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MONADIC SECOND-ORDER DEFINABLE GRAPH ORDERINGS

Achim Blumensath and Bruno Courcelle

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1 INTRODUCTION

When studying the expressive power of monadic second-order logic (MSO) often the question arises of whether one can define an order on a certain set of vertices. For instance, the property that a set has even cardinality can, in general, not be expressed in MSO. If, however, this set is linearly ordered, we can easily write down a corresponding MSO-formula. The same holds for every predicate $\text{Card}_q(X)$ expressing that the cardinality of the set X is a multiple of q . It follows that the extension of MSO by all these counting predicates $\text{Card}_q(X)$, called *counting monadic second-order logic* (CMSO), is no more powerful than MSO on every class of structures on which we can define an order.

Another example¹ of a situation where a linear order increases the expressive power of monadic second-order logic is the construction of graph decompositions like the modular decomposition of a graph. It is shown in [3] that modular decompositions are definable in MSO if the graph is equipped with a linear order. Finally, although we will not address complexity questions in this article, let us recall that, over linearly ordered structures, the complexity class PTIME is captured by least fixed-point logic [9, 12].

¹Yet another example is the construction of (a combinatorial description of) a plane embedding of a connected planar graph. Such embeddings are definable in MSO if we can order the neighbours of each vertex (see [4]). For 3-connected graphs such an ordering is always definable, but for graphs that are not 3-connected this is not always the case.

Recall that a formula $\varphi(x, y)$ with two free first-order variables x and y defines a (linear) order on a relational structure \mathfrak{A} if the binary relation consisting of all pairs (a, b) of elements of \mathfrak{A} satisfying $\mathfrak{A} \models \varphi(a, b)$ is a linear order on A . We say that $\varphi(x, y)$ defines an order on a class of structures if it defines a (linear) order on each structure of that class. Our objective is to provide combinatorial characterisations of classes of finite graphs whose representing structures are MSO-orderable, i.e., on which one can define an order by an MSO-formula. (The question of whether a partial order is definable is trivial since equality is a partial order. Therefore, we only consider linear orders.)

As defined above the notion of an MSO-orderable class is too restrictive. To get interesting results, we allow in the above definitions formulae with parameters. That is, we take a formula $\varphi(x, y; \vec{Z})$ with additional free set variables $\vec{Z} = \langle Z_0, \dots, Z_{n-1} \rangle$ and, for each structure \mathfrak{A} in the given class, we choose values $P_0, \dots, P_{n-1} \subseteq A$ for these variables such that the binary relation

$$\{ (a, b) \mid \mathfrak{A} \models \varphi(a, b; \vec{P}) \}$$

is a linear order on A .

There is no MSO-formula (even with parameters) that defines a linear order on all finite graphs. (This is even the case for all finite sets, i.e., finite graphs without edges.) On the other hand, to take an easy example, the class of all finite connected graphs of degree at most d (for fixed d) is MSO-orderable.

If graphs are replaced by their incidence graphs, MSO-formulae become more powerful, because they can quantify over sets of edges. In this case we speak of MSO₂-orderable classes. Otherwise, we call the class MSO₁-orderable. Due to the greater expressive power, the MSO₂-orderable classes properly include the MSO₁-orderable ones. This means that, in the combinatorial characterisations below, the conditions for MSO₁-orderability must be stronger than those for MSO₂-orderability. A simple example of a class that is MSO₂-orderable but not MSO₁-orderable is the class of all cliques.

Our main results are the following ones. We first give a necessary condition for MSO₂-orderability based on the number of connected components resulting from the removal of n vertices. We prove that this condition is sufficient for every proper minor closed class of graphs. We also show that it is sufficient for complete d -partite graphs. We then study the MSO₁-orderability in a similar way and we exhibit a necessary condition that is stronger than the previous one. This condition is sufficient for classes where some variant of clique-width is bounded.

2 PRELIMINARIES

Let us fix our notation and terminology. We write $[n] := \{0, \dots, n-1\}$, for $n \in \mathbb{N}$. We denote tuples $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ with a bar. The empty tuple is $\langle \rangle$.

Trees will always be rooted and directed, i.e., every edge is oriented away from the root. The *tree-order* associated with a tree T is the partial order defined by

$$x \leq y \quad : \text{iff} \quad \text{the path from the root to } y \text{ contains } x.$$

The n -th level of a tree T consists of all vertices at distance n from the root. The *height* of T is the maximal level of its vertices.

We consider purely relational structures $\mathfrak{A} = \langle A, R_0^{\mathfrak{A}}, \dots, R_{n-1}^{\mathfrak{A}} \rangle$ with finite signatures $\Sigma = \{R_0, \dots, R_{n-1}\}$. The universe A will always be finite, and we allow it to be empty as this convention is common in graph theory. In some places we will also allow relational structures with constants, but when doing so it will always be mentioned explicitly. For a relation R and a set X , we write $R \upharpoonright X$ for the restriction of R to X . For a tuple \vec{R} of relations, we denote by $\vec{R} \upharpoonright X$ the corresponding tuple of restrictions.

For the most part, we will consider graphs instead of arbitrary relational structures. Graphs will always be simple, loop-free, and undirected. We will denote the edge between vertices u and v by (u, v) . Note that the same edge can also be written as (v, u) . There are two ways to represent a graph $G = \langle V, E \rangle$ by a structure. Both of them will be used. We can use structures of the form $[G] := \langle V, \text{edg} \rangle$ where the universe V consists of the set of vertices and we have a binary edge relation $\text{edg} \subseteq V \times V$, or we can use structures of the form $[G] := \langle V \cup E, \text{inc} \rangle$ where the universe contains both, the vertices and the edges of the graph and we have a binary incidence relation $\text{inc} \subseteq V \times E$ telling us which vertices belong to which edges. If \mathcal{C} is a class of graphs, we denote the corresponding classes of relational structures by, respectively, $[\mathcal{C}]$ and $[\mathcal{C}]$.

Definition 2.1. A graph $G = \langle V, E \rangle$ is r -sparse² if, for every subset $X \subseteq V$,

$$|E \upharpoonright X| \leq r \cdot |X|.$$

We denote by $\mathfrak{A} \oplus \mathfrak{B}$ the disjoint union of the structures \mathfrak{A} and \mathfrak{B} . For structures \mathfrak{A} and \mathfrak{B} encoding graphs, we also use a dual operation $\mathfrak{A} \otimes \mathfrak{B}$ that, after forming the disjoint union of \mathfrak{A} and \mathfrak{B} , adds all possible edges connecting an element of \mathfrak{A} to an element of \mathfrak{B} . For a set $S \subseteq A$ of elements, we write $\mathfrak{A} - S$ for

²In [5] such graphs are called *uniformly r-sparse*.

the substructure of \mathfrak{A} with universe $A - S$. Similarly, for a graph G , we denote by $G - S$ the subgraph of G induced by the complement of S .

Monadic second-order logic (MSO) is the extension of first-order logic by set variables and quantifiers over such variables. The *quantifier-rank* $\text{qr}(\varphi)$ of an MSO-formula φ is the maximal number of nested quantifiers in φ , where we count both, first-order and second-order quantifiers. The *monadic second-order theory* of quantifier rank h of a structure \mathfrak{A} is the set of all MSO-formulae of quantifier rank h satisfied by \mathfrak{A} . We denote it by $\text{MTh}_h(\mathfrak{A})$. Frequently, we are interested not in the theory of the structure \mathfrak{A} itself, but in the theory of an expansion $\langle \mathfrak{A}, \bar{P}, \bar{a} \rangle$ by unary predicates \bar{P} and constants \bar{a} . In this case we write $\text{MTh}_h(\mathfrak{A}, \bar{P}, \bar{a})$ omitting the brackets. Note that situations like this are the only time we allow constants in structures.

Let $\varphi(\bar{x}, \bar{Y}; \bar{Z})$ be an MSO-formula with free first-order variables \bar{x} and free second-order variables \bar{Y}, \bar{Z} . Given a structure \mathfrak{A} and sets $P_i \subseteq A$, we can assign the values \bar{P} to the variables \bar{Z} . This way we obtain a formula $\varphi(\bar{x}, \bar{Y}; \bar{P})$ with partially assigned variables. The values \bar{P} are called the *parameters* of this formula. The *relation defined* by a formula $\varphi(\bar{x}; \bar{P})$ in a structure \mathfrak{A} is the set

$$\varphi(\bar{x}; \bar{P})^{\mathfrak{A}} := \{ \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}; \bar{P}) \}.$$

In this article we will only make limited use of monadic second-order transductions. The following simple version suffices.

Definition 2.2. Let Σ and Γ be signatures. A *quantifier-free transduction* τ is an operation on structures that is specified by a list

$$\langle \delta(x), (\varphi_R(\bar{x}))_{R \in \Sigma} \rangle$$

of quantifier-free formulae over the signature Γ where δ has one free variable x and the numbers of free variables of the formulae $\varphi_R(\bar{x})$ correspond to the arities of the relations R . A Γ -structure \mathfrak{A} is mapped by τ to the Σ -structure

$$\tau(\mathfrak{A}) := \langle \delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R \in \Sigma} \rangle$$

where the universe is the set defined by δ and each relation R is defined by the corresponding formula φ_R .

Lemma 2.3 (Backwards Translation). *Let τ be a quantifier-free transduction. For every MSO-sentence φ , there exists an MSO-sentence φ^τ , of the same quantifier-rank as φ , such that*

$$\tau(\mathfrak{A}) \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi^\tau, \quad \text{for all structures } \mathfrak{A}.$$

Corollary 2.4. *Let τ be a quantifier-free transduction and \mathfrak{A} and \mathfrak{B} structures.*

$$\text{MTh}_h(\mathfrak{A}) = \text{MTh}_h(\mathfrak{B}) \quad \text{implies} \quad \text{MTh}_h(\tau(\mathfrak{A})) = \text{MTh}_h(\tau(\mathfrak{B})).$$

One important tool to compute monadic theories is the so-called Composition Theorem (see, e.g. [13, 1, 5]), which allows one to compute the theory of a structure composed from smaller parts from the theories of these parts. There are several variants of the Composition Theorem. For our needs the following version suffices. Let $\mathfrak{A}_0, \dots, \mathfrak{A}_{m-1}$ be structures and $\vec{a}^i = \langle a_{o_0}^i, \dots, a_{n-1}^i \rangle \in A_i^n$ n -tuples, for $i < m$. The *amalgamation* of the structures \mathfrak{A}_i over the parameters \vec{a}^i is the structure $\langle \mathfrak{A}', \vec{a}' \rangle$ obtained from the disjoint union $\mathfrak{A}_0 \oplus \dots \oplus \mathfrak{A}_{m-1}$ by, for every $k < n$, merging the elements $a_{k_0}^0, \dots, a_{k_0}^{m-1}$ into a single element a'_{k_0} . The tuple $\vec{a}' = \langle a'_{o_0}, \dots, a'_{n-1} \rangle$ consists of the elements resulting from the merging.

Theorem 2.5 (Composition Theorem). *Let $\mathfrak{A}_0, \dots, \mathfrak{A}_{m-1}, \mathfrak{B}_0, \dots, \mathfrak{B}_{m-1}$ be structures and, for $i < m$, let $\vec{a}_i \in A_i^n$ and $\vec{b}_i \in B_i^n$ be n -tuples, and $\vec{c}_i \in A_i^{l_i}$ and $\vec{d}_i \in B_i^{l_i}$ l_i -tuples such that*

$$\text{MTh}_h(\mathfrak{A}_i, \vec{a}_i \vec{c}_i) = \text{MTh}_h(\mathfrak{B}_i, \vec{b}_i \vec{d}_i).$$

Let $\langle \mathfrak{A}', \vec{a}' \rangle$ and $\langle \mathfrak{B}', \vec{b}' \rangle$ be the amalgamations of, respectively, the structures \mathfrak{A}_i over \vec{a}_i and the structures \mathfrak{B}_i over \vec{b}_i . Then

$$\text{MTh}_h(\mathfrak{A}', \vec{a}' \vec{c}_0 \dots \vec{c}_{m-1}) = \text{MTh}_h(\mathfrak{B}', \vec{b}' \vec{d}_0 \dots \vec{d}_{m-1}), \quad \text{for all } i < m.$$

3 DEFINABLE ORDERS

Throughout the article we use the term *order* for linear orders. Otherwise we will speak of a *partial order*.

Definition 3.1. Let Σ be a relational signature and \mathcal{C} a class of Σ -structures.

(a) We say that an MSO-formula $\varphi(x, y; \vec{Z})$ *defines an order* on \mathcal{C} if, for every structure $\mathfrak{A} \in \mathcal{C}$, there are sets $P_0, \dots, P_{n-1} \subseteq A$ such that the formula $\varphi(x, y; \vec{P})$ defines a (linear) order on \mathfrak{A} .

(b) The class \mathcal{C} is *MSO-orderable* if there is an MSO-formula φ defining an order on \mathcal{C} .

(c) We call a class \mathcal{C} of graphs *MSO₁-orderable* if the class $\lfloor \mathcal{C} \rfloor$ is MSO-orderable, and we call it *MSO₂-orderable* if $\lceil \mathcal{C} \rceil$ is MSO-orderable.

Lemma 3.2. *Let \mathcal{C} and \mathcal{K} be classes of Σ -structures.*

- (a) $\mathcal{C} \cup \mathcal{K}$ is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.
- (b) $\mathcal{C} \oplus \mathcal{K} := \{ \mathfrak{A} \oplus \mathfrak{B} \mid \mathfrak{A} \in \mathcal{C}, \mathfrak{B} \in \mathcal{K} \}$ is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.

Proof. (a) Clearly, if φ defines an order on $\mathcal{C} \cup \mathcal{K}$, it also defines orders on \mathcal{C} and on \mathcal{K} . Conversely, let $\varphi(x, y; \vec{Z})$ and $\psi(x, y; \vec{Z}')$ be MSO-formulae defining an order on, respectively, \mathcal{C} and \mathcal{K} . Let $\text{ord}_\varphi(\vec{Z})$ be a formula stating that the relation defined by φ with parameters \vec{Z} is an order. Then we can order $\mathcal{C} \cup \mathcal{K}$ by the formula

$$\vartheta(x, y; \vec{Z}, \vec{Z}') := [\text{ord}_\varphi(\vec{Z}) \wedge \varphi(x, y; \vec{Z})] \vee [-\text{ord}_\varphi(\vec{Z}) \wedge \psi(x, y; \vec{Z}')].$$

(b) First, suppose that \mathcal{C} and \mathcal{K} are ordered by the formulae $\varphi(x, y; \vec{Z})$ and $\psi(x, y; \vec{Z}')$, respectively. We order $\mathcal{C} \oplus \mathcal{K}$ as follows. Consider $\mathfrak{A} \oplus \mathfrak{B} \in \mathcal{C} \oplus \mathcal{K}$ and let \vec{P} and \vec{Q} be the parameters used by φ and ψ to order \mathfrak{A} and \mathfrak{B} , respectively. Using one additional set $S := B$ as parameter we can define the order

$$\begin{aligned} x \leq y \quad & \text{iff} \quad x, y \in A \text{ and } \mathfrak{A} \models \varphi(x, y; \vec{P}) \\ & \text{or } x, y \in B \text{ and } \mathfrak{B} \models \psi(x, y; \vec{P}) \\ & \text{or } x \in A \text{ and } y \in B. \end{aligned}$$

Conversely, suppose that there is a formula $\varphi(x, y; \vec{Z})$ ordering $\mathcal{C} \oplus \mathcal{K}$. We construct a formula ordering \mathcal{C} . (The orderability of \mathcal{K} follows by symmetry.) Let $\mathfrak{A} \in \mathcal{C}$ and fix an arbitrary structure $\mathfrak{B} \in \mathcal{K}$. Let \vec{P} be the parameters used to order $\mathfrak{A} \oplus \mathfrak{B}$. Using the Composition Theorem, there exist two finite lists p_0, \dots, p_{n-1} and q_0, \dots, q_{n-1} of MSO-theories of quantifier-rank $h := \text{qr}(\varphi)$ such that, for $a, b \in A$,

$$\begin{aligned} \mathfrak{A} \oplus \mathfrak{B} \models \varphi(a, b; \vec{P}) \quad & \text{iff} \quad \text{MTh}_h(\mathfrak{A}, \vec{P} \upharpoonright A, a, b) = p_i \text{ and} \\ & \text{MTh}_h(\mathfrak{B}, \vec{P} \upharpoonright B) = q_i, \text{ for some } i < n. \end{aligned}$$

Let $I := \{ i < n \mid \text{MTh}_h(\mathfrak{B}, \vec{P} \upharpoonright B) = q_i \}$. It follows that we can order \mathfrak{A} by the formula

$$\psi(x, y; \vec{Z}) := \bigvee_{i \in I} \vartheta_i(x, y; \vec{Z}),$$

where ϑ_i is the conjunction of all formulae in p_i . □

Remark 3.3. (a) Every class consisting of a single (finite) structure is obviously MSO-orderable. By this lemma, it follows that all finite classes are MSO-orderable.

(b) For every MSO-formula $\varphi(x, y; \vec{Z})$ there exists a largest class \mathcal{C}_φ of Σ -structures that is ordered by φ . This class can be defined by

$$\exists \vec{Z} \text{ord}_\varphi(\vec{Z}),$$

where $\text{ord}_\varphi(\vec{Z})$ is the formula from the proof of Lemma 3.2.

Fixing an enumeration $\varphi_0(x, y; \vec{Z}), \dots, \varphi_{n-1}(x, y; \vec{Z})$ of all MSO-formulae of quantifier-rank m with k parameters Z_0, \dots, Z_{k-1} (up to logical equivalence, there are only finitely many such formulae, see Section 5.6 of [5] for details), we obtain the class $\mathcal{C}_{m,k}$ of all Σ -structures ordered by some of these formulae. $\mathcal{C}_{m,k}$ is defined by

$$\exists \vec{Z} \bigvee_{i < n} \text{ord}_{\varphi_i}(\vec{Z}).$$

This class can be ordered by the formula

$$\psi_{m,k}(x, y; \vec{Z}) := \bigvee_{i < n} \left[\text{ord}_{\varphi_i}(\vec{Z}) \wedge \bigwedge_{j < i} \neg \text{ord}_{\varphi_j}(\vec{Z}) \wedge \varphi_i(x, y; \vec{Z}) \right].$$

It follows that any MSO-orderable class \mathcal{C} can be ordered by $\psi_{m,k}$ for sufficiently large m and k .

Remark 3.4. Let \mathcal{C} be a class of graphs and let $\varphi(x, y; \vec{Z})$ be an MSO-formula defining an order on $[\mathcal{C}]$. The class \mathcal{C}_+ of all graphs obtained from graphs in \mathcal{C} by adding edges arbitrarily can be ordered by the formula $\varphi_+(x, y; \vec{Z}, Z')$ obtained from $\varphi(x, y; \vec{Z})$ by replacing every atomic formula of the form $\text{inc}(u, v)$ by the formula $\text{inc}(u, v) \wedge v \in Z'$, and by relativising every quantifier to the set Z' . (If \vec{P} are parameters such that $\varphi(x, y; \vec{P})$ orders the graph $G = \langle V, E \rangle$, then $\varphi_+(x, y; \vec{P}, E)$ orders every supergraph $G_+ = \langle V, E_+ \rangle$ with $E_+ \supseteq E$.)

Remark 3.5. Definition 3.1 can be formulated in terms of monadic second-order transductions (for details and definitions, see, e.g., Chapter 7 of [5]). A class \mathcal{C} of Σ -structures is MSO-orderable if, and only if, there exists a noncopying, domain-preserving transduction mapping each structure $\mathfrak{A} \in \mathcal{C}$ to an expansion $\langle \mathfrak{A}, \leq \rangle$ by a linear order \leq . With respect to the transduction hierarchy (cf. [2]), it follows that, if \mathcal{C} is infinite (up to isomorphism) and MSO-orderable, there exists an MSO-transduction mapping \mathcal{C} to the class of all finite paths.

The opposite of an orderable class is a class where no infinite subclass can be ordered. We call such classes *hereditarily unorderable*.

Definition 3.6. A class \mathcal{C} of structures is *hereditarily MSO-unorderable*, if it is infinite and no infinite subclass $\mathcal{C}_o \subseteq \mathcal{C}$ is MSO-orderable. For classes of graphs, we define the terms *hereditarily MSO₁-unorderable* and *hereditarily MSO₂-unorderable* analogously.

Example 3.7. (a) The class $\mathcal{C} = \{K_n \mid n \in \mathbb{N}\}$ of all complete graphs is MSO₂-orderable and hereditarily MSO₁-unorderable.

(b) The class \mathcal{T}_n of all trees of height at most n is both, hereditarily MSO₁-unorderable and hereditarily MSO₂-unorderable.

4 MSO₂-DEFINABLE ORDERINGS

In this section we derive characterisations for MSO₂-orderable classes. MSO₁-orderability will be considered in Section 5.

4.1 NECESSARY CONDITIONS

We start by providing a necessary condition for MSO₂-orderability. Below we will then show that, for certain classes of graphs, this condition is also sufficient.

Definition 4.1. Let $\mathfrak{A} = \langle A, \bar{R} \rangle$ be a relational structure.

(a) We call \mathfrak{A} *connected* if it cannot be written as a disjoint union $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ of two nonempty substructures. A *connected component* of \mathfrak{A} is a maximal substructure that is connected and nonempty.

(b) For a number $k \in \mathbb{N}$, we denote by $\text{Sep}(\mathfrak{A}, k)$ the maximal number of connected components of $\mathfrak{A} - S$, where $S \subseteq A$ ranges over all sets of size at most k .

(c) For a function $f : \mathbb{N} \rightarrow \mathbb{N}$, we say that a class \mathcal{C} of structures has property $\text{SEP}(f)$ if

$$\text{Sep}(\mathfrak{A}, k) \leq f(k), \quad \text{for all } \mathfrak{A} \in \mathcal{C} \text{ and all } k \in \mathbb{N}.$$

We say that \mathcal{C} has property SEP , if it has property $\text{SEP}(f)$, for some function $f : \mathbb{N} \rightarrow \mathbb{N}$.

Example 4.2. For complete bipartite graphs $K_{n,m}$ with $n \leq m$ we have

$$\text{Sep}(K_{n,m}, k) = \begin{cases} 1 & \text{if } k < n, \\ m & \text{if } k \geq n. \end{cases}$$

For complete d -partite graphs $K_{m_0, \dots, m_{d-1}}$ with $m_0 \geq \dots \geq m_{d-1}$ and $d \geq 2$, we have

$$\text{Sep}(K_{m_0, \dots, m_{d-1}}, k) = \begin{cases} 1 & \text{if } k < m_1 + \dots + m_{d-1}, \\ m_0 & \text{if } k \geq m_1 + \dots + m_{d-1}. \end{cases}$$

We leave the straightforward verification to the reader.

Example 4.3. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function and let $n \in \mathbb{N}$. We construct a graph $G_n(f)$ such that

$$\text{Sep}(G_n(f), k) \geq f(k), \quad \text{for all } k \leq n.$$

Let T be the tree of height n , where every vertex v on level k has $f(k)$ successors. That is,

$$T := \{ w \in \mathbb{N}^{\leq n} \mid w(k) < f(k) \text{ for all } k \}.$$

The desired graph $G_n(f)$ is obtained from this tree by adding all edges (x, y) with $x < y$.

Let us show that having property SEP is a necessary condition for a class to be MSO_2 -orderable. Recall that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *elementary* if it is bounded by a function of the form \exp_k , for some $k \in \mathbb{N}$, where

$$\exp_0(n) := n \quad \text{and} \quad \exp_{k+1}(n) := 2^{\exp_k(n)}.$$

Lemma 4.4. *There exists a function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that $\text{Sep}(G, k) \leq f(n, m, k)$ for every graph G such that $[G]$ can be ordered by an MSO -formula of the form $\varphi(x, y; \vec{P})$ where $\text{qr}(\varphi) \leq m$ and $\vec{P} = P_0 \dots P_{n-1}$ are parameters. Furthermore, the function $f(n, m, k)$ is effectively elementary in the argument k , that is, there exists a computable function g such that $f(n, m, k) \leq \exp_{g(n, m)}(k)$.*

Proof. Fixing $k, m, n \in \mathbb{N}$, we define $f(n, m, k) := d$ where d is the number of MSO -theories of the form

$$\text{MTh}_m([H], P_0, \dots, P_{n-1}, v_0, \dots, v_k)$$

where H is a graph, P_0, \dots, P_{n-1} are parameters, and v_0, \dots, v_k are vertices of H . Note that, for fixed n and m , the number of such theories is elementary in k (see Section 5.6 of [5] for a detailed calculation).

Let $\varphi(x, y; \bar{Z})$ be an MSO-formula of quantifier-rank at most m , let G be a graph with $\text{Sep}(G, k) > f(n, m, k)$, and let P_0, \dots, P_{n-1} parameters from G . We have to show that $\varphi(x, y; \bar{P})$ does not order $[G]$. Fix a set $S = \{s_0, \dots, s_{k-1}\}$ of vertices such that $G - S$ has more than d connected components. Fix distinct connected components C_0, \dots, C_d of $G - S$ and vertices $a_i \in C_i$. By choice of d , there are indices $i < j$ such that

$$\begin{aligned} & \text{MTh}_m([G[C_i \cup S]], \bar{P} \upharpoonright C_i \cup S, s_0, \dots, s_{k-1}, a_i) \\ &= \text{MTh}_m([G[C_j \cup S]], \bar{P} \upharpoonright C_j \cup S, s_0, \dots, s_{k-1}, a_j). \end{aligned}$$

As the structure $\langle [G], \bar{P}, s_0, \dots, s_{k-1}, a_i, a_j \rangle$ is the amalgamation of the structures

$$\begin{aligned} & \langle [G[C_i \cup S]], \bar{P} \upharpoonright C_i \cup S, s_0, \dots, s_{k-1}, a_i \rangle, \\ & \langle [G[C_j \cup S]], \bar{P} \upharpoonright C_j \cup S, s_0, \dots, s_{k-1}, a_j \rangle, \\ \text{and } & \langle [G[C_l \cup S]], \bar{P} \upharpoonright C_l \cup S, s_0, \dots, s_{k-1} \rangle, \quad \text{for } l \neq i, j, \end{aligned}$$

over the tuple $\langle s_0, \dots, s_{k-1} \rangle$, it therefore follows by Theorem 2.5 that

$$\text{MTh}_m([G], \bar{P}, s_0, \dots, s_{k-1}, a_i, a_j) = \text{MTh}_m([G], \bar{P}, s_0, \dots, s_{k-1}, a_j, a_i).$$

In particular,

$$G \models \varphi(a_i, a_j; \bar{P}) \quad \text{iff} \quad G \models \varphi(a_j, a_i; \bar{P}).$$

Hence, $\varphi(x, y; \bar{P})$ does not define an order. \square

Corollary 4.5. *An MSO_2 -orderable class of graphs \mathcal{C} has property $\text{SEP}(f)$, for an elementary function f .*

The converse does not hold. For instance, according to Theorem 4.30 below, the class of all bipartite graphs of the form $K_{n, 2^{2^n}}$ is not MSO_2 -orderable, while we have seen in Example 4.2 that it has property $\text{SEP}(f)$ for the elementary function $f(n) = 2^{2^n}$. Our objective therefore is to get converse results for particular classes of graphs satisfying certain combinatorial conditions.

Remark 4.6. We have noted in Remark 3.4 that, if an MSO_2 -orderable graph G is obtained from a graph H by deleting edges, then H is also MSO_2 -orderable. In this case, we further have $\text{Sep}(H, k) \leq \text{Sep}(G, k)$, for all k .

Remark 4.7. All results of Section 4 also hold for directed graphs since every orientation of an undirected graph can be defined by an MSO_2 -formula with two parameters (see Proposition 9.46 of [5]). It follows that a class of directed graphs is MSO_2 -orderable if, and only if, the corresponding class of undirected graphs is. This is different for MSO_1 -orderability.

As a simple introductory example let us consider classes of trees.

Theorem 4.8. *Let \mathcal{T} be a class of trees. The following statements are equivalent:*

- (1) \mathcal{T} is MSO_1 -orderable.
- (2) \mathcal{T} is MSO_2 -orderable.
- (3) \mathcal{T} has property SEP.
- (4) There exists a number $d \in \mathbb{N}$ such that every tree in \mathcal{T} has maximal degree at most d .

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3) has been shown in Corollary 4.5.

(3) \Rightarrow (4) Suppose that \mathcal{T} has property $\text{SEP}(f)$ and let $T \in \mathcal{T}$. Every vertex $v \in T$ has at most $f(1)$ neighbours since $T - \{v\}$ has at most $f(1)$ connected components. Consequently, the maximal outdegree of T is bounded by $f(1)$.

(4) \Rightarrow (1) Let T be a tree with maximal degree at most d . We use d parameters P_0, \dots, P_{d-1} to order T . Fixing a vertex $r \in T$ as root, there exists an injective embedding $g : T \rightarrow d^{<m}$, for some number $m \in \mathbb{N}$. We set

$$P_i := \{v \in T \mid g(v) = wi \text{ for some } w\}.$$

Note that r is the only vertex of T that is not contained in any of these sets. Hence, using \bar{P} , we can define the tree order \leq on T . We can also define the lexicographic ordering:

$$u \leq v \quad \text{: iff} \quad u \leq v, \text{ or } u_o \in P_i, v_o \in P_k, \text{ for } i < k, \text{ where } u_o, v_o \text{ are the immediate successors of the longest common prefix of } u \text{ and } v \text{ with } u_o \leq u \text{ and } v_o \leq v. \quad \square$$

Corollary 4.9. *Let $k \in \mathbb{N}$. The class of trees of depth at most k is hereditarily MSO_2 -unorderable.*

Proof. For any given depth k , there are only finitely many trees (up to isomorphism) satisfying condition (4) of the theorem. \square

4.2 OMITTING A MINOR

We start by presenting a characterisation for classes of graphs omitting a minor. Recall that we can orient a spanning forest F of a graph G by fixing a root in each connected component. This defines a tree-order \leq_F on F . A spanning forest F is *normal* if the ends of every edge of G are comparable with respect to \leq_F (see, e.g., Section 1.5 of [8]).

Definition 4.10. Let G be a graph and $F \subseteq G$ a normal spanning forest of G .

(a) We denote by \leq_F the tree-order associated with F and the set of predecessors by

$$\downarrow_F x := \{ y \mid y <_F x \}.$$

(b) For $x \in G$, we define

$$B_F(x) := \{ v <_F x \mid \text{there is an edge } (u, v) \text{ of } G \text{ with } x \leq_F u \}.$$

Lemma 4.11. Let G be a graph, F a normal spanning forest of G , $x \in G$, and $B \subseteq \downarrow_F x$.

- (a) If $|B| \geq p$ and there are p immediate successors y of x with $B_F(y) = B \cup \{x\}$, then $K_{p,p} \leq G$.
- (b) If $|B| < p$ and $\text{Sep}(G, p) \leq d$, then there are at most d immediate successors y of x with $B_F(y) = B \cup \{x\}$.

Proof. (a) Suppose that there are p distinct immediate successors y_0, \dots, y_{p-1} of x with $B(y_i) = B \cup \{x\}$ and fix distinct vertices $b_0, \dots, b_{p-1} \in B$. Let H be the minor of G obtained by contracting the subtrees below y_0, \dots, y_{p-1} to single vertices $\tilde{y}_0, \dots, \tilde{y}_{p-1}$ and by removing all remaining vertices except for $\tilde{y}_0, \dots, \tilde{y}_{p-1}$ and b_0, \dots, b_{p-1} . Then $H \cong K_{p,p}$.

(b) Set $S := B \cup \{x\}$ and let y_0, \dots, y_{n-1} be an enumeration of all immediate successors of x with $B(y_i) = S$. Then y_0, \dots, y_{n-1} lie in different connected components of $G - S$. Hence, $n \leq \text{Sep}(G, p) = d$. \square

Theorem 4.12. Let $\mathcal{C}_{p,d}$ be the class of all graphs G such that $\text{Sep}(G, p) \leq d$ and G does not contain $K_{p,p}$ as a minor. Then $\mathcal{C}_{p,d}$ is MSO_2 -orderable, for every p, d .

Proof. Let F be a normal spanning forest of G . Since G has $\text{Sep}(G, 0) \leq d$ connected components, the forest F has at most d roots. We regard F as oriented

with edges pointing away from the root. Note that we can encode F by two parameters: its set of edges and its set of roots. (Since the first set consists of edges and the second one of vertices, we could take their union as a single parameter. For simplicity, we have refrained from doing so.) We shall use a lexicographic order on F to order G , based on orderings (i) of the roots of F and (ii) of the successors of every vertex of F .

Consider a vertex $x \in F$ with successors y_0, \dots, y_{m-1} . Since each set $B_F(y_i)$ is linearly ordered by \leq_F , we can define a preorder on the successors by using the lexicographic ordering of the sets $B_F(y_i)$:

$$y_i \sqsubseteq y_k \quad \text{iff} \quad B_F(y_i) \leq_{\text{lex}} B_F(y_k).$$

To prove that there is a definable order extending this preorder, it is sufficient to show that the equivalence classes of this preorder have bounded cardinality. Let $k := \max\{p, d\}$. For every set $B \subseteq \downarrow_F x$, there are at most k successors y_i of x with $B_F(y_i) = B \cup \{x\}$: for $|B| \geq p$, this follows from Lemma 4.11 (a); for $|B| < p$, it follows from Lemma 4.11 (b).

The parameters needed to define the desired linear order are: the set of edges of the spanning forest F and $k+1$ parameters to distinguish (i) the roots of F and (ii) the successors y of a vertex x with the same set $B_F(y)$. \square

Theorem 4.13. *Let \mathcal{C} be a class of graphs omitting a minor H . The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) \mathcal{C} has property SEP .
- (3) \mathcal{C} has property $\text{SEP}(f)$ for some elementary function f .

Furthermore, given H we can compute a number k such that we can choose the function f in (3) to be \exp_k .

Proof. (1) \Rightarrow (3) follows by Corollary 4.5 and (3) \Rightarrow (2) is trivial.

For (2) \Rightarrow (1), suppose that \mathcal{C} has property $\text{SEP}(f)$. By Theorem 4.12, the classes $\mathcal{C}_{p,d}$ are MSO_2 -orderable, for all p, d . Let p be large enough such that H is a minor of $K_{p,p}$ and set $d := f(p)$. Then $\mathcal{C} \subseteq \mathcal{C}_{p,d}$ and it follows that \mathcal{C} is also MSO_2 -orderable. \square

Remark 4.14. (a) For each $k \in \mathbb{N}$, the class of all graphs of tree-width at most k excludes some planar graph as a minor and, hence, it satisfies the conditions of Theorem 4.13.

(b) Grohe has proved that every class of graphs excluding a minor is orderable in least fixed-point logic. It follows that least fixed-point logic captures PTIME on these classes [11, 10].

In contrast to Remark 4.14 (a), we have the following result for classes of graphs of bounded n -depth tree-width (where we only allow tree decompositions with index trees of height at most n). This graph complexity measure was introduced in [2].

Proposition 4.15. *Let $n, k \in \mathbb{N}$. The class of all graphs of n -depth tree-width at most k is hereditarily MSO_2 -unorderable.*

Proof. Let \mathcal{C} be an infinite class of graphs of n -depth tree-width at most k . If it were MSO_2 -orderable, we could define an MSO_2 -transduction mapping this class to the class of all finite paths. This is not possible by Theorem 6.4 of [2]. \square

In the following we try to compute a better bound on the function f in Theorem 4.13 (3).

Lemma 4.16. *Let G be a graph with $\text{Sep}(G, p) \leq d$ such that $K_{p,p}$ is not a minor of G . Let F be a normal spanning forest of G and S a set of at most k vertices of G . For every vertex $x \in S$, at most $k + 2^k \cdot \max\{p, d\}$ connected components of $G - S$ contain an immediate successor of x (in F).*

Proof. Let $s_0 <_F \dots <_F s_{m-1} = x$ be an enumeration of all elements $s \in S$ with $s \preceq_F x$. For an immediate successor y of x , we define

$$I(y) := \{ i < m \mid \text{there is some } z \in B_F(y) \text{ such that} \\ z <_F s_i, \text{ and } i = 0 \text{ or } s_{i-1} <_F z \}.$$

If y and y' are immediate successors of x in different connected components of $G - S$, then $I(y) \cap I(y') = \emptyset$. Consequently, there are at most $m \leq k$ connected components of $G - S$ containing an immediate successor y of x with $I(y) \neq \emptyset$.

It remains to show that there are at most $2^k \cdot \max\{p, d\}$ components of $G - S$ containing an immediate successor y with $I(y) = \emptyset$. Note that every such immediate successor y satisfies $B(y) \subseteq S$. Hence, $B(y)$ can take at most $2^m \leq 2^k$ values and, according to Lemma 4.11, for each such value $B \subseteq S$ there are at most $\max\{p, d\}$ immediate successors y with $B(y) = B$. \square

Proposition 4.17. *Let G be a graph with $\text{Sep}(G, p) \leq d$ such that $K_{p,p}$ is not a minor of G . Then*

$$\text{Sep}(G, k) \leq d + k^2 + k2^k \cdot \max\{p, d\}, \quad \text{for } k \geq p.$$

Proof. Let F be a normal spanning forest of G and S a set of at most k vertices of G . We have seen in Lemma 4.16 that, for every vertex $x \in S$, at most $k + 2^k \cdot \max\{p, d\}$ connected components of $G - S$ contain an immediate successor of x . Since every connected component of $G - S$ contains a root of F or the immediate successor of some $x \in S$, there are at most $d + k(k + 2^k \cdot \max\{p, d\})$ such components. \square

Corollary 4.18. *Let \mathcal{C} be a class of graphs omitting a minor H and let p be some number such that H is a minor of $K_{p,p}$. Then \mathcal{C} is MSO_2 -orderable if, and only if,*

$$\sup \{ \text{Sep}(G, p) \mid G \in \mathcal{C} \} < \infty.$$

Remark 4.19. Graphs omitting a minor H are r -sparse, for some number r depending on H . Since, for r -sparse graphs, the expressive powers of MSO_1 and MSO_2 coincide, it follows that the criterion in Corollary 4.18 also characterises MSO_1 -orderability.

Remark 4.20. The proof technique of Theorem 4.12 can be extended to order certain classes of graphs that do not exclude any graph as a minor. We give two examples.

(a) First, let us consider the class of all graphs H_p , for $p \geq 1$, defined as follows. The set of vertices of H_p is

$$V := \{*\} \cup [p] \cup [p] \times S_p,$$

where S_p is the set of all permutations of $[p]$. H_p has the following edges:

$$\begin{aligned} & (*, o) \\ & (*, (o, \sigma)) \quad \text{for } \sigma \in S_p, \\ & (i, i+1) \quad \text{for } i \in [p], i < p, \\ & ((i, \sigma), (i+1, \sigma)) \quad \text{for } i \in [p], \sigma \in S_p, i < p-1, \\ & (i, (\sigma(i), \sigma)) \quad \text{for } i \in [p], \sigma \in S_p, i < p-1. \end{aligned}$$

The graph H_2 is shown in Figure 1. Note that the vertex $*$ has degree $1 + p!$.

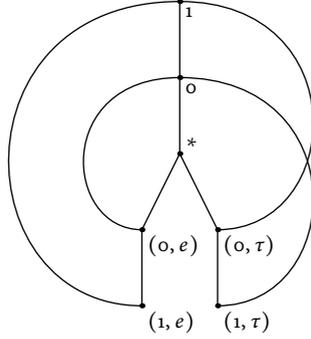


Figure 1: The graph H_2 .

Clearly, H_p contains $K_{p,p!}$ as a minor. Nevertheless, the class of all graphs H_p is MSO_2 -orderable. Since each H_p is 2-sparse (it has an orientation of indegree 2, see Chapter 9 of [5]), it follows that the class is even MSO_1 -orderable.

(b) Another example is the class of all cliques. It is MSO_2 -orderable and does not omit a minor. If we replace each edge by a path of length 2, we obtain a class of 2-sparse graphs that is MSO_2 -orderable and that still does not omit a minor.

Remark 4.21. It is not possible to extend Theorem 4.13 to r -sparse graphs. A counterexample is given by the class \mathcal{C} of all graphs obtained from some bipartite graph $K_{n,f(n)}$ by replacing every edge by a path of length 2, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a fixed non-elementary function. This is a class of 2-sparse graphs with property SEP that is not MSO_1 -orderable.

4.3 DECIDING MSO_2 -ORDERABILITY

We have presented above a combinatorial property characterising MSO_2 -orderability for classes of graphs omitting a minor. A natural question is whether this property is decidable. Of course, this question does only make sense for classes of graphs that can be described in a finite way. This is the case for equational classes of graphs that generalise context-free languages. Let us recall some of their basic properties. For a more detailed treatment we refer the reader to [5].

An equational class is defined by a system of equations. Depending on the graph operations allowed in these equations we obtain an HR-equational class or a VR-equational one. Every HR-equational class has bounded tree-width and

a bound can be computed from a system of equations for the class. Furthermore, for every $k \in \mathbb{N}$, the class of all graphs of tree-width at most k is HR-equational. Finally, every HR-equational class has a decidable MSO_2 -theory.

VR-equational classes enjoy similar properties with clique-width replacing tree-width, and MSO_1 replacing MSO_2 . Every HR-equational class is VR-equational (as we only consider simple graphs, this follows from Theorem 4.49 of [5]). For an example of a VR-equational class we can take the class of cographs which we will consider below in more detail. A *cograph* is a graph that can be constructed from single vertices using the operations of disjoint union \oplus and complete join \otimes . Each cograph can be denoted by a term over \oplus , \otimes , and a constant 1 that denotes an isolated vertex. For instance, $(1 \oplus 1) \otimes (1 \oplus 1 \oplus 1)$ is a term for $K_{2,3}$, and $1 \otimes 1 \otimes \dots \otimes 1$ is a term for a clique. Since \oplus and \otimes are associative and commutative we consider them of variable arity and we do ignore the order of the arguments. The class \mathcal{C} of cographs is VR-equational. It can be defined by the equation

$$\mathcal{C} = \mathcal{C} \oplus \mathcal{C} \cup \mathcal{C} \otimes \mathcal{C} \cup \{1\}.$$

The following result is Theorem 7.42 of [5] (the fact that one can compute a representation of the semilinear set is not stated explicitly in [5], but it follows from the proof since all of its steps are effective).

Theorem 4.22 (Semi-Linearity Theorem). *Let \mathcal{C} be a VR-equational class of graphs and $\varphi(X_0, \dots, X_{n-1})$ an MSO -formula. The set*

$$M_\varphi(\mathcal{C}) := \left\{ (|P_0|, \dots, |P_{n-1}|) \mid \exists G = \langle V, E \rangle \in \mathcal{C} \text{ and } P_0, \dots, P_{n-1} \subseteq V \right\}$$

is semilinear, and a finite representation of this set can be computed from φ and a system of equations for \mathcal{C} .

Proposition 4.23. *It is decidable whether a VR-equational class \mathcal{C} has property SEP.*

Proof. Let \mathcal{C} be a VR-equational class and let $\varphi(X, Y)$ be an MSO -formula expressing, for a graph G , that the set Y contains exactly one vertex of each connected component of $G - X$. The class \mathcal{C} has property SEP if, and only if, there exists a function f such that, for all $G = \langle V, E \rangle \in \mathcal{C}$ and $P, Q \subseteq V$,

$$G \models \varphi(P, Q) \quad \text{implies} \quad |Q| \leq f(|P|).$$

According to the Semi-Linearity Theorem, the set

$$M(\mathcal{C}) := \{ (|P|, |Q|) \mid G \models \varphi(P, Q) \text{ for some } G = \langle V, E \rangle \in \mathcal{C} \text{ and } P, Q \subseteq V \}$$

is semi-linear and an effective description of $M(\mathcal{C})$ can be computed from a system of equations for \mathcal{C} . Using this description, we can check whether or not, for every $n \in \mathbb{N}$, the set of $p \in \mathbb{N}$ with $(n, p) \in M(\mathcal{C})$ is bounded. This is the case if, and only if, \mathcal{C} has property SEP. \square

Corollary 4.24. *For an HR-equational class \mathcal{C} , it is decidable whether \mathcal{C} is MSO_2 -orderable.*

Proof. An HR-equational class \mathcal{C} has bounded tree-width (Proposition 4.7 of [5]) and, hence, omits some $K_{p,p}$ as a minor. Since HR-equational classes are VR-equational, it follows from Theorem 4.13 that \mathcal{C} is MSO_2 -orderable if, and only if, it has property SEP. The latter is decidable by the above proposition. \square

Remark 4.25. An alternative decidability proof can be based on Corollary 4.18. As the tree-width of $K_{p,p}$ is p , every class \mathcal{C} of tree-width at most $p-1$ omits $K_{p,p}$ as a minor. Furthermore, an upper bound on the tree-width of an HR-equational class \mathcal{C} can be computed from a system of equations for \mathcal{C} (see Proposition 4.7 of [5]). By Corollary 4.18, \mathcal{C} is MSO_2 -orderable if, and only if, there is a number m such that $\text{Sep}(G, p) \leq m$, for all $G \in \mathcal{C}$. To check this condition, we consider the formula $\varphi(X)$ expressing that there exists a set S of size $|S| \leq p$ such that X contains exactly one vertex of each connected component of $G - S$. By the Semi-Linearity Theorem, we can compute a representation of the semi-linear set

$$M(\mathcal{C}) := \{ |P| \mid G \models \varphi(P) \text{ for some } G = \langle V, E \rangle \in \mathcal{C} \text{ and } P \subseteq V \}.$$

Using this representation we can check whether or not $M(\mathcal{C})$ is finite.

For VR-equational classes we do not obtain decidability since we cannot apply Theorem 4.13. We conjecture that a corresponding statement holds for these classes.

Conjecture 4.26. *Every VR-equational class with SEP is MSO_2 -orderable.*

Below we shall prove this conjecture for particular classes of cographs.

4.4 DENSE GRAPHS

We have characterised MSO_2 -orderability in Theorem 4.13 for classes excluding a minor. The graphs in such classes are sparse. In this section we consider the opposite extreme of certain dense graphs, in particular, cographs and chordal graphs.

Lemma 4.27. *Let $s, r \in \mathbb{N}$ and let \mathcal{C} be a class of graphs such that each $G \in \mathcal{C}$ is obtained from some $K_{n,m}$ with $n \leq m \leq 2^{s+n+r}$ by possibly adding new edges. Then \mathcal{C} is MSO_2 -orderable.*

Proof. Consider a graph $G = \langle V, E \rangle \in \mathcal{C}$ obtained from $K_{n,m}$ with $n \leq m \leq 2^{s+n+r}$ by adding some new edges (see also Remark 3.4). Since every graph with at most 2^r vertices can be ordered using r parameters, we may assume that G has more than 2^r vertices. Hence, $n > 0$. Since $m \leq 2^{s+n+r} \leq 2^{(s+r)n}$ there exists an injective function $\mu : [m] \rightarrow \wp([s+r]n)$. Fix enumerations a_0, \dots, a_{n-1} and b_0, \dots, b_{m-1} of the two vertex classes of $K_{n,m}$. We define an ordering of G using the following parameters.

$$\begin{aligned} A &:= \{ a_i \mid i < n \} \subseteq V, \\ B &:= \{ b_i \mid i < m \} \subseteq V, \\ S &:= \{ (a_i, b_j) \mid i \leq j \} \subseteq E, \\ R_k &:= \{ (a_i, b_j) \mid ki \in \mu(j) \} \subseteq E, \quad \text{for } k < s+r. \end{aligned}$$

First, we define an order $<_A$ on A by

$$u <_A v \quad : \text{iff} \quad \text{for all } x \in B, (u, x) \in S \Rightarrow (v, x) \in S.$$

Note that, by definition of S , this order is linear. We extend this order to all of G by defining $u < v$ if, and only if, one of the following conditions holds:

- ◆ $u, v \in A$ and $u <_A v$.
- ◆ $u \in A$ and $v \in B$.
- ◆ $u, v \in B$ and, if k is the minimal number such that, for some $x \in A$,

$$(x, u) \in R_k \Leftrightarrow (x, v) \notin R_k,$$

and if $x \in A$ is the $<_A$ -least element with this property, then $(x, u) \in R_k$ and $(x, v) \notin R_k$. \square

The technique employed in this proof will be used several times in this article. Given an already defined order on a set A we order vertices not in A by considering the lexicographic ordering on their set of neighbours in A .

Lemma 4.28. *A class \mathcal{C} of complete bipartite graphs is MSO_2 -orderable if, and only if, there exists a constant s such that*

$$K_{n,m} \in \mathcal{C} \text{ with } n \leq m \text{ implies } m \leq 2^{s(n+1)}.$$

Proof. (\Leftarrow) is a special case of Lemma 4.27.

(\Rightarrow) Suppose that \mathcal{C} is ordered by an MSO -formula $\varphi(x, y; \vec{Z})$ with s set variables Z_0, \dots, Z_{s-1} . We claim that there is no $K_{n,m} \in \mathcal{C}$ with $m > 2^{s(n+1)}$.

For a contradiction, suppose that there is such a graph $K_{n,m} \in \mathcal{C}$. Let \bar{P} be the parameters such that $\varphi(x, y; \bar{P})$ orders $[K_{n,m}]$. We enumerate the two vertex classes of $K_{n,m}$ as a_0, \dots, a_{n-1} and b_0, \dots, b_{m-1} . Since $m > 2^{s(n+1)}$ there is a subset $I \subseteq [m]$ of size $|I| > 2^{s(n+1)}/2^s = 2^{sn}$ such that

$$b_i \in P_l \Leftrightarrow b_j \in P_l \quad \text{for all } i, j \in I \text{ and all } l < s.$$

Similarly, there is a subset $J \subseteq I$ of size $|J| > 2^{sn}/2^{sn} = 1$ such that

$$(a_k, b_i) \in P_l \Leftrightarrow (a_k, b_j) \in P_l \quad \text{for all } i, j \in J \text{ and all } l < s \text{ and } k < n.$$

Hence, there are at least two indices $i < j$ in J . The mapping $\pi : K_{n,m} \rightarrow K_{n,m}$ that interchanges b_i and b_j and leaves every other vertex fixed is an automorphism of the structure $\langle [K_{n,m}], \bar{P} \rangle$. Hence,

$$[K_{n,m}] \models \varphi(b_i, b_j; \bar{P}) \quad \text{iff} \quad [K_{n,m}] \models \varphi(b_j, b_i; \bar{P}).$$

A contradiction. □

Lemma 4.29. *Let \mathcal{C} be a class of graphs such that every graph in \mathcal{C} is of the form $K_{m_0, \dots, m_{d-1}}$ where*

$$d > 2 \quad \text{and} \quad m_1 + \dots + m_{d-1} \geq m_0 \geq m_1 \geq \dots \geq m_{d-1} \geq 1.$$

Then \mathcal{C} is MSO_2 -orderable.

Proof. Consider $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $m_0 \geq \dots \geq m_{d-1} \geq 1$. Let A_0, \dots, A_{d-1} be the classes of this graph and let $a_0^k, \dots, a_{m_k-1}^k$ be an enumeration of A_k . Using the parameter

$$R := \{ (a_0^k, a_0^{k+1}) \mid 0 \leq k < d-1 \}$$

we can define the preorder

$$u \sqsubseteq v \quad : \text{iff} \quad u \in A_i \text{ and } v \in A_k \text{ for } i \leq k.$$

As usual, we write

$$\begin{aligned} u \equiv v & : \text{iff} \quad u \sqsubseteq v \text{ and } v \sqsubseteq u, \\ u \sqsubset v & : \text{iff} \quad u \sqsubseteq v \text{ and } v \not\sqsubseteq u. \end{aligned}$$

Using the parameter

$$S := \{ (a_i^k, a_j^{k+1}) \mid i \leq j \},$$

and \sqsubseteq , we can define a linear order \leq_B on $B := A_1 \cup \dots \cup A_{d-1}$ by setting $u \leq_B v$ if, and only if,

- ◆ $u \sqsubset v$ or
- ◆ $u \equiv v$ and, for all $x \sqsubset u$, $(x, u) \in S$ implies $(x, v) \in S$.

Hence, it remains to define a linear order \leq_A on A_0 . Since $m_0 \leq m_1 + \dots + m_{d-1}$, we can fix an enumeration b_0, \dots, b_{n-1} of B and use the parameter

$$S_0 := \{ (a_i^0, b_j) \mid i \leq j \}$$

to define such an order. □

Theorem 4.30. *Let \mathcal{C} be a class of graphs such that, every graph in \mathcal{C} is complete d -partite for some $d \in \mathbb{N}$. (We do not require the number d to be the same for every graph.) The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) There exists a constant s such that \mathcal{C} has property $\text{SEP}(f)$ where $f(k) = 2^{s(k+1)}$.
- (3) There exists a constant s such that

$$K_{m_0, \dots, m_{d-1}} \in \mathcal{C} \quad \text{implies} \quad M \leq 2^{s(N-M+1)}$$

where $M := \max_{i < d} m_i$ and $N := \sum_{i < d} m_i$.

Proof. (3) \Rightarrow (1) Consider $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $m_0 \geq \dots \geq m_{d-1} \geq 1$. We distinguish several cases.

- If $d \leq 2$, the claim follows by Lemma 4.27.
- If $d > 2$ and $M \geq N - M$, we have $K_{N-M, M} \subseteq K_{m_0, \dots, m_{d-1}}$ and the claim follows again by Lemma 4.27.
- If $d > 2$ and $M < N - M$ the claim follows by Lemma 4.29.

(1) \Rightarrow (3) Suppose that $[\mathcal{C}]$ is ordered by an MSO-formula $\varphi(x, y; \bar{Z})$ with s set variables Z_0, \dots, Z_{s-1} . We claim that there is no $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $M > 2^{s(N-M)+s}$.

For a contradiction, suppose that there is such a graph $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$. Let \bar{P} be the parameters such that $\varphi(x, y; \bar{P})$ orders $K_{m_0, \dots, m_{d-1}}$. Let A be a vertex class of $K_{m_0, \dots, m_{d-1}}$ of size M and let B be its complement. We enumerate A and B as a_0, \dots, a_{M-1} and b_0, \dots, b_{N-M-1} , respectively. Since $M > 2^{s(N-M)+s}$ there is a subset $I \subseteq [M]$ of size $|I| > 2^{s(N-M)+s} / 2^s = 2^{s(N-M)}$ such that

$$a_i \in P_l \Leftrightarrow a_j \in P_l \quad \text{for all } i, j \in I \text{ and all } l < s.$$

Similarly, there is a subset $J \subseteq I$ of size $|J| > 2^{s(N-M)} / 2^{s(N-M)} = 1$ such that

$$(a_i, b_k) \in P_l \Leftrightarrow (a_j, b_k) \in P_l \quad \text{for all } i, j \in J, l < s, \text{ and } k < N - M.$$

Hence, there are at least two indices $i < j$ in J . The mapping $\pi : K_{m_0, \dots, m_{d-1}} \rightarrow K_{m_0, \dots, m_{d-1}}$ interchanging a_i and a_j and leaving every other vertex fixed is an automorphism of the structure $\langle [K_{m_0, \dots, m_{d-1}}], \bar{P} \rangle$. Hence,

$$[K_{m_0, \dots, m_{d-1}}] \models \varphi(a_i, a_j; \bar{P}) \quad \text{iff} \quad [K_{m_0, \dots, m_{d-1}}] \models \varphi(a_j, a_i; \bar{P}).$$

A contradiction.

(3) \Rightarrow (2) Let $K_{m_0, \dots, m_{d-1}}$ be a complete d -partite graph and set $M := \max_{i < d} m_i$

and $N := \sum_{i < d} m_i$. If $M \leq 2^{s(N-M+1)}$, then

$$\begin{aligned} \text{Sep}(K_{m_0, \dots, m_{d-1}}, k) &= \begin{cases} 1 & \text{if } k < N - M \\ M & \text{if } k \geq N - M \end{cases} \\ &\leq \begin{cases} 2^{s(k+1)} & \text{if } k < N - M \\ 2^{s(N-M+1)} & \text{if } k \geq N - M \end{cases} \\ &\leq \begin{cases} 2^{s(k+1)} & \text{if } k < N - M \\ 2^{s(k+1)} & \text{if } k \geq N - M \end{cases} \\ &= 2^{s(k+1)}. \end{aligned}$$

(2) \Rightarrow (3) Suppose that \mathcal{C} has property SEP(f) where $f(k) = 2^{s(k+1)}$. Note that

$$\text{Sep}(K_{m_0, \dots, m_{d-1}}, k) = \begin{cases} 1 & \text{if } k < N - M, \\ M & \text{if } k \geq N - M, \end{cases}$$

where M and N are as above. It follows that

$$M = \text{Sep}(K_{m_0, \dots, m_{d-1}}, N - M) \leq f(N - M) = 2^{s(N-M+1)}. \quad \square$$

As a corollary we obtain a special case of Conjecture 4.26 for particular classes of cographs.

Corollary 4.31. *Let \mathcal{C} be a VR-equational class of graphs that are complete d -partite for some d . Then \mathcal{C} is MSO₂-orderable if, and only if, it has property SEP. This property is decidable.*

Proof. For every $d \in \mathbb{N}$, there is an MSO-formula $\varphi_d(X_0, \dots, X_{d-1})$ stating that X_0, \dots, X_{d-1} are the vertex classes of a complete d -partite graph. By the Semi-Linearity Theorem, it follows that the set

$$M_d := \{ (m_0, \dots, m_{d-1}) \mid K_{m_0, \dots, m_{d-1}} \in \mathcal{C} \}$$

is semi-linear.

Suppose that \mathcal{C} has property SEP. By Example 4.2, it follows that, for every choice of m_0, \dots, m_{d-2} , there are only finitely many m_{d-1} with $K_{m_0, \dots, m_{d-2}, m_{d-1}} \in$

\mathcal{C} . Semi-linearity of M_d therefore implies that there are numbers $a, b \in \mathbb{N}$ such that

$$m_{d-1} \leq a(m_0 + \cdots + m_{d-2}) + b, \quad \text{for all } K_{m_0, \dots, m_{d-1}} \in \mathcal{C}.$$

By Theorem 4.30 it follows that \mathcal{C} is MSO_2 -orderable. \square

4.5 SPLIT GRAPHS AND CHORDAL GRAPHS

As the next step towards the Conjecture 4.26, the case of an VR-equational class of cographs suggests itself, but, so far, we were unable to find a proof. Instead, we consider split graphs and, more generally, chordal graphs.

Definition 4.32. Let G be a graph.

(a) G is a *split graph* if there exists a partition of its vertex set V into two parts A and B such that A induces a clique whereas B is independent.

(b) Let F be a rooted spanning forest of G with tree order \leq_F . We call F a *perfect spanning forest* if it is normal (cf. Section 4.2) and, for every vertex $v \in F$, the set of all neighbours u of v with $u <_F v$ induces a clique in G .

(c) G is *chordal* if it has a perfect spanning forest.

Every split graph is chordal. There are many equivalent definitions of chordal graphs. See Proposition 2.72 of [5] for an overview and a proof of their equivalence.

Theorem 4.33. *A class \mathcal{C} of split graphs is MSO_2 -orderable if, and only if, there is some $s \in \mathbb{N}$ such that \mathcal{C} has property $\text{SEP}(f)$ for $f(n) = 2^{s(n+1)}$.*

Proof. (\Leftarrow) Given s , we construct an MSO_2 -formula $\varphi(x, y; \bar{Z})$ with $s + 1$ parameters that orders every split graph G such that $\text{Sep}(G, n) \leq 2^{s(n+1)}$, for all n . Let $G = \langle V, E \rangle$ be such a split graph and let $V = A \cup B$ be the partition of V into a clique A and an independent set B . We use one parameter P to define an order on A as follows. Fixing an enumeration a_0, \dots, a_{n-1} of A we set

$$P := \{a_0\} \cup \{(a_i, a_{i+1}) \mid i < n - 1\}.$$

Then we can write down an MSO_2 -formula $\psi(x, y; P)$ stating that every path that connects the unique vertex in P to y and that only uses edges in P contains the vertex x . This defines a linear order \leq_A on A .

We use this order to define an order on B as follows. For $b \in B$ let

$$N(b) := \{ a \in A \mid (a, b) \in E \}.$$

We can define a preorder \sqsubseteq on B by

$$b \sqsubseteq b' \quad \text{iff} \quad N(b) = N(b') \text{ or the } \leq_A\text{-least element of } N(b) \Delta N(b') \\ \text{belongs to } N(b).$$

Since this preorder is linear, it is sufficient to define an order on each of the equivalence classes of the equivalence relation associated with \sqsubseteq . Given $b \in B$, let b_0, \dots, b_{m-1} be an enumeration of all vertices $b_i \in B$ with $N(b_i) = N(b)$ and let a_0, \dots, a_{n-1} be a \leq_A -increasing enumeration of $N(b)$. Then

$$m \leq \text{Sep}(G, n) \leq 2^{s(n+1)}.$$

Fix an injective function $\pi : [m] \rightarrow \wp([s(n+1)])$ and, for $k < s$, set

$$Q_k := \{ (b_i, a_l) \mid k(n+1) + l \in \pi(i) \} \cup \{ b_i \mid k(n+1) + n \in \pi(i) \}.$$

Using the parameters Q_0, \dots, Q_{s-1} we can order b_0, \dots, b_{m-1} by

$$b_i <_B b_j \quad \text{iff} \quad \text{the least element of } \pi(i) \Delta \pi(j) \text{ belongs to } \pi(i).$$

Finally, combining the (pre-)orders \leq_A , \sqsubseteq , and $<_B$, we can define an order on all of G .

(\Rightarrow) Suppose that a split graph $G = \langle V, E \rangle$ is ordered by a formula $\varphi(x, y; \bar{P})$ with s parameters P_0, \dots, P_{s-1} . We will prove that $\text{Sep}(G, n) \leq 2^{(s+1)(n+1)}$. Let $V = A \cup B$ be the partition of V into a clique A and an independent set B . For $b \in B$ let

$$N(b) := \{ a \in A \mid (a, b) \in E \}.$$

We start by showing that, for every $b \in B$, there are at most $2^{s(|N(b)|+1)}$ vertices $b' \in B$ with $N(b') = N(b)$. Let b_0, \dots, b_{m-1} be a list of distinct vertices of B with $N(b_0) = \dots = N(b_{m-1})$. For a contradiction, suppose that $m > 2^{s|N(b_0)|+s}$. Then there are indices $i < j$ such that

$$\begin{array}{lll} b_i \in P_k & \text{iff} & b_j \in P_k, & \text{for all } k < s, \\ (b_i, a) \in P_k & \text{iff} & (b_j, a) \in P_k, & \text{for all } k < s \text{ and } a \in N(b_0). \end{array}$$

It follows that the mapping that interchanges b_i and b_j and that fixes every other vertex of $\langle G, \bar{P} \rangle$ is an automorphism. Hence,

$$[G] \equiv \varphi(b_i, b_j; \bar{P}) \quad \text{iff} \quad [G] \equiv \varphi(b_j, b_i; \bar{P}).$$

A contradiction.

To compute $\text{Sep}(G, n)$ consider a set $S \subseteq V$ of size $|S| \leq n$. We have seen above that, for every set $X \subseteq S \cap A$, there are at most $2^{s(|X|+1)}$ vertices $b \in B$ with $N(b) = X$. Setting $k := |S \cap A|$, it follows that there are at most $2^k \cdot 2^{s(k+1)}$ vertices $b \in B$ with $N(b) \subseteq S \cap A$. Consequently, $G - S$ has at most

$$1 + 2^k \cdot 2^{s(k+1)} \leq 2^{sk+s+k+1} = 2^{(s+1)(k+1)} \leq 2^{(s+1)(n+1)}$$

connected components. □

Lemma 4.34. *For every increasing and unbounded function $g : \mathbb{N} \rightarrow \mathbb{N}$ there exists a class of split graphs that is not MSO_2 -orderable and that has property $\text{SEP}(f)$ for $f(n) := 2^{g(n)}$.*

Proof. For $k \in \mathbb{N}$ let $G_k := K_k \otimes D_{2g(k)}$ where D_n denotes the graph with n vertices and no edges. We claim that $\mathcal{C} := \{G_k \mid k \in \mathbb{N}\}$ has the desired properties. Note that

$$\text{Sep}(G_k, n) \leq \begin{cases} 1 & \text{if } n < k, \\ 2^{g(n)} & \text{if } n \geq k. \end{cases}$$

Hence, \mathcal{C} has property SEP , but it does not have property $\text{SEP}(f)$, for any function of the form $f(n) = 2^{s(n+1)}$. By Theorem 4.33, it follows that \mathcal{C} is not MSO_2 -orderable. □

Remark 4.35. The class in the preceding lemma is not VR-equational since it does not satisfy the Semi-Linearity Theorem. Hence, it does not provide a counterexample to Conjecture 4.26.

It would be interesting to extend Theorem 4.33 to classes of chordal graphs. At this point, we are only able to present a sufficient condition for MSO_2 -orderability. But there are examples showing that it is not necessary. We start with a technical lemma.

Lemma 4.36. *Let F be a perfect spanning forest of a chordal graph G with tree order \leq_F . If $u <_F v \leq_F w$ are vertices then*

$$(u, w) \in E \quad \text{implies} \quad (u, v) \in E.$$

Proof. Let $x_n <_F \dots <_F x_0$ be the path in F from $v = x_n$ to $w = x_0$. We show by induction on i , that $(u, x_i) \in E$. For $i = 0$, there is nothing to do. Hence, suppose that $i > 0$ and that we have already shown that $(u, x_{i-1}) \in E$. Then u and x_i are both neighbours of x_{i-1} . Since $u, x_i <_F x_{i-1}$, it follows by definition of a perfect spanning forest that $(u, x_i) \in E$. \square

Proposition 4.37. *Let \mathcal{C} be a class of chordal graphs with property $\text{SEP}(f)$ where $f(n) = 2^{s(n+1)}$, for some $s \in \mathbb{N}$. Then \mathcal{C} is MSO_2 -orderable.*

Proof. Let $G = \langle V, E \rangle$ be a chordal graph with $\text{Sep}(G, n) \leq 2^{s(n+1)}$. We order G as follows. Fix a perfect spanning forest F of G . It is sufficient to define, for every vertex v , an order on the immediate successors of v in F . Then we can use the lexicographic ordering on F to order G . Fix a vertex v and let u_0, \dots, u_{n-1} be the immediate successors of v in F . For $i < n$, we define

$$B_i := \{ w \preceq_F v \mid (w, u_i) \in E \}.$$

We start by showing that, for every set $B \subseteq V$, there are at most $2^{s(|B|+1)}$ indices i such that $B_i = B$. Given B , let I be the set of all $i < n$ with $B_i = B$. By Lemma 4.36, it follows that, for each $i \in I$ and every edge $(x, y) \in E$ with $x <_F u_i \preceq_F y$, we have $x \in B_i = B$. Hence,

$$|I| \leq \text{Sep}(G, |B|) \leq 2^{s(|B|+1)}$$

as desired. As in the proof of Theorem 4.33, we can use $s+1$ parameters Q_0, \dots, Q_s to colour the edges of the subgraphs $B_i \otimes u_i$ such a way that we can define the ordering

$$u_i < u_k \quad \text{iff} \quad i < k, \quad \text{for } i, k \in I.$$

Consequently, we can order all immediate successors of v by

$$u_i \leq u_k \quad \text{: iff} \quad B_i = B_k \text{ and } i \leq k, \text{ or} \\ \text{the } <_F\text{-least element of } B_i \Delta B_k \text{ belongs to } B_i. \quad \square$$

Corollary 4.38. *Let \mathcal{C} be a VR-equational class of chordal graphs. The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) \mathcal{C} has property SEP .

- (3) *There are constants $r, s \in \mathbb{N}$ such that \mathcal{C} has property $\text{SEP}(f)$ where $f(n) = rn + s$.*

These properties are decidable.

Since we have already proved (3) \Rightarrow (1) and (1) \Rightarrow (2) in Proposition 4.37 and Corollary 4.5, only the implication (2) \Rightarrow (3) remains to be proved. We leave this proof to the reader, it is similar to that of Corollary 4.31.

5 MSO_1 -DEFINABLE ORDERS

5.1 NECESSARY CONDITIONS

During our investigation of MSO_1 -orderability we will employ tools related to the notion of clique-width. We consider graphs with *ports* in a finite set $[k]$, that is, graphs $G = \langle V, E, \chi \rangle$ equipped with a function $\chi : V \rightarrow [k]$. We say that a vertex $a \in V$ has *port label* a if $\chi(v) = a$. The notion of clique-width is defined in terms of the following operations on graphs with ports:

- ◆ for each $a \in [k]$, a constant a denoting the graph with a single vertex that has port label a ;
- ◆ the disjoint union \oplus of two graphs with ports;
- ◆ the edge addition operation $\text{add}_{a,b}$, for $a, b \in [k]$, adding all edges between some vertex with port label a and some vertex with port label b that do not already exist;
- ◆ the port relabelling operation relab_h , for $h : [k] \rightarrow [k]$, changing each port label a to the port label $h(a)$.

Each term using these operations defines a graph with ports in $[k]$. The clique-width of a graph $G = \langle V, E \rangle$ is the least number k such that, for some function $\chi : V \rightarrow [k]$, there exists a term denoting $\langle G, \chi \rangle$ (for details cf. [5, 6, 7]). We denote the clique width of G by $\text{cwd}(G)$.

Below we will not use the operations defining clique-width, but some related operations that are more convenient in our context.

Definition 5.1. Let $k \in \mathbb{N}$ and $R \subseteq [k] \times [k]$.

(a) For undirected graphs G and H with ports in $[k]$, we construct the undirected graph $G \otimes_R H$ by adding to the disjoint union $G \oplus H$ all edges (x, y) such that

- ◆ either $x \in G$ and $y \in H$, or $x \in H$ and $y \in G$,
- ◆ x has port label a and y has port label b , for some $(a, b) \in R$.

Similarly, we define $G \otimes_R H$ for graphs G and H expanded by additional unary predicates and constants.

(b) For a graph G with ports, we denote by $\text{Un}(G)$ the graph obtained from G by forgetting all port labels.

Remark 5.2. (a) The operation \otimes_R is associative and commutative with the empty graph as neutral element. Furthermore, $\otimes_R = \otimes_{R \cup R^{-1}}$.

(b) With only 1 port label, there are two operations of the form \otimes_R : the operations \oplus and \otimes used to build cographs.

(c) We have $\overline{G \otimes_R H} = \overline{G} \otimes_{R'} \overline{H}$ where $R' := ([k] \times [k]) \setminus R$ and \overline{G} denotes the edge complement of G .

(d) We can express \otimes_R as a combination of the operations defining clique-width in the following way:

$$G \otimes_R H = \text{relab}_{h_-}(\text{add}_{a_o, b_o}(\cdots \text{add}_{a_n, b_n}(G \oplus \text{relab}_{h_+}(H)) \cdots)),$$

for suitable functions $h_+ : [k] \rightarrow [2k]$ and $h_- : [2k] \rightarrow [k]$ and ports labels $a_o, b_o, \dots, a_n, b_n \in [2k]$. (h_+ is needed to make the port labels appearing in H distinct from those appearing in G .)

Remark 5.3. (a) Similar to Lemma 3.2 (b), one can show that

$$\mathcal{C} \otimes_R \mathcal{K} := \{ G \otimes_R H \mid G \in \mathcal{C}, H \in \mathcal{K} \}$$

is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.

(b) $\overline{\mathcal{C}} := \{ \overline{G} \mid G \in \mathcal{C} \}$ is MSO-orderable if, and only if, \mathcal{C} is MSO-orderable.

To give a necessary condition for MSO₁-orderability we introduce a combinatorial property similar to SEP, but based on the operation \otimes_R .

Definition 5.4. Let G be a graph and $k \in \mathbb{N}$.

(a) We denote by $\text{Cut}(G, k)$ the maximal number n such that there exist non-empty graphs H_0, \dots, H_{n-1} with ports in $[k]$ and a relation $R \subseteq [k] \times [k]$ such that

$$G \cong \text{Un}(H_0 \otimes_R \cdots \otimes_R H_{n-1}).$$

(b) We say that a class \mathcal{C} of graphs has property $\text{CUT}(f)$, for a function $f : \mathbb{N} \rightarrow \mathbb{N}$, if

$$\text{Cut}(G, k) \leq f(k), \quad \text{for all } G \in \mathcal{C} \text{ and all } k \in \mathbb{N}.$$

We say that \mathcal{C} has property CUT , if it has property $\text{CUT}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$.

Remark 5.5. Note that $\text{Cut}(G, k) = \text{Cut}(\overline{G}, k)$.

For the proof that property CUT is necessary for MSO_1 -orderability, we use the following technical lemma.

Lemma 5.6. *Let G, G', H, H' be labelled graphs, $\bar{P}, \bar{P}', \bar{Q}, \bar{Q}'$ tuples of sets of vertices of the respective graphs, and $\bar{a}, \bar{a}', \bar{b}, \bar{b}'$ tuples of vertices. For each port label c , let C_c, C'_c, D_c, D'_c be the sets of all vertices of the respective graph labelled by c . Then*

$$\begin{aligned} \text{MTh}_m(\lfloor G \rfloor, \bar{P}, \bar{C}, \bar{a}) &= \text{MTh}_m(\lfloor G' \rfloor, \bar{P}', \bar{C}', \bar{a}') \\ \text{MTh}_m(\lfloor H \rfloor, \bar{Q}, \bar{D}, \bar{b}) &= \text{MTh}_m(\lfloor H' \rfloor, \bar{Q}', \bar{D}', \bar{b}') \end{aligned}$$

implies that

$$\text{MTh}_m(\lfloor G \otimes_R H \rfloor, \bar{S}, \bar{a}\bar{b}) = \text{MTh}_m(\lfloor G' \otimes_R H' \rfloor, \bar{S}', \bar{a}'\bar{b}'),$$

where $S_i := P_i \cup Q_i$ and $S'_i = P'_i \cup Q'_i$.

Proof. Let σ be a quantifier-free transduction that maps a structure \mathfrak{A} to its expansion $\langle \mathfrak{A}, I \rangle$ where $I := A \times A$ is the equivalence relation on A with a single class. Given R , we can write down a quantifier-free transduction τ such that

$$\begin{aligned} \langle \lfloor G \otimes_R H \rfloor, \bar{S}, \bar{a}\bar{b} \rangle &= \tau(\sigma(\langle \lfloor G \rfloor, \bar{P}, \bar{C}, \bar{a} \rangle) \oplus \sigma(\langle \lfloor H \rfloor, \bar{Q}, \bar{D}, \bar{b} \rangle)) \\ \text{and } \langle \lfloor G' \otimes_R H' \rfloor, \bar{S}', \bar{a}'\bar{b}' \rangle &= \tau(\sigma(\langle \lfloor G' \rfloor, \bar{P}', \bar{C}', \bar{a}' \rangle) \oplus \sigma(\langle \lfloor H' \rfloor, \bar{Q}', \bar{D}', \bar{b}' \rangle)). \end{aligned}$$

Consequently, the claim follows from the Composition Theorem and the Backwards Translation Lemma. \square

Lemma 5.7. *There exists a function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that $\text{Cut}(G, k) \leq f(n, m, k)$ for every graph G such that $\lfloor G \rfloor$ can be ordered by an MSO -formula of the form $\varphi(x, y; \bar{P})$ where $\text{qr}(\varphi) \leq m$ and $\bar{P} = P_0 \dots P_{n-1}$ are parameters. Furthermore, the function $f(n, m, k)$ is effectively elementary in the argument k , that is, there exists a computable function g such that $f(n, m, k) \leq \exp_{g(n, m)}(k)$.*

Proof. Fixing $k, m, n \in \mathbb{N}$, we define $f(n, m, k)$ as the number of MSO-theories of the form

$$\text{MTh}_m(\lfloor H \rfloor, v, P_0, \dots, P_{n-1}, Q_0, \dots, Q_{k-1})$$

where H is a graph, v is a vertex of H and P_0, \dots, Q_0, \dots are parameters. Note that, for fixed m , the number of such theories is elementary in k (see Section 5.6 of [5] for a detailed calculation of an upper bound).

Let $\varphi(x, y; \bar{Z})$ be an MSO-formula of quantifier-rank at most m , let G be a graph with $\text{Cut}(G, k) > f(n, m, k)$, and P_0, \dots, P_{n-1} parameters from G . We have to show that $\varphi(x, y; \bar{P})$ does not order G . Fix graphs H_0, \dots, H_{d-1} with $d = \text{Cut}(G, k)$ and a relation $R \subseteq [k] \times [k]$ such that

$$G = \text{Un}(H_0 \otimes_R \dots \otimes_R H_{d-1}).$$

For $c < k$, let

$$C_c := \{x \in G \mid x \in H_i, \text{ for some } i < d, \text{ and } x \text{ has port label } c \text{ in } H_i\}.$$

Since $d > f(n, m, k)$, there are indices $i < j$ such that

$$\begin{aligned} & \text{MTh}_m(\lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i) \\ &= \text{MTh}_m(\lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j). \end{aligned}$$

As there exists a graph F such that

$$\begin{aligned} & \langle \lfloor G \rfloor, a_i a_j, \bar{P}, \bar{Q} \rangle \\ &= \langle \lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i \rangle \otimes_R \langle \lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j \rangle \otimes_R F \\ \text{and } & \langle \lfloor G \rfloor, a_j a_i, \bar{P}, \bar{Q} \rangle \\ &= \langle \lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j \rangle \otimes_R \langle \lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i \rangle \otimes_R F, \end{aligned}$$

it follows by Lemma 5.6 that

$$\text{MTh}_m(\lfloor G \rfloor, a_i a_j, \bar{P}, \bar{C}) = \text{MTh}_m(\lfloor G \rfloor, a_j a_i, \bar{P}, \bar{C}).$$

In particular, we have

$$\lfloor G \rfloor \models \varphi(a_i, a_j; \bar{P}) \quad \text{iff} \quad \lfloor G \rfloor \models \varphi(a_j, a_i; \bar{P}).$$

Hence, $\varphi(x, y; \bar{P})$ does not define an order on G . □

Corollary 5.8. *An MSO_1 -orderable class of graphs \mathcal{C} has property $\text{CUT}(f)$, for an elementary function f .*

Example 5.9. The following classes are not MSO_1 -orderable:

- ◆ the class of all cliques K_n ;
- ◆ the class of all complete bipartite graphs $K_{n,m}$;
- ◆ any class of graphs of the form $G \otimes (H_0 \oplus \cdots \oplus H_n)$ where the number n is unbounded and each H_i is nonempty.

As MSO_1 -orderability implies MSO_2 -orderability, we can expect that the property CUT implies SEP . The following lemma proves this fact.

Lemma 5.10. *A class \mathcal{C} of graphs with property $\text{CUT}(f)$ has property $\text{SEP}(g)$ where $g(n) := f(n + 2^n) - 1$.*

Proof. Let $G = \langle V, E \rangle \in \mathcal{C}$ and consider a set $S \subseteq V$ of size $|S| \leq n$. Let C_0, \dots, C_{d-1} be an enumeration of the connected components of $G - S$. We claim that $d \leq g(n)$.

We define colourings $\rho : S \rightarrow D$ and $\chi_i : C_i \rightarrow D$, for $i < d$ as follows. The set of colours is $D := S \cup \wp(S)$. (To be formally correct, we have to take the set $[k]$ where $k := |S \cup \wp(S)|$. To simplify notation, we will use $S \cup \wp(S)$ instead.) We set

$$\rho(s) := s \quad \text{and} \quad \chi_i(v) := \{s \in S \mid (v, s) \in E\}.$$

It follows that

$$G = \text{Un}(\langle S, \rho \rangle \otimes_R \langle C_0, \chi_0 \rangle \otimes_R \cdots \otimes_R \langle C_{d-1}, \chi_{d-1} \rangle),$$

where

$$R := \{(s, X) \in S \times \wp(S) \mid s \in X\}.$$

Consequently, $\text{Cut}(G, |D|) \geq d + 1$. Since $|D| \leq n + 2^n$, it follows that

$$d + 1 \leq \text{Cut}(G, n + 2^n) \leq f(n + 2^n) = g(n) + 1. \quad \square$$

The converse obviously does not hold. A special case, where it *does* hold is the case of r -sparse graphs (cf. Definition 2.1). This case is of particular interest since, for r -sparse graphs, the expressive powers of MSO_1 and MSO_2 coincide (see Theorem 9.38 of [5]).

Lemma 5.11. *The graph $K_{m,n}$ is r -sparse if, and only if, $r \geq \left(\frac{1}{m} + \frac{1}{n}\right)^{-1}$.*

Proof. Every induced subgraph of $K_{m,n}$ is of the form $K_{m',n'}$ with $m' \leq m$ and $n' \leq n$. Such a subgraph has $m' + n'$ vertices and $m'n'$ edges. The ratio is

$$\frac{m'n'}{m' + n'} = \frac{1}{\frac{1}{m'} + \frac{1}{n'}} \leq \frac{1}{\frac{1}{m} + \frac{1}{n}}. \quad \square$$

Lemma 5.12. *A class \mathcal{C} of r -sparse graphs with property $\text{SEP}(f)$ has property $\text{CUT}(g)$ where $g(k) := f(2k^2r(2r+1))$.*

Proof. Let $G \in \mathcal{C}$. Suppose that

$$G = \text{Un}\left((H_0, \chi_0) \otimes_R \cdots \otimes_R (H_{d-1}, \chi_{d-1})\right),$$

where $R \subseteq [k] \times [k]$. W.l.o.g. we may assume that R is symmetric. We have to show that $d \leq g(k)$.

Set $I_a := \{i < d \mid \chi_i^{-1}(a) \neq \emptyset\}$. First, let us show that

$$|I_a| \leq 2r + 1 \quad \text{or} \quad |I_b| \leq 2r + 1, \quad \text{for every } (a, b) \in R.$$

For a contradiction, suppose that there is some $(a, b) \in R$ that $|I_a| \geq 2r + 2$ and $|I_b| \geq 2r + 2$. Choose subsets $I'_a \subseteq I_a$ and $I'_b \subseteq I_b$ of size $m := 2r + 2$ and select vertices $x_i \in \chi_i^{-1}(a)$, for $i \in I'_a$, and $y_i \in \chi_i^{-1}(b)$, for $i \in I'_b$. The subgraph induced by these vertices has $m^2 - |I_a \cap I_b| \geq m^2 - m$ edges and $2m$ vertices. Since

$$\frac{m^2 - m}{2m} = \frac{m - 1}{2} = \frac{2r + 1}{2} > r,$$

it follows that G is not r -sparse. A contradiction.

For $a, b \in [k]$, set

$$\begin{aligned} S_{ab} &:= \bigcup \left\{ \chi_i^{-1}(a) \mid i \in I_a, |\chi_i^{-1}(a)| \leq 2r \right\}, \\ S &:= \bigcup \left\{ S_{ab} \mid (a, b) \in R, |I_a| \leq 2r + 1 \right\}. \end{aligned}$$

Note that

$$|S_{ab}| \leq 2r|I_a| \quad \text{and} \quad |S| \leq |R| \cdot (2r + 1) \cdot (2r) \leq 2k^2r(2r + 1).$$

We claim that every connected component of $G - S$ is contained in $H_i - S$, for some i . For a contradiction, suppose that there is a connected component C of

$G - S$ containing vertices from both $H_i - S$ and $H_j - S$. Then there exists an edge (x, y) of G with $x \in H_i - S$ and $y \in H_j - S$. Let $a := \chi_i(x)$ and $b := \chi_j(y)$. Then $(a, b) \in R$. We have shown above that $|I_a| \leq 2r + 1$ or $|I_b| \leq 2r + 1$. In the first case, we have $x \in \chi_i^{-1}(a) \subseteq S_{ab} \subseteq S$, in the second case, we have $y \in \chi_i^{-1}(b) \subseteq S_{ba} \subseteq S$. Hence, both cases lead to a contradiction.

It follows that $G - S$ has at least d connected components. Consequently,

$$d \leq \text{Sep}(G, |S|) \leq \text{Sep}(G, 2k^2 r(2r + 1)) \leq f(2k^2 r(2r + 1)) = g(k). \quad \square$$

5.2 COGRAPHS

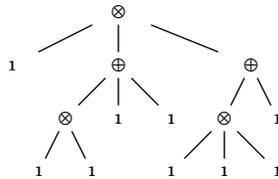
Recall from Section 4.3 that cographs are constructed by the operations \oplus , \otimes , and 1 . It follows that a cograph G with more than one vertex is either disconnected and of the form $G = H_0 \oplus \cdots \oplus H_n$ for connected cographs H_0, \dots, H_n , or it is connected and of the form $G = H_0 \otimes \cdots \otimes H_n$ for cographs H_0, \dots, H_n each of which is either disconnected or a single vertex. Furthermore, these decompositions of G are unique, up to the ordering of H_0, \dots, H_n . Using this observation, we can associate with every cograph a unique term as follows.

Definition 5.13. A term t of the cograph-operations \oplus , \otimes , 1 (where we consider \oplus and \otimes as many-ary operations with unordered arguments) is a *cotree* if there is no node that is labelled by the same operation as one of its immediate successors. Note that every graph has a unique cotree. The *depth* of a cograph is the height of this cotree.

Example 5.14. The cograph G defined by the term

$$(1 \otimes (1 \oplus (1 \oplus (1 \otimes 1)))) \otimes ((1 \otimes (1 \otimes 1)) \oplus 1)$$

has the cotree



Note that the leaves correspond to the vertices of G and that every subtree corresponds to an induced subgraph of G .

Recall (see, e.g., [3]) that a *module* of a graph $G = \langle V, E \rangle$ is a set M of vertices such that every vertex in $V \setminus M$ is either adjacent to all elements of M , or to none of them. A module M is called *strong* if there is no module N such that $M \setminus N$ and $N \setminus M$ are both nonempty (cf. [3]). Clearly, being a module and being a strong module are expressible in MSO_1 . In a cograph we can distinguish between two types of strong modules: the connected and the disconnected ones.

Definition 5.15. A \oplus -*module* of a cograph G with cotree t is the value of a subterm s of t where the root of s is labelled with \oplus . Similarly, a \otimes -*module* is the value of a subterm whose root is labelled by \otimes .

Theorem 5.16. *Let \mathcal{C} be a class of cographs. The following statements are equivalent.*

- (1) \mathcal{C} is MSO_1 -orderable.
- (2) \mathcal{C} has property CUT.
- (3) There exists a constant $d \in \mathbb{N}$ such that the cotree of every graph in \mathcal{C} has outdegree at most d .

Proof. (3) \Rightarrow (1) is Corollary 6.12 from [3] and (1) \Rightarrow (2) was shown in Corollary 5.8.

For (2) \Rightarrow (3), suppose that, for every $d \in \mathbb{N}$, there exists a graph $G_d \in \mathcal{C}$ with a cotree of maximal outdegree at least d . It is sufficient to show that $\text{Cut}(G_d, 3) > d$.

By assumption, there is a strong module A of G_d containing strong submodules B_0, \dots, B_{n-1} , for $n > d$, such that either (i) $A = B_0 \oplus \dots \oplus B_{n-1}$, or (ii) $A = B_0 \otimes \dots \otimes B_{n-1}$. Let $C := G - A$ be the graph induced by the complement of A . Every vertex $v \in C$ is either connected to all vertices of A , or to none of them. We assign the port label 0 to the former vertices and the port label 1 to the latter ones. Each vertex of A gets port label 2. It follows that

$$G_d = C \otimes_R B_0 \otimes_R \dots \otimes_R B_{n-1}$$

where $R = \{(0, 2), (2, 0)\}$ or $R = \{(0, 2), (2, 0), (2, 2)\}$. □

Corollary 5.17. *Let $k \in \mathbb{N}$. The class of cographs of depth at most k is hereditarily MSO_1 -unorderable.*

Proof. For any given depth k , there are only finitely many cographs (up to isomorphism) satisfying condition (3) of Theorem 5.16. □

Corollary 5.18. *For VR-equational classes of cographs, MSO_1 -orderability is decidable.*

Proof. Let \mathcal{C} be a VR-equational class of cographs. By Theorem 5.16, it is sufficient to decide whether there is a constant d such that every cotree of a graph in \mathcal{C} has maximal outdegree at most d . Let $\varphi(X)$ be an MSO_1 -formula stating that there exists a strong module Z such that

- ◆ $X \subseteq Z$ and
- ◆ every strong module $Y \subset Z$ contains at most one element of X .

Given a cograph G , it follows that the maximal outdegree of the cotree of G is equal to the maximal size of a set X satisfying φ in G . Using the Semi-Linearity Theorem, we can decide whether this size is bounded. \square

Remark 5.19. If a class \mathcal{C} of cographs is MSO_1 -orderable, there exists an MSO -transduction (see [3] or Chapter 7 of [5]) mapping each graph in \mathcal{C} to its cotree. But, conversely, the existence of such an MSO -transduction is not enough to ensure MSO_1 -orderability: there exists an MSO -transduction from the class of all cographs of depth k to their respective cotrees (this is a routine construction). But, as we have just seen, this class is hereditarily MSO_1 -unorderable.

5.3 \otimes -DECOMPOSITIONS

Cographs are precisely the graphs of clique-width 2. A natural aim is thus to extend the equivalence (1) \Leftrightarrow (2) of Theorem 5.16 to classes of graphs of bounded clique-width. However, we must leave this as a conjecture. Instead we only consider the special case of graphs where the height of the decomposition (as defined below) is bounded. These generalise the cographs of bounded depth, and we show that classes of such graphs are hereditarily MSO_1 -unorderable.

We start by introducing a kind of decomposition associated with the notion of clique-width. Since we are only interested in decompositions of bounded height, a simplified version, called a \otimes -decomposition, suffices.

Definition 5.20. Let $G = \langle V, E \rangle$ be a graph.

(a) A \otimes -decomposition of G of width k is a family $(H_v)_{v \in T}$ of labelled graphs $H_v = \langle U_v, F_v, \chi_v \rangle$ with $\chi_v : U_v \rightarrow [k]$ such that

- ◆ the index set T is a rooted tree,

- $H_{\langle \rangle} = \langle V, E, \chi_{\langle \rangle} \rangle$, for some labelling $\chi_{\langle \rangle}$,
- $|U_v| = 1$, for every leaf $v \in T$,
- for every internal vertex $v \in T$ with successors u_0, \dots, u_{d-1} , there is some $R_v \subseteq [k] \times [k]$ such that

$$\text{Un}(H_v) = \text{Un}(H_{u_0} \otimes_{R_v} \dots \otimes_{R_v} H_{u_{d-1}}).$$

We call \otimes_{R_v} the *operation at v* . Note that the port labels of H_v and $H_{u_0}, \dots, H_{u_{d-1}}$ are unrelated.

(b) A *strong \otimes -decomposition* of G is a \otimes -decomposition $(H_v)_{v \in T}$ with $H_v = \langle U_v, F_v, \chi_v \rangle$ such that, for each internal vertex $v \in T$ with successors u_0, \dots, u_{d-1} , there is some $R_v \subseteq [k] \times [k]$ and some function $\rho : [k] \rightarrow [k]$ such that

$$H_v = \text{relab}_{\rho}(H_{u_0} \otimes_{R_v} \dots \otimes_{R_v} H_{u_{d-1}}).$$

(c) The *height* of a \otimes -decomposition $(H_v)_{v \in T}$ is the height of the tree T .

(d) We define $\text{wd}_n^{\otimes}(G)$ as the least number k such that G has a \otimes -decomposition of width at most k and height at most n . Similarly, we define $\text{swd}_n^{\otimes}(G)$ as the least number k such that G has a strong \otimes -decomposition of width at most k and height at most n . We call $\text{wd}_n^{\otimes}(G)$ the *n -depth \otimes -width* of G and $\text{swd}_n^{\otimes}(G)$ its *strong n -depth \otimes -width*.

Remark 5.21. (a) Note that, for every graph G and all n, m with $m < n$, we have

$$\begin{aligned} \text{wd}_n^{\otimes}(G) &\leq \text{swd}_n^{\otimes}(G) \leq |V|, \\ \text{wd}_n^{\otimes}(G) &\leq \text{wd}_m^{\otimes}(G), \\ \text{swd}_n^{\otimes}(G) &\leq \text{swd}_m^{\otimes}(G). \end{aligned}$$

(b) Recall the definition of clique-width at the beginning of Section 5.1. Since the operation \otimes_R can be expressed by the operations clique-width is based on, but using twice as many port labels, it follows that the clique-width of a graph is at most twice its strong n -depth \otimes -width (for any n). Since, conversely, for sufficiently large n , the strong n -depth \otimes -width of a graph G is at most its clique-width, it follows that, for every graph G and all sufficiently large n ,

$$\text{swd}_n^{\otimes}(G) \leq \text{cwd}(G) \leq 2 \cdot \text{swd}_n^{\otimes}(G).$$

If we define $\text{swd}_n^\otimes(G)$ as the minimal value of $\text{swd}_n^\otimes(G)$, for $n \in \mathbb{N}$, we therefore obtain a nontrivial width measure that is equivalent to clique-width.

(c) Note that $\text{wd}_n^\otimes(G) \leq 2$, for every graph G with n vertices. Hence, the width $\text{wd}_n^\otimes(G)$ is only of interest if there is a bound on n .

Because of its relation to clique-width, the strong \otimes -width is of more interest than the \otimes -width (which becomes trivial for large depths). We have introduced the simpler notion of \otimes -width since, in the special case we consider, there exists a bound on the depth of \otimes -decompositions. In this case we can use the following lemma to transform a bound on the \otimes -width of a class into a bound on its strong \otimes -width.

Lemma 5.22. *For every graph G and every $n \in \mathbb{N}$,*

$$\text{wd}_n^\otimes(G) \leq \text{swd}_n^\otimes(G) \leq [\text{wd}_n^\otimes(G)]^n.$$

Proof. The first inequality being trivial, we only prove the second one. Given a \otimes -decomposition $(H_v)_{v \in T}$ of G of height n and width $k := \text{wd}_n^\otimes(G)$, we construct a strong \otimes -decomposition $(H'_v)_{v \in T}$ of G of the same height and width k^n . Consider $v \in T$ and let v_o, \dots, v_k be the path in T from the root $\langle \rangle = v_o$ to $v = v_k$. Suppose that $H_v = \langle U_v, F_v, \chi_v \rangle$. We set $H'_v := \langle U_v, F_v, \chi'_v \rangle$ where

$$\chi'_v(x) := \langle \chi_{v_o}(x), \dots, \chi_{v_k}(x) \rangle.$$

Then

$$H'_v = \text{relab}_\rho(H'_{u_o} \otimes_{R_v} \dots \otimes_{R_v} H'_{u_{d-1}}),$$

where the function ρ maps $\langle a_o, \dots, a_k, a_{k+1} \rangle$ to $\langle a_o, \dots, a_k \rangle$. □

Lemma 5.23. *Let G be a graph and $(H_v)_{v \in T}$ a \otimes -decomposition of G of width at most k . Every vertex of T has less than $\text{Cut}(G, k + 2^k)$ successors.*

Proof. Suppose that $H_v = \langle U_v, F_v, \chi_v \rangle$. Let $v \in T$ be a vertex with successors u_o, \dots, u_{m-1} . Hence,

$$H_v = H_{u_o} \otimes_R \dots \otimes_R H_{u_{m-1}},$$

where \otimes_R is the operation at v . Let $C := G - H_v$, i.e., the subgraph induced by the complement of the set of vertices of H_v . We claim that

$$G = C \otimes_{R'} H_{u_o} \otimes_{R'} \dots \otimes_{R'} H_{u_{m-1}},$$

for a suitable labelling $\rho : C \rightarrow [k + 2^k]$ of C and a suitable relation $R' \subseteq [k + 2^k] \times [k + 2^k]$. This implies that $m + 1 \leq \text{Cut}(G, k + 2^k)$, as desired.

It remains to define ρ and R' . Fix a bijection $\pi_o : \wp([k]) \rightarrow [2^k]$ and set $\pi(B) := \pi_o(B) + k$, for $B \subseteq [k]$. We define

$$\rho(x) := \pi(\{\chi_v(y) \mid y \in U_v, (x, y) \in E\}), \quad \text{for } x \in C,$$

and $R' := R \cup \{(a, \pi(B)) \mid a \in [k], B \subseteq [k], a \in B\}$. □

We obtain the following characterisation of MSO_1 -orderable classes of bounded n -depth \otimes -width.

Theorem 5.24. *Let \mathcal{C} be a class of graphs such that, for some $n, k \in \mathbb{N}$,*

$$\text{wd}_n^\otimes(G) \leq k, \quad \text{for all } G \in \mathcal{C}.$$

The following statements are equivalent:

- (1) \mathcal{C} is MSO_1 -orderable.
- (2) \mathcal{C} has property CUT.
- (3) *There is a constant $d \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a \otimes -decomposition $(H_v)_{v \in T}$ of height at most n and width at most k where every vertex of T has at most d successors.*
- (4) \mathcal{C} is finite.

Proof. (4) \Rightarrow (1) is trivial and (1) \Rightarrow (2) follows from Corollary 5.8.

(2) \Rightarrow (3) Suppose that \mathcal{C} has property $\text{CUT}(f)$. Let $G \in \mathcal{C}$ and let $(H_v)_{v \in T}$ be a reduced \otimes -decomposition of G of height at most n and width at most k . Then it follows by Lemma 5.23 that every vertex of T has less than $d := f(k + 2^k)$ successors.

(3) \Rightarrow (4) Since every tree of height at most n with degree at most d has at most $1 + d + d^2 + \dots + d^{n-1} < d^n$ vertices, it follows that every graph in \mathcal{C} has at most that many elements. □

We obtain the following extension of Corollary 5.17.

Corollary 5.25. *Let $n, k \in \mathbb{N}$. The class of all graphs G of n -depth \otimes -width at most k is hereditarily MSO_1 -unorderable.*

6 REDUCTIONS BETWEEN DIFFICULT CASES

In this section we consider classes of graphs where the question of orderability is as hard as in the general case.

Definition 6.1. Let $G = \langle V, E \rangle$ be a graph.

(a) The *incidence graph* of G is the graph $\text{Inc}(G) := \langle V \cup E, I, P \rangle$ where the edge relation

$$\begin{aligned} I &:= \text{inc} \cup \text{inc}^{-1} \\ &= \{ (x, y) \mid x \text{ is an end-point of } y \text{ or } y \text{ is an end-point of } x \} \end{aligned}$$

is the symmetric version of the incidence relation and $P := V$ is a unary relation identifying the vertices of G .

(b) The *incidence split graph* of G is the graph $\text{IS}(G) := \langle V \cup E, J \rangle$ where

$$J := I \cup \{ (x, y) \in V \times V \mid x \neq y \}$$

and I is the symmetric incidence relation from (a).

(c) For a class \mathcal{C} of graphs we set

$$\begin{aligned} \text{Inc}(\mathcal{C}) &:= \{ \text{Inc}(G) \mid G \in \mathcal{C} \}, \\ \text{IS}(\mathcal{C}) &:= \{ \text{IS}(G) \mid G \in \mathcal{C} \}. \end{aligned}$$

Note that $\text{IS}(G)$ is a split graph. The proposition below suggests that a characterisation of MSO_1 -orderability for classes of split graphs is as hard as a characterisation of MSO_2 -orderability for arbitrary classes of graphs. We start with a technical lemma.

Lemma 6.2. *Let \mathcal{C} be a class of graphs.*

(a) \mathcal{C} has property SEP if, and only if, $\text{Inc}(\mathcal{C})$ has property SEP.

(b) $\text{Inc}(\mathcal{C})$ has property CUT if, and only if, $\text{IS}(\mathcal{C})$ has property CUT.

Proof. (a) (\Leftarrow) Suppose that $\text{Inc}(\mathcal{C})$ has property $\text{SEP}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. We claim that \mathcal{C} also has property $\text{SEP}(f)$. Let $G = \langle V, E \rangle$ be a graph in \mathcal{C} . To compute $\text{Sep}(G, k)$ consider a set $S \subseteq V$ of size $|S| \leq k$. Let C_0, \dots, C_{m-1} be the connected components of $G - S$. Then the connected components of $\text{Inc}(G) - S$ are

$$C'_0, \dots, C'_{m-1}, e_0, \dots, e_{n-1}$$

where e_0, \dots, e_{n-1} are all edges of G between vertices in S and C'_i is the graph obtained from $\text{Inc}(C_i)$ by adding all edges of G connecting a vertex in S to some vertex of C_i . It follows that

$$\text{Sep}(G, k) \leq \text{Sep}(\text{Inc}(G), k) \leq f(k).$$

(\Rightarrow) Suppose that \mathcal{C} has property $\text{SEP}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $G = \langle V, E \rangle$ be a graph in \mathcal{C} with $\text{Inc}(G) = \langle V \cup E, I, P \rangle$. To compute $\text{Sep}(\text{Inc}(G), k)$ consider a set $S \subseteq V \cup E$ of size $|S| \leq k$. For each edge $e \in S \cap E$, we select one end-point. Let X be the set of these end-points and set $S' := (S \setminus E) \cup X$. Then $\text{Inc}(G) - S'$ has at least as many connected components as $\text{Inc}(G) - S$. Since $S' \subseteq V$ it follows by what we have seen above that $\text{Inc}(G) - S'$ has at most $m + \binom{k}{2}$ connected components, where m is the number of connected components of $G - S'$. Consequently,

$$\text{Sep}(\text{Inc}(G), k) \leq \text{Sep}(G, k) + \frac{k}{2}(k-1).$$

It follows that $\text{Inc}(\mathcal{C})$ has property $\text{SEP}(f')$ where $f'(k) = f(k) + \frac{k}{2}(k-1)$.

(b) (\Rightarrow) Suppose that $\text{Inc}(\mathcal{C})$ has property $\text{CUT}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $\text{Inc}(G) = \langle V \cup E, I, P \rangle$ be a graph in $\text{Inc}(\mathcal{C})$ and let $\text{IS}(G) = \langle V \cup E, J \rangle$. To compute $\text{Cut}(\text{IS}(G), k)$ suppose that

$$\text{IS}(G) = \text{Un}(H_0 \otimes_R \dots \otimes_R H_{m-1}),$$

for k -labelled graphs H_0, \dots, H_{m-1} and a relation $R \subseteq [k] \times [k]$. Suppose that $H_i = \langle U_i, J_i \rangle$, for $i < m$, and let χ_i be the labelling of H_i . We set $H'_i := \langle U_i, I_i, P_i \rangle$ where $I_i := J_i \setminus (V \times V)$ and $P_i := U_i \cap V$. We label H'_i by

$$\chi'_i(v) := \begin{cases} \chi_i(v) & \text{if } v \notin V, \\ \chi_i(v) + k & \text{if } v \in V. \end{cases}$$

Then

$$\text{Inc}(G) = \text{Un}(H'_0 \otimes_{R'} \dots \otimes_{R'} H'_{m-1}),$$

where

$$R' := \{ (x, y), (x+k, y), (x, y+k) \mid (x, y) \in R \}.$$

Consequently, $\text{Cut}(\text{IS}(G), k) \leq \text{Cut}(\text{Inc}(G), 2k) \leq f(2k)$.

(\Leftarrow) Suppose that $\text{IS}(\mathcal{C})$ has property $\text{CUT}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $\text{Inc}(G) = \langle V \cup E, I, P \rangle$ be a graph in $\text{Inc}(\mathcal{C})$ and let $\text{IS}(G) = \langle V \cup E, J \rangle$. To compute $\text{Cut}(\text{Inc}(G), k)$ suppose that

$$\text{Inc}(G) = \text{Un}(H_0 \otimes_R \cdots \otimes_R H_{m-1}),$$

for k -labelled graphs H_0, \dots, H_{m-1} and a relation $R \subseteq [k] \times [k]$. Suppose that $H_i = \langle U_i, I_i, P_i \rangle$, for $i < m$, and let χ_i be the labelling of H_i . We define the graph $H'_i := \langle U_i, J_i \rangle$ where $J_i := I_i \cup \{ (x, y) \mid x, y \in P_i, x \neq y \}$ with labelling

$$\chi'_i(v) := \begin{cases} \chi_i(v) & \text{if } v \in V, \\ \chi_i(v) + k & \text{if } v \notin V. \end{cases}$$

Then

$$\text{IS}(G) = \text{Un}(H'_0 \otimes_{R'} \cdots \otimes_{R'} H'_{m-1}),$$

where

$$R' := \{ (x, y), (x + k, y), (x, y + k), (x + k, y + k) \mid (x, y) \in R \} \\ \cup [k] \times [k].$$

Consequently, $\text{Cut}(\text{Inc}(G), k) \leq \text{Cut}(\text{IS}(G), 2k) \leq f(2k)$. □

Proposition 6.3. *Let \mathcal{C} be a class of graphs.*

- (a) \mathcal{C} is MSO_2 -orderable if, and only if, $\text{IS}(\mathcal{C})$ is MSO_1 -orderable.
- (b) \mathcal{C} has property SEP if, and only if, $\text{IS}(\mathcal{C})$ has property CUT .

Proof. (a) is a routine construction. (b) follows by the preceding lemma since $\text{Inc}(\mathcal{C})$ is 2-sparse and, by Lemmas 5.10 and 5.12, such a class has property SEP if, and only if, it has property CUT . □

Corollary 6.4. *Let \mathcal{P} be a graph property such that a class of split graphs is MSO_1 -orderable if, and only if, it has properties CUT and \mathcal{P} . Then a class of arbitrary graphs is MSO_2 -orderable if, and only if, it has properties SEP and $\text{IS}^{-1}(\mathcal{P})$.*

Remark 6.5. (a) Characterising MSO_2 -orderable classes therefore amounts to a characterisation of MSO_1 -orderable classes of split graphs contained in the image of the function IS.

(b) If \mathcal{C} is a class of graphs with property SEP that is not MSO_2 -orderable, then $\text{IS}(\mathcal{C})$ is a class of split graphs with property CUT that is not MSO_1 -orderable.

We also present a lemma suggesting that a characterisation of MSO_1 -orderability for classes of bipartite graphs is as hard as a characterisation of MSO_1 -orderability for arbitrary classes of graphs. We leave the proof – which is similar to the one above – to the reader.

Definition 6.6. For a graph $G = \langle V, E \rangle$ we define

$$\text{BP}(G) := \langle V \times [4], E' \rangle$$

where

$$E' := \left\{ \begin{aligned} &((x, 0), (y, 3)) \mid (x, y) \in E \\ &\cup \{((x, i), (x, i + 1)) \mid x \in V, 0 \leq i < 3\}. \end{aligned} \right.$$

For classes \mathcal{C} of graphs, we define $\text{BP}(\mathcal{C})$ in the usual way.

Lemma 6.7. *Let \mathcal{C} be a class of graphs.*

- (a) \mathcal{C} is MSO_1 -orderable if, and only if, $\text{BP}(\mathcal{C})$ is MSO_1 -orderable.
- (b) \mathcal{C} has property CUT if, and only if, $\text{BP}(\mathcal{C})$ has property CUT.

7 CONCLUSION

For arbitrary classes of graphs, it is difficult to obtain necessary and sufficient conditions for MSO_i -orderability, as there are many different ways to construct MSO -definable orderings depending on many different structural properties of the considered graphs. General conditions should thus cover simultaneously a large number of possibilities. It is therefore necessary to consider particular graph classes. We have obtained necessary and sufficient conditions in Theorems 4.13, 4.30, 4.33, and 5.16 with corresponding decidability results for VR-equational classes of graphs.

Concerning future work we think that the following questions can be fruitfully investigated:

(a) Does Conjecture 4.26 hold? We have already proved several special cases and more cases seem to be within reach. It remains to be seen whether the full conjecture can be resolved.

(b) Which condition must be added to the property SEP to yield a necessary and sufficient condition for MSO_2 -orderability of a class of cographs? And more generally, for graph classes of bounded clique-width?

(c) What could be an extension of Theorem 5.16, say, for classes of ‘bounded strong \otimes -width’?

(d) Which operations do preserve MSO_i -orderability? Candidates include the operations defining tree-width or clique-width, graph substitutions, and monadic second-order transductions. We presented a few simple results in Lemma 3.2 and Remark 5.3, but it should not be too hard to develop a more comprehensive theory.

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