)) = Q(A) \circ \rho_{\mathbb{C}}(\sigma)$  for all permutation  $\sigma$ , so that it is also a projection. For all  $A \in \text{End}((\mathbb{C}^N)^{\otimes n})$ , the endomorphism  $Q(P(A))$  is on one hand equal to  $Q(A)$ , because  $\rho_{\mathbb{C}}(\sigma)$  and  $U^{\otimes n}$  commute for all  $\sigma \in \mathfrak{S}_n$  and all  $U \in U(N)$ . On the other hand,  $Q(P(A))$  is equal to  $P(A)$ , because the range of  $P$  is contained in the range of  $\rho_{\mathbb{C}}$ . Altogether, the representation  $\rho_{\mathbb{C}}$  being understood on the right-hand side, we have for all  $A \in \text{End}((\mathbb{C}^N)^{\otimes n})$  the formula

$$\int_{U(N)} U^{\otimes n} \circ A \circ (U^{-1})^{\otimes n} dU = \text{Wg} \sum_{\sigma \in \mathfrak{S}_n} \text{Tr}(A \circ \sigma^{-1}) \sigma.$$

From this equation it follows that for all  $A_1, \dots, A_n$  and  $B_1, \dots, B_n$  in  $M_N(\mathbb{C})$ , one has

$$(103) \quad \begin{aligned} & \int_{U(N)} \text{tr}(U A_1 U^{-1} B_1 \dots U A_n U^{-1} B_n) dU = \\ &= \frac{1}{N} \int_{U(N)} \text{Tr}(U^{\otimes n} \circ A_1 \otimes \dots \otimes A_n \circ (U^{-1})^{\otimes n} \circ B_1 \otimes \dots \otimes B_n \circ (n \dots 1)) dU \\ &= \frac{1}{N} \sum_{\sigma, \tau \in \mathfrak{S}_n} \text{Wg}(\sigma \tau (n \dots 1)) \text{Tr}(A_1 \otimes \dots \otimes A_n \circ \sigma^{-1}) \text{Tr}(B_1 \otimes \dots \otimes B_n \circ \tau^{-1}). \end{aligned}$$

From the equation (102), it is possible to infer that for all  $\sigma \in \mathfrak{S}_n$  with cycle lengths  $m_1, \dots, m_r$ , one has

$$\text{Wg}(\sigma) = \frac{(-1)^{|\sigma|}}{N^{n+|\sigma|}} \prod_{i=1}^r C_{m_i-1} + O(N^{-n-|\sigma|-1}),$$

where  $C_m = \frac{1}{m+1} \binom{2m}{m}$  is the  $m$ -th Catalan number, which satisfies  $C_0 = 1$  and  $C_{m+1} = \sum_{k=0}^m C_k C_{m-k}$ , and  $|\sigma| = n - \ell(\sigma)$  is the distance between  $\sigma$  and the identity. Thus, the highest power of  $N$  which appears in the generic term of the sum (103) above is

$$\ell(\sigma) + \ell(\tau) - 1 - n - |\sigma \tau (n \dots 1)| = d(\text{id}, (1 \dots n)) - d(\text{id}, \sigma) - d(\sigma, \sigma \tau) - d(\sigma \tau, (1 \dots n)).$$

This power is nonpositive, and it is zero if and only if  $\text{id}$ ,  $\sigma$ ,  $\sigma \tau$  and  $(1 \dots n)$  are located in this order on a geodesic. We shall use the notation  $\sigma_1 \preceq \sigma_2$  to indicate that  $\sigma_1$  is located on a geodesic from  $\text{id}$  to  $\sigma_2$ . To the highest order, the sum is thus restricted to the sublattice of  $\mathfrak{S}_n$  formed by the permutations  $\sigma$  such that  $\text{id} \preceq \sigma \preceq (1 \dots n)$ . This lattice is isomorphic to the lattice  $\text{NC}_n$  of non-crossing partitions (see [5]). Moreover, for  $\sigma_1, \sigma_2$  in this lattice, with  $\sigma_1 \preceq \sigma_2$ ,

the Möbius function  $\mu(\sigma_1, \sigma_2)$  is equal to  $(-1)^{d(\sigma_1, \sigma_2)} \prod_{i=1}^r C_{m_i-1}$ , where the product runs over the cycles of  $\sigma_2 \sigma_1^{-1}$  (see [35]).

Then, using the notation

$$p_\sigma(A_1, \dots, A_n) = \prod_{\substack{c \text{ cycle of } \sigma \\ c=(i_1 \dots i_r)}} \text{tr}(A_{i_1} \dots A_{i_r}) \text{ and } \kappa_\sigma(A_1, \dots, A_n) = \prod_{\substack{c \text{ cycle of } \sigma \\ c=(i_1 \dots i_r)}} \kappa_r(A_{i_1}, \dots, A_{i_r}),$$

where  $\kappa_r$  denotes the free cumulant of order  $r$ , we have

$$\begin{aligned} \int_{U(N)} \text{tr}(U A_1 U^{-1} B_1 \dots U A_n U^{-1} B_n) dU &= \\ &= \sum_{\substack{\sigma \preccurlyeq (1 \dots n) \\ \tau \preccurlyeq \sigma^{-1}(1 \dots n)}} \mu(\tau, \sigma^{-1}(1 \dots n)) p_\sigma(A_1, \dots, A_n) p_\tau(B_1, \dots, B_n) + O(N^{-1}) \\ &= \sum_{\sigma \preccurlyeq (1 \dots n)} p_\sigma(A_1, \dots, A_n) \kappa_{\sigma^{-1}(1 \dots n)}(B_1, \dots, B_n) + O(N^{-1}) \\ &= \sum_{\pi \in \text{NC}_n} \tau_\pi(A_1, \dots, A_n) \kappa_{\pi^\vee}(B_1, \dots, B_n) + O(N^{-1}), \end{aligned}$$

where in the last line we used the classical notation  $\tau_\pi$  for the non-commutative moments and  $\pi^\vee$  for the Kreweeras complement of a non-crossing partition  $\pi$ . The last equation which we have obtained implies classically the asymptotic freeness of the families  $\{U A_1 U^{-1}, \dots, U A_n U^{-1}\}$  and  $\{B_1, \dots, B_n\}$ .

**A.2. Orthogonal case.** In the orthogonal case, things are slightly different. We start nevertheless in the same way, by defining the endomorphism  $P$  of  $\text{End}((\mathbb{R}^N)^{\otimes n})$  by

$$\forall A \in \text{End}((\mathbb{R}^N)^{\otimes n}), \quad P(A) = \int_{\text{SO}(N)} R^{\otimes n} \circ A \circ (R^{-1})^{\otimes n} dR.$$

Instead of  $\rho_{\mathbb{C}}$ , we shall naturally use the homomorphism of algebras  $\rho_{\mathbb{R}} : B_{n,N} \rightarrow \text{End}((\mathbb{R}^N)^{\otimes n})$  (see (32)). The main assertion of Schur-Weyl duality in this case is that the range of  $P$  is contained in the range of  $\rho_{\mathbb{R}}$ .

The definition of  $Q$  is however slightly different. For all  $A \in \text{End}((\mathbb{R}^N)^{\otimes n})$  we define an element  $Q_0(A)$  of  $B_{n,N}$  by setting

$$Q_0(A) = \sum_{\pi \in \mathfrak{B}_n} \text{Tr}({}^t A \circ \rho_{\mathbb{R}}(\pi)) \pi.$$

We shall prove in a moment that for  $N$  large enough, the restriction of  $Q_0$  to the range of  $\rho_{\mathbb{R}}$  is injective, hence bijective. It is proved in [7] that this is true for  $N \geq n$ . We denote by  $\text{Wg}_N$  its inverse, so that

$$\text{Wg}_N = (Q_0|_{\rho_{\mathbb{R}}(B_{n,N})})^{-1}.$$

We shall use the notation  $\text{Wg}(\pi) = \sum_{\pi' \in \mathfrak{B}_n} \text{Wg}_N(\pi, \pi') \rho_{\mathbb{R}}(\pi')$ .

Now for all endomorphism  $A$ , we have on one hand  $(\text{Wg}_N \circ Q_0)(P(A)) = (\text{Wg} \circ Q_0)(A)$ , because  $\rho_{\mathbb{R}}(\pi)$  and  $R^{\otimes n}$  commute for all  $\pi \in \mathfrak{B}_n$  and all  $R \in \text{SO}(N)$ . On the other hand, we have  $(\text{Wg}_N \circ Q_0)(P(A)) = P(A)$  because  $P(A)$  belongs to the range of  $\rho_{\mathbb{R}}$ . Hence, the formula in the orthogonal case is

$$\int_{\text{SO}(N)} R^{\otimes n} \circ A \circ (R^{-1})^{\otimes n} dU = \sum_{\pi \in \mathfrak{B}_n} \text{Tr}({}^t A \circ \pi) \text{Wg}_N(\pi).$$



As in the unitary case, it follows from this equation that for all  $A_1, \dots, A_n$  and  $B_1, \dots, B_n$  in  $M_N(\mathbb{R})$ , one has

$$(104) \quad \int_{\text{SO}(N)} \text{tr}(RA_1R^{-1}B_1 \dots RA_nR^{-1}B_n) dR = \frac{1}{N} \sum_{\pi, \pi' \in \mathfrak{B}_n} \text{Wg}_N(\pi, \pi') \text{Tr}({}^tA_1 \otimes \dots \otimes {}^tA_n \circ \pi) \text{Tr}(B_1 \otimes \dots \otimes B_n \circ (n \dots 1)\pi').$$

We need to determine the highest order in  $N$  of  $\text{Wg}_N(\pi, \pi')$ . The key point is that  $\mathfrak{B}_n$  is endowed with a natural distance, which can be defined in several ways. Let us pause briefly to clarify this point.

We have defined the elements of  $\mathfrak{B}_n$  combinatorially, as the partitions of  $\{1, \dots, 2n\}$  by pairs, but there are other natural ways to define them. In particular, given an element  $\pi$  of  $\mathfrak{B}_n$ , there is a unique element  $i_\pi$  of  $\mathfrak{S}_{2n}$  whose cycles are the pairs of  $\pi$ , and the correspondence  $\pi \mapsto i_\pi$  is a bijection between  $\mathfrak{B}_n$  and the set  $\mathcal{I}_{2n}$  of fixed point free involutions of  $\{1, \dots, 2n\}$ , which is a conjugacy class in  $\mathfrak{S}_{2n}$ . The group  $\mathfrak{S}_{2n}$  acts on  $\mathfrak{B}_n$  through its natural action on  $\{1, \dots, 2n\}$ , a permutation  $\alpha$  transforming a partition  $\pi = \{\{i, j\}, \dots\}$  into  $\alpha \cdot \pi = \{\{\alpha(i), \alpha(j)\}, \dots\}$ . It also acts by conjugation on  $\mathcal{I}_{2n}$  and the map  $i : \mathfrak{B}_n \rightarrow \mathcal{I}_{2n}$  is equivariant, in the sense that  $i_{\alpha \cdot \pi} = \alpha i_\pi \alpha^{-1}$ . In  $\mathfrak{B}_n$ , there is a distinguished element  $\text{id}$ , which is the unit of the algebra  $\mathfrak{B}_{n, \lambda}$ , and which satisfies  $i_{\text{id}} = (1 \ n+1) \dots (n \ 2n)$ . The stabiliser of  $\text{id}$  under the action of  $\mathfrak{S}_{2n}$  on  $\mathfrak{B}_n$  is the hyperoctahedral group  $H_n$  (see Section 2.7). The group  $H_n$  is also the centraliser of  $i_{\text{id}}$ . The choice of  $\text{id} \in \mathfrak{B}_n$  thus determines a bijection between  $\mathfrak{B}_n$  and the set  $\mathfrak{S}_{2n}/H_n$  of left  $H_n$ -cosets in  $\mathfrak{S}_{2n}$ . We denote by  $\pi \mapsto C_\pi$  this correspondence.

Recall that we defined in Section 2.6 the number  $\ell(\pi)$ . Recall also the operations  $S_{a,b}$  and  $F_{a,b}$  which we defined in Section 3.4. Finally, let us denote by  ${}^t\pi$  the pairing  $i_{\text{id}} \cdot \pi$ , obtained by turning the box representing  $\pi$  upside down.

**Lemma 6.26.** *Let  $\pi$  and  $\pi'$  be two elements of  $\mathfrak{B}_n$ . The following numbers are equal.*

1. *The minimal length of a chain  $\pi = \pi_0, \pi_1, \dots, \pi_r = \pi'$  such that each element is obtained from the preceding one by an operation  $S_{a,b}$ .*
2. *The smallest distance in  $\mathfrak{S}_{2n}$  between the identity and an element  $\alpha$  such that  $\alpha \cdot \pi = \pi'$ .*
3. *The smallest distance in  $\mathfrak{S}_{2n}$  between an element of  $C_\pi$  and an element of  $C_{\pi'}$ .*
4. *The number  $n - \ell({}^t\pi\pi')$ .*
5. *The half of the distance in  $\mathfrak{S}_{2n}$  between  $i_\pi$  and  $i_{\pi'}$ .*

We denote these five numbers by  $d(\pi, \pi')$ . The function  $d$  is a distance on  $\mathfrak{B}_n$ , which makes it a metric space of diameter  $n - 1$ . The action of  $\mathfrak{S}_{2n}$  on  $\mathfrak{B}_n$  induced by its natural action on  $\{1, \dots, 2n\}$  and the actions of  $\mathfrak{S}_n$  by left and right multiplication on  $\mathfrak{B}_n \subset B_{n, \lambda}$  preserve the distance  $d$ . The inclusion  $\mathfrak{S}_n \subset \mathfrak{B}_n$  is an isometry. Finally, any shortest path in  $\mathfrak{B}_n$  between two elements of  $\mathfrak{S}_n$  stays in  $\mathfrak{S}_n$ .

*Proof.* Let us denote by  $d_1, \dots, d_5$  the five numbers as they are defined in the statement.

The equality  $d_1 = d_2$  follows from the identity  $S_{a,b}(\pi) = (ab) \cdot \pi$ , which hold for all  $\pi \in \mathfrak{B}_n$  and all  $a, b \in \{1, \dots, 2n\}$ .

Let us think of the set  $C_\pi$  as the set  $\{\alpha \in \mathfrak{S}_{2n} : \alpha \cdot i_{\text{id}} = \pi\}$ . From the equality

$$\{\alpha \in \mathfrak{S}_{2n} : \alpha \cdot \pi = \pi'\} = \{\sigma_2 \sigma_1^{-1} : \sigma_1 \in C_\pi, \sigma_2 \in C_{\pi'}\}$$

of subsets of  $\mathfrak{S}_{2n}$ , it follows that  $d_2$ , which is the distance of  $\text{id}$  to the subset on the left-hand side, is equal to  $d_3$ , which is the distance of  $\text{id}$  to the subset on the left-hand side.

The number  $\ell({}^t\pi\pi')$  is the number of loops formed by the superposition of the diagrams of  $\pi$  and  $\pi'$ . In this picture, each loop contains an even number of edges, and since there are  $2n$

edges altogether, there are at most  $n$  loops. Moreover, there are  $n$  loops only if each loop has length 2, and this happens only if  $\pi = \pi'$ . Now let us prove that  $d_2 \leq d_4$ . If  $d_4 = 0$ , this follows from our last remark. If  $d_4 > 0$ , then  $\pi \neq \pi'$  and there is at least one loop of length at least 4. There exists  $i, j, k, l \in \{1, \dots, 2n\}$  such that  $\{i, j\}$  and  $\{k, l\}$  belong to  $\pi$  and  $\{j, k\}$  belongs to  $\pi'$ . Then  $\ell({}^t(S_{j,l}(\pi)), \pi') = \ell({}^t\pi\pi') - 1$ . Iterating this argument, we find that we can go from  $\pi$  to  $\pi'$  in  $d_4$  applications of an operator  $S_{a,b}$ . Hence,  $d_2 \leq d_4$ . On the other hand, it is even easier to check that the application of an operator  $S_{a,b}$  cannot increase or decrease  $\ell({}^t\pi\pi')$  by more than 1. Following a minimal chain of applications of the operators  $S_{a,b}$  leading from  $\pi$  to  $\pi'$ , we find  $d_4 \leq d_2$ . Finally,  $d_2 = d_4$ .

The permutations  $i_\pi$  and  $i_{\pi'}$  are involutions, so that their distance in  $\mathfrak{S}_{2n}$  is equal to  $2n$  minus the number of cycles of their product  $i_\pi i_{\pi'}$ . The image of an integer  $i$  by this product is easily computed on the diagram formed by the superposition of those of  $\pi$  and  $\pi'$ , by following first the edge of  $\pi'$  issued from  $i$ , thus arriving at an integer  $j$ , and then following the other edge issued from  $j$ . The permutation  $i_\pi i_{\pi'}$  has thus exactly twice as many cycles as the superposition of the diagrams of  $\pi$  and  $\pi'$ . Hence,  $d_4 = d_5$ .  $\square$

Note that, up to the multiplicative factor involved in the definition of  $F_{a,b}$ , the effect of an operator  $F_{a,b}$  on a diagram can always be obtained by the action of an operator  $S_{a,b}$ . Thus, in the definition of the first number above, we could have replaced  $S$  by  $F$  without altering its definition.

The matrix in the basis  $\mathfrak{B}_n$  of the restriction of  $Q_0$  to the range of  $\rho_{\mathbb{R}}$  is the Gram matrix of  $\mathfrak{B}_n$  for the scalar product  $\langle \pi_1, \pi_2 \rangle = \text{Tr}({}^t\pi_1\pi_2)$ . Lemma 6.26 allows us to rewrite this matrix as  $N^n(N^{-d(\pi,\pi')})_{\pi,\pi' \in \mathfrak{B}_n}$ . In particular,  $N^{-n}$  times this matrix tends to the identity matrix, so that it is invertible for  $N$  large enough. More to the point, this way of writing this matrix implies

$$(105) \quad \text{Wg}_N(\pi, \pi') = N^{-n} \sum_{r \geq 0} (-1)^r \sum_{\pi_1, \dots, \pi_{r-1} \in \mathfrak{B}_n} N^{-d(\pi, \pi_1) - d(\pi_1, \pi_2) - \dots - d(\pi_{r-1}, \pi')},$$

where the sum is taken over all the chains  $\pi, \pi_1, \dots, \pi_{r-1}, \pi'$  in which each term is different from the next. The term of highest order is provided by chains for which the exponent of  $N$  is  $-d(\pi, \pi')$ . The highest power of  $N$  which appears in the generic term of (104) is thus

$$\ell(\pi) + \ell((n \dots 1)\pi') - n - 1 - d(\pi, \pi') = d(\text{id}, (1 \dots n)) - d(\text{id}, \pi) - d(\pi, \pi') - d(\pi', (1 \dots n)),$$

using the invariance of  $d$  under left multiplication by  $(1 \dots n)$ . This power is nonpositive and equal to 0 only if  $\pi$  and  $\pi'$  are located in this order on a same geodesic from  $\text{id}$  to  $(1 \dots n)$ . Lemma 6.26 asserts that a necessary condition for this is that  $\pi$  and  $\pi'$  belong to  $\mathfrak{S}_n$ . We must then have  $\text{id} \preceq \pi \preceq \pi' \preceq (1 \dots n)$ . Moreover, in this case, we recognise in the expression of the term of highest order of  $\text{Wg}_N(\pi, \pi')$  given by (105) the value  $\mu(\pi, \pi')$  of the Möbius function of the lattice  $\text{NC}_n$ . We thus obtain

$$\begin{aligned} \int_{\text{SO}(N)} \text{tr}(RA_1R^{-1}B_1 \dots RA_nR^{-1}B_n) dR &= \\ &= \sum_{\sigma \preceq \sigma' \preceq (1 \dots n)} \mu(\sigma, \sigma') p_\sigma({}^tA_1, \dots, {}^tA_n) p_{(n \dots 1)\sigma'}(B_1, \dots, B_n) + O(N^{-1}) \\ &= \sum_{\sigma' \preceq (1 \dots n)} \kappa_{(\sigma')^{-1}}(A_1, \dots, A_n) p_{(n \dots 1)\sigma'}(B_1, \dots, B_n) + O(N^{-1}) \\ &= \sum_{\pi \in \text{NC}_n} \tau_\pi(A_1, \dots, A_n) \kappa_{\pi^\vee}(B_1, \dots, B_n) + O(N^{-1}), \end{aligned}$$

and conclude as in the unitary case to the asymptotic freeness of the families  $\{RA_1R^{-1}, \dots, RA_nR^{-1}\}$  and  $\{B_1, \dots, B_n\}$ .

**A.3. Symplectic case.** The symplectic case is very similar to the orthogonal case. Let us define the endomorphism  $P$  of the real algebra  $M_N(\mathbb{H})^{\otimes n}$  by setting

$$\forall A \in M_N(\mathbb{H})^{\otimes n}, \quad P(A) = \int_{\text{Sp}(N)} S^{\otimes n} \circ A \circ (S^{-1})^{\otimes n} dS.$$

We shall now consider the homomorphism of algebras  $\rho_{\mathbb{H}} : B_{n,-2N} \rightarrow M_N(\mathbb{H})^{\otimes n}$  (see (41)). The Schur-Weyl duality asserts again in this case that the range of  $P$  is contained in the range of  $\rho_{\mathbb{H}}$ .

For all  $A \in M_N(\mathbb{H})^{\otimes n}$ , we define an element  $Q_0(A)$  of  $B_{n,-2N}$  by setting

$$Q_0(A) = \sum_{\pi \in \mathfrak{B}_n} (-2\Re\text{Tr})^{\otimes n}(A^* \circ \rho_{\mathbb{H}}(\pi))\pi.$$

According to Lemma 2.6, the matrix in the basis  $\mathfrak{B}_n$  of the restriction of  $Q_0$  to  $B_{n,-2N}$  is  $(-2N)^n((-2N)^{-d(\pi,\pi')})_{\pi,\pi' \in \mathfrak{B}_n}$ . As  $N$  tends to infinity,  $(-2N)^{-n}$  times this matrix tends to the identity matrix, so that it is invertible for  $N$  large enough. We denote again by  $\text{Wg}_{-2N}$  its inverse and shall use the notation  $\text{Wg}_{-2N}(\pi) = \sum_{\pi' \in \mathfrak{B}_n} \text{Wg}_{-2N}(\pi, \pi')\rho_{\mathbb{R}}(\pi')$ .

Consider  $A \in M_N(\mathbb{H})^{\otimes n}$ . On one hand,  $(\text{Wg}_{-2N} \circ Q_0)(P(A)) = (\text{Wg}_{-2N} \circ Q_0)(A)$ , because  $\rho_{\mathbb{H}}(\pi)$  and  $S^{\otimes n}$  commute for all  $\pi \in \mathfrak{B}_n$  and all  $S \in \text{Sp}(N)$ . On the other hand, we have  $(\text{Wg}_{-2N} \circ Q_0)(P(A)) = P(A)$  because  $P(A)$  belongs to the range of  $\rho_{\mathbb{H}}$ . Hence, the formula in the orthogonal case is

$$\int_{\text{Sp}(N)} S^{\otimes n} \circ A \circ (S^{-1})^{\otimes n} dS = \sum_{\pi \in \mathfrak{B}_n} (-2\Re\text{Tr})^{\otimes n}(A^* \circ \pi)\text{Wg}_{-2N}(\pi).$$

We find, for all  $A_1, \dots, A_n$  and  $B_1, \dots, B_n$  in  $M_N(\mathbb{H})$ ,

$$\begin{aligned} \int_{\text{Sp}(N)} \Re\text{tr}(SA_1S^{-1}B_1 \dots SA_nS^{-1}B_n) dS = \\ - \frac{1}{2N} \sum_{\pi, \pi' \in \mathfrak{B}_n} \text{Wg}_{-2N}(\pi, \pi')(-2\Re\text{Tr})^{\otimes n}(A_1^* \otimes \dots \otimes A_n^* \circ \pi) \\ (106) \quad (-2\Re\text{Tr})^{\otimes n}(B_1 \otimes \dots \otimes B_n \circ (n \dots 1)\pi'). \end{aligned}$$

The same computation as in the orthogonal case, with  $N$  replaced by  $-2N$  shows that the highest order of  $N$  in  $\text{Wg}_{-2N}(\pi, \pi')$  is  $-d(\pi, \pi')$ . The dominant terms of (106) are thus of order 0 in  $N$ , so that the constant  $-2$  disappears, and the coefficients are, for the same reason as in the orthogonal case, given by the Möbius function of  $\text{NC}_n$ .

Let us modify the definition of  $p_{\sigma}$  to suit the symplectic case, and set

$$p_{\sigma}(A_1, \dots, A_n) = \prod_{\substack{c \text{ cycle of } \sigma \\ c=(i_1 \dots i_r)}} \Re\text{tr}(A_{i_1} \dots A_{i_r}).$$

In particular,  $p_{\sigma}(\overline{A_1}, \dots, \overline{A_n}) = p_{\sigma}(A_1, \dots, A_n)$ , where  $\overline{A}$  is the conjugate matrix defined by  $(\overline{A})_{ij} = (A_{ij})^*$ . The cumulants are defined by the usual relation  $\kappa_{\sigma} = \sum_{\sigma' \preceq \sigma} \mu(\sigma', \sigma)p_{\sigma'}$  and are also unchanged if each matrix is replaced by the conjugate matrix. In particular,

$\kappa_\sigma(A_1^*, \dots, A_n^*) = \kappa_{\sigma^{-1}}(A_1, \dots, A_n)$ . We thus find

$$\begin{aligned} \int_{\mathrm{Sp}(N)} \mathrm{tr}(SA_1S^{-1}B_1 \dots SA_nS^{-1}B_n) dS &= \\ &= \sum_{\sigma \preccurlyeq (1\dots n)} \kappa_\sigma(A_1^*, \dots, A_n^*) p_{(n\dots 1)\sigma}(B_1, \dots, B_n) + O(N^{-1}) \\ &= \sum_{\sigma \preccurlyeq (1\dots n)} \kappa_{\sigma^{-1}}(A_1, \dots, A_n) p_{(n\dots 1)\sigma}(B_1, \dots, B_n) + O(N^{-1}) \end{aligned}$$

and conclude as before to the asymptotic freeness of the families  $\{SA_1S^{-1}, \dots, SA_nS^{-1}\}$  and  $\{B_1, \dots, B_n\}$ .

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