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# Optimal posting distance of limit orders: a stochastic algorithm approach

SOPHIE LARUELLE <sup>\*</sup>      CHARLES-ALBERT LEHALLE <sup>†</sup>      GILLES PAGÈS <sup>‡</sup>

## Abstract

This paper presents a stochastic recursive procedure under constraints to find the optimal distance at which an agent must post his order to minimize his execution cost. We prove the *a.s.* convergence of the algorithm under assumptions on the cost function and give some practical criteria on model parameters to ensure that the conditions to use the algorithm are fulfilled (using notably principle of opposite monotony). We illustrate our results with numerical experiments on simulated data but also by using a financial market dataset.

**Keywords** *Stochastic approximation, order book, limit order, market impact, statistical learning, high-frequency optimal liquidation, compound Poisson process, co-monotony principle.*

*2010 AMS classification:* 62L20, secondary: 62P05, 60G55, 65C05.

## 1 Introduction

In the recent years, with the growth of electronic trading, most of the transactions in the markets occur in *Limit Order Books*. During the matching of electronic orders, traders send orders of two kinds to the market: passive (*i.e.* limit or patient orders) which will not give birth to a trade but will stay in the order book (sell orders at a higher price than the *higher bid price* or buy orders at a higher price than the *lower ask price* are passive orders) and aggressive orders (*i.e.* market or impatient orders) which will generate a trade (sell orders at a lower price than the higher remaining buy price or buy orders at a higher price than the lowest remaining price). When a trader has to buy or sell a large number of shares, he cannot just send this large order at once (the price moves during the execution procedure); he has first to schedule his trading rate to make balance between the market risk and the market impact cost of being too aggressive ! (too many orders exhaust the order book and makes the price move). Several theoretical frameworks have been proposed for optimal scheduling of large orders (see [3], [6], [19], [2]). Once this optimal trading rate is known, the trader has to send smaller orders in the electronic book by alternating limit (*i.e.* patient) orders and market (*i.e.* urgent) orders. The optimal mix of limit and market orders for a trader has not been studied in the quantitative literature even if it has been studied from a global economic efficiency viewpoint (see for instance [8] and not from the viewpoint of one trader trying to optimize its own interactions with other market participants). One of the difficulties from the trader prospective is that transactions obtained by

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inserting a limit order in an electronic book is a functional of its distance to the mid-price, giving birth to a large number of possible tactics in terms of re! assessment of the price of such orders.

In this paper, we study the optimal distance to submit limit orders in an electronic order book, without needing a model of the limit order book dynamics (see [1] or [10] for such models of limit order books).

Optimal submission strategies have been studied in the microstructure literature using utility framework and optimal control (see [4], [9], [5], [11] and [12]). The authors consider an agent who plays the role of a dealer, *i.e.* he provides liquidity on the exchange by quoting bid and ask prices at which he is willing to buy and sell a specific quantity of assets. Strategies for bid and ask orders are derived by maximizing his utility function.

Our approach is different: we consider an agent who wants to buy (or sell) on a short period  $[0, T]$  a quantity  $Q_T$  of traded assets and we look for the optimal distance where he must post his order to minimize the execution cost.

We are typically at a smaller time scale than in optimal liquidation frameworks (or any market making framework coming from a backward stochastic control process) and order posting strategies derived from the viewpoint presented here can be “plugged” into any larger scale strategy.

If a stochastic algorithm approach has been already proposed by the authors for optimal spacial split of an order across different Dark Pools, here the purpose is not to control fractions of the size of an order, but to adjust successive posting prices to converge to an optimal price. Qualitatively, this framework can be used as soon as a trader wants to trade a given quantity  $Q_T$  over a given time interval  $[0, T]$ . The trader can post his order very close to the “*fair price*” (that can be seen as the fundamental price, the mid price of the available trading venues or any other reference price). In this case he will be exposed to the risk to trade too fast at a “bad price”, or he can post it far away from the fair price, and in this case he will be exposed to never obtain a transaction for the quantity  $Q_T$ , but only a part of it (say the positive part of  $Q_T - N_T$ , where  $N_T$  is the quantity that the trading flow allowed to trade). He will then have to consume ! aggressively liquidity with the remaining quantity, disturbing the market and paying not only the current market price  $S_T$ , but also market impact (*i.e.*  $S_T \Phi(Q_T - N_T)$ ).

The approach presented here follows the mechanism of a “learning trader”. He will try to guess the optimal posting distance to the fair price achieving the balance between being too demanding in price and too impatient, by trials and errors. The optimal recursive procedure derived from our framework gives the best price adjustment to apply to an order given the observed past on the market. We provide proofs of the convergence of the procedure and of its optimality.

To this end we model the execution process of orders by a Poisson process  $(N_t^{(\delta)})_{0 \leq t \leq T}$  whose intensity  $\Lambda_T(\delta, S)$  depends on the fair price  $(S_t)_{t \geq 0}$  and the distance of order submission  $\delta$ . The execution cost results from the sum of the price of the executed quantity and a penalization function depending on the remaining quantity to be executed at the end of the period  $[0, T]$ . This penalty models the over-cost induced by crossing the spread and the market impact of this execution. The aim is to find the optimal distance  $\delta^* \in [0, \delta_{\max}]$ , where  $\delta_{\max}$  is the depth of the limit order book, which minimize the execution cost. This leads to an optimization problem under constraints which we solve by using a recursive stochastic procedure with projection (This particular class of algorithm is studied in [16] and [17]). We prove the *a.s.* convergence of the constrained algorithm under additional assumptions on the execution cost function. From a practical point of view, it is not easy to check the conditions on the cost function. So we give sufficient criteria on model parameters to ensure the viability of the algorithm which relies on a principle of opposite and co-monotony for diffusion process. Our approach consists to start from the co-monotony principle for  $n$ -tuples of independent variables established for functions where marginals are co-monotonic component by component. We first apply this result to the Euler scheme of the diffusion, under appropriate conditions on the drift and the

diffusion coefficient. As a second step, we use a weak functional convergence theorem to transfer the principle to co(or anti)-monotonic functionals of the diffusion (for a given functional  $F$  on  $\mathcal{C}([0, T], \mathbb{R})$ , monotonic should be understood hence as

$$(\forall t \in [0, T], \quad \alpha(t) \leq \beta(t)) \implies F(\alpha) \leq F(\beta).$$

We conclude this paper by some numerical experiments with simulated and real data. We consider the Poisson intensity presented in [4] and use a Brownian motion to model the fair price dynamics. We plot the cost function and its derivative and show the convergence of the algorithm to its target  $\delta^*$ .

The paper is organized as follows: In Section 2, we first propose a model for the execution process of posted orders, then we define a penalized cost function (including the market impact at the terminal execution date). Then we devise the stochastic recursive procedure under constraint to solve the resulting optimization problem in terms of optimal posting distance on the limit order book. We state the main convergence result and provide operating criteria that ensure this convergence, based on a monotony principle for one dimensional diffusions. Section 3 establishes the representations as expectations of the cost function and its derivatives which allow to define the mean function of the algorithm. Section B presents the principle of opposite monotony which is used in Section 4 to derive the convergence criteria (which ensure that the optimization is well-posed). Finally Section 5 illustrates with numerical experiments the convergence result of the recursive procedure towards its target.

**Notations.** •  $(x)_+ = \max\{x, 0\}$  denotes the positive part of  $x$ ,  $\lfloor x \rfloor = \max\{k \in \mathbb{N} : k \leq x\}$ .

•  $\mathcal{C}([0, T], A) := \{f : [0, T] \rightarrow A \text{ continuous}\}$  and  $\mathbb{D}([0, T], A) := \{f : [0, T] \rightarrow A \text{ càdlàg}\}$  where  $A = \mathbb{R}^q, \mathbb{R}_+^q$ , etc.

## 2 Design of the execution procedure and main results

### 2.1 Modeling and design of the algorithm

We consider on a short period  $T$ , say a dozen of seconds, a Poisson process modeling the execution of posted passive buy orders on the market

$$(N_t^{(\delta)})_{0 \leq t \leq T} \quad \text{with intensity} \quad \Lambda_T(\delta, S) := \int_0^T \lambda(S_t - (S_0 - \delta)) dt \quad (2.1)$$

where  $0 \leq \delta \leq \delta_{\max}$  with  $\delta_{\max} \in (0, S_0)$  denotes the depth of the order book and  $(S_t)_{t \geq 0}$  is a stochastic process modeling the dynamics of the “fair price” of a security stock (from an economic point of view). In practice one may consider that  $S_t$  represents the best opposite price. We assume that the function  $\lambda$  is defined on the whole real line as a finite non-increasing convex function. Its specification will rely on parametric or non parametric statistical estimation based on former transactions (see Figure 1 below and Section 5). At time  $t = 0$ , buy orders are posted in the limit order book at price  $S_0 - \delta$ . Between  $t$  and  $t + \Delta t$ , the probability for such an order to be executed is  $\lambda(S_t - (S_0 - \delta))\Delta t$  where  $S_t - (S_0 - \delta)$  is the distance to the current fair price of our posted order at time  $t$ . The further the order is at time  $t$ , the lower is ! the probability for this order to be executed since  $\lambda$  is decreasing on  $[-S_0, +\infty)$ . Empirical tests strongly confirm this kind of relationship with a convex function  $\lambda$  (even close to an exponential shape, see Figure 1). Over the period  $[0, T]$ , we aim at executing a portfolio of size  $Q_T \in \mathbb{N}$  invested in the asset  $S$ . The execution cost for a distance  $\delta$  is  $\mathbb{E} \left[ (S_0 - \delta) \left( Q_T \wedge N_T^{(\delta)} \right) \right]$ . We add to this execution cost a penalization depending on the remaining quantity to execute, namely at the end of the period

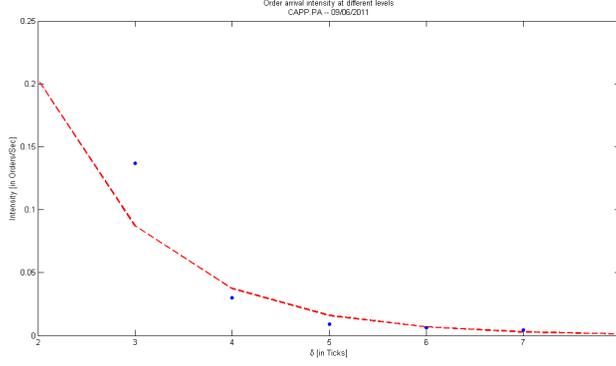


Figure 1: Empirical probabilities of execution (blue stars) and its fit with an exponential law (red dotted line) with respect to the distance to the “fair price”.

$T$ , we want to have  $Q_T$  assets in the portfolio, so we buy the remaining quantity  $(Q_T - N_T^{(\delta)})_+$  at price  $S_T$ . Then we introduce a *market impact penalization function*  $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$ , nondecreasing and convex, with  $\Phi(0) = 0$  to model the additional cost of the execution of the remaining quantity (including the market impact). Then the resulting cost of execution on a period  $[0, T]$  reads

$$C(\delta) := \mathbb{E} \left[ (S_0 - \delta) (Q_T \wedge N_T^{(\delta)}) + \kappa S_T \Phi \left( (Q_T - N_T^{(\delta)})_+ \right) \right] \quad (2.2)$$

where  $\kappa > 0$  is free tuning parameter. When  $\Phi = \text{id}$ , we just consider that we buy the remaining quantity at the end price  $S_T$ , but introducing a market impact penalization function  $\Phi(x) = (1 + \eta(x))x$ , where  $\eta \geq 0$ ,  $\eta \not\equiv 0$ , models the market impact induced by the execution of  $(Q_T - N_T^{(\delta)})_+$  at the end of the period whereas we neglect the market impact of the execution process over  $[0, T]$ . Our aim is then to minimize this cost, namely to solve the following optimization problem

$$\min_{0 \leq \delta \leq \delta_{\max}} C(\delta). \quad (2.3)$$

Our strategy to solve numerically (2.3) using a large enough dataset is to take advantage of the representation of  $C$  and its first two derivatives as expectations to devise a recursive stochastic algorithm, namely a stochastic gradient procedure, to find the minimum of the (penalized) cost function. To be more precise we will show that under natural assumptions on the quantity  $Q_T$  to be executed and on the parameter  $\kappa$ , the function  $C$  is twice differentiable, *strictly convex on  $[0, \delta_{\max}]$  with  $C'(0) < 0$* . Consequently,

$$\operatorname{argmin}_{\delta \in [0, \delta_{\max}]} C(\delta) = \{\delta^*\}, \quad \delta^* \in (0, \delta_{\max}]$$

and

$$\delta^* = \delta_{\max} \quad \text{iff} \quad C \text{ is non-increasing on } [0, \delta_{\max}].$$

Criteria involving  $\kappa$  and based on both the risky asset  $S$  and the trading process especially the execution intensity  $\lambda$ , are established further on in Proposition 4.1 and Proposition 4.2. We specify representations as expectations of the function  $C$  and its derivatives  $C'$  and  $C''$ . In particular we will show that there exists a Borel functional

$$H : [0, \delta_{\max}] \times \mathbb{D}([0, T], \mathbb{R}) \longrightarrow \mathbb{R}$$

such that

$$\forall \delta \in [0, \delta_{\max}], \quad C'(\delta) = \mathbb{E} \left[ H(\delta, (S_t)_{t \in [0, T]}) \right].$$

The functional  $H$  has an explicit form given in Proposition 3.2, Equations (3.15) or (3.17)), involving integrals over  $[0, T]$  of the intensity  $\lambda(S_t - S_0 + \delta)$  of the Poisson process  $(N_t^{(\delta)})_{t \in [0, T]}$ . In particular, any quantity  $H(\delta, (S_t)_{t \in [0, T]})$  can be simulated, up to a natural time discretization, either from a true dataset (of past executed orders) or from the stepwise constant Euler scheme (with step  $\frac{T}{m}$ ) of a formerly calibrated diffusion process modeling  $(S_t)_{t \in [0, T]}$ . This will lead us to replace in our implementations the continuous time process  $(S_t)_{t \in [0, T]}$  over  $[0, T]$ , by a finite dimensional  $\mathbb{R}^{m+1}$ -valued random vector  $(S_{t_i})_{0 \leq i \leq m}$  (where  $t_0 = 0$  and  $t_m = T$ ) with the implicit assumption that  $S_t = S_{t_i}$  on  $[t_i, t_{i+1})$ ,  $i = 0, \dots, m$ .

Based on this representation (3.15) of  $C'$ , we can formally devise a recursive stochastic gradient descent *a.s.* converging toward  $\delta^*$ . However to make it consistent, we need to introduce constrain so that it lives in  $[0, \delta_{\max}]$ . In the classical literature on Stochastic Approximation Theory (see [16] and [17]) this amounts to consider a variant *with projection on the "order book depth interval"*  $[0, \delta_{\max}]$ , namely

$$\delta_{n+1} = \text{Proj}_{[0, \delta_{\max}]} \left( \delta_n - \gamma_{n+1} H \left( \delta_n, (\bar{S}_{t_i}^{(n+1)})_{0 \leq i \leq m} \right) \right), \quad n \geq 0, \quad \delta_0 \in [0, \delta_{\max}], \quad (2.4)$$

where  $\text{Proj}_{[0, \delta_{\max}]}$  denotes the projection on the (nonempty closed convex)  $[0, \delta_{\max}]$ ,  $(\gamma_n)_{n \geq 1}$  is a positive step sequence satisfying at least the minimal *decreasing step assumption*

$$\sum_{n \geq 1} \gamma_n = +\infty \quad \text{and} \quad \gamma_n \rightarrow 0. \quad (2.5)$$

The input sequence  $\{(\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}, n \geq 0\}$  showing up in the above procedure is the innovation process and needs to be specified. In a first (naive) approach one may imagine that it is a sequence of i.i.d. copies  $(\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}$  of the true underlying dynamics of  $(S_t)_{0 \leq t \leq T}$  or at least of its Euler scheme. Of course this not at all realistic in the perspective of performing this procedure on real data. An alternative is to assume that, at least within a laps of a few minutes, the dynamics of the asset  $S$  is *stationary* and, say,  $\alpha$ -*mixing*. In such a framework, one can consider backward shifted samples of  $(S_t)_{0 \leq t \leq T}$ : if  $\Delta t > 0$  denotes a fixed time shift parameter such that  $t_i - t_{i-1} = \Delta t$ , we set

$$\forall t \in [0, T], \quad \bar{S}_{t_i}^{(n)} = \bar{S}_{t_i - n\Delta t} = \bar{S}_{t_{i-n}}$$

so that the sequence  $\bar{S}^{(n)} = (\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}$  shares some averaging properties in the sense of [18]. Here  $\bar{S}$  may represent either a stepwise constant Euler scheme when thinking of simulated data or a historical high frequency data base of best opposite prices of the asset  $S$ .

## 2.2 Main convergence results

The following theorems give *a.s.* convergence result for the stochastic procedure (2.4): the first one for i.i.d. sequences and the second one for "averaging" sequences (see [18]).

### 2.2.1 I.i.d. simulated data from a formerly calibrated model

**Theorem 2.1.** *Assume that  $C$  is strictly convex with  $C'(0) < 0$ ,  $((\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m})_{n \geq 1}$  is an  $\mathbb{R}^{m+1}$ -valued sequence of i.i.d. copies of the stepwise constant Euler scheme of  $(S_t)_{0 \leq t \leq T}$  with step  $\frac{T}{m}$  and (2.5) is satisfied. Furthermore, assume that the step sequence satisfies the standard*

$$\sum_{n \geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n \geq 1} \gamma_n^2 < +\infty. \quad (2.6)$$

Then the recursive procedure defined by (2.4) converges a.s. towards its target  $\delta^* = \operatorname{argmin}_{\delta \in [0, \delta_{\max}]} C(\delta)$ :

$$\delta_n \xrightarrow{a.s.} \delta^*.$$

This theorem is a straightforward application of the classical a.s. convergence for constrained stochastic algorithms (see Appendix A).

### 2.2.2 Direct implementation on a dataset sharing averaging properties

In this case, we assume that the sequence  $(\bar{S}_{t_i}^{(n)})_{0 \leq i \leq m}$  shares an averaging properties with respect to a distribution  $\nu$  on  $(\mathbb{R}^{m+1}, \mathcal{B}(\mathbb{R}^{m+1}))$  as developed in [18].

**Definition 2.1.** Let  $q \in \mathbb{N}$ . A  $[0, L]^q$ -valued sequence  $(\xi_n)_{n \geq 1}$  is  $\nu$ -averaging if

$$\frac{1}{n} \sum_{k=1}^n \delta_{\xi_k} \xrightarrow{(\mathbb{R}^q)} \nu \quad \text{as } n \rightarrow \infty.$$

Then  $(\xi_n)_{n \geq 1}$  satisfies

$$D_n^*(\xi) := \sup_{x \in [0, L]^q} \left| \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{[0, x]}(\xi_k) - \nu([0, x]) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where  $D_n^*(\xi)$  is called the discrepancy at the origin or star discrepancy.

In this setting, we will require the existence of a *pathwise* Lyapunov function, which means in this one dimensional setting that  $H(\cdot, (y_{t_i})_{0 \leq i \leq m})$  is monotonic with a monotony independent of  $(y_{t_i})_{0 \leq i \leq m} \in \mathbb{R}_+^{m+1}$ .

**Theorem 2.2.** Let  $\lambda(x) = Ae^{-kx}$ ,  $A > 0$ ,  $k > 0$ . Assume that  $C$  is strictly convex with  $C'(0) < 0$  and  $C'(\delta_{\max}) > 0$ ,  $(\bar{S}^{(n)})_{n \geq 1}$  is a  $[0, L]^q$ -valued  $\nu$ -averaging sequence and  $(\gamma_n)_{n \geq 1}$  is a positive non-increasing sequence satisfying

$$\sum_{n \geq 1} \gamma_n = +\infty, \quad nD_n^*(\xi)\gamma_n \xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \quad \sum_{n \geq 1} nD_n^*(\xi) \max(\gamma_n^2, |\Delta\gamma_{n+1}|) < +\infty. \quad (2.7)$$

Furthermore, assume that

$$Q_T \geq 2T\lambda(-S_0) \quad \text{and} \quad \begin{cases} \kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k \|S\|_\infty} & \text{if } \Phi \neq \text{id} \\ \kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k \|S\|_\infty (\Phi(Q_T) - \Phi(Q_T - 1))} & \text{if } \Phi = \text{id} \end{cases} \quad (2.8)$$

Then the recursive procedure defined by (2.4) converges a.s. towards its target  $\delta^* = \operatorname{argmin}_{\delta \in [0, \delta_{\max}]} C(\delta)$ :

$$\delta_n \xrightarrow{a.s.} \delta^*.$$

**Proof.** We will apply Theorem 2.1 in a QMC framework similar to Section 3 c.f. [18]. First we set the integrability parameter  $p$  to  $p = 1$ . Note that  $\delta^* \in (0, \delta_{\max})$  since  $C'(0) < 0$  and  $C'(\delta_{\max}) > 0$  so we can extend  $C'$  as a convex function on the whole real line. Moreover, by using the proof of Proposition 4.1, we prove that if  $Q_T \geq 2T\lambda(-S_0)$  and (2.8) is satisfied, then  $H$  is nondecreasing in  $\delta$

so  $H$  satisfies the strict pathwise Lyapunov assumption with  $L(\delta) = \frac{1}{2}|\delta - \delta^*|^2$ . It remains to check the averaging rate assumption for  $H(\delta^*, \cdot)$ . As it is a nondecreasing function on  $[0, L]^a$ , then it has finite variation and by using the Koksma-Hlawka Inequality, we get

$$\left| \frac{1}{n} \sum_{k=1}^n H(\delta^*, \xi_k) - \int_{[0, L]^a} H(\delta^*, u) \nu(du) \right| \leq (H(\delta^*, L) - H(\delta^*, 0)) D_n^*(\xi),$$

so that  $H(\delta^*, \cdot)$  is  $\nu$ -averaging at rate  $\varepsilon_n = D_n^*(\xi)$ . Finally, Theorem 2.1 from [18] yields

$$\delta_n \xrightarrow{a.s.} \delta^*. \quad \square$$

### 2.3 Criteria for the convexity and monotony at the origin

Checking that the assumptions on the function  $C$  (*i.e.*  $C$  convex with  $C'(0) < 0$ ) in Theorem 2.1 are satisfied on  $[0, \delta_{\max}]$  is a nontrivial task: in fact, as emphasized further on in Figures 2 and 7 in Section 5, the function  $C$  in (2.2) is never convex on the whole nonnegative real line, so we need reasonably simple criteria involving the market impact function  $\Phi$ ,  $Q_T$  and the parameter  $\kappa$  and others quantities related to the asset dynamics which ensure that the required conditions are fulfilled by the function  $C$ . These criteria should take the form of upper bounds on the free parameter  $\kappa$ .

Their original form, typically those derived by simply writing  $C'(0) < 0$  and  $C''(0) \geq 0$ , are not really operating since they involve ratios of expectations of functionals combining both the dynamics of the asset  $S$  and the execution parameters in a highly nonlinear way. A large part of this paper is devoted to establish simpler criteria (although slightly more conservative) based on a monotony principle for one-dimensional diffusions introduced in further details in Appendix B. Since the statement of this criteria do not require any knowledge on this monotony principle, we present them in the theorem 2.3 below.

We still need an additional assumption, this time on the function  $\lambda$ . Roughly speaking we need that the functional  $\Lambda$  depends on the distance parameter  $\delta$  *essentially exponentially* in the following sense:

$$0 < \underline{k}_1 := \inf_{\delta \in [0, \delta_{\max}]} \left( -\frac{\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)}{\Lambda_T(\delta, S)} \right) \leq \bar{k}_1 := \sup_{\delta \in [0, \delta_{\max}]} \left( -\frac{\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)}{\Lambda_T(\delta, S)} \right) < +\infty, \quad (2.9)$$

$$0 < \underline{k}_2 := \inf_{\delta \in [0, \delta_{\max}]} \left( -\frac{\frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S)}{\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)} \right) \leq \bar{k}_2 := \sup_{\delta \in [0, \delta_{\max}]} \left( -\frac{\frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S)}{\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)} \right) < +\infty. \quad (2.10)$$

Note that the above assumption implies

$$k_0 := \inf_{\delta \in [0, \delta_{\max}]} \left( -\frac{\frac{\partial}{\partial \delta} \Lambda_T(0, S)}{\Lambda_T(0, S)} \right) \geq \underline{k}_1 > 0$$

Although this assumption is stated on the functional  $\Lambda$  (and subsequently depends on  $S$ ), this is mainly an assumption on the intensity function  $\lambda$ . In particular, both above assumptions are satisfied by intensity functions of the form

$$\lambda_k(x) = e^{-kx}, \quad x \in \mathbb{R}, \quad k \in (0, +\infty).$$

For  $\lambda_k$ , one checks that  $\underline{k}_1 = \bar{k}_1 = \underline{k}_2 = \bar{k}_2 = k$ .

**Theorem 2.3.** *Assume that the asset dynamics  $(S_t)_{t \geq 0}$  of the asset  $S$  is a  $(0, \infty)$ -Brownian diffusion*

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 > 0,$$

admissible for this open interval in the sense of Definition 2.3 below (the geometric Brownian motion is admissible, see Appendix B). Assume that the function  $\lambda$  is essentially exponential in the sense of (2.9) and (2.10) above. Then the following criteria hold true.

(a) MONOTONY AT THE ORIGIN: The derivative  $C'(0) < 0$  as soon as

$$Q_T \geq 2T\lambda(-S_0) \quad \text{and} \quad \kappa \leq \frac{S_0}{\mathbb{E}[S_T](\Phi(Q_T) - \Phi(Q_T - 1))} + \frac{1}{k_0 \mathbb{E}[S_T](\Phi(Q_T) - \Phi(Q_T - 1))}.$$

(b) CONVEXITY. Let  $\rho_Q \in \left(0, 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \Big|_{\mu = T\lambda(-S_0)}\right)$ . If  $\Phi \neq \text{id}$ , assume that  $\Phi$  satisfies

$$\forall x \in [1, Q_T - 1], \quad \Phi(x) - \Phi(x - 1) \leq \rho_Q(\Phi(x + 1) - \Phi(x)).$$

If  $Q_T \geq \left(2\sqrt{1 + \frac{\bar{k}_1^2}{k_1 k_2}}\right) T\lambda(-S_0)$  and  $\kappa \leq \frac{2k_1}{k_1 \bar{k}_2 \mathbb{E}[S_T] \Phi'(Q_T)}$ ,

then  $C''(\delta) \geq 0$ ,  $\delta \in [0, \delta_{\max}]$ , so that  $C$  is convex on  $[0, \delta_{\max}]$ .

**Remark.** These conditions on the model parameters are conservative. Indeed, “sharper” criteria can be given whose bounds involve ratios of expectation which can be evaluated only by Monte Carlo simulations:

$$C'(0) < 0 \iff 0 < \kappa < b_2,$$

where 
$$b_2 = \frac{\mathbb{E}[-Q_T \mathbb{P}^{(0)}(N^\mu > Q_T) + (S_0 \frac{\partial}{\partial \delta} \Lambda_T(0, S) - \Lambda_T(0, S)) \mathbb{P}^{(0)}(N^\mu \leq Q_T - 1)]}{\mathbb{E}[S_T \frac{\partial}{\partial \delta} \Lambda_T(0, S) \varphi^{(0)}(\mu)]}$$

and 
$$C \text{ is convex on } [0, \delta_{\max}] \iff 0 < \kappa < \min_{\delta \in \mathcal{D}_+} \frac{A(\delta)}{B(\delta)}$$

where 
$$A(\delta) = \mathbb{E} \left[ \left( (S_0 - \delta) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - (S_0 - \delta) \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right],$$

$$B(\delta) = \mathbb{E} \left[ S_T \left( \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) - \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \psi^{(\delta)}(\mu) \right) \right] \quad \text{and} \quad \mathcal{D}_+ = \{\delta \in [0, \delta_{\max}] \mid B(\delta) > 0\}.$$

## 2.4 Introduction to monotony principle for diffusions

Let  $I$  be a non-empty open interval of  $\mathbb{R}$ . Consider the real-valued diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0 \in I, \quad t \in [0, T]. \quad (2.11)$$

**Definition 2.2.** The Lamperti transform of the diffusion process (2.11) is the function  $L : [0, T] \times I \rightarrow \mathbb{R}$  defined for every  $(t, x) \in [0, T] \times I$  by

$$L(t, x) := \int_{x_1}^x \frac{d\xi}{\sigma(t, \xi)} \quad (2.12)$$

where  $x_1$  is an arbitrary fixed value lying in  $I$ . Let  $t \in [0, T]$ . The inverse of  $L(t, \cdot)$  will be denoted  $L^{-1}(t, \cdot)$ .

**Definition 2.3.** *The diffusion process (2.11) is admissible if*

- (i)  $\sigma \in \mathcal{C}^1([0, T] \times I, I)$ ,
- (ii)  $\forall (t, x) \in [0, T] \times I, |b(t, x)| \leq C(1 + |x|)$  and  $0 < \sigma(t, x) \leq C(1 + |x|)$ ,
- (iii)  $\forall x \in I, \int_{(-\infty, x] \cap I} \frac{d\xi}{\sigma(t, \xi)} = \int_{[x, +\infty) \cap I} \frac{d\xi}{\sigma(t, \xi)} = +\infty$ ,
- (iv) for every starting value  $x_0 \in I$ , (2.11) has a unique weak solution which lives in  $I$  up to  $t = +\infty$  (see Proposition B.3 for criteria),
- (v) the function  $\beta$  defined by

$$\beta(t, y) := \left( \frac{b}{\sigma} - \int_{x_1} \frac{1}{\sigma^2(t, \xi)} \frac{\partial \sigma}{\partial t}(t, \xi) d\xi - \frac{1}{2} \frac{\partial \sigma}{\partial x} \right) (t, L^{-1}(t, y)),$$

is continuous on  $[0, T] \times \mathbb{R}$ , nondecreasing in  $y$  for every  $t \in [0, T]$  and satisfies

$$\exists K > 0 \text{ such that } |\beta(t, y)| \leq K(1 + |y|), \quad t \in [0, T], \quad y \in \mathbb{R}.$$

**Definition 2.4.** *Let  $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a functional.*

- (i) *The functional  $F$  is nondecreasing (resp. non-increasing) on  $\mathbb{D}([0, T], \mathbb{R})$  if*

$$\forall \alpha_1, \alpha_2 \in \mathbb{D}([0, T], \mathbb{R}), \quad (\forall t \in [0, T], \alpha_1(t) \leq \alpha_2(t)) \Rightarrow F(\alpha_1) \leq F(\alpha_2) \text{ (resp. } F(\alpha_1) \geq F(\alpha_2)).$$

- (ii) *The functional  $F$  is continuous at  $\alpha \in \mathcal{C}([0, T], \mathbb{R})$  if*

$$\forall \alpha_n \in \mathbb{D}([0, T], \mathbb{R}), \quad \alpha_n \xrightarrow{U} \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad F(\alpha_n) \rightarrow F(\alpha).$$

where  $U$  denotes the uniform convergence of functions on  $[0, T]$ . The functional  $F$  is  $C$ -continuous if it is continuous at every  $\alpha \in \mathcal{C}([0, T], \mathbb{R})$ .

- (iii) *The functional  $F$  has polynomial growth if there exists a positive real number  $r > 0$  such that*

$$\forall \alpha \in \mathbb{D}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq K \left( 1 + \|\alpha\|_{\text{sup}}^r \right). \quad (2.13)$$

**Remark.** Any  $C$ -continuous functional in the above sense is in particular  $\mathbb{P}_Z$ -a.s. continuous for every process  $Z$  with continuous paths.

**Theorem 2.4.** *Assume that the diffusion process (2.11) is admissible. Let  $F, G : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be two functionals  $C$ -continuous with polynomial growth and opposite monotony. Then*

$$\text{Cov} \left( F \left( (X_t)_{t \in [0, T]} \right), G \left( (X_t)_{t \in [0, T]} \right) \right) \leq 0.$$

### 3 Representations as expectations of $C$ and its derivatives

First we briefly recall for convenience few basic facts on Poisson distributed variables that will be needed to compute the cost function  $C$  and its derivatives  $C'$  and  $C''$  (proofs are left to the reader).

**Proposition 3.1.** (Classical formulas). *Let  $(N^\mu)_{\mu>0}$  be a family of Poisson distributed random variables with parameter  $\mu > 0$ .*

(i) *For every function  $f : \mathbb{N} \rightarrow \mathbb{R}_+$  such that  $\log f(n) = O(n)$ ,*

$$\frac{d}{d\mu} \mathbb{E}[f(N^\mu)] = \mathbb{E}[f(N^\mu + 1) - f(N^\mu)] = \mathbb{E}\left[f(N^\mu) \left(\frac{N^\mu}{\mu} - 1\right)\right].$$

*In particular, for any  $k \in \mathbb{N}$ ,  $\frac{d}{d\mu} \mathbb{P}(N^\mu \leq k) = -\mathbb{P}(N^\mu = k)$ .*

For any  $k \in \mathbb{N}^*$ ,

(ii)  $\mathbb{E}[k \wedge N^\mu] = k\mathbb{P}(N^\mu > k) + \mu\mathbb{P}(N^\mu \leq k - 1)$  and  $\frac{d}{d\mu} \mathbb{E}[k \wedge N^\mu] = \mathbb{P}(N^\mu \leq k - 1)$ ,

(iii)  $\mathbb{E}[(k - N^\mu)_+] = k\mathbb{P}(N^\mu \leq k) - \mu\mathbb{P}(N^\mu \leq k - 1)$ ,

(iv)  $k\mathbb{P}(N^\mu = k) = \mu\mathbb{P}(N^\mu = k - 1)$ .

To compute the cost function (or its gradient), it is convenient to proceed a pre-conditioning with respect to  $\mathcal{F}_T^S := \sigma(S_t, 0 \leq t \leq T)$ . We come down to compute the above quantity when  $N^{(\delta)}$  is replaced by  $N^\mu$ , a standard Poisson random variable with parameter  $\mu$ . Therefore we have

$$\begin{aligned} C(\delta) &= \mathbb{E}\left[(S_0 - \delta)(Q_T \wedge N_T^{(\delta)}) + \kappa S_T \Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right)\right] \\ &= \mathbb{E}\left[(S_0 - \delta)\mathbb{E}\left[\left(Q_T \wedge N_T^{(\delta)}\right) \mid \mathcal{F}_T^S\right] + \kappa S_T \mathbb{E}\left[\Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right) \mid \mathcal{F}_T^S\right]\right] \\ &= \mathbb{E}\left[(S_0 - \delta)\mathbb{E}[Q_T \wedge N^\mu]_{|\mu=\Lambda_T(\delta, S)} + \kappa S_T \mathbb{E}\left[\Phi\left(\left(Q_T - N^\mu\right)_+\right)_{|\mu=\Lambda_T(\delta, S)}\right]\right] \\ &= \mathbb{E}\left[\tilde{C}\left(\delta, \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T}\right)\right], \end{aligned} \tag{3.14}$$

where for every  $x \in \mathcal{C}([0, T], \mathbb{R}_+)$  and every  $\mu \in \mathbb{R}_+$ ,

$$\tilde{C}(\delta, \mu, x) = (x_0 - \delta)(Q_T \mathbb{P}(N^\mu > Q_T) + \mu \mathbb{P}(N^\mu \leq Q_T - 1)) + \kappa x_T \mathbb{E}\left[\Phi\left(\left(Q_T - N^\mu\right)_+\right)\right].$$

We introduce some notations for reading convenience: we set

$$\mathbb{P}^{(\delta)}(N^\mu > Q_T) = \mathbb{P}(N^\mu > Q_T)_{|\mu=\Lambda_T(\delta, S)} \quad \text{and} \quad \mathbb{E}^{(\delta)}[f(\mu)] = \mathbb{E}[f(\mu)]_{|\mu=\Lambda_T(\delta, S)}.$$

Now we are in position to compute the first and second derivatives of the cost function  $C$ .

**Proposition 3.2.** (a) *If  $\Phi \neq \text{id}$ , then  $C'(\delta) = \mathbb{E}[H(\delta, S)]$  with*

$$\begin{aligned} H(\delta, S) &= -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T) + \left(\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)(S_0 - \delta) - \Lambda_T(\delta, S)\right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \\ &\quad - \kappa S_T \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) \end{aligned} \tag{3.15}$$

where  $\varphi^{(\delta)}(\mu) = \mathbb{E}^{(\delta)} [(\Phi(Q_T - N^\mu) - \Phi(Q_T - N^\mu - 1)) \mathbb{1}_{\{N^\mu \leq Q_T - 1\}}]$  and

$$\begin{aligned} C''(\delta) &= \mathbb{E} \left[ \left( (S_0 - \delta) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - \kappa S_T \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) \right. \\ &\quad \left. - (S_0 - \delta) \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) + \kappa S_T \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \psi^{(\delta)}(\mu) \right] \end{aligned} \quad (3.16)$$

where  $\psi^{(\delta)}(\mu) = \mathbb{E}^{(\delta)} [\Phi((Q_T - N^\mu - 2)_+) - 2\Phi((Q_T - N^\mu - 1)_+) + \Phi((Q_T - N^\mu)_+)]$ .

(b) If  $\Phi = \text{id}$ , then  $C'(\delta) = \mathbb{E}[H(\delta, S)]$  with

$$H(\delta, S) = -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T) + \left( (S_0 - \delta - \kappa S_T) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) - \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \quad (3.17)$$

$$\begin{aligned} \text{and } C''(\delta) &= \mathbb{E} \left[ \left( (S_0 - \delta - \kappa S_T) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \right. \\ &\quad \left. - (S_0 - \delta - \kappa S_T) \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right] \end{aligned} \quad (3.18)$$

**Proof.** Interchanging derivation and expectation in the representation (3.14) implies

$$C'(\delta) = \mathbb{E} \left[ \frac{\partial}{\partial \delta} \tilde{C}(\delta, \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T}) + \frac{\partial}{\partial \mu} \tilde{C}(\delta, \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T}) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right].$$

(a) We come down to compute the partial derivatives of  $\tilde{C}(\delta, \mu, x)$ .

$$\begin{aligned} \frac{\partial \tilde{C}}{\partial \delta}(\delta, \mu, x) &= -\mathbb{E}[(Q_T \wedge N^\mu)] = -Q_T \mathbb{P}(N^\mu > Q_T) - \mu \mathbb{P}(N^\mu \leq Q_T - 1), \\ \frac{\partial \tilde{C}}{\partial \mu}(\delta, \mu, x) &= -(x_0 - \delta) \frac{\partial}{\partial \mu} \mathbb{E}[(Q_T - N^\mu)_+] + \kappa x_T \frac{\partial}{\partial \mu} \mathbb{E}[\Phi((Q_T - N^\mu)_+)]. \end{aligned}$$

$$\begin{aligned} \text{We have } \frac{\partial}{\partial \mu} \mathbb{E}[(Q_T - N^\mu)_+] &= -Q_T \mathbb{P}(N^\mu = Q_T) - \mathbb{P}(N^\mu \leq Q_T - 1) + \mu \mathbb{P}(N^\mu = Q_T - 1) \\ &= -\mathbb{P}(N^\mu \leq Q_T - 1) \quad \text{thanks to (iv) in Proposition 3.1} \end{aligned}$$

$$\begin{aligned} \text{and } \frac{\partial}{\partial \mu} \mathbb{E}[\Phi((Q_T - N^\mu)_+)] &= \mathbb{E}[\Phi((Q_T - N^\mu - 1)_+) - \Phi((Q_T - N^\mu)_+)] \\ &= \mathbb{E}[(\Phi(Q_T - N^\mu - 1) - \Phi(Q_T - N^\mu)) \mathbb{1}_{\{N^\mu \leq Q_T - 1\}}] := -\varphi(\mu) \end{aligned}$$

owing to (v) in Proposition 3.1. Therefore

$$\frac{\partial \tilde{C}}{\partial \mu}(\delta, \mu, x) = (x_0 - \delta) \mathbb{P}(N^\mu \leq Q_T - 1) - \kappa x_T \varphi(\mu).$$

Consequently

$$\begin{aligned} C'(\delta) &= \mathbb{E} \left[ -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T) + \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) (S_0 - \delta) - \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \right. \\ &\quad \left. - \kappa S_T \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) \right] \\ &= \mathbb{E} \left[ \hat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \right], \end{aligned} \quad (3.19)$$

where  $\varphi^{(\delta)}(\mu) := \varphi(\mu)|_{\mu=\Lambda_T(\delta,S)}$  and for every  $x \in \mathcal{C}([0, T], \mathbb{R}_+)$  and every  $\mu, \nu \in \mathbb{R}_+$ ,

$$\widehat{C}(\delta, \mu, \nu, x) = -Q_T \mathbb{P}(N^\mu > Q_T) + (\nu(x_0 - \delta) - \mu) \mathbb{P}(N^\mu \leq Q_T - 1) - \kappa x_T \nu \varphi(\mu).$$

Interchanging derivation and expectation in the representation (3.19) implies

$$\begin{aligned} C''(\delta) &= \mathbb{E} \left[ \frac{\partial}{\partial \delta} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \right. \\ &\quad + \frac{\partial}{\partial \mu} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \\ &\quad \left. + \frac{\partial}{\partial \nu} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \right]. \end{aligned}$$

We deal now with the partial derivatives of  $\widehat{C}(\delta, \mu, \nu, x)$ .

$$\begin{aligned} \frac{\partial \widehat{C}}{\partial \delta}(\delta, \mu, \nu, x) &= -\nu \mathbb{P}(N^\mu \leq Q_T - 1), \\ \frac{\partial \widehat{C}}{\partial \mu}(\delta, \mu, \nu, x) &= -\mathbb{P}(N^\mu \leq Q_T - 1) - (x_0 - \delta) \nu \mathbb{P}(N^\mu = Q_T - 1) + \kappa x_T \nu \psi(\mu), \\ \frac{\partial \widehat{C}}{\partial \nu}(\delta, \mu, \nu, x) &= (x_0 - \delta) \mathbb{P}(N^\mu \leq Q_T - 1) - \kappa x_T \varphi(\mu). \end{aligned}$$

Consequently

$$\begin{aligned} C''(\delta) &= \mathbb{E} \left[ \left( (S_0 - \delta) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - \kappa S_T \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) \right. \\ &\quad \left. - (S_0 - \delta) \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) + \kappa S_T \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \psi^{(\delta)}(\mu) \right]. \end{aligned}$$

(b) If  $\Phi = \text{id}$  so that  $\frac{\partial}{\partial \mu} \mathbb{E}[\Phi((Q_T - N^\mu)_+)] = -\mathbb{P}(N^\mu \leq Q_T - 1)$ . Therefore

$$\frac{\partial \widetilde{C}}{\partial \delta}(\delta, \mu, x) = -Q_T \mathbb{P}(N^\mu > Q_T) - \mu \mathbb{P}(N^\mu \leq Q_T - 1) \quad \text{and} \quad \frac{\partial \widetilde{C}}{\partial \mu}(\delta, \mu, x) = (x_0 - \delta - \kappa x_T) \mathbb{P}(N^\mu \leq Q_T - 1).$$

Consequently

$$\begin{aligned} C'(\delta) &= \mathbb{E} \left[ -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T) + \left( (S_0 - \delta - \kappa S_T) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) - \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \right] \\ &= \mathbb{E} \left[ \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \right], \end{aligned} \tag{3.20}$$

where for every  $x \in \mathcal{C}([0, T], \mathbb{R}_+)$  and every  $\mu, \nu \in \mathbb{R}_+$ ,

$$\widehat{C}(\delta, \mu, \nu, x) = -Q_T \mathbb{P}(N^\mu > Q_T) + ((x_0 - \delta - \kappa x_T) \nu - \mu) \mathbb{P}(N^\mu \leq Q_T - 1).$$

Interchanging derivation and expectation in the the representation (3.20) implies

$$\begin{aligned} C''(\delta) &= \mathbb{E} \left[ \frac{\partial}{\partial \delta} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \right. \\ &\quad + \frac{\partial}{\partial \mu} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \\ &\quad \left. + \frac{\partial}{\partial \nu} \widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \right]. \end{aligned}$$

We come down to compute the partial derivatives of  $\widehat{C} \left( \delta, \Lambda_T(\delta, S), \frac{\partial}{\partial \delta} \Lambda_T(\delta, S), (S_t)_{0 \leq t \leq T} \right)$ .

$$\frac{\partial \widehat{C}}{\partial \delta}(\delta, \mu, \nu, x) = -\nu \mathbb{P}(N^\mu \leq Q_T - 1), \quad \frac{\partial \widehat{C}}{\partial \nu}(\delta, \mu, \nu, x) = (x_0 - \delta - \kappa x_T) \mathbb{P}(N^\mu \leq Q_T - 1),$$

$$\frac{\partial \widehat{C}}{\partial \mu}(\delta, \mu, \nu, x) = -\mathbb{P}(N^\mu \leq Q_T - 1) - (x_0 - \delta - \kappa x_T) \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \mathbb{P}(N^\mu = Q_T - 1).$$

Consequently

$$\begin{aligned} C''(\delta) &= \mathbb{E} \left[ \left( (S_0 - \delta - \kappa S_T) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \right. \\ &\quad \left. - (S_0 - \delta - \kappa S_T) \left( \frac{\partial}{\partial \delta} \Lambda_T(\delta, S) \right)^2 \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right]. \quad \square \end{aligned}$$

## 4 Convexity and monotony criteria for the cost function $C$

To ensure that the optimization problem is well-posed, namely that the cost function  $C$  has a minimum on  $[0, \delta_{\max}]$ , we need some additional assumptions: the cost function  $C$  must be convex with  $C'(0) < 0$ . This leads to define bounds for the parameter  $\kappa$  and this section is devoted to give sufficient condition on  $\kappa$  to ensure that this two properties are satisfied. The computations of the bounds given below rely on the co- (and opposite) monotony principle introduced in the previous Section 2.4 and Appendix B.

### 4.1 Criteria for local and global monotony

The above proposition gives bounds for the parameter  $\kappa$  which ensure that the cost function has a minimum. The aim of this subsection is to obtain sufficient bounds, easy to compute, namely depending only of the model parameters.

**Proposition 4.1.** (a) Monotony at the origin.  $C'(0) < 0$  as soon as  $Q_T \geq 2T\lambda(-S_0)$ ,  $k_0 = \infess \left( -\frac{\frac{\partial}{\partial \delta} \Lambda_T(0, S)}{\Lambda_T(0, S)} \right) > 0$  and

$$\kappa \leq \frac{S_0}{\mathbb{E}[S_T] (\Phi(Q_T) - \Phi(Q_T - 1))} + \frac{1}{k_0 \mathbb{E}[S_T] (\Phi(Q_T) - \Phi(Q_T - 1))}.$$

In particular, when  $\Phi \equiv \text{id}$ , the condition reduces to

$$\kappa \leq \frac{S_0}{\mathbb{E}[S_T]} + \frac{1}{k_0 \mathbb{E}[S_T]}.$$

(b) Global monotony (exponential intensity). Assume that  $s^* := \|\sup_{t \in [0, T]} S_t\|_{L^\infty} < +\infty$ . If  $\lambda(x) = Ae^{-kx}$ ,  $A > 0$ ,  $k > 0$ ,  $Q_T \geq 2T\lambda(-S_0)$  and

$$\kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k s^*} \quad \text{if } \Phi \neq \text{id}, \quad \kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k s^* (\Phi(Q_T) - \Phi(Q_T - 1))} \quad \text{if } \Phi = \text{id}, \quad (4.21)$$

then  $H(\cdot, (y_i)_{0 \leq i \leq m})$  is nondecreasing on  $[0, \delta_{\max}]$  for every  $(y_i)_{0 \leq i \leq m} \in [0, s^*]^{m+1}$ .

To prove this result, we need to establish the monotony of several functions of  $\mu$  which appear in the expression of  $C'$ .

**Lemma 4.1.** (i) The function  $\mu \mapsto \mu \mathbb{P}(N^\mu \leq Q)$  is nondecreasing on  $\left[0, \left\lfloor \frac{Q+1}{2} \right\rfloor\right]$ .

(ii) The function  $\mu \mapsto \Theta(Q, \mu) := \mathbb{E}[\Phi(Q - N^\mu) - \Phi(Q - N^\mu - 1) \mid N^\mu \leq Q - 1]$  is non-increasing.

**Proof of Lemma 4.1.** (i) We have  $\frac{d}{d\mu}(\mu \mathbb{P}(N^\mu \leq Q)) = \mathbb{P}(N^\mu \leq Q) - \mu \mathbb{P}(N^\mu = Q)$ . Consequently

$$\frac{d}{d\mu}(\mu \mathbb{P}(N^\mu \leq Q)) \geq 0 \quad \text{iff} \quad \sum_{k=0}^Q \frac{\mu^k}{k!} \geq \mu \frac{\mu^Q}{Q!}.$$

But  $k \mapsto \frac{\mu^k}{k!}$  is nondecreasing on  $\{0, 1, \dots, \lfloor \mu \rfloor\}$  and non-increasing on  $\{\lfloor \mu \rfloor, \dots\}$ .

Hence  $\sum_{k=0}^Q \frac{\mu^k}{k!} \geq \sum_{k=\lfloor \mu \rfloor}^Q \frac{\mu^k}{k!} = (Q - \lfloor \mu \rfloor) \frac{\mu^Q}{Q!}$ , so that  $\sum_{k=0}^Q \frac{\mu^k}{k!} \geq \mu \frac{\mu^Q}{Q!}$  as soon as  $Q \geq 2\lfloor \mu \rfloor + 1$ .

(ii) The function  $\Phi$  is nondecreasing, non-negative and convex with  $\Phi(0) = 0$ . If we look at the representation of  $\mu \mapsto N^\mu$  by

$$N^\mu(\omega) = \max \left\{ n \in \mathbb{N} \mid \prod_{i=1}^n U_i(\omega) > e^{-\mu} \right\},$$

where  $U_i$  are i.i.d. uniformly distributed random variables on the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , then  $\mu \mapsto N^\mu$  is clearly nondecreasing, so  $\mu \mapsto Q - N^\mu$  is non-increasing and  $\mu \mapsto \varphi(\mu) = \Phi(Q - N^\mu) - \Phi(Q - N^\mu - 1)$  too (because of the convexity of  $\Phi$ ).  $\square$

**Remark.** If  $\mu \in (0, 1)$ , then  $\mu \mapsto \mu \mathbb{P}(N^\mu \leq Q)$  is always nondecreasing. If  $\mu \in [1, 2)$ , then the function  $\mu \mapsto \mu \mathbb{P}(N^\mu = 0) = \mu e^{-\mu}$  is not always nondecreasing clearly, but only on  $[0, 1]$ .

**Proof of Proposition 4.1.** (a) In our problem the intensity parameter  $\mu = \int_0^T \lambda(S_t - S_0 + \delta) dt$  is continuous, non-increasing to zero when  $\delta$  tends to  $+\infty$  and bounded by assumption ( $\lambda(-S_0) < +\infty$ ). Hence  $\mu \in [0, \lambda(-S_0)T]$ .

(i) From (3.19), we have for  $\delta = 0$ ,

$$\begin{aligned} & \widehat{C} \left( 0, \Lambda_T(0, S), \frac{\partial}{\partial \delta} \Lambda_T(0, S), (S_t)_{0 \leq t \leq T} \right) \\ & \leq \left( S_0 \frac{\partial}{\partial \delta} \Lambda_T(0, S) - \Lambda_T(0, S) \right) \mathbb{P}^{(0)}(N^\mu \leq Q_T - 1) - \kappa S_T \frac{\partial}{\partial \delta} \Lambda_T(0, S) \varphi^{(0)}(\mu) \\ & = \left( \frac{\partial}{\partial \delta} \Lambda_T(0, S) (S_0 - \kappa S_T \Theta(Q_T, \Lambda_T(0, S))) - \Lambda_T(0, S) \right) \mathbb{P}^{(0)}(N^\mu \leq Q_T - 1), \end{aligned}$$

because  $-Q_T \mathbb{P}(N^\mu > Q_T) |_{\mu = \Lambda_T(0, S)} < 0$  and small if  $Q_T$  is large. Set  $k_0 = \text{infess} \left( -\frac{\frac{\partial}{\partial \delta} \Lambda_T(0, S)}{\Lambda_T(0, S)} \right) > 0$  a.s. by assumption, i.e.  $\frac{\partial}{\partial \delta} \Lambda_T(0, S) \leq -k_0 \Lambda_T(0, S)$  a.s.. Then

$$\begin{aligned} & \widehat{C} \left( 0, \Lambda_T(0, S), \frac{\partial}{\partial \delta} \Lambda_T(0, S), (S_t)_{0 \leq t \leq T} \right) \\ & \leq - (1 + k_0 (S_0 - \kappa S_T \Theta(Q_T, \Lambda_T(0, S)))) (\mu \mathbb{P}(N^\mu \leq Q_T - 1)) |_{\mu = \Lambda_T(0, S)}. \end{aligned}$$

Now, by Lemma 4.1,  $\mu \mapsto \Theta(Q_T, \mu)$  is non-increasing.

Consequently  $\Theta(Q_T, \mu) \leq \Theta(Q_T, 0) = \Phi(Q) - \Phi(Q-1) = \varphi(0)$ .

Therefore  $\widehat{C}\left(0, \Lambda_T(0, S), \frac{\partial}{\partial \delta} \Lambda_T(0, S), (S_t)_{0 \leq t \leq T}\right)$   
 $\leq -(1 + k_0(S_0 - \kappa S_T(\Phi(Q_T) - \Phi(Q_T - 1)))) (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(0, S)}.$

By Lemma 4.1, if  $Q_T \geq 2T\lambda(-S_0)$ , then  $\mu \mapsto \mu\mathbb{P}(N^\mu \leq Q - 1)$  is nondecreasing. Moreover, the functional

$$F : \mathbb{D}([0, T, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\alpha \mapsto \Lambda_T(0, \alpha) = \int_0^T \lambda(\alpha(t) - S_0 + \delta) dt$$

is non-increasing and

$$\alpha(T) \mapsto (-1 - k_0(S_0 - \kappa\alpha(T)(\Phi(Q_T) - \Phi(Q_T - 1)))) \text{ is nondecreasing.}$$

Therefore, by opposite monotony principle for diffusion (see Theorem B.2)

$$\begin{aligned} & \mathbb{E} \left[ (-1 - k_0(S_0 - \kappa S_T(\Phi(Q_T) - \Phi(Q_T - 1)))) (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(0, S)} \right] \\ & \leq \mathbb{E} [-1 - k_0(S_0 - \kappa S_T(\Phi(Q_T) - \Phi(Q_T - 1)))] \mathbb{E} \left[ (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(0, S)} \right], \end{aligned}$$

and we obtain

$$C'(0) \leq \mathbb{E} [-1 - k_0(S_0 - \kappa S_T(\Phi(Q_T) - \Phi(Q_T - 1)))] \mathbb{E} \left[ (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(0, S)} \right].$$

As  $\mathbb{E} \left[ (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(0, S)} \right] \geq 0$ , then  $C'(0) \leq 0$  as soon as

$$\mathbb{E} [-1 - k_0(S_0 - \kappa S_T(\Phi(Q_T) - \Phi(Q_T - 1)))] \leq 0,$$

$$i.e. \quad \kappa < \frac{S_0}{\mathbb{E}[S_T](\Phi(Q_T) - \Phi(Q_T - 1))} + \frac{1}{k_0 \mathbb{E}[S_T](\Phi(Q_T) - \Phi(Q_T - 1))}.$$

(b) From (3.19), the form of  $\lambda$  and (a), we get that for every  $\delta \in [0, \delta_{\max}]$ ,  $S = (S_i)_{1 \leq i \leq m} \in \mathbb{R}^{m+1}$ ,

$$H(\delta, S) = -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T) + f(\delta, S) (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(\delta, S)},$$

where  $f(\delta, S) = -1 - k(S_0 - \delta - \kappa S_T \Theta(Q_T - 1, \Lambda_T(\delta, S)))$  if  $\Phi \neq \text{id}$  and  $f(\delta, S) = -1 - k(S_0 - \delta - \kappa S_T)$  if  $\Phi = \text{id}$ . Since  $\delta \mapsto -Q_T \mathbb{P}^{(\delta)}(N^\mu > Q_T)$  is nondecreasing,  $\delta \mapsto (\mu\mathbb{P}(N^\mu \leq Q_T - 1))|_{\mu=\Lambda_T(\delta, S)}$  is non-increasing and non-positive owing to Lemma 4.1 (i) and  $\delta \mapsto f(\delta, S)$  is nondecreasing owing to Lemma 4.1 (ii),  $\delta \mapsto H(\delta, S)$  is nondecreasing if  $f(\delta, S) > 0$ ,  $\delta \in [0, \delta_{\max}]$ ,  $S \in \mathbb{R}^{m+1}$ , which leads to (4.21).  $\square$

## 4.2 Sufficient condition for the convexity condition

**Proposition 4.2.** (i) If  $\Phi \neq \text{id}$ , assume that there exists  $\rho_Q \in \left(0, 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)}|_{\mu=T\lambda(-S_0)}\right)$  such that

$$\forall x \in [1, Q_T - 1], \quad \Phi(x) - \Phi(x - 1) \leq \rho_Q (\Phi(x + 1) - \Phi(x)).$$

$$\text{If} \quad Q_T \geq \left(2 \vee \left(1 + \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2}\right)\right) T \lambda(-S_0) \quad \text{and} \quad \kappa \leq \frac{2\bar{k}_1}{\bar{k}_1 \bar{k}_2 \mathbb{E}[S_T] \Phi'(Q_T)},$$

then  $C''(\delta) \geq 0$ ,  $\delta \in [0, \delta_{\max}]$ , so that  $C$  is convex on  $[0, \delta_{\max}]$ .

(ii) When  $\Phi = \text{id}$ , the condition reads

$$\kappa \leq \frac{2\bar{k}_1}{\bar{k}_1 \bar{k}_2 \mathbb{E}[S_T]}.$$

To prove the above Proposition, we need the following results

**Lemma 4.2.** *If  $\mu \leq Q - 1$ , then  $\mu \mapsto \frac{\mathbb{P}(N^\mu = Q - 1)}{\mathbb{P}(N^\mu \leq Q - 1)}$  is nondecreasing.*

**Proof of Lemma 4.2.**

$$\begin{aligned} \frac{d}{d\mu} \frac{\mathbb{P}(N^\mu = Q - 1)}{\mathbb{P}(N^\mu \leq Q - 1)} &= \frac{\mathbb{P}(N^\mu = Q - 2) - \mathbb{P}(N^\mu = Q - 1)}{\mathbb{P}(N^\mu \leq Q - 1)} + \frac{\mathbb{P}(N^\mu = Q - 1)^2}{\mathbb{P}(N^\mu \leq Q - 1)^2} \\ &= \frac{\mathbb{P}(N^\mu = Q - 1) \left( \mathbb{P}(N^\mu \leq Q - 1) \left( \frac{Q-1}{\mu} - 1 \right) + \mathbb{P}(N^\mu = Q - 1) \right)}{\mathbb{P}(N^\mu \leq Q - 1)^2} \\ &\geq 0 \quad \text{if } \mu \leq Q - 1. \quad \square \end{aligned}$$

**Lemma 4.3.** *Assume that  $\Phi \neq \text{id}$ . If there exists  $\rho_Q \in \left( 0, 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \Big|_{\mu = T\lambda(-S_0)} \right)$  such that*

$$\forall x \in [1, Q_T - 1], \quad \Phi(x) - \Phi(x - 1) \leq \rho_Q (\Phi(x + 1) - \Phi(x)),$$

*then  $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$  is non-increasing.*

**Remark.** If  $\Phi = \text{id}$ , then  $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \equiv 1$ , therefore we do not need the previous lemmas.

**Proof of Lemma 4.3.** We have

$$\frac{d}{d\mu} \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} = \frac{\mathbb{P}(N^\mu = Q_T - 1) \varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)^2} - \frac{\psi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \leq 0$$

$$\text{iff} \quad -\psi(\mu) \mathbb{P}(N^\mu \leq Q_T - 1) + \mathbb{P}(N^\mu = Q_T - 1) \varphi(\mu) \leq 0$$

$$\text{iff} \quad \mathbb{P}(N^\mu \leq Q_T - 1) \mathbb{E} [\Phi((Q_T - N^\mu - 1)_+) - \Phi((Q_T - N^\mu - 2)_+)] \leq \mathbb{P}(N^\mu \leq Q_T - 2) \varphi(\mu).$$

But

$$\begin{aligned} \Phi((Q_T - N^\mu - 1)_+) &- \Phi((Q_T - N^\mu - 2)_+) \\ &\leq \rho_Q (\Phi((Q_T - N^\mu)_+) - \Phi((Q_T - N^\mu - 1)_+)) \mathbf{1}_{\{N^\mu \leq Q_T - 2\}} \\ &= \rho_Q (\Phi((Q_T - N^\mu)_+) - \Phi((Q_T - N^\mu - 1)_+) - (\Phi(1) - \Phi(0)) \mathbf{1}_{\{N^\mu = Q_T - 1\}}) \\ &\leq \rho_Q (\Phi((Q_T - N^\mu)_+) - \Phi((Q_T - N^\mu - 1)_+)) \end{aligned}$$

$$\text{since} \quad (\Phi(1) - \Phi(0)) \mathbf{1}_{\{N^\mu = Q_T - 1\}} \geq 0 \quad a.s.$$

Consequently

$$\begin{aligned} &\mathbb{P}(N^\mu \leq Q_T - 1) \mathbb{E} [\Phi((Q_T - N^\mu - 1)_+) - \Phi((Q_T - N^\mu - 2)_+)] - \mathbb{P}(N^\mu \leq Q_T - 2) \varphi(\mu) \\ &\leq (\rho_Q \mathbb{P}(N^\mu \leq Q_T - 1) - \mathbb{P}(N^\mu \leq Q_T - 2)) \varphi(\mu) \\ &= \left( \rho_Q - \left( 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \right) \right) \mathbb{P}(N^\mu \leq Q_T - 1) \varphi(\mu) \leq 0 \quad \text{if } \rho_Q \leq 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)}. \quad \square \end{aligned}$$

**Proof of Proposition 4.2.** By using the notation (2.9)-(2.10), we obtain the following minoration for the second derivative of the cost function

$$\begin{aligned} C''(\delta) &\geq \mathbb{E} \left[ 2\underline{k}_1 \Lambda_T(\delta, S) \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) \right. \\ &\quad \left. + (S_0 - \delta) \underline{k}_1 \underline{k}_2 \Lambda_T(\delta, S) \left( \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - \frac{\overline{k}_1^2}{\underline{k}_1 \underline{k}_2} \Lambda_T(\delta, S) \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right) \right. \\ &\quad \left. - \kappa S_T \Lambda_T(\delta, S) \left( \overline{k}_1 \overline{k}_2 \phi^{(\delta)}(\mu) - \underline{k}_1^2 \Lambda_T(\delta, S) \psi^{(\delta)}(\mu) \right) \right]. \end{aligned}$$

By adapting the result of Lemma 4.1, we obtain that, if  $Q_T \geq \left(1 + \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2}\right) T \lambda(-S_0)$ , then

$$\mathbb{E} \left[ \left( \mathbb{P}(N^\mu \leq Q_T - 1) - \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2} \mu \mathbb{P}(N^\mu = Q_T - 1) \right) \Big|_{\mu = \Lambda_T(\delta, S)} \right] \geq 0$$

and by convexity of the penalty function  $\Phi$ , we have  $\psi^{(\delta)}(\mu) \geq 0$  *a.s.* Then we obtain the following upper bound for  $\kappa$ ,

$$\kappa \leq \frac{2\underline{k}_1 \mathbb{E} \left[ (\mu \mathbb{P}(N^\mu \leq Q_T - 1)) \Big|_{\mu = \Lambda_T(\delta, S)} \right]}{\bar{k}_1 \bar{k}_2 \mathbb{E} [S_T \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu)]}.$$

By Lemma 4.3,  $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$  is non-increasing and by Lemma 4.1  $\mu \mapsto \mu \mathbb{P}(N^\mu \leq Q_T - 1)$  is nondecreasing for  $Q_T \geq 2 \lfloor \mu \rfloor - 1$ . Furthermore  $\alpha \mapsto \Lambda_T(\delta, \alpha)$  is non-increasing. By applying the principle of opposite monotony (see Theorem B.2), we then have, for  $Q_T \geq 2T \lambda(-S_0)$ , that

$$\mathbb{E} [S_T \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu)] \leq \mathbb{E} [(\mu \mathbb{P}(N^\mu \leq Q_T - 1)) \Big|_{\mu = \Lambda_T(\delta, S)}] \mathbb{E} \left[ \left( \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \right) \Big|_{\mu = \Lambda_T(\delta, S)} \right].$$

Therefore

$$\kappa \leq \frac{2\underline{k}_1}{\bar{k}_1 \bar{k}_2 \mathbb{E} \left[ S_T \left( \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \right) \Big|_{\mu = \Lambda_T(\delta, S)} \right]}.$$

As, by Lemma 4.3,  $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$  is non-increasing, then

$$\left( \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \right) \Big|_{\mu = \Lambda_T(\delta, S)} \leq \left( \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \right) \Big|_{\mu=0} = \Phi((Q)_+) - \Phi((Q-1)_+) \leq \Phi'(Q),$$

*i.e.*

$$\kappa \leq \frac{2\underline{k}_1}{\bar{k}_1 \bar{k}_2 \mathbb{E} [S_T] \Phi'(Q)}. \quad \square$$

**Remark.** As  $\delta \in [0, \delta_{\max}]$ , then  $(S_0 - \delta) \in [S_0 - \delta_{\max}, S_0]$  and

$$\begin{aligned} & (S_0 - \delta) \underline{k}_1 \underline{k}_2 \Lambda_T(\delta, S) \left( \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2} \Lambda_T(\delta, S) \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right) \\ & \geq (S_0 - \delta_{\max}) \underline{k}_1 \underline{k}_2 \Lambda_T(\delta, S) \left( \mathbb{P}^{(\delta)}(N^\mu \leq Q_T - 1) - \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2} \Lambda_T(\delta, S) \mathbb{P}^{(\delta)}(N^\mu = Q_T - 1) \right) \\ & = (S_0 - \delta_{\max}) \underline{k}_1 \underline{k}_2 \left[ \mu \left( \mathbb{P}(N^\mu \leq Q_T - 1) - \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2} \mu \mathbb{P}(N^\mu = Q_T - 1) \right) \right] \Big|_{\mu = \Lambda_T(\delta, S)}. \end{aligned}$$

Unfortunately we cannot use the opposite monotony principle for diffusion to improve the bound because, for  $Q \geq \left(2 \vee \left(1 + \frac{\bar{k}_1^2}{\underline{k}_1 \underline{k}_2}\right)\right) T \lambda(-S_0)$ , the function  $\mu \mapsto \mu \mathbb{P}(N^\mu \leq Q_T - 1)$  is nondecreasing and  $\mu \mapsto 1 - \frac{\mu \mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$  is non-increasing, and we need to obtain a lower bound for this expression but we get an upper bound.

## 5 Numerical experiments

In this section, we present numerical results with simulated data. We first present the chosen model for the price dynamic and the penalization function. Within the numerical examples, we are modeling

the optimal behaviour of a “learning trader” reassessing the price of his passive order every 5 units of time (can be seconds or minutes) in the order books to adapt to the characteristics of the market (fair price moves  $S_t$  and order flow dynamics  $N_t$ ). During each  $n^{\text{th}}$  slice of 5 seconds, he posts his order of size  $Q_5$  in the book at a distance  $\delta$  of the best opposite price ( $\delta$  lower than the best ask for a buy order), and waits 5 seconds. If the order is not completely filled after these 5 seconds (say at time  $T$ ), the trader cancel the remaining quantity  $(Q_5 - N_5)_+$  and buys it using a market order at price  $S_T$  plus a market impact; he will buy at  $\kappa S_T (1 + \eta((Q_5 - N_5)_+))$ . Then he can reproduce the experiment choosing another value for the distance to the best opposite  $\delta$ .

The reassessment procedure used here is the one of formula (2.4) using the expectation representation of  $C'$  given by property 3.2 as the proper form for  $H$ .

Then we plot the cost function and its derivative for the chosen penalization function and for the identity. We conclude by the results of the recursive procedure for each case of  $\Phi$  on one hand simulated data and on the other hand real data obtained by replaying the market.

## 5.1 Simulated data

We assume that  $dS_t = \sigma dW_t$ ,  $S_0 = s_0$  and  $\Lambda_T(\delta, S) = A \int_0^T e^{-k(S_t - S_0 + \delta)} dt$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion and  $\sigma, A, k > 0$  (this means that  $\lambda(x) = Ae^{-kx}$ ). We denote by  $(\bar{S}_t)_{t \geq 0}$  the Euler scheme with step  $\frac{T}{m}$  of  $(S_t)_{t \geq 0}$  defined by

$$\bar{S}_{k+1} := \bar{S}_k + \sigma \sqrt{\frac{T}{m}} Z_{k+1}, \quad \bar{S}_0 = s_0, \quad Z_{k+1} \sim \mathcal{N}(0, 1), \quad k \geq 0,$$

and we approximate  $\Lambda_T(\delta, S)$  by  $\bar{\Lambda}_T(\delta, S) = A \frac{T}{m} \sum_{k=0}^m e^{-k(\bar{S}_k - S_0 + \delta)}$ . The market impact penalization function is  $\Phi(x) = (1 + \eta(x))x$  with  $\eta(x) = A' e^{k'x}$ . Now we present the cost function and its derivative for the following parameters:

- parameters of the asset dynamics:  $s_0 = 100$  and  $\sigma = 0.01$ ,
- parameters of the intensity of the execution process:  $A = 5$  and  $k = 1$ ,
- parameters of the execution:  $T = 5$  and  $Q = 10$ ,
- parameters of the penalization function:  $\kappa = 1$ ,  $A' = 0.1$  and  $k' = 0.05$ .

We use  $N = 20$  for the Euler scheme and  $M = 10000$  simulations Monte Carlo.

**Setting 1** ( $\eta \neq 0$ )

**Setting 2** ( $\eta \equiv 0$ )

Now we present the results of our stochastic recursive procedure for the two cases with

$$n = 100 \quad \text{and} \quad \gamma_n = \frac{1}{100n}.$$

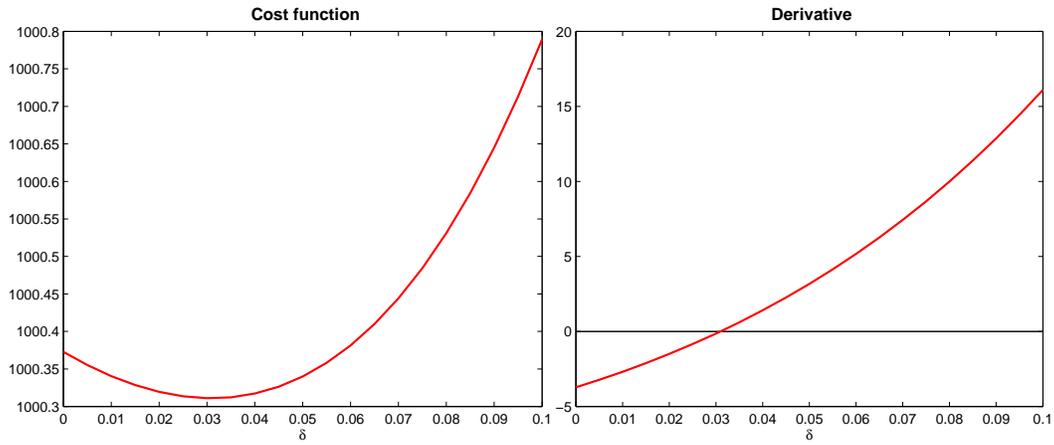


Figure 2:  $\eta \neq 0$ :  $T = 5$ ,  $A = 5$ ,  $k = 1$ ,  $s_0 = 100$ ,  $\sigma = 0.01$ ,  $Q = 10$ ,  $\kappa = 6$ ,  $A' = 1$ ,  $k' = 0.01$ ,  $N = 20$  and  $M = 10000$ .

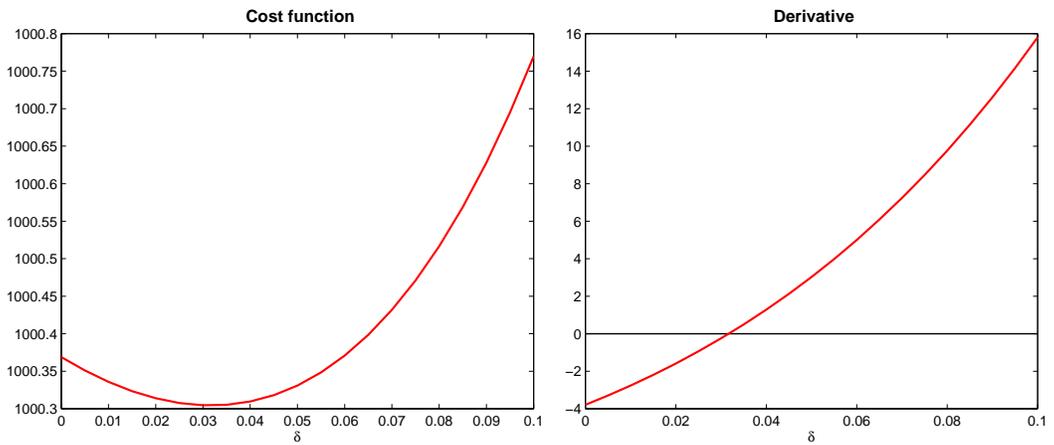


Figure 3:  $\eta \equiv 0$ :  $T = 5$ ,  $A = 5$ ,  $k = 1$ ,  $s_0 = 100$ ,  $\sigma = 0.01$ ,  $Q = 10$ ,  $\kappa = 12$ ,  $N = 20$  and  $M = 10000$ .

**Setting 1** ( $\eta \neq 0$ )

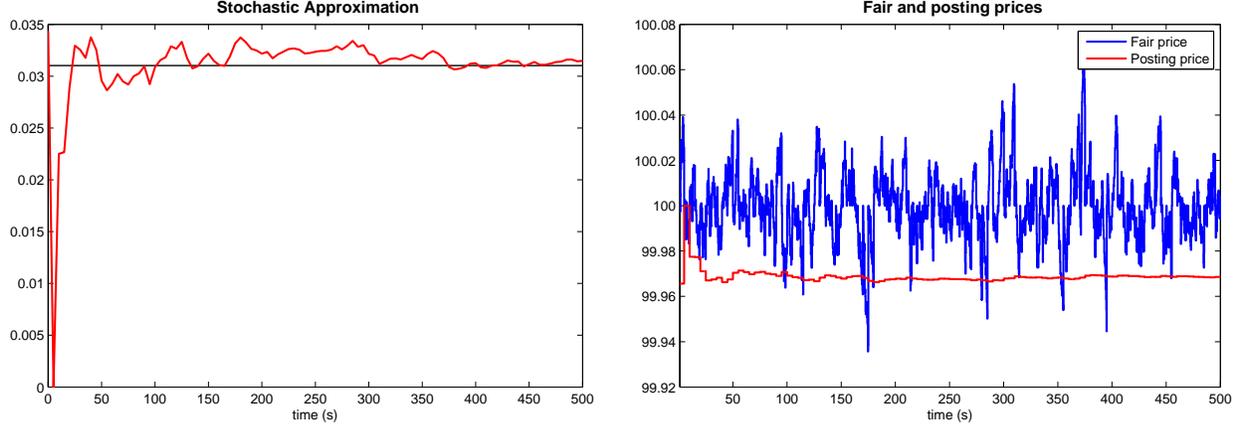


Figure 4:  $\eta \neq 0$ :  $T = 5$ ,  $A = 5$ ,  $k = 1$ ,  $s_0 = 100$ ,  $\sigma = 0.01$ ,  $Q = 10$ ,  $\kappa = 6$ ,  $A' = 1$ ,  $k' = 0.01$ ,  $N = 20$  and  $n = 100$

**Setting 2** ( $\eta \equiv 0$ )

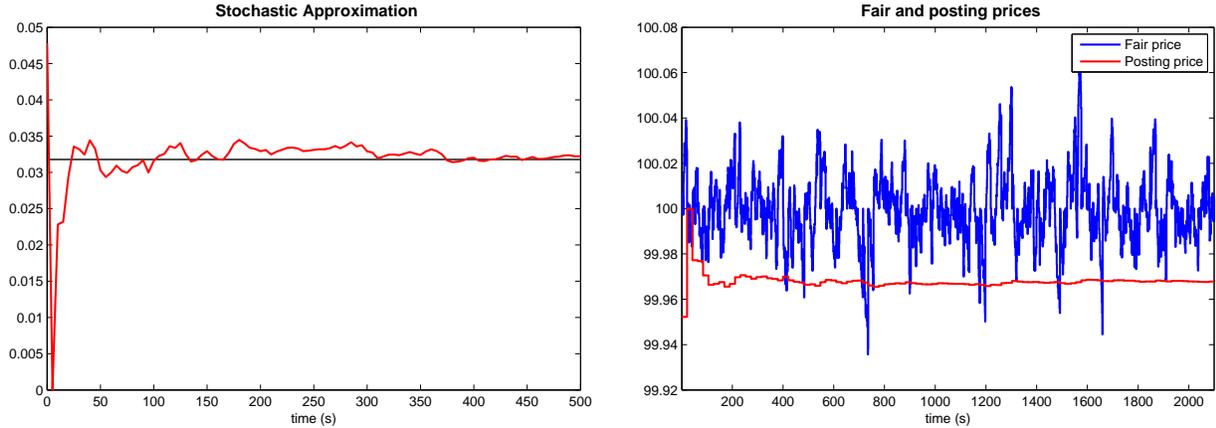


Figure 5:  $\eta \equiv 0$ :  $T = 5$ ,  $A = 5$ ,  $k = 1$ ,  $s_0 = 100$ ,  $\sigma = 0.01$ ,  $Q = 10$ ,  $\kappa = 12$ ,  $N = 20$  and  $n = 100$ .

## 5.2 Market data

The auto adaptiveness nature of this recurrence procedure allows to use it on real data, even if they are not exactly following the models.

In the numerical example of this section, the trader reassess his order using the previously exposed recurrence procedure not on simulated data following exactly the models, but on real data on which the parameters of the models have been fit.

As market data, we use the bid prices of Accor SA (ACCP.PA) of 11/11/2010 for the fair price process  $(S_t)_{t \in [0, T]}$ . We divide the day into periods of 15 trades which will denote steps of the stochastic procedure. Let  $M_{\text{cycles}}$  be the number of these periods. For every  $m \in M_{\text{cycles}}$ , we have a sequence of bid prices  $(S_{t_i}^m)_{1 \leq i \leq 15}$  and we approximate the jump intensity of the Poisson process  $\Lambda_{T^m}(\delta, S)$ , where  $T^m = \sum_{i=1}^{15} t_i$ , by

$$\forall m \in M_{\text{cycles}}, \quad \Lambda_{T^m}(\delta, S) = A \sum_{i=2}^{15} e^{-k(S_{t_i}^m - S_{t_1} + \delta)} (t_i - t_{i-1}).$$

The empirical mean of the intensity function

$$\bar{\Lambda}(\delta, S) = \frac{1}{M_{\text{cycles}}} \sum_{n=1}^{M_{\text{cycles}}} \Lambda_{T^n}(\delta, S)$$

is plotted on Figure 6.

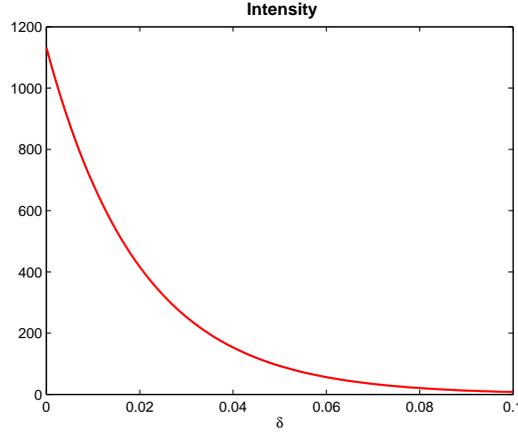


Figure 6: Fit of the exponential model on real data (Accor SA (ACCP.PA) 11/11/2010):  $A = 1/50$ ,  $k = 50$  and  $M_{\text{cycles}} = 220$ .

The penalization function is of the following form

$$\Phi(x) = (1 + \eta(x))x \quad \text{with} \quad \eta(x) = A'e^{k'x}.$$

Now we present the cost function and its derivative for the following parameters:  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $A' = 0.001$  and  $k' = 0.0005$ .

**Setting1** ( $\eta \neq 0$ )

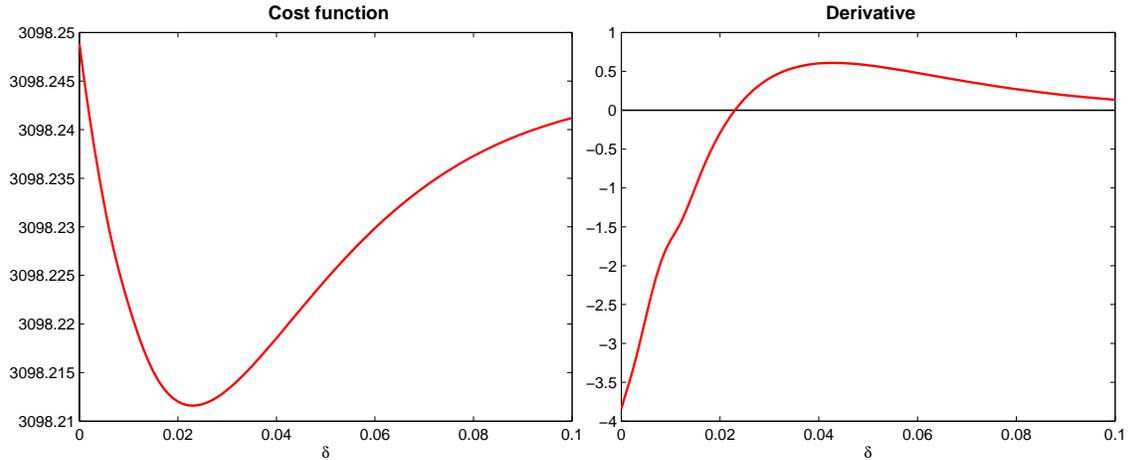


Figure 7:  $\eta \neq 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1$ ,  $A' = 0.001$ ,  $k' = 0.0005$  and  $M_{\text{cycles}} = 220$ .

**Setting2** ( $\eta \equiv 0$ )

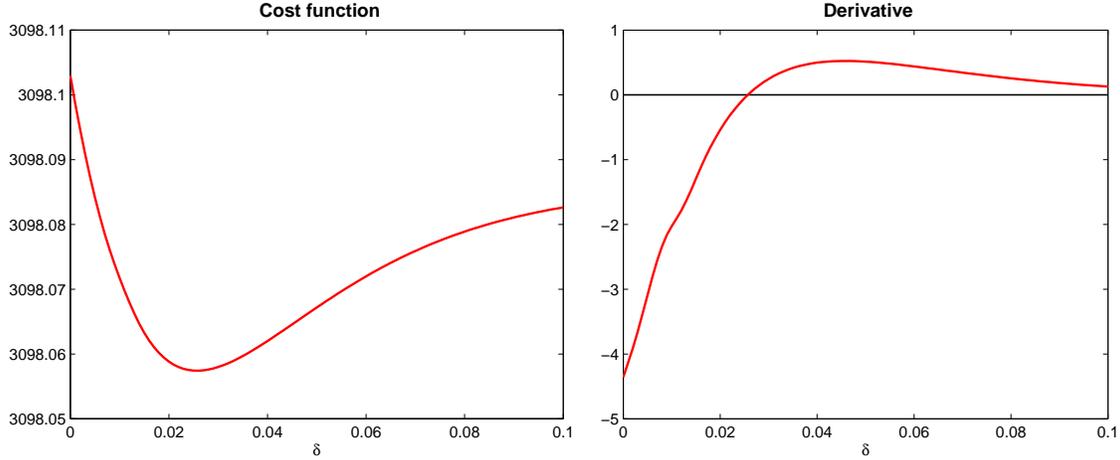


Figure 8:  $\eta \equiv 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1.001$  and  $M_{\text{cycles}} = 220$ .

Now we present the results of the stochastic recursive procedure for two cases. To smoothen the behaviour of the stochastic algorithm, we use the averaging principle of Ruppert and Poliak (see [7]). In short, this principle is two-folded:

- *Phase 1*: Implement the original zero search procedure with  $\gamma_n = \frac{\gamma_1}{n^\rho}$ ,  $\frac{1}{2} < \rho < 1$ ,  $\gamma_1 > 0$ ,
- *Phase 2*: Compute the arithmetic mean at each step  $n$  of all the past values of the procedure, namely

$$\bar{\delta}_n = \frac{1}{n+1} \sum_{k=0}^n \delta_k, \quad n \geq 1.$$

It has been shown by several authors that this procedure under appropriate assumptions is ruled by a *CLT* having a minimal asymptotic variance (among recursive procedures).

**Setting1** ( $\eta \neq 0$ )

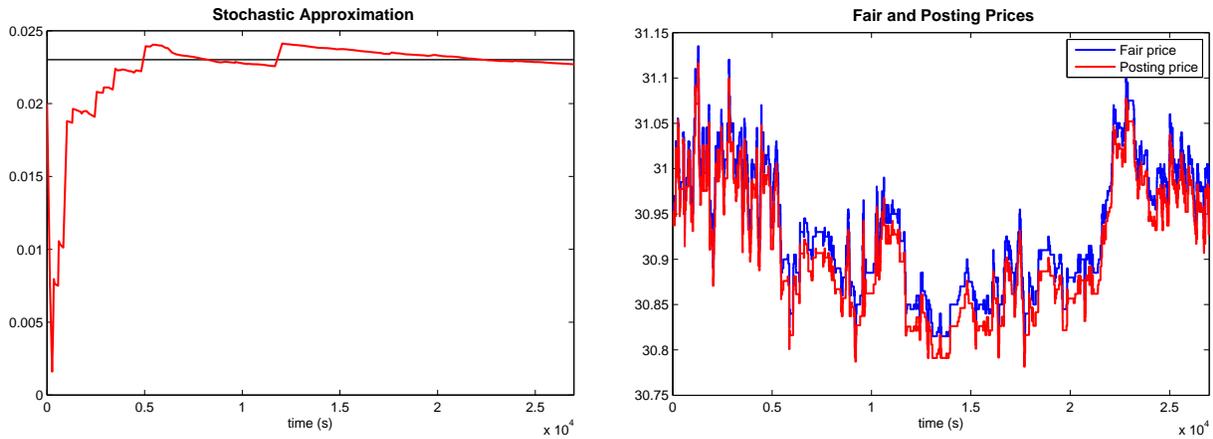


Figure 9:  $\eta \neq 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1$ ,  $A' = 0.001$ ,  $k' = 0.0005$  and  $M_{\text{cycles}} = 220$ . Crude algorithm with  $\gamma_n = \frac{1}{550n}$ .

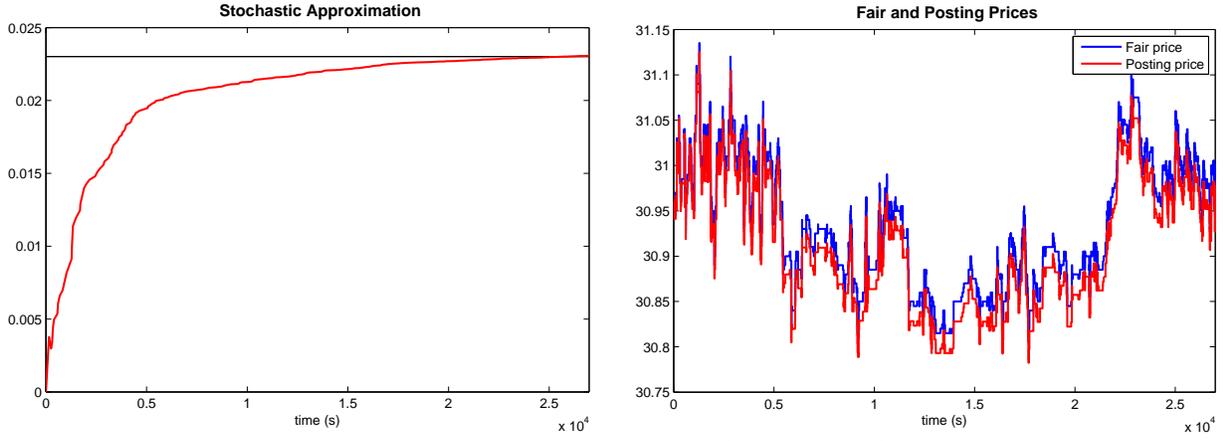


Figure 10:  $\eta \neq 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1$ ,  $A' = 0.001$ ,  $k' = 0.0005$  and  $M_{\text{cycles}} = 220$ . Averaging algorithm (Ruppert and Poliak) with  $\gamma_n = \frac{1}{550n^{0.95}}$ .

**Setting 2** ( $\eta \equiv 0$ )

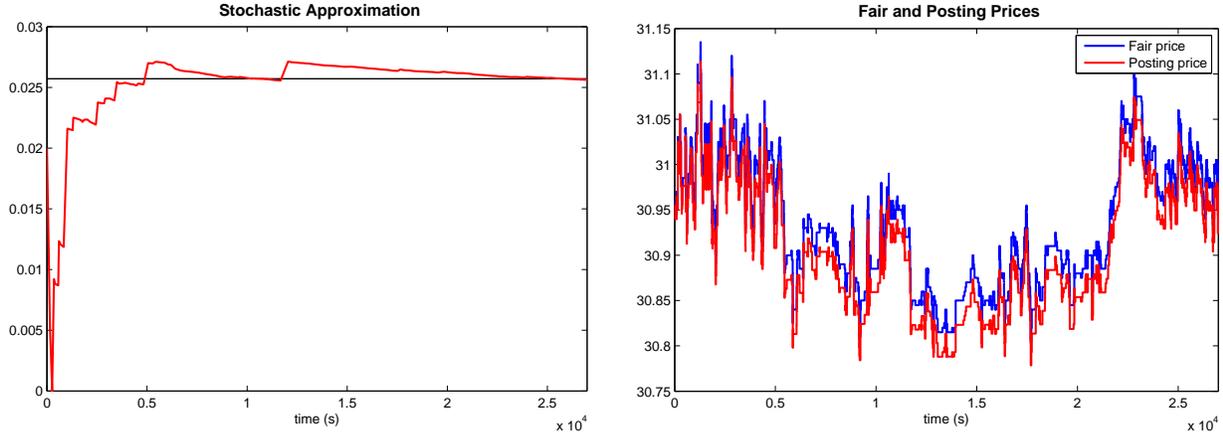


Figure 11:  $\eta \equiv 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1.001$ ,  $\gamma_n = \frac{1}{450n}$  and  $M_{\text{cycles}} = 220$ .

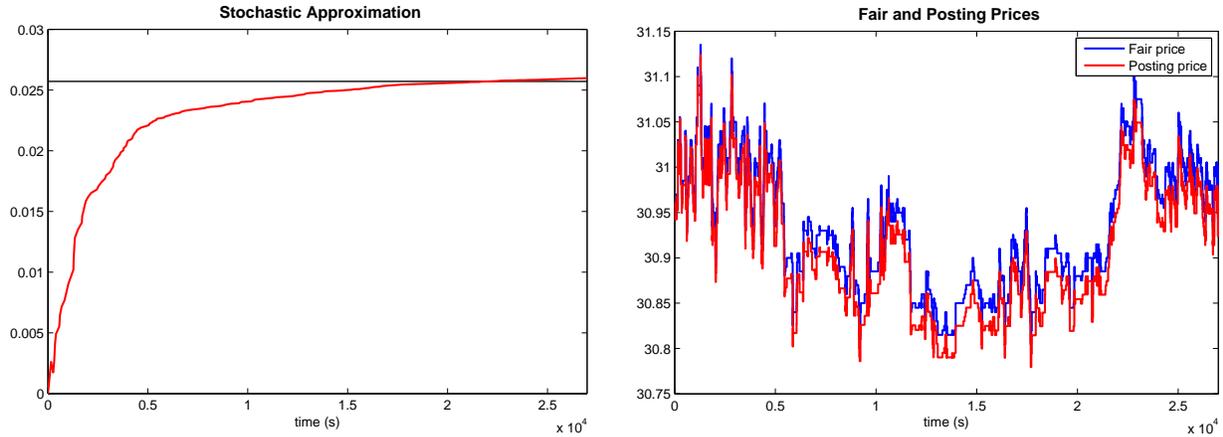


Figure 12:  $\eta \neq 0$ :  $A = 1/50$ ,  $k = 50$ ,  $Q = 100$ ,  $\kappa = 1.001$  and  $M_{\text{cycles}} = 220$ . Averaging algorithm (Ruppert and Poliak) with  $\gamma_n = \frac{1}{450n^{0.95}}$ .

We see on Figures 10 (for  $\eta \neq 0$ ) and 12 (for  $\eta \equiv 0$ ) that the recursive procedures converge toward their respective targets, namely the minimum of the execution cost functions presented in Figures 7 (for  $\eta \neq 0$ ) and 8 (for  $\eta \equiv 0$ ).

# Appendix

## A Convergence theorem for constrained algorithms

The aim is to determine an element of the set  $\{\theta \in \Theta : h(\theta) = \mathbb{E}[H(\theta, Y)] = 0\}$  (zeros of  $h$  in  $\Theta$ ) where  $\Theta \subset \mathbb{R}^d$  is a closed convex set,  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $H : \mathbb{R}^d \times \mathbb{R}^q \rightarrow \mathbb{R}^d$ . For  $\theta_0 \in \Theta$ , we consider the  $\mathbb{R}^d$ -valued sequence  $(\theta_n)_{n \geq 0}$  defined by

$$\theta_{n+1} = \text{Proj}_{\Theta}(\theta_n - \gamma_{n+1}H(\theta_n, Y_{n+1})), \quad (\text{A.22})$$

where  $(Y_n)_{n \geq 1}$  is an i.i.d. sequence with the same law as  $Y$ ,  $(\gamma_n)_{n \geq 1}$  is a positive sequence of real numbers and  $\text{Proj}_{\Theta}$  denotes the Euclidean projection on  $\Theta$ . The recursive procedure (A.22) can be rewritten as follows

$$\theta_{n+1} = \theta_n - \gamma_{n+1}h(\theta_n) - \gamma_{n+1}\Delta M_{n+1} + \gamma_{n+1}p_{n+1}, \quad (\text{A.23})$$

where  $\Delta M_{n+1} = H(\theta_n, Y_{n+1}) - h(\theta_n)$  is a martingale increment and

$$p_{n+1} = \frac{1}{\gamma_{n+1}} \text{Proj}_{\Theta}(\theta_n - \gamma_{n+1}H(\theta_n, Y_{n+1})) - \frac{\theta_n}{\gamma_{n+1}} + H(\theta_n, Y_{n+1}).$$

**Theorem A.1.** (see [16] and [17]) *Let  $(\theta_n)_{n \geq 0}$  be the sequence defined by (A.23). Assume that there exists a unique  $\theta^* \in \Theta$  such that  $h(\theta^*) = 0$  and that the mean function satisfies on  $\Theta$  the following mean-reverting property, namely*

$$\forall \theta \neq \theta^* \in \Theta, \quad \langle h(\theta) | \theta - \theta^* \rangle > 0. \quad (\text{A.24})$$

*Assume that the gain parameter sequence  $(\gamma_n)_{n \geq 1}$  satisfies*

$$\sum_{n \geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n \geq 1} \gamma_n^2 < +\infty. \quad (\text{A.25})$$

*If the function  $H$  satisfies*

$$\exists K > 0 \text{ such that } \forall \theta \in \Theta, \quad \mathbb{E} \left[ |H(\theta, Y)|^2 \right] \leq K(1 + |\theta|^2), \quad (\text{A.26})$$

*then*

$$\theta_n \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \theta^*.$$

**Remark.** If  $\Theta$  is bounded (A.26) reads  $\sup_{\theta \in \Theta} \mathbb{E} \left[ |H(\theta, Y)|^2 \right] < +\infty$ , which is always satisfied if  $\Theta$  is compact and  $\theta \mapsto \mathbb{E} \left[ |H(\theta, Y)|^2 \right]$  is continuous.

## B Monotony principle for a class of one-dimensional diffusions

In this section, we present the principle of co- and opposite monotony, first for random vectors taking values in a nonempty interval  $I$ , then for one-dimensional diffusions lying in  $I$ .

## B.1 Case of random variables and random vectors

First we recall a classical result for random variables.

**Proposition B.1.** *Let  $f, g : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be two monotonic functions with opposite monotony. Let  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow I$  be a real valued random variable such that  $f(X), g(X) \in L^2(\mathbb{P})$ . Then*

$$\text{Cov}(f(X), g(X)) \leq 0.$$

**Proof.** Let  $X, Y$  be two independent random variables defined on the same probability space with the same distribution  $\mathbb{P}_X$ . Then

$$(f(X) - f(Y))(g(X) - g(Y)) \leq 0$$

hence its expectation. Consequently

$$\mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)g(Y)] - \mathbb{E}[f(Y)g(X)] + \mathbb{E}[f(Y)g(Y)] \leq 0$$

so using that  $Y \stackrel{(d)}{=} X$  and  $Y, X$  are independent

$$2\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)]\mathbb{E}[g(Y)] + \mathbb{E}[f(Y)]\mathbb{E}[g(X)] = 2\mathbb{E}[f(X)]\mathbb{E}[g(X)]$$

that is  $\text{Cov}(f(X), g(X)) \leq 0$ . □

**Proposition B.2.** *Let  $F, G : \mathbb{R}^d \rightarrow \mathbb{R}$  be two monotonic functions with opposite monotony in each of their variables, i.e. for every  $i \in \{1, \dots, d\}$ ,  $x_i \mapsto F(x_1, \dots, x_i, \dots, x_n)$  and  $x_i \mapsto G(x_1, \dots, x_i, \dots, x_n)$  are monotonic with an opposite monotony which may depend on  $i$  (but does not depend on  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{d-1}$ ). Let  $X_1, \dots, X_d$  be independent real valued random variables defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that  $F(X_1, \dots, X_d), G(X_1, \dots, X_d) \in L^2(\mathbb{P})$ . Then*

$$\text{Cov}(F(X_1, \dots, X_d), G(X_1, \dots, X_d)) \leq 0.$$

**Proof.** The proof of the above proposition is made by induction on  $d$ . The case  $d = 1$  is given by Proposition B.1. We give here the proof for  $d = 2$  for notational convenience, but the general case of dimension  $d$  follows straightforwardly. By opposite monotonic assumption on  $F$  and  $G$ , we have for every  $x_2 \in \mathbb{R}$ ,  $X'_1 \stackrel{d}{=} X_1$  with  $X'_1, X_1$  independent, that

$$(F(X_1, x_2) - F(X'_1, x_2))(G(X_1, x_2) - G(X'_1, x_2)) \leq 0,$$

which implies that (see Proposition B.1)

$$\text{Cov}(F(X_1, x_2)G(X_1, x_2)) \leq 0.$$

If  $X_1$  and  $X_2$  are independent, using Fubini Theorem and what precedes, we have

$$\begin{aligned} \mathbb{E}[F(X_1, X_2)G(X_1, X_2)] &= \int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2) \mathbb{E}[F(X_1, x_2)G(X_1, x_2)] \\ &\leq \int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2) \mathbb{E}[F(X_1, x_2)] \mathbb{E}[G(X_1, x_2)]. \end{aligned}$$

By setting  $\varphi(x_2) = \mathbb{E}[F(X_1, x_2)]$  and  $\psi(x_2) = \mathbb{E}[G(X_1, x_2)]$  and using the monotonic assumptions on  $F$  and  $G$ , we have

$$\int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2) \mathbb{E}[F(X_1, x_2)] \mathbb{E}[G(X_1, x_2)] = \mathbb{E}[\varphi(X_2)\psi(X_2)] \leq \mathbb{E}[\varphi(X_2)] \mathbb{E}[\psi(X_2)],$$

i.e.  $\text{Cov}(F(X_1, X_2)G(X_1, X_2)) \leq 0$ . □

## B.2 Case of (one-dimensional) diffusions

This framework corresponds to the infinite dimensional case and we can not apply straightforwardly the result of Proposition B.1: indeed, if we define the following order relation on the process of  $\mathbb{D}([0, T], \mathbb{R})$ , namely

$$\forall \alpha_1, \alpha_2 \in \mathbb{D}([0, T], \mathbb{R}), \quad \alpha_1 \leq \alpha_2 \iff (\forall t \in [0, T], \alpha_1(t) \leq \alpha_2(t)),$$

this order relation is partial and not total which makes the formal proof of Proposition B.1 collapse. To establish a principle of opposite monotony on diffusions, we proceed in two steps: first, we use the Lamperti transform to “force” the diffusion coefficient to be equal to 1 and we establish the opposite monotony principle for this kind of diffusions. Then, by the inverse Lamperti transform, we go back to the original process.

In this section, we first present the framework in which we place. Then we recall some weak convergence results for diffusion with diffusion coefficient equal to 1. Afterwards we present the Lamperti transform and we conclude by the general result on opposite monotony principle.

Let  $I$  be a nonempty open interval of  $\mathbb{R}$ . One considers a real-valued Brownian diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0 \in I, \quad t \in [0, T], \quad (\text{B.27})$$

where  $b, \sigma : [0, T] \times I \rightarrow \mathbb{R}$  are Borel functions with at most linear growth such that the above Equation (B.27) admits at least one (weak) solution over  $[0, T]$  and  $W$  is a Brownian motion defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We assume that the diffusion  $X$  *a.s.* does not explode and lives in the interval  $I$ . This implies assumptions on the function  $b$  and  $\sigma$  especially in the neighbourhood (in  $I$ ) of the endpoints of  $I$  that we will not detail here. At a finite endpoint of  $I$ , these assumptions are strongly connected with the Feller classification for which we refer to [15] with  $\sigma(t, \cdot) > 0$  for every  $t \in [0, T]$ . We will simply make some classical linear growth assumption on  $b$  and  $\sigma$  (which prevent explosion at a finite time) that will be used for different purpose in what follows.

To “remove” the diffusion coefficient of the diffusion  $X$ , we will introduce the so-called *Lamperti transform* which requires additional assumptions on the drift  $b$  and the diffusion coefficient  $\sigma$ , namely assumptions (i)-(iii) in Definition 2.3.

**Remark.** Condition (iii) clearly does not depend on  $x \in I$ . Furthermore, if  $I = \mathbb{R}$ , (iii) follows from (ii) since  $\frac{1}{\sigma(t, \xi)} \geq \frac{1}{C(1+|\xi|)}$ .

Before passing to a short background on the Lamperti transform which will lead to study a new diffusion deduced from (B.27) whose diffusion coefficient is equal to 1, we need to recall (and adapt) some background on solution and discretization of *SDE*.

### B.2.1 Background on diffusions with $\sigma \equiv 1$ (weak solution, discretization).

The following proposition gives condition on the drift for the existence and the uniqueness of a weak solution of a SDE with  $\sigma \equiv 1$  (see [14] Proposition 3.6, Chap. 5, p. 303 and Corollary 3.11, Chap. 5, p. 305).

**Proposition B.3.** *Consider the stochastic differential equation*

$$dY_t = \beta(t, Y_t)dt + dW_t, \quad t \in [0, T], \quad (\text{B.28})$$

where  $T$  is a fixed positive number,  $W$  is a one-dimensional Brownian motion and  $\beta : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a Borel-measurable function satisfying

$$|\beta(t, y)| \leq K(1 + |y|), \quad t \in [0, T], \quad y \in \mathbb{R}, \quad K > 0.$$

For any probability measure  $\nu$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , equation (B.28) has a weak solution with initial distribution  $\nu$ .

If, furthermore, the drift term  $b$  satisfies one of the following conditions

- (i)  $\beta$  is bounded on  $[0, T] \times \mathbb{R}$ ,
- (ii)  $\beta$  is continuous, locally Lipschitz in  $y \in \mathbb{R}$  uniformly in  $t \in [0, T]$ ,

then this weak solution is unique (in fact (ii) is a uniqueness assumption for the existing strong solution).

Now we introduce the stepwise constant (Brownian) Euler scheme  $\bar{Y}^n = \left(\bar{Y}_{\frac{kT}{n}}\right)_{0 \leq k \leq n}$  with step  $\frac{T}{n}$  of the process  $Y = (Y_t)_{t \geq 0}$  defined by (B.28). It is defined by

$$\bar{Y}_{t_{k+1}^n} = \bar{Y}_{t_k^n} + \beta(t_k^n, \bar{Y}_{t_k^n}) \frac{T}{n} + \sqrt{\frac{T}{n}} U_{k+1}, \quad \bar{Y}_0 = Y_0 = y_0, \quad k = 0, \dots, n-1, \quad (\text{B.29})$$

where  $t_k^n = \frac{kT}{n}$ ,  $k = 0, \dots, n$ , and  $(U_k)_{0 \leq k \leq n}$  denotes a sequence of i.i.d.  $\mathcal{N}(0, 1)$ -distributed random variables given by

$$U_k = \sqrt{\frac{n}{T}} \left( W_{t_k^n} - W_{t_{k-1}^n} \right), \quad k = 1, \dots, n.$$

The following theorem gives a weak convergence result for the stepwise constant Euler scheme (B.29). Its proof is a straightforward consequence of the functional limit theorems for semi-martingales (see Theorem 3.39, Chap. IX, p. 551 in [13]).

**Theorem B.1.** *Let  $\beta : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying*

$$\exists K > 0, \quad |\beta(t, y)| \leq K(1 + |y|), \quad t \in [0, T], \quad y \in \mathbb{R}.$$

*Assume that the weak solution of equation (B.28) is unique. Then, the stepwise constant Euler scheme of (B.28) with step  $\frac{T}{n}$  satisfies*

$$\bar{Y}^n \xrightarrow{\mathcal{L}} Y \quad \text{for the Skorokhod topology as } n \rightarrow \infty.$$

*In particular, for every  $\mathbb{P}_Y$ -a.s. continuous functional  $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  with polynomial growth, we have*

$$\mathbb{E}F(\bar{Y}^n) \xrightarrow{n \rightarrow \infty} \mathbb{E}F(Y).$$

## B.2.2 Background on the Lamperti transform

We will introduce a new diffusion  $Y_t := L(t, X_t)$  which will satisfy a new SDE whose diffusion coefficient will be constant equal to 1. This function  $L$  defined on  $[0, T] \times I$  is known in the literature as the Lamperti transform. It is defined for every  $(t, x) \in [0, T] \times I$  by

$$L(t, x) := \int_{x_1}^x \frac{d\xi}{\sigma(t, \xi)} \quad (\text{B.30})$$

where  $x_1$  is an arbitrary fixed value lying in  $I$ . The Lamperti transform clearly depends on the choice of  $x_1$  in  $I$  but not its properties of interest. First, under Definition 2.3 (i)-(ii),  $L \in C^{1,2}([0, T] \times I)$  with

$$\frac{\partial L}{\partial t}(t, x) = - \int_{x_1}^x \frac{1}{\sigma^2(t, \xi)} \frac{\partial \sigma}{\partial t}(t, \xi) d\xi, \quad \frac{\partial L}{\partial x}(t, x) = \frac{1}{\sigma(t, x)} > 0 \quad \text{and} \quad \frac{\partial^2 L}{\partial x^2}(t, x) = - \frac{1}{\sigma^2(t, x)} \frac{\partial \sigma}{\partial x}(t, x).$$

Let  $t \in [0, T]$ ,  $L(t, \cdot)$  is an increasing  $\mathcal{C}^2$ -diffeomorphism from  $I$  onto  $\mathbb{R} = L(t, I)$  (the last claim follows from Definition 2.3 (iii)). Its inverse will be denoted  $L^{-1}(t, \cdot)$ .

Notice that,  $(t, y) \mapsto L^{-1}(t, y)$  is continuous on  $[0, T] \times I$  since both sets

$$\{(t, y) \in [0, T] \times I : L^{-1}(t, y) \leq c\} = \{(t, y) \in [0, T] \times \mathbb{R} : L(t, c) \geq y\}$$

and

$$\{(t, y) \in [0, T] \times I : L^{-1}(t, y) \geq c\} = \{(t, y) \in [0, T] \times \mathbb{R} : L(t, c) \leq y\}$$

are both closed for every  $c \in \mathbb{R}$ . Therefore, if Definition 2.3 (i)-(iii) holds, the function  $\beta : [0, T] \times I \mapsto \mathbb{R}$  defined by

$$\beta(t, y) := \left( \frac{b}{\sigma} - \int_{x_1}^{\cdot} \frac{1}{\sigma^2(t, \xi)} \frac{\partial \sigma}{\partial t}(t, \xi) d\xi - \frac{1}{2} \frac{\partial \sigma}{\partial x} \right) (t, L^{-1}(t, y)). \quad (\text{B.31})$$

is a Borel function, continuous as soon as  $b$  is.

Now, we set

$$\forall t \in [0, T], \quad Y_t := L(t, X_t).$$

Itô formula straightforwardly yields

$$dY_t = \beta(t, Y_t)dt + dW_t, \quad Y_0 = L(0, x_0) =: y_0 \in \mathbb{R}. \quad (\text{B.32})$$

**Remarks.** • In the homogeneous case, which is the most important case for our applications,

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \in [0, T], \quad (\text{B.33})$$

we have

$$L(x) := \int_{x_1}^x \frac{d\xi}{\sigma(\xi)},$$

then by setting  $Y_t := L(X_t)$ , we obtain

$$dY_t = \beta(Y_t)dt + dW_t, \quad Y_0 = L(x_0) =: y_0 \quad \text{with} \quad \beta := \left( \frac{b}{\sigma} - \frac{\sigma'}{2} \right) \circ L^{-1}.$$

Note that  $\beta$  is bounded as soon as  $\frac{b}{\sigma} - \frac{\sigma'}{2}$  is.

• If the partial derivative  $b'_x$  exists on  $[0, T] \times I$ , one easily checks that, using  $(L^{-1})'_y(t, y) = \sigma(t, L^{-1}(t, y))$ , for every  $(t, y) \in [0, T] \times I$ ,

$$\beta'_y(t, y) = \left( b'_x - \frac{b\sigma'_x + \sigma'_t}{\sigma} - \frac{\sigma\sigma''_{x^2}}{2} \right) (t, L^{-1}(t, y)). \quad (\text{B.34})$$

As a consequence, one derives that  $\beta$  satisfies the linear growth assumption as soon as the function

$$b'_x - \frac{b\sigma'_x + \sigma'_t}{\sigma} - \frac{\sigma\sigma''_{x^2}}{2} \quad \text{is bounded on} \quad [0, T] \times I \quad (\text{B.35})$$

**Definition B.1.** *The functional Lamperti transform, denoted  $\Lambda$ , is a functional from  $\mathcal{C}([0, T], I)$  to  $\mathcal{C}([0, T], \mathbb{R})$  defined by*

$$\forall \alpha \in \mathcal{C}([0, T], I), \quad \Lambda(\alpha) = L(\cdot, \alpha(\cdot)).$$

**Proposition B.4.** *If the diffusion coefficient  $\sigma$  satisfies Definition 2.3 (i)-(iii), the functional Lamperti transform is an homeomorphism from  $\mathcal{C}([0, T], I)$  onto  $\mathcal{C}([0, T], \mathbb{R})$ .*

**Proof.** Let  $\alpha \in \mathcal{C}([0, T], I)$ . Since  $\sigma$  is bounded away from 0 on the compact set  $[0, T] \times \alpha([0, T])$ , standard arguments based on Lebesgue domination theorem, imply that  $\Lambda(\alpha) \in \mathcal{C}([0, T], \mathbb{R})$ .

Conversely, as  $L(t, \cdot) : I \rightarrow \mathbb{R}$  is an homeomorphism for every  $t \in [0, T]$ ,  $\Lambda$  admits an inverse defined by

$$\forall \xi \in \mathcal{C}([0, T], \mathbb{R}), \quad \Lambda^{-1}(\xi) := (t \mapsto L^{-1}(t, \xi(t))) \in \mathcal{C}([0, T], I).$$

Let  $U_K$  denote the topology of the convergence on compact sets of  $I$  on  $\mathcal{C}([0, T], I)$ .

$\triangleright U_K$ -Continuity of  $\Lambda$ : If  $\alpha_n \xrightarrow{U_K} \alpha_\infty$ , the set  $K = [0, T] \times \bigcup_{n \in \mathbb{N}} \alpha_n([0, T])$  is a compact set included in  $I$ . Hence  $\sigma$  is bounded away from 0 on  $K$  so that

$$\forall t \in [0, T], \quad |L(t, \alpha_n(t)) - L(t, \alpha_\infty(t))| \leq \frac{1}{\inf_K \sigma} |\alpha_n(t) - \alpha_\infty(t)|$$

*i.e.* 
$$\|\Lambda(\alpha_n) - \Lambda(\alpha_\infty)\|_{\text{sup}} \leq \frac{1}{\inf_K \sigma} \|\alpha_n - \alpha_\infty\|_{\text{sup}}.$$

$\triangleright U_K$ -Continuity of  $\Lambda^{-1}$ : by using Definition 2.3 (ii), we have for a fixed  $t \in [0, T]$ ,

$$\forall x, x' \in I, \quad |L(t, x) - L(t, x')| \geq \frac{1}{C} \int_{x \wedge x'}^{x \vee x'} \frac{d\xi}{1 + |\xi|} = \frac{1}{C} |\Phi(x) - \Phi(x')|,$$

where  $\Phi(z) = \text{sign}(z) \log(1 + |z|)$ . Thus,

$$\forall y, y' \in \mathbb{R}, \quad |\Phi(L^{-1}(t, y)) - \Phi(L^{-1}(t, y'))| \leq C |y - y'|.$$

Let  $(\xi_n)_{n \geq 1}$  be a sequence of functions of  $\mathbb{D}([0, T], \mathbb{R})$  such that  $\xi_n \xrightarrow[n \rightarrow +\infty]{U} \xi \in \mathcal{C}([0, T], \mathbb{R})$ . Then, for every  $t \in [0, T]$  and  $n \geq 1$ ,

$$|\Phi(L^{-1}(t, \xi_n(t))) - \Phi(L^{-1}(t, 0))| \leq C |\xi_n(t)| \leq C (\|\xi_n(t) - \xi\| + \|\xi\|) + |\Phi(x_0)| \leq C',$$

since  $L^{-1}(t, 0) = x_0$ . Consequently, for every  $t \in [0, T]$  and every  $n \geq 1$ ,  $L^{-1}(t, \xi_n(t)) \in K' := \Phi^{-1}([-C', C'])$ . The set  $K'$  is compact (because the function  $\Phi$  is continuous and proper ( $\lim_{|z| \rightarrow \infty} |\Phi(z)| = +\infty$ )). As  $\inf_{K'} \Phi' > 0$ , we deduce that there exists  $\eta_0 > 0$  such that

$$\forall x, y \in I, \quad |\Phi(x) - \Phi(y)| > \eta_0 |x - y|,$$

*i.e.*

$$\forall t \in [0, T], \forall u, v \in L(t, I), \quad |L^{-1}(t, u) - L^{-1}(t, v)| \leq C'' |u - v|, \quad C'' > 0.$$

Hence, one concludes that

$$\|\Lambda^{-1}(\xi_n) - \Lambda^{-1}(\xi_\infty)\|_{\text{sup}} \leq C'' \|\xi_n - \xi_\infty\|_{\text{sup}}. \quad \square$$

### B.2.3 Opposite monotony principle for diffusion

**Theorem B.2.** Assume that the real-valued diffusion process (B.27) is admissible (see Defintion 2.3). Let  $F, G : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be two  $C$ -continuous functionals, satisfying (2.13), with opposite monotony. Then

$$\text{Cov} \left( F((X_t)_{t \in [0, T]}), G((X_t)_{t \in [0, T]}) \right) \leq 0.$$

**Remark.** In the homogeneous case (see (B.33)), as  $L$  is increasing,

$$\beta \text{ is nondecreasing iff } \frac{b}{\sigma} - \frac{\sigma'}{2} \text{ is nondecreasing.}$$

If  $b, \sigma' \in D(I)$ , this condition is equivalent to

$$\left(\frac{b}{\sigma}\right)' \geq \frac{\sigma''}{2}.$$

Before passing to the proof, we state few lemmas: one is a key step to transfer opposite monotony from the Euler scheme to the diffusion process, the other aims at transferring uniqueness property for weak solutions.

**Lemma B.1.** *For every  $\alpha \in \mathbb{D}([0, T], \mathbb{R})$ , set*

$$\alpha^{(n)} = \sum_{k=0}^{n-1} \alpha(t_k^n) \mathbb{1}_{[t_k^n, t_{k+1}^n)} + \alpha(T) \mathbb{1}_{\{T\}}, \quad n \geq 1, \quad (\text{B.36})$$

with  $t_k^n := \frac{kT}{n}$ ,  $k = 0, \dots, n$ . Then  $\alpha^{(n)} \xrightarrow{U} \alpha$  as  $n \rightarrow \infty$ .

If  $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  is  $C$ -continuous and nondecreasing (resp. non-increasing), then the unique function  $F_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  satisfying  $F(\alpha^{(n)}) = F_n(\alpha(t_k^n), k = 0, \dots, n)$  is continuous and nondecreasing (resp. non-increasing) in each of its variables. Furthermore, if  $F$  satisfies a polynomial growth assumption of the form

$$\forall \alpha \in \mathbb{D}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r)$$

then, for every  $n \geq 1$ ,

$$|F_n(x_0, \dots, x_n)| \leq C(1 + \max_{0 \leq k \leq n} |x_k|^r)$$

with the same real constant  $C > 0$ .

**Lemma B.2.** *Let  $(S, d)$ ,  $(T, \delta)$  be two Polish spaces and let  $\Phi : S \mapsto T$  be a continuous injective function. Let  $\mu$  and  $\mu'$  be two probability measures on  $(S, \mathcal{B}or(S))$ . If  $\mu \circ \Phi^{-1} = \mu' \circ \Phi^{-1}$ , then  $\mu = \mu'$ .*

**Proof of Lemma B.2.** For every Borel set  $A$  of  $S$ ,  $\mu(A) = \sup\{\mu(K), K \subset A, K \text{ compact}\}$ . Let  $A \in \mathcal{B}or(S)$  such that  $\mu(A) \neq \mu'(A)$ . Then there exists a compact set  $K$  of  $A$  such that  $\mu(K) \neq \mu'(K)$ . But  $\Phi(K)$  is a compact set of  $T$  because  $\Phi$  is continuous, so  $\Phi^{-1}(\Phi(K))$  is a Borel set of  $S$  which contains  $K$ . As  $\Phi$  is injective,  $\Phi^{-1}(\Phi(K)) = K$ . Therefore  $\mu(\Phi^{-1}(\Phi(K))) \neq \mu'(\Phi^{-1}(\Phi(K)))$ . We deduce that  $\mu \circ \Phi^{-1} \neq \mu' \circ \Phi^{-1}$ .  $\square$

**Proof of Theorem B.2.** First we consider the Lamperti transform  $(Y_t)_{t \geq 0}$  (see (B.30)) of the diffusion  $X$  solution to (B.28) with  $X_0 = x_0 \in I$ . Using the homeomorphism property of  $\Lambda$  and calling upon the above Lemma B.2 with  $\Lambda^{-1}$  and  $\Lambda$ , we see that existence and uniqueness assumptions on Equation (B.28) can be transferred to (B.32) since  $\Lambda$  is a one-to-one mapping between the solutions of these two SDE's.

Then we introduce the stepwise constant (Brownian) Euler scheme  $\bar{Y}^n = (\bar{Y}_{\frac{kT}{n}})_{0 \leq k \leq n}$  with step  $\frac{T}{n}$  (defined by (B.29)) of  $Y = (Y_t)_{t \geq 0}$ . It is clear by induction on  $k$  that there exists for every  $k \in \{1, \dots, n\}$  a function  $\Theta_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  such that

$$\bar{Y}_{t_k^n} = \Theta_k(y_0, \Delta W_{t_1^n}, \dots, \Delta W_{t_k^n})$$

where for  $(y_0, z_1, \dots, z_k) \in \mathbb{R}^{k+1}$ ,

$$\Theta_k(y_0, z_1, \dots, z_k) = \Theta_{k-1}(y_0, z_1, \dots, z_{k-1}) + \beta(t_{k-1}^n, \Theta_{k-1}(y_0, z_1, \dots, z_{k-1})) \frac{T}{n} + z_k.$$

Thus for every  $i \in \{1, \dots, k\}$ ,  $z_i \mapsto \Theta_k(y_0, z_1, \dots, z_i, \dots, z_k)$  is nondecreasing because  $\beta$  is nondecreasing. We deduce that if  $F_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is nondecreasing in each variables, then, for every  $i \in \{1, \dots, k\}$ ,

$$z_i \mapsto F_n(y_0, \Theta_1(y_0, z_1), \dots, \Theta_n(y_0, z_1, \dots, z_n)) \text{ is nondecreasing.}$$

By the same reasoning, we deduce that for  $G_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , non-increasing in each variables, we have for every  $i \in \{1, \dots, k\}$ ,

$$z_i \mapsto G_n(y_0, \Theta_1(y_0, z_1), \dots, \Theta_n(y_0, z_1, \dots, z_n)) \text{ is non-increasing.}$$

Let  $F_n$  and  $G_n$  be the functions defined on  $\mathbb{R}^{n+1}$  associated to  $F$  and  $G$  respectively by Lemma B.1. As  $\beta$  has linear growth,  $Y$  and its Euler scheme have polynomial moments at any order  $p > 0$ . Then we can apply Proposition B.2 to deduce that

$$\begin{aligned} \mathbb{E} [FG(\bar{Y}^n)] &= \mathbb{E} \left[ F_n \left( \left( \bar{Y}_{\frac{kT}{n}} \right)_{0 \leq k \leq n} \right) G_n \left( \left( \bar{Y}_{\frac{kT}{n}} \right)_{0 \leq k \leq n} \right) \right] \\ &\leq \mathbb{E} \left[ F_n \left( \left( \bar{Y}_{\frac{kT}{n}} \right)_{0 \leq k \leq n} \right) \right] \mathbb{E} \left[ G_n \left( \left( \bar{Y}_{\frac{kT}{n}} \right)_{0 \leq k \leq n} \right) \right] = \mathbb{E} [F(\bar{Y}^n)] \mathbb{E} [G(\bar{Y}^n)]. \end{aligned}$$

Note that if  $F$  and  $G$  are  $C$ -continuous with polynomial growth, then  $FG$  too. We derive from Theorem B.1 that

$$\mathbb{E} [FG(\bar{Y}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E} FG(Y), \quad \mathbb{E} [F(\bar{Y}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E} F(Y), \quad \mathbb{E} [G(\bar{Y}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E} G(Y),$$

therefore

$$\text{Cov}(F(Y), G(Y)) \leq 0.$$

To conclude the proof, we need to go back to the process  $X$  by using the inverse Lamperti transform. Indeed, for every  $t \in [0, T]$ ,  $X_t = L^{-1}(t, Y_t)$ , where  $Y$  satisfies (B.32). Let  $F : \mathbb{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ . Set

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad \tilde{F}(\alpha) := F \left( (L^{-1}(t, \alpha_t))_{t \in [0, T]} \right).$$

Assume first that  $F$  and  $G$  are bounded. The functional  $\tilde{F}$  is  $C$ -continuous owing to Proposition B.4, nondecreasing (resp. non-increasing) since  $L^{-1}(t, \cdot)$  is for every  $t \in [0, T]$  and is bounded. Consequently,

$$\text{Cov}(F(X), G(X)) = \text{Cov}(\tilde{F}(Y), \tilde{G}(Y)) \leq 0.$$

To conclude one may approximate in a robust way with respect to the constraints on the functionals,  $F$  and  $G$  by a canonical truncation procedure, say

$$F_M := \max \left( (-M), \min(F, M) \right), \quad M \in \mathbb{N}.$$

If  $F$  and  $G$  have polynomial growth, it is clear that  $\text{Cov}(F_M(X), G_M(X)) \rightarrow \text{Cov}(F(X), G(X))$  as  $M \rightarrow \infty$ .  $\square$

**Examples of admissible diffusions.** • *The Bachelier model:* This simply means that  $X_t = \mu t + \sigma W_t$ ,  $\sigma > 0$ , clearly fulfills the assumptions of Theorem B.2.

• *The Black-Scholes model:* The diffusion process  $X$  is a geometric Brownian motion, solution to the SDE

$$dX_t = rX_t dt + \vartheta X_t dW_t, \quad X_0 = x_0 > 0,$$

where  $r \in \mathbb{R}$  and  $\vartheta > 0$  are real numbers. The geometric Brownian motion lives in the open interval  $I = (0, +\infty)$  and  $\beta(y) = \frac{r}{\vartheta} - \frac{\vartheta}{2}$  is constant. One checks that  $L(x) = \frac{1}{\sigma} \log\left(\frac{x}{x_1}\right)$  where  $x_1 \in (0, +\infty)$  is fixed.

• *The Hull-White model:* It is an elementary improvement of the Black-Scholes model where  $\vartheta : [0, T] \rightarrow (0, +\infty)$  is a deterministic positive function *i.e.* the diffusion process  $X$  is a geometric Brownian motion solution the SDE

$$dX_t = rX_t dt + \vartheta(t)X_t dW_t, \quad X_0 = x_0 > 0.$$

Then, elementary stochastic calculus shows that

$$X_t = x_0 e^{rt - \frac{1}{2} \int_0^t \vartheta^2(s) ds + \int_0^t \vartheta(s) dW_s} = x_0 e^{rt - \frac{1}{2} \int_0^t \vartheta^2(s) ds + B_{\int_0^t \vartheta^2(s) ds}}$$

where  $(B_u)_{u \geq 0}$  is a standard Brownian motion (the second equality follows from the Dambins-Dubins-Schwarz theorem).

Consequently  $X_t = \varphi\left(t, B_{\int_0^t \vartheta^2(s) ds}\right)$  where the functional  $\xi \mapsto \left(t \mapsto \varphi\left(t, \xi\left(\int_0^t \vartheta^2(s) ds\right)\right)\right)$  defined on  $\mathbb{D}([0, T_\vartheta], \mathbb{R})$ ,  $T_\vartheta = \int_0^T \vartheta^2(t) dt$ , is  $C$ -continuous on  $\mathcal{C}([0, T_\vartheta], \mathbb{R})$ . Hence for any  $C$ -continuous  $\mathbb{R}$ -functional on  $\mathbb{D}([0, T], \mathbb{R})$ , the  $\mathbb{R}$ -valued functional  $\tilde{F}$  defined by  $\tilde{F}(\xi) = F\left(\varphi\left(t, \xi\left(\int_0^t \vartheta^2(s) ds\right)\right)\right)$  is  $C$ -continuous on  $\mathbb{D}([0, T_\vartheta], \mathbb{R})$ . Then, one can transfer the opposite monotony property from  $B$  to  $X$ .

• *Local volatility model (elliptic case):* More generally, it applies still with  $I = (0, +\infty)$  to some usual extensions like the models with local volatility

$$dX_t = rX_t dt + \vartheta(X_t)X_t dW_t, \quad X_0 = x_0 > 0,$$

where  $\vartheta : \mathbb{R} \rightarrow (\vartheta_0, +\infty)$ ,  $\vartheta_0 > 0$ , is a bounded, non-increasing, twice differentiable function satisfying  $x \mapsto x\vartheta(x)$  is concave,  $|\vartheta'(x)| \leq \frac{C}{1+x}$  and  $|\vartheta''(x)| \leq \frac{C}{1+x^2}$ ,  $x \in (0, +\infty)$ .

Note that the family of functions defined for every  $x \in (0, +\infty)$  by

$$\vartheta_{\vartheta_0, \eta_0, a, \alpha}(x) = \vartheta_0 + \frac{a}{(x + \eta_0)^\alpha}, \quad \eta_0, a > 0, \alpha \in (0, 1)$$

satisfy the above assumptions.

In this case  $I = (0, +\infty)$  and,  $x_1 \in I$  being fixed, one has for every  $x \in I$ ,

$$L(x) = \int_{x_1}^x \frac{d\xi}{\xi \vartheta(\xi)}$$

which clearly defines an increasing homeomorphism from  $I$  onto  $\mathbb{R}$  since  $\vartheta$  is bounded. Then  $\beta = \tilde{\beta} \circ L^{-1}$  with

$$\tilde{\beta}(x) = \frac{r}{\vartheta(x)} - \frac{1}{2}(I_{(0, +\infty)} \times \vartheta)'(x)$$

where  $I_{(0, +\infty)}$  denotes the identity function in  $(0, +\infty)$ . As a consequence

$$\beta \text{ is nondecreasing on } \mathbb{R} \text{ iff } \frac{r}{\vartheta} - \frac{1}{2}(I_{(0, +\infty)} \times \vartheta)' \text{ is nondecreasing on } (0, +\infty).$$

Now  $\vartheta$  non-increasing implies that  $\frac{r}{\vartheta}$  is nondecreasing and  $I_{(0,\infty)} \times \vartheta$  being concave,  $(I_{(0,\infty)} \times \vartheta)' \leq 0$ . Finally, under the above assumptions,  $\beta$  is nondecreasing.

Furthermore, one easily derives from the explicit form (B.34) and the condition (B.35) that  $\beta$  has linear growth as soon as the function

$$x \mapsto rx \frac{\vartheta'}{\vartheta}(x) + \frac{x^2 \vartheta \vartheta''(x)}{2} + x \vartheta \vartheta'(x) \text{ is bounded on } (0, \infty)$$

which easily follows from the assumptions made on  $\vartheta$ .

**Extension to other classes of diffusions and models.** This general approach does not embody all situations: thus the true CEV model does not fulfill the above assumptions. The *CEV* model is a diffusion process  $X$  following the SDE

$$dX_t = rX_t dt + \vartheta X_t^\alpha dW_t, \quad X_0 = x_0,$$

where  $\vartheta > 0$  and  $0 < \alpha < 1$  are real numbers.

So this *CEV* model, for which  $I = (0, +\infty)$ , does not fulfill Definition 2.3 (iii). As a consequence  $L(t, I) \neq \mathbb{R}$  is an open interval (depending on the choice of  $x_1$ ). To be precise, if  $x_1 \in (0, +\infty)$  is fixed,

$$L(x) = \frac{1}{\vartheta(1-\alpha)}(x^{1-\alpha} - x_1^{1-\alpha}), \quad x \in (0, +\infty)$$

so that, if we set

$$J_{x_1} := L(I) = \left( -\frac{x_1^{1-\alpha}}{\vartheta(1-\alpha)}, +\infty \right),$$

$L$  defines an homeomorphism from  $I = (0, +\infty)$  onto  $J_{x_1}$ . Finally the function  $\beta$  defined by

$$\beta(y) = \frac{r}{\vartheta}(\vartheta(1-\alpha)y + x_1^{1-\alpha}) - \frac{\alpha\vartheta}{2} \frac{1}{(\vartheta(1-\alpha)y + x_1^{1-\alpha})}, \quad y \in J_{x_1}$$

is nondecreasing with linear growth at  $+\infty$ . Now, tracing the lines of the above proof, in particular establishing weak existence and uniqueness of the solution of the EDS (B.28) in that setting, leads to the same positive conclusion concerning the covariance inequalities for co-monotonic or anti-monotonic functionals.

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