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Functional composition is a generalized symmetry

Alexander K. Zvonkin*

Abstract

We argue that the well-known and thoroughly studied phenomenon of symmetry in the world of maps (which are finite tessellations of two-dimensional surfaces) is a particular case of a much vaster phenomenon of “hidden symmetries” represented by compositions of ramified coverings.

1 Introduction

1.1 What is a tessellation?

According to the Wikipedia article,

A tessellation or a tiling of the plane is a collection of plane figures that fills the plane with no overlaps and no gap. One may also speak of tessellations of the parts of the plane or of other surfaces.

This definition taken literally, a tessellation may be considered as a purely topological object. However, the subsequent examples given in the Wikipedia article apparently presume not only a topological but also a geometric structure of the subdivisions in question: tiles are considered to be regular polygons, or irregular ones but congruent to each other, or having other rigidity properties. For surfaces different from the plane or the sphere this approach may create some technical difficulties since, for example, a torus obtained by gluing the opposite sides of the square cannot be isometrically embedded in \mathbb{R}^3 . A very important aspect in the study of tessellations is symmetry.

In this paper we will consider another type of rigidity, namely: our tessellations will be considered not up to isometry but up to conformal equivalence. The fundamental difference between the two is that in many cases the conformal structure is fully determined by the combinatorial/topological one. Also, it turns out that symmetry is a particular case of a much vaster phenomenon which is composition of coverings.

We consider only compact oriented two-dimensional manifolds, and only finite tessellations, that is, decompositions of these surfaces into a finite number of simply connected domains. Such tessellations are commonly called *maps*.

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1.2 Maps and hypermaps

Mathematicians have studied maps for almost three centuries but it seems that the objects called *hypermaps* are in many respects more fundamental.

Convention 1.1 We consider only connected graphs; loops and multiple edges are allowed.

Definition 1.2 (Map) A *map* is a graph embedded in a compact oriented two-dimensional manifold in such a way that:

- The edges do not intersect.
- The complement of the graph in the surface is a disjoint union of regions homeomorphic to open disks.

These regions are called *faces* of the map. The *genus* of a map is, by definition, the genus of the underlying surface.

Definition 1.3 (Hypermap) A *hypermap* is a bicolored map: its vertices are colored in black and white in such a way that the adjacent vertices have opposite colors.

The above definition suggests that a hypermap is a particular case of a map. However, hypermaps were first introduced as a generalization of maps [6]. Indeed, every map can be transformed into a hypermap by simply inserting a white vertex in the middle of every edge. In this way we obtain a particular case of hypermaps: those whose white vertices are all of degree 2.

Convention 1.4 (Terminology) Our conventions will be as follows:

- When it is not stated otherwise, we always work with hypermaps.
- If it so happens that all the white vertices of a given hypermap are of degree 2, we erase them in order to simplify the picture and draw the hypermap as a map – and quite often even call it a map.
- In order to be coherent with the previous definitions, a segment in a hypermap joining a black vertex and a white one is called not an edge but a *germ of an edge*, or simply a *germ*, the term “edge” being reserved for maps (an edge of a map consists of two germs).

Definition 1.5 (Degree) The degree of a hypermap is the number of its germs of edges. The degree of a vertex is the number of germs incident with it. The degree of a face is half the number of germs surrounding it. (The reason for this “half” will become clear in a moment.)

Let M be a hypermap of degree m and of genus g , F be the number of its faces, B and W the numbers of black and white vertices, respectively, and $V = B + W$ the total number of vertices. Then the Euler characteristic of M is

computed as follows:

$$\chi(M) = 2 - 2g = B + W + F - m = V + F - m.$$

According to tradition, we use the adjective *plane* while treating maps and hypermaps on the *sphere*.

1.3 Encoding hypermaps by permutations

Hypermaps admit an encoding by triples of permutations. In constructing this encoding we twice use the fact that the surface on which the hypermap is drawn is oriented: first, moving along a germ of an edge in a specific direction we can distinguish the left bank and the right one; second, the surface itself is endowed with a positive (counterclockwise) and a negative (clockwise) orientation.

Let M be a hypermap of degree m . We label the germs from 1 to m , and place the label of a germ near its left bank when we go from its black end to the white one. Then we associate to M the following triple of permutations $(\sigma, \alpha, \varphi)$ on the set of m labels:

- A cycle of σ contains the labels of the germs incident with a black vertex, taken in the positive direction around this vertex; thus, there are as many cycles in σ as there are black vertices, and the degree of a vertex is equal to the length of the corresponding cycle.
- The cycles of α correspond, in the same way, to the white vertices (the direction of the germs is also taken to be positive).
- A cycle of φ contains the labels placed inside a face; the corresponding germs are considered to be incident with this face, and their number is equal to the face degree (cf. Definition 1.5: exactly half of the germs surrounding a face are incident with it). The labels are taken in the positive direction when we look at them from the inside of the face; thus, there are as many cycles in φ as there are faces, and the degree of a face is equal to the length of the corresponding cycle.

Example 1.6 For the hypermap of Fig. 1 we obtain the following permutations (we show also the cycles of length 1 which correspond to vertices or faces of degree 1):

$$\begin{aligned}\sigma &= (1, 2, 3)(4, 5)(6)(7, 8, 9), \\ \alpha &= (1, 4)(2, 9, 3)(5, 6, 7)(8), \\ \varphi &= (1, 5, 9)(2)(3, 8, 7, 6, 4).\end{aligned}$$

For the outer face, one may think that the corresponding cycle $(3, 8, 7, 6, 4)$ turns in the negative direction, but in fact we must look “from the opposite side of the sphere”, or, in other words, “from the inside of the outer face”, and then the direction becomes positive.

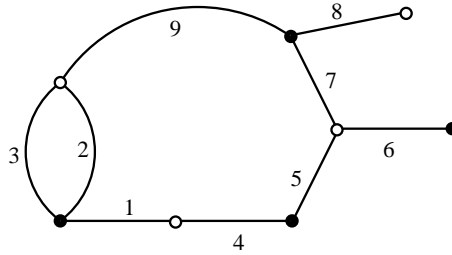


Figure 1: A labeling of the germs of edges of a hypermap.

Remark 1.7 The following very important observation is true for any hypermap (the proof is a simple exercise¹):

$$\sigma\alpha\varphi = 1.$$

Therefore, in order to encode a hypermap, we may take any two of these three permutations. The representation by a triple of permutations is, however, more symmetric.

The correspondence between hypermaps and triples of permutations works also in the opposite direction.

Proposition 1.8 *To any triple of permutations $(\sigma, \alpha, \varphi)$ such that*

- *the permutation group $G = \langle \sigma, \alpha, \varphi \rangle$ generated by σ, α, φ is transitive,*
- *$\sigma\alpha\varphi = 1,$*

there corresponds a hypermap.

The condition of transitivity ensures the connectivity of the corresponding graph.

Example 1.9 (Genus 4 embedding of the icosahedron) Take the icosahedron, which is usually considered as a plane map. Write down the corresponding permutations σ and α , and then replace σ by $\sigma' = \sigma^2$ preserving α as it is (that is, taking $\alpha' = \alpha$). Naturally, φ' is now computed as $(\sigma'\alpha')^{-1}$. The triple of permutations $(\sigma', \alpha', \varphi')$ represents an embedding of the graph of icosahedron in the surface of genus $g = 4$ in such a way that the corresponding map has 12 vertices of degree 5, 30 edges, and 12 faces of degree 5. Contrary to the usual plane icosahedron, this “icosahedron of genus 4” is self-dual.

Notation 1.10 (Two icosahedra) We denote the usual icosahedron—that of genus 0—by I_0 , and the icosahedron of genus 4, by I_4 .

¹We multiply permutations from left to right; this is the usual convention in the systems of symbolic calculations, like Maple.

There are two beautiful ways to visualize I_4 ; they are shown in Fig. 2. The left-hand one is called Great Dodecahedron. It is easily seen that its one-dimensional spine is a usual icosahedron. The pentagonal “faces” are also easily seen. They are obtained in the following way: we come to a vertex along an edge, and then take not the next edge in the cyclic order proper to I_0 but the one after the next, thus realizing the permutation σ^2 instead of σ . As to vertices, one might have an impression that they are surrounded by ten triangles, but in fact each pentagonal face appears twice near the same vertex.

For the right-hand figure, which is called Small Stellated Dodecahedron, the vertices look like usual polyhedron vertices, but the faces are now represented by pentagrams. This time, after coming to a vertex along an edge we must take the next edge in the cyclic order of edges around the vertex. Also, these two figures are dual to each other in a purely geometric sense. What is extremely difficult to imagine in both figures is the fact that they are not embeddings but *immersions* (i. e., with self-intersections) of the surface of genus 4 into \mathbb{R}^3 .

The automorphism group of I_0 consists of permutations commuting with σ and α ; for I_4 , it consists of permutations commuting with σ^2 and α . Therefore, the group is the same, namely, A_5 . Note, however, that, while an axis of the symmetry of order 5 for I_0 passes through two opposite vertices, the same symmetry for the I_4 “passes through” two vertices and two faces.

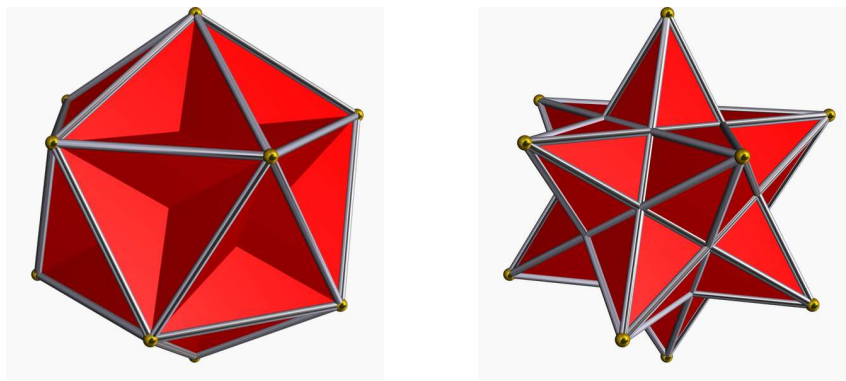


Figure 2: Two ways to visualize the icosahedron of genus 4. The images are borrowed from the Wikipedia. They are created using Robert Webb’s Great Stella software, see the Stella website: <http://www.software3d.com/Stella.html>.

The discovery of these figures is usually attributed to Kepler (1619), though the Small Stellated Dodecahedron can already be found on the mosaic floor, attributed to Paolo Ucello (around 1430), of St. Mark’s Basilica in Venice, see Fig. 13 at the end of the paper. Later on they were rediscovered by Poincaré, and then encountered by Felix Klein in his study of equations of fifth degree (see Example 4.5). The Great Dodecahedron is commercialized as a variation of Rubik’s cube under the name of Alexander’s Star.

1.4 Cartographic group

A fact which rarely attracts attention is that a simple picture drawn on a piece of paper generates, via the triple of permutations described above, a permutation group. We call this group $G = \langle \sigma, \alpha, \varphi \rangle$ the *cartographic group* of the corresponding (hyper)map. Of course, more often than not (see [7]) the group thus obtained is either S_m or A_m . But there exist also many other, more interesting and sometimes very subtle examples.

Example 1.11 (Mathieu group M_{22}) The hypermap shown in Fig. 3 is of degree 22. If we write down the corresponding permutations σ , α , and φ , we find out that the permutation group $\langle \sigma, \alpha, \varphi \rangle$ generated by them is isomorphic to the Mathieu group M_{22} (a sporadic simple group of order 443 520, see, for example, Chapter 6 of [8]). Note that the set of degrees of the black vertices, of the white vertices, and of the faces is the same, corresponding to the partition $4^4 2^2 1^2$ of 22. It turns out that in the group M_{22} there are two conjugacy classes with the cycle structure $4^4 2^2 1^2$, and they are *not* inverse to each other, a situation which is rather unusual; even the sizes of these classes are different. In the Atlas of Finite Groups [5] these classes are denoted by $4A$ and $4B$. One may verify² (see also [12]) that in this particular example all the three permutations belong to the class $4B$. There also exist hypermaps with the same set of vertex- and face degrees which generate the same group but have other combinations of conjugacy classes (like, for example, $4A, 4B, 4B$).

Cartographic groups will play very important role in what follows.

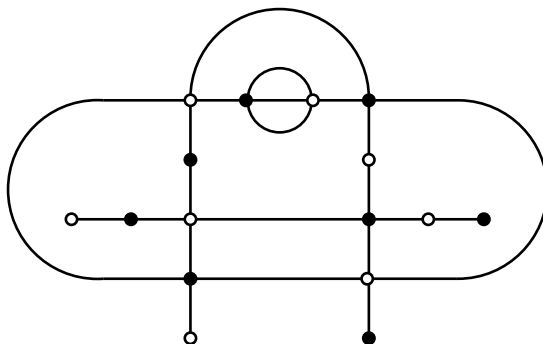


Figure 3: A hypermap representing the Mathieu group M_{22} . All the three permutations σ , α , and φ belong to the conjugacy class $4B$; see [5] for notation.

²The Maple package `group` is unable to compute the centralizers of either of permutations σ, α, φ : it replies “too many levels of recursion”. But the following easy trick permits to overcome this difficulty: compute first the centralizer of the *involution* σ^2 , and then compute the centralizer of σ inside the centralizer of σ^2 .

2 Belyi functions

2.1 Planar case

Definition 2.1 (Belyi function, planar case) Let M be a plane hypermap of degree m . A rational function f of degree m is a *Belyi function* corresponding to M if M may be embedded in the complex Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in such a way that:

1. All black vertices of M are roots of the equation $f(x) = 0$, the multiplicity of each root being equal to the degree of the corresponding vertex.
2. All white vertices of M are roots of the equation $f(x) = 1$, the multiplicity of each root being equal to the degree of the corresponding vertex.
3. The hypermap itself is obtained as $M = f^{-1}([0, 1])$.
4. Inside each face of M there exists a single pole of f (or, if you like, a root of the equation $f(x) = \infty$), the multiplicity of the pole being equal to the degree of the face. We call this pole the *center* of the face (of course, it is in no way its “geometric center”).
5. Beside 0, 1, and ∞ , there are no other critical values of f .

Remark 2.2 (Terminology) One must distinguish between *critical values* and *critical points*. A critical point of f is a root of its derivative (with a standard change of variables when it comes to infinity) while a critical value is the value of f at its critical point.

Example 2.3 (Explicit computation) The following example is borrowed from a recent paper [28]. Let us consider the hypermap shown in Fig. 4.

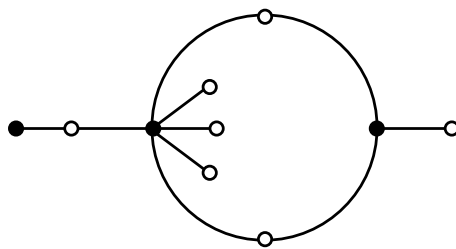


Figure 4: One more example of a hypermap.

We claim that the corresponding Belyi function is as follows:

$$f(x) = \frac{50000}{27} \cdot \frac{x^6 (x-1)^3 (x+1)}{(x^2 + 4x - 1)^5}. \quad (1)$$

Indeed, this function has a root $x = 0$ of multiplicity 6, a root $x = 1$ of multiplicity 3, and a simple root $x = -1$ (compare with the degrees of the black vertices). It also has two poles of degree 5 each (compare with the face degrees): they are the roots of the polynomial $x^2 + 4x - 1$. Now, factoring the function $f(x) - 1$ we obtain

$$f(x) - 1 = \frac{1}{27} \cdot \frac{(11x^3 + x^2 - 3x + 3)^2 (7x - 1) (59x^3 - 121x^2 + 33x - 3)}{(x^2 + 4x - 1)^5}. \quad (2)$$

We see that this function has three double roots (they are the roots of the polynomial $11x^3 + x^2 - 3x + 3$) and four simple roots; compare with the degrees of the white vertices.

These observations are not yet conclusive since there exist seven non-isomorphic hypermaps having the same set $(6^1 3^1 1^1, 2^3 1^4, 5^2)$ of vertex-and-face degrees (see below, Remark 2.4). But the Maple plot shown in Fig. 5 convinces us that the answer is correct.

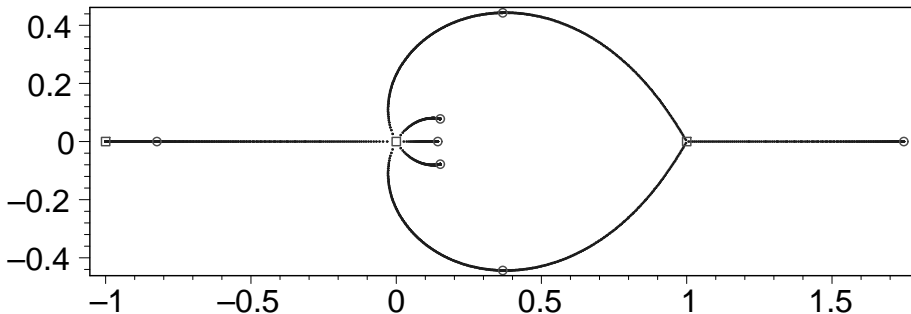


Figure 5: A Maple plot of the hypermap of Fig. 4 obtained as a preimage of the segment $[0, 1]$ via the Belyi function (1). Black vertices are marked by little squares.

This example is not a “random” one. If we send, via a linear fractional transformation, the two poles of the above Belyi function f to 0 and ∞ , respectively, we get a Laurent polynomial L with rather unusual properties with respect to the so-called moment problem (see [28]): it has a huge space of Laurent polynomials orthogonal to all powers of L on the unit circle. This phenomenon is at least partly explained by unusual behavior of the cartographic group. While acting on 10 points, this group is isomorphic to S_5 . The representation of any permutation group by permutation matrices always has at least two invariant subspaces: the space of dimension 1 spanned by the vector $(1, 1, \dots, 1)$, and the space of dimension $n - 1$ containing all vectors (x_1, x_2, \dots, x_n) with $\sum_{i=1}^n x_i = 0$. But in this particular example the second space splits further into two invariant subspaces.

Remark 2.4 Above, we claimed that there exist seven non-isomorphic hypermaps with the set of vertex-and-face degrees $(6^1 3^1 1^1, 2^3 1^4, 5^2)$. To prove this

sort of claim is not an easy task. First, we may draw the hypermaps in question: see Fig. 6. Then, in order to prove that their list is exhaustive, we may use Frobenius's formula, see Proposition 2.5.

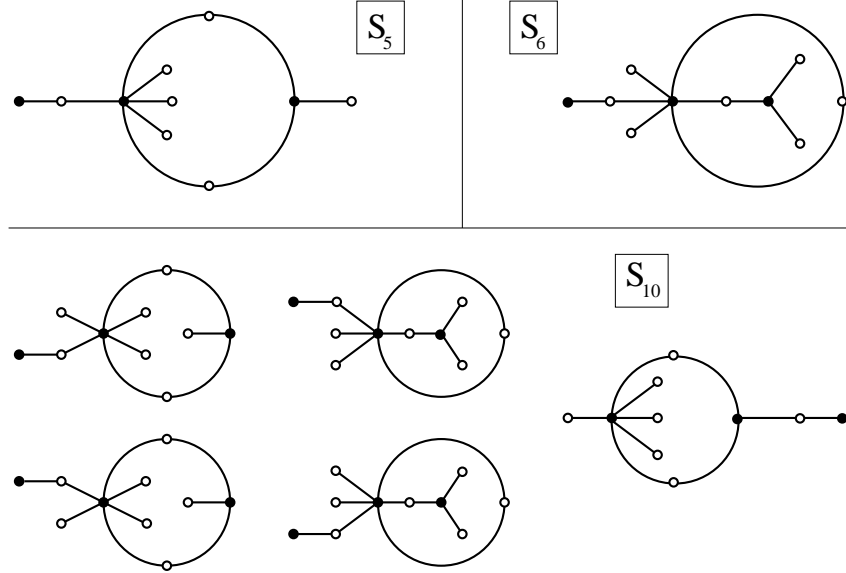


Figure 6: All the seven hypermaps with the set of vertex-and-face degrees equal to $(6^1 3^1 1^1, 2^3 1^4, 5^2)$. Cartographic groups are also indicated.

Proposition 2.5 (Frobenius's formula) *Let C_1, C_2, \dots, C_k be conjugacy classes in a finite group G . Then the number $\mathcal{N}(G; C_1, C_2, \dots, C_k)$ of k -tuples (x_1, x_2, \dots, x_k) of elements $x_i \in G$ such that each $x_i \in C_i$ and $x_1 x_2 \dots x_k = 1$, is equal to*

$$\mathcal{N}(G; C_1, C_2, \dots, C_k) = \frac{|C_1| \cdot |C_2| \cdot \dots \cdot |C_k|}{|G|} \cdot \sum_{\chi} \frac{\chi(C_1) \chi(C_2) \dots \chi(C_k)}{(\dim \chi)^{k-2}},$$

where the sum is taken over the set of all irreducible characters of the group G .

Applying this formula to the group $G = S_{10}$, $k = 3$, and the conjugacy classes C_1, C_2, C_3 determined by the cycle structures $6^1 3^1 1^1$, $2^2 1^4$, and 5^2 , respectively, and computing the irreducible characters of S_{10} using the Maple package `combinat`, we get

$$\mathcal{N}(G; C_1, C_2, C_3) = 25\,401\,600 = 7 \cdot 10!.$$

None of the maps shown in Fig. 6 has a non-trivial orientation-preserving automorphism; therefore, each of them admits $10!$ different labelings, and we are done.

But the situation is not always that simple. First, to find “by hand” all figures one is looking for may very soon become extremely difficult, and no reasonable algorithm, let alone software is known to do this job. Second, the presence of symmetric figures may obscure the counting: when the formula gives “seven” as the answer, it may as well mean “six asymmetric figures plus two symmetric ones having the symmetry of order 2”. And, third, Frobenius’s formula also counts k -tuples of permutations which generate non-transitive groups. Thus, the situation may easily become inextricable.

Theorem 2.6 (Riemann’s existence theorem, planar case) *For any plane hypermap M there exists a Belyi function $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $M = f^{-1}([0, 1])$. This function is unique up to an isomorphism, that is, up to a linear fractional transformation, of the preimage sphere.*

2.2 Greater genera

For greater genera the situation is more complicated. For the genus $g = 0$ there exists only one Riemann surface, namely, the Riemann sphere $\overline{\mathbb{C}}$. But for every fixed genus $g \geq 1$ there exist infinitely many Riemann surfaces of genus g . All of them are homeomorphic as topological surfaces, but they are not isomorphic as complex analytic manifolds. This is why we need a notion of a Belyi pair.

For what follows it is also important to note that the concept of the Riemann surface and that of the complex algebraic curve are equivalent: see, for example, [19] or [24].

Definition 2.7 (Belyi pair) A pair (X, f) , where X is a Riemann surface, and $f : X \rightarrow \overline{\mathbb{C}}$ is a meromorphic function, is called a *Belyi pair* if all critical values of f belong to $\{0, 1, \infty\}$.

Theorem 2.8 (Riemann’s existence theorem, nonplanar case) *For any hypermap M of genus $g \geq 1$ there exists a Belyi pair (X, f) such that $M = f^{-1}([0, 1])$. The correspondence between black vertices and f -preimages of 0, white vertices and f -preimages of 1, and so on, is the same as in Definition 2.1. The pair (X, f) is unique up to an automorphism of the surface X .*

Example 2.9 (Fermat curve) Let us consider the Fermat curve

$$\mathcal{F} = \{(x:y:z) \mid x^n + y^n = z^n\} \subset \mathbb{CP}^2$$

and its projection on the first coordinate

$$p : \mathcal{F} \rightarrow \overline{\mathbb{C}} : (x:y:z) \mapsto (x:z)$$

(the projection is not defined at the point $(0 : 1 : 0)$ but this point does not belong to the curve \mathcal{F}). In the affine part of $\overline{\mathbb{C}}$ (that is, for $z = 1$) the only critical values of p are the n th roots of unity since only for them the equation $x^n + y^n = 1 \Leftrightarrow y^n = 1 - x^n$, considered as an equation in y , has multiple

roots. It is easy to verify that $\infty = (1:0)$ is not a critical value either. Then, composing the projection p with the mapping $x \mapsto x^n$ we send all the above critical values to 1, and create two new critical values, 0 and ∞ . The function f thus obtained is a Belyi function on \mathcal{F} . The degree of f is n^2 since f is obtained as a composition of two functions of degree n each.

In this example it is interesting to take the f -preimage of the triangulation of the sphere $\overline{\mathbb{C}}$ consisting of three vertices positioned at 0, 1, and ∞ , of three edges $[0, 1]$, $[1, \infty]$, and $[\infty, 0]$, and of two triangles which are the upper and the lower half-spheres. It is clear that the preimage will have $3n^2$ edges and $2n^2$ triangular faces (since there is no critical values inside the faces or the edges), while there are $3n$ vertices of three different “colors”, namely, n preimages of 0, n preimages of 1, and n preimages of ∞ . The graph thus obtained is the complete tripartite graph $K_{n,n,n}$ and, since its embedding is a triangulation, it is also the one of the least genus, which can be easily calculated: $g = (n-1)(n-2)/2$. The fact that the least genus of the graph $K_{n,n,n}$ is equal to $(n-1)(n-2)/2$ was first proved in [34] and [30] in a purely combinatorial way; without doubt, the authors of both papers were not aware of the fact that their work was related to Fermat’s equation.

The preimage $f^{-1}([0, 1])$ provides an example of a regular embedding of the complete bipartite graph $K_{n,n}$. This example was a starting point for a series of beautiful and profound results concerning other *regular* embeddings of complete bipartite graphs, their classification, and the corresponding Belyi functions: see [15], [16], [18].

Unfortunately, the margins of this paper are too narrow in order to present the proof of Fermat’s Last Theorem which would be based on the above construction.

Not every Riemann surface can support a Belyi function, but the class of those that can is maybe the most interesting one.

Theorem 2.10 (Belyi theorem) *A Belyi function $f : X \rightarrow \overline{\mathbb{C}}$ exists if and only if the Riemann surface X is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. In this case the function f can also be represented over $\overline{\mathbb{Q}}$.*

Remark 2.11 In fact, Theorems 2.6 and 2.8 are just particular cases of Riemann’s existence theorem. More generally, if we fix k points $y_1, y_2, \dots, y_k \in \overline{\mathbb{C}}$ and k permutations $\sigma_1, \sigma_2, \dots, \sigma_k$ generating a transitive group on m points and such that $\sigma_1 \sigma_2 \dots \sigma_k = 1$, then there exists, and is unique up to an isomorphism, a pair (X, f) such that X is a Riemann surface and $f : X \rightarrow \overline{\mathbb{C}}$ is a meromorphic function of degree m having y_1, y_2, \dots, y_k as its critical values (or, in another terminology, its *ramification points*), and having $\sigma_1, \sigma_2, \dots, \sigma_k$ as its monodromy permutations (for the definition of the monodromy, see next section).

Making a linear fractional transformation of $\overline{\mathbb{C}}$ we can put any three critical values in any three prescribed positions—for example, in 0, 1, and ∞ , thus reducing the number of continuous parameters y_i from k to $k-3$. If now $k=3$

then, in order to represent the pair (X, f) , we don't need any more continuous parameters but only a triple of permutations.

The pair (X, f) is defined over $\overline{\mathbb{Q}}$ when all critical values of f belong to $\overline{\mathbb{Q}}$ (infinity is considered as being “defined over \mathbb{Q} ”). This is the so-called “obvious” part of the Belyi theorem, though it is obvious only for specialists in algebraic geometry; see in this respect [37], [21], and also Sec. 2.6 of [22]. The “difficult” part, which is, in fact, rather elementary, consists in taking all these critical values to $\{0, 1, \infty\}$ by successive compositions. We see, even at this stage, the fundamental role played by *composition* which is the main subject of this paper.

Taking a Belyi pair (X, f) (or just a Belyi function f when $X = \overline{\mathbb{C}}$) and replacing all the algebraic numbers involved in their definition by their conjugates under some automorphism of the field of algebraic numbers, we obtain another Belyi pair and thus another hypermap. Therefore, the Galois group $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, that is, the group of automorphisms of the field $\overline{\mathbb{Q}}$, acts on hypermaps. This fact hugely impressed Alexandre Grothendieck who wrote:

I do not believe that a mathematical fact has ever struck me so strongly as this one, nor had a comparable psychological impact.

(This and other remarks by Grothendieck concerning Belyi theorem, both mathematical and personal, can be found in [11], pages 252–253 and 280. It is also Grothendieck who called hypermaps embedded in Riemann surfaces via Belyi functions “*dessins d'enfants*”, which means “children’s drawings” in French.)

Example 2.12 (Galois orbit) In Fig. 7 we see two maps, each having six edges, or 12 germs of edges. The cartographic group of both of them is the Mathieu group M_{12} (a sporadic simple group of order 95 040; once again, for the definition of Mathieu groups see Chapter 6 of [8]). Computing the corresponding Belyi functions (see, for example, [39]) we find out that their coefficients belong to the field $\mathbb{Q}(\sqrt{-11})$ (explicit expressions are too cumbersome to be written here in full). Taking one of the values of the square root we get one map, taking the other one we get the other map.

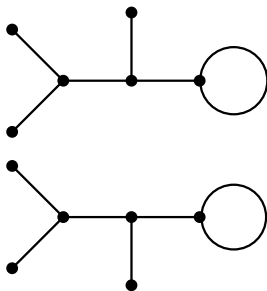


Figure 7: Two maps whose cartographic group is the Mathieu group M_{12} , and whose Belyi functions are defined over the field $\mathbb{Q}(\sqrt{-11})$.

Notice that the only irrational entries in the character table of M_{12} (see [5]) also belong to the field $\mathbb{Q}(\sqrt{-11})$: they are the values of certain characters on the two mutually inverse conjugacy classes with the cycle structure $(11, 1)$. This is, by the way, the cycle structure of the faces of our maps. This phenomenon, observed in many examples, is based on some general results on the inverse problem of Galois theory (see, for example, Chapter 8 of [32]), but no effective version of it is known which would be helpful in computation of Belyi functions.

As another example we may consider the family of seven dessins shown in Fig. 6. This family splits into three Galois orbits: the dessins with the cartographic groups S_5 and S_6 are defined over \mathbb{Q} and thus form two separate orbits, while the five dessins with the cartographic group S_{10} constitute a single orbit defined over an extension of degree 5 of \mathbb{Q} .

The examples given up to now show the important and sometimes very subtle role played by the cartographic group.

One of the most interesting parts of the theory of dessins d'enfants is the search for combinatorial invariants of the Galois action on hypermaps. Among these invariants let us mention the following (the list is incomplete):

- the cartographic group;
- the automorphism group (which is nothing else but the centralizer of the cartographic group in S_m);
- the set of conjugacy classes of σ , α , and φ , as in Example 1.11, or the set of vertex-and-face degrees as a loose analogue of the above;
- last but not least, invariants related to the functional composition.

For the proofs, see [17]. Some interesting Galois invariants related to composition are considered in [38]. In the next section we will see that the composition itself is also related to the cartographic group.

3 Coverings and Ritt's theorem

A meromorphic function is a ramified covering of the Riemann sphere $\overline{\mathbb{C}}$ by a Riemann surface X . Ramified coverings are usually constructed as compactifications of *unramified* coverings. Below we repeat some well-known facts about unramified coverings.

3.1 Unramified coverings and fundamental groups

Let X and Y be two path-connected topological spaces and $f : X \rightarrow Y$ a continuous mapping.

Definition 3.1 (Unramified covering) The mapping $f : X \rightarrow Y$ is an *unramified covering* of Y by X if it is locally homeomorphic and if every point $y \in Y$ has a neighborhood U such that the preimage $f^{-1}(U) \subset X$ is homeomorphic to a product $U \times D$ where D is a discrete set. We will consider only *finite coverings*, when $|D| = m < \infty$; then m is called the *degree* of f and denoted by $\deg f$. Fixing a base point $y_0 \in Y$ we may take $D = f^{-1}(y_0)$.

Let us take as a base point in X an arbitrary element $x_0 \in D$; therefore $f : x_0 \mapsto y_0$. It is clear that a loop in X attached to x_0 is taken to a loop in Y attached to y_0 , and that the product of loops in X is taken to a product of loops in Y . Thus, we have a homomorphism of fundamental groups

$$\phi : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

Lemma 3.2 (Lemma of covering homotopy) *The homomorphism ϕ is injective. Therefore, to any unramified covering $f : X \rightarrow Y$ there corresponds a subgroup of $\pi_1(Y, y_0)$ which is the image of $\pi_1(X, x_0)$ under ϕ and which is isomorphic to $\pi_1(X, x_0)$.*

A proof of the lemma can be found in virtually every topology textbook.

Construction 3.3 (Monodromy) Denote for simplicity $\pi_1(X, x_0) = P$ and $\pi_1(Y, y_0) = Q$. In a slightly loose way, we may consider P to be a subgroup of Q ; the inclusion of groups is denoted as $P \leq Q$. A loop $\gamma \in Q$ starts at y_0 and finishes also at y_0 . Therefore, its preimage $f^{-1}(\gamma)$ starts at $x_i \in D$ and finishes at $x_j \in D$. Thus, the group Q acts on D ; this action is transitive since X is path-connected. It is clear that P is the stabiliser of $x_0 \in D$. The permutation group F created by the above action is called the *monodromy group* of the covering. Since $|D| = m$ we may consider F as a subgroup of S_m ; this subgroup is the image of Q under the group homomorphism $Q \rightarrow S_m$. It is also clear that the set D is in a bijection with the right cosets of P in Q (hence m is the index of P in Q), and that the action itself is equivalent to the action of Q on the right cosets of P by right multiplication.

Now, in the opposite direction, to any subgroup $P \leq Q = \pi_1(Y, y_0)$ of a finite index m there corresponds a covering of Y of degree m . It is constructed as follows. Take the set \mathcal{X} of pairs (y, γ) where $y \in Y$ and γ is an oriented path from y_0 to y . Then X is obtained as a quotient of \mathcal{X} by the following equivalence relation: two pairs (y_1, γ_1) and (y_2, γ_2) are equivalent if $y_1 = y_2$ and $\gamma_1 \gamma_2^{-1} \in P$. The covering function f associates to a pair $x = (y, \gamma)$ its end-point y . We leave it to the reader to verify that all the conditions of the monodromy construction are satisfied.

We conclude that there is a correspondence between the coverings of Y and the subgroups (in fact, conjugacy classes of subgroups—see below) of its fundamental group, which works in both directions.

“Up to...” What if we change certain elements of our construction?

1. Replacing the base point $y_0 \in Y$ with another point $y_1 \in Y$ we obtain a group $\pi_1(Y, y_1)$ which is isomorphic to $\pi_1(Y, y_0)$. This isomorphism is *not canonical*; however, two such isomorphisms are obtained from one another by an *inner* automorphism of one of the groups. In other words, one may say that a free loop which is not attached to a specific base point represents not an element of the fundamental group but an entire conjugacy class.
2. Replacing the point $x_0 \in D = f^{-1}(y_0)$ with another point $x_1 \in D$ we get a subgroup $P' \leq \pi_1(Y, y_0)$ conjugate to P . (This is a general algebraic fact: the stabilizers of two points in a transitive action are conjugate.)
3. Two coverings $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ are called *isomorphic* if there exists a homeomorphism $u : X_1 \rightarrow X_2$ such that the following diagram

$$\begin{array}{ccc}
 X_1 & \xrightarrow{u} & X_2 \\
 & \searrow f_1 & \swarrow f_2 \\
 & & Y
 \end{array}$$

is commutative. Then two coverings are isomorphic if and only if the corresponding subgroups $P_1, P_2 \leq \pi_1(Y, y_0)$ are conjugate.

Relation to dessins d’enfants. Let us take as Y the Riemann sphere punctured at k points: $Y = \overline{\mathbb{C}} \setminus \{y_1, \dots, y_k\}$. Then Construction 3.3 gives us, as a covering space X , a Riemann surface punctured at the set $f^{-1}(\{y_1, \dots, y_k\})$. Take the natural generators $\gamma_1, \dots, \gamma_k$ of $\pi_1(Y, y_0)$, namely, the loops from y_0 going around the punctures y_1, \dots, y_k . Then their images under the group homomorphism $\pi_1(Y, y_0) \rightarrow S_m$ are k permutations $\sigma_1, \dots, \sigma_k \in S_m$. Since the product $\gamma_1 \dots \gamma_k$ is a loop going around all the points y_1, \dots, y_k and is therefore contractible on the sphere, we get $\gamma_1 \dots \gamma_k = 1$ and therefore $\sigma_1 \dots \sigma_k = 1$. Since $\gamma_1 \dots \gamma_k$ generate $\pi_1(Y, y_0)$, the permutations $\sigma_1, \dots, \sigma_k$ generate the monodromy group $F \leq S_m$.

In order to construct a *ramified* covering out of unramified one we must compactify both X and Y . For $Y = \overline{\mathbb{C}} \setminus \{y_1, \dots, y_k\}$ it is easy: we just put back the missing points. For X it needs some standard technique of introducing the complex structure in the neighborhoods of the added points; we won’t speak about it here. The points $y_1, \dots, y_k \in \overline{\mathbb{C}}$ are now *ramification points*, or *critical values* of the covering. We still denote the compactified surfaces by X and Y .

If $k = 3$ and the three ramification points are chosen to be $0, 1$, and ∞ , then instead of the notation $\sigma_1, \sigma_2, \sigma_3$ we use σ, α, φ . These three permutations used to act on the preimages of the segment $[0, 1]$; however, taking the base point $y_0 \in [0, 1]$ we may consider them acting on the preimages of y_0 . Thus, a *dessin d’enfant* is a covering of the Riemann sphere unramified outside $\{0, 1, \infty\}$, and the *cartographic group* is a particular case of the monodromy group.

Remark 3.4 (Origamis) Different topological spaces may have isomorphic fundamental groups. The fundamental group of the thrice-punctured sphere is the free group F_2 of rank 2. The fundamental group of the once-punctured torus is the same. Thus, we obtain two entirely parallel theories: the theory of dessins d'enfants, and the theory of coverings of the once-punctured torus. The latter one is called *theory of origamis*. It is difficult to choose a particular reference since they are numerous; as an example take [13]. Note that while the sphere has only one complex structure, the torus may have infinitely many of them; the corresponding Riemann surfaces are called elliptic curves.

3.2 Ritt's theorem

We are now ready to prove the main theorem of our paper. It characterizes the coverings which can be decomposed in a successive application of two (or more) coverings.

Definition 3.5 (Imprimitive permutation group) A permutation group H acting on a set D is called *imprimitive* if D can be subdivided into nontrivial disjoint blocks D_1, \dots, D_n of equal size (nontrivial means $1 < |D_i| < |D|$) such that the image of a block under the action of any element of H is once again a block. In other words, H respects a nontrivial equivalence relation on D .

Theorem 3.6 (Ritt's theorem) *An unramified covering $h : X \rightarrow Z$ of degree $\deg h = mn$ is a composition of two coverings,*

$$h : X \xrightarrow{f} Y \xrightarrow{g} Z, \quad \deg f = m, \quad \deg g = n, \quad m, n > 1, \quad (3)$$

if and only if the monodromy group H of h is imprimitive and has n blocks of size m .

Proof. Suppose that the decomposition (3) takes place, and choose the base points $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$ such that $x_0 \xrightarrow{f} y_0 \xrightarrow{g} z_0$. Then there is an inclusion of fundamental groups $P < Q < R$ where $P = \pi_1(X, x_0)$, $Q = \pi_1(Y, y_0)$, and $R = \pi_1(Z, z_0)$. The action of R on the right cosets of P is imprimitive, the blocks being the right cosets of Q .

Now, in the opposite direction, let an action of R on the right cosets of P be imprimitive. Recall that in the action of R on the set $D = h^{-1}(z_0)$ the subgroup P acts as the stabilizer of x_0 . Then take Q as the stabilizer of the block B containing x_0 . The group P is a proper subgroup of Q since Q permutes the points inside B while P leaves x_0 fixed. (In fact, Q acts transitively on B .) Also, Q is a proper subgroup of R since R is transitive on D while Q leaves B fixed. The existence of this intermediate subgroup permits us to construct two coverings $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ along the lines of the second part of Construction 3.3. \square

Remark 3.7 (Context) Ritt’s theorem is general and fundamental in studying compositions. As we have seen, it is also very simple, on the level of an undergraduate student exercise. However, it has had an unlucky history. In topology books they usually explain the relations between coverings and subgroups of fundamental groups; but they take no interest in compositions. Group theorists know pretty well that in order to obtain a *primitive* action of a group one must consider its action on the cosets of a *maximal subgroup* (Corollary 1.5A of [8]). However, they are not interested in compositions either. Or, it would be better to say that they did study compositions in the form of imprimitive permutation groups, some of whose aspects are highly non-trivial. But still the function-theoretic nature of the question in this approach was not explicit. Finally, in complex function theory they do take interest in compositions but usually prove the corresponding statement only for some particular cases. Ritt himself first proved the theorem for polynomials (see [31]), and ten years later used it for rational functions. In the book [22] the theorem is proved for two-dimensional surfaces. Also, many proofs include some unnecessary elements related to the Reidemeister–Schreier construction whose only effect is to make the proof longer and more obscure. In fact, I have never seen this theorem being stated and proved in all its generality and simplicity, though many specialists certainly “know” the above proof.

Construction 3.8 (Checking imprimitivity) There exists a simple algorithm that checks the imprimitivity of a group H and calculates the blocks ([2]; we follow [8]). Take $x_0, x_1 \in D$ and construct a digraph whose edges constitute the orbit of (x_0, x_1) under the action of H . (In constructing the orbit it suffices to act only by generators of H). Then forget the orientation of the graph and calculate its connected component containing x_0 . This component is the minimal “block” which contains both x_0 and x_1 , though the “block” may turn out to be trivial, that is, it may coincide with the whole set D . Repeat this procedure for all $x_1 \neq x_0$. If every time you get the whole set D then the action is primitive; otherwise, you will find a nontrivial block.

The complexity of the calculation of connected components of a graph is proportional to the number of edges, and the number of edges is $O(|D|^2)$. Hence, the complexity of the algorithm is $O(|D|^3)$.

4 Various manifestations of composition

We return here to the world of dessins d’enfants, hypermaps, and Belyi functions and enumerate a number of phenomena related to compositions.

4.1 Symmetry

Symmetric hypermaps are almost inevitably imprimitive, the blocks being the orbits of the action of the symmetry group. However, if a hypermap is *regular*, that is, if its symmetry group acts transitively on the set of germs of edges,

then this procedure leads to a single trivial block. In this case we can take as blocks the orbits of a subgroup of the symmetry group. Therefore, the only exception to the rule is when the symmetry group does not have nontrivial subgroups. This is the case, for example, for the polynomial x^p with p prime; the corresponding dessin is the star-tree with its center at 0.

From the covering point of view, returning to the notation $P = \pi_1(X, x_0)$, $R = \pi_1(Z, z_0)$, we say that a covering $h : X \rightarrow Z$ is regular if P is a normal subgroup of R . In this case, the symmetry group of the dessin drawn on X is the factor R/P . If P is not normal we take its normalizer $Q = N_R(P)$ in R , that is, the biggest subgroup of R in which P is a normal subgroup. Then, the symmetry group is the factor Q/P . Thus, if Q does not coincide either with P or with R , this procedure provides us with an intermediate subgroup between P and R and therefore with a composition of coverings.

(In the general setting of coverings outside the world of dessins d'enfants we must specify, the symmetry of what objects we consider. These objects are preimages in X of an "elementary" object in Z which should not have any symmetry itself.)

Computing Belyi functions is a very challenging task. In certain cases they may be computed for an infinite series of dessins depending on a few parameters, like star trees, chain trees, etc. But for generic dessins the frontier of feasibility lies somewhere between the degrees 10 and 20; I know of only two computations of functions of degree 23 and 24, both using very sophisticated techniques based on LLL algorithm. However, in [23] we were able to compute Belyi functions for all the maps corresponding to the Archimedean solids, the degrees of the functions in question going as far as 360. It goes without saying that these computations were feasible only because of the symmetry properties of the corresponding maps; thus, these complicated functions were found as compositions of simpler ones.

Belyi functions for Platonic solids were in fact computed by Felix Klein in 1875; see [20]. Very recently, it was discovered in [25] that some of these functions admit significantly different decompositions in functions of smaller degrees—a phenomenon that Ritt [31] proved to be impossible for polynomials (with two notable exceptions: polynomials x^n , and Chebyshev polynomials of the first kind T_n : for them non-equivalent decompositions do exist). I find it incredible that in the 21st century it is still possible to say something new about Platonic solids.

Symmetry is a thoroughly studied subject, so we won't say any more on it. We only note that composition is a much wider phenomenon.

4.2 Operations with maps

Many standard operations with maps can very easily be expressed in terms of Belyi functions. For example, if M is a map and f is its Belyi function then for the map M^* dual to M the Belyi function is $1/f$. Indeed, $1/f$ exchanges 0 and ∞ while leaving 1 fixed; therefore, vertices become face centers and vice versa.

Example 4.1 (A map together with its dual) If we want to draw the original map and its dual on the same picture we should find a function taking both 0 and ∞ to 0 while leaving 1 fixed. It is easy to verify that the function which does the job is $g(y) = 4y/(y + 1)^2$. Thus, the Belyi function for $H = M \cup M^*$ is $h = g \circ f$. Note that the result of the operation is a hypermap even if the original figure M was a map: white vertices of M were all of degree 2 but white vertices of $H = M \cup M^*$ are of degree 4. All the faces of $M \cup M^*$ are surrounded by four germs of edges and are therefore of degree 2.

In fact, in order to make an operation with a dessin drawn on the surface X , we must first make the same operation with the elementary hypermap $[0, 1]$ on the sphere Y , and then send this newly obtained dessin on Y to the segment $[0, 1]$ on Z .

Example 4.2 (Truncation) When applied to a polyhedron, truncation means “cutting vertices”. For an arbitrary hypermap, it means replacing each black vertex by a face of the same degree, and inserting a new black vertex of degree 3 inside each germ of an edge.

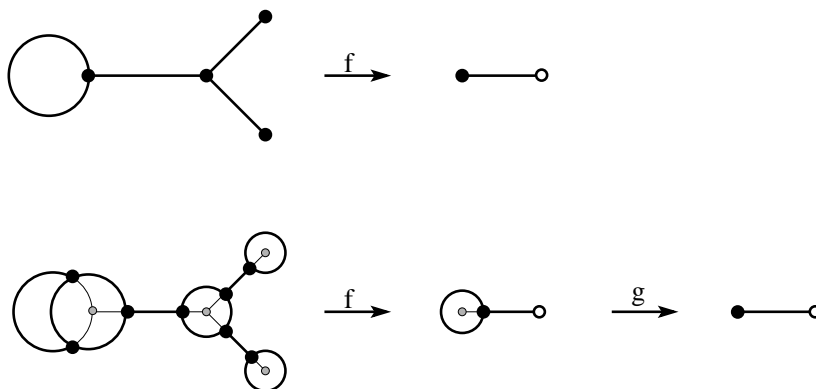


Figure 8: Truncation of a map. In order to simplify the picture we do not show white vertices of degree 2: one should imagine them inside each edge whose both ends are black. White vertices of degree 1 are shown explicitly.

Let us consider a simple map with four edges, thus eight germs of edges, shown in Fig. 8 in upper left corner. (By the way, this map is interesting in itself: its cartographic group is $\text{PSL}_3(2)$.) Its Belyi function can easily be computed:

$$f(x) = -\frac{1}{1728} \frac{(x^2 - 5x + 1)^3 (x^2 - 13x + 49)}{x}$$

while

$$f(x) - 1 = -\frac{1}{1728} \frac{(x^4 - 14x^3 + 63x^2 - 70x - 7)^2}{x}.$$

Below in the figure, the truncation of this map is shown. The same Belyi function takes it to the truncation of the elementary hypermap $[0, 1]$. The latter is a hypermap of degree 3: it has one simple edge consisting of two germs (the “circle”), and the third germ is attached to it (the “tail”). The Belyi function g which takes this hypermap to $[0, 1]$ is

$$g(y) = \frac{(4y - 1)^3}{27y}, \quad g(y) - 1 = \frac{(8y + 1)^2 (y - 1)}{27y}.$$

The composition $h = g \circ f$ is the Belyi function for the truncated map.

Example 4.3 (More operations) Practically all reasonable cartographic operations can be represented as compositions. By “reasonable” and “cartographic” we mean the following informal condition: if a transformation is made with faces (for example), then it must act in the same way on *all* faces and not on a particular subset of them; the same condition should also be valid for vertices.

1. Transforming a hypermap into a map. We would like to color every white vertex in black, thus transforming every germ of an edge into an edge (and inserting inside it a new white vertex of degree 2). For that end, we only have to compose the Belyi function of the hypermap with the function $g(y) = 4y(1 - y)$.
2. Doubling edges. We want to replace every edge of a map by a pair of two parallel edges. The function which makes this operation is $g(y) = y^2/4(y - 1)$.
3. Medial map. Each white vertex becomes a black one; we join them successively inside each face (thus, all new vertices are of degree 4); former black vertices become faces centers. The operation is made by the function $g(y) = -(y - 1)^2/4y$.
4. Subdivision of edges. We want to subdivide every edge of a map in k parts by inserting in it $k - 1$ new black vertices; taking the ends into account, an edge will contain $k + 1$ black vertices. Chebyshev polynomials of the first kind T_n are almost Belyi functions: they have only three critical values but these values, instead of being 0, 1, and ∞ are ± 1 and ∞ . The preimage of the segment $[-1, 1]$ is a chain tree consisting of n germs of edges going also from -1 to 1, with preimages of 1 and -1 going in turns. Therefore, we need a little adjustment of coordinates: we first take $y = 2f - 1$ (here f is a Belyi function of the original map) and apply to it the polynomial $(1 - T_{2k}(y))/2$.
5. Triangulation of faces. Insert a new vertex inside each face of a map and connect it by edges with all the vertices adjacent with this face; thus, a face of degree k is subdivided into k triangles. For the elementary hypermap $[0, 1] \subset \overline{\mathbb{C}}$ this operation means inserting a new vertex in the

point ∞ and connecting it with 0 by an edge going along the negative part of the real axis. The Belyi function for the resulting hypermap is $g(y) = -27y^2/(y-4)^3$.

Inclusions between triangle groups classified by D. Singerman [33] give rise to interesting compositions of Belyi functions. Other examples may be found in [14] and [36]. Our list is in no way exhaustive.

4.3 Hidden symmetries

If a map is symmetric, or if it is obtained by one of the operations described above, this fact can easily be observed by the naked eye. However, the world of compositions is much vaster than a toolbox of simple operations; more often than not the decomposability of a map or of a hypermap does not jump to the eye. I like to call this property of a hypermap a “hidden symmetry”, a terminology borrowed from theoretical physics.

Example 4.4 (A non-obvious composition) This example was briefly mentioned elsewhere but without details. Consider a simple map with six edges shown in Fig. 9. It is decomposable, and I believe that this fact is not evident.

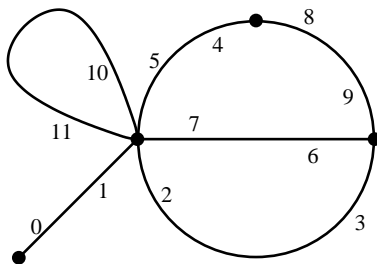


Figure 9: A decomposable map.

The only thing we need in order to be convinced that this is indeed true is to find a system of blocks. Four blocks, denoted a, b, c, d , are shown in Table 1.

a	b	c	d
0	1	2	3
4	5	6	7
8	9	10	11

Table 1: Blocks of imprimitivity of the cartographic group of the map of Fig. 9.

The permutations σ, α, φ being reduced to their action on blocks look as follows:

$$\bar{\sigma} = (b, c, d), \quad \bar{\alpha} = (a, b)(c, d), \quad \bar{\varphi} = (a, d, b).$$

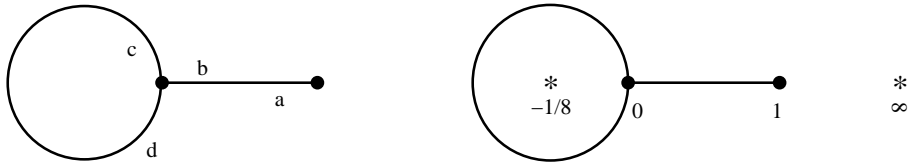


Figure 10: A map whose germs are blocks of the map of Fig. 9.

We can now construct a map encoded by these permutations. This map is shown twice in Fig. 10: to the left, the labels a, b, c, d are written near the corresponding germs of edges; to the right, the coordinates of the vertices and face centers are given: they correspond to the Belyi function

$$g(y) = -\frac{64y^3(y-1)}{8y+1}, \quad g(y) - 1 = -\frac{(8y^2 - 4y - 1)^2}{8y+1}.$$

In order for the function $h : X \xrightarrow{f} Y \xrightarrow{g} Z$ to be a Belyi function, the critical values of f must be taken to critical points of g (which, in its turn, takes them to $\{0, 1, \infty\}$). The first surprise is that f itself is not a Belyi function: it has four critical values. These values correspond to the cycles of $\bar{\sigma}, \bar{\alpha}, \bar{\varphi}$ which become longer while being lifted to σ, α, φ . For example, a fixed point c in $\bar{\sigma}$ is lifted to a fixed point 0 and a cycle $(4, 8)$ in σ . Therefore, the vertex of degree 1 of the map of Fig. 10 is a critical value of f , with its preimages being one generic point (the vertex of degree 1) and one critical point of multiplicity 2 (the vertex of degree 2). Pursuing these considerations we find out that both vertices and both face centers of this map are critical values of f . Looking at their positions we conclude that the critical values of f are $0, 1, -1/8$, and ∞ (in their preimages one point is always generic and the other one is critical, of multiplicity 2).

The second surprise: there are not one but four such functions f , and they form a Galois orbit. All of them have the form

$$f(x) = K \cdot \frac{(x-1)^2(x-a)}{x}$$

where a is one of the roots of the polynomial

$$a^4 - 138a^3 + 678a^2 + 26a + 162, \quad (4)$$

while K is expressed in terms of a as

$$K = -\frac{1}{20412} (13a^3 - 1779a^2 + 9508a - 3564).$$

We conclude that the map of Fig. 9 is a member of a Galois orbit containing four elements and is defined over the splitting field of the polynomial (4). The three remaining members of the orbit are shown in Fig. 11.

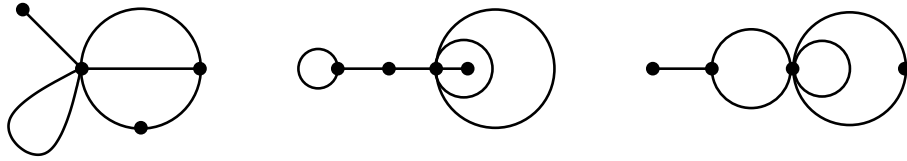


Figure 11: Galois orbit containing the map of Fig. 9 contains also these three maps.

There are, in total, 18 maps with six edges having the set of vertex-and-face degrees $(6^1 3^1 2^1 1^1, 2^6, 6^1 3^1 2^1 1^1)$. They split into three Galois orbits, of sizes 8, 6, and 4, respectively. The orbit of size 4, having an imprimitive cartographic group, has been constructed above. For the orbit of size 8, the cartographic group is the Mathieu group M_{12} , and for the orbit of size 6 the group is A_{12} .

4.4 Belyi pairs

Computing Belyi pairs is an incredibly challenging problem. A recent calculation of them for the maps of genus 1 up to four edges [1] should be considered as a real accomplishment. Such an enterprise is fraught both with fundamental and technical difficulties.

The main problem is to find a Riemann surface which corresponds to a particular dessin. It turns out that for the surfaces of smaller genera there exist canonical representations: for example, Riemann surfaces of genus 1 can be represented as elliptic curves. But for higher genera such representations do not any longer exist. Thus, for the dessins of higher genera we are not only unable to solve the corresponding systems of algebraic equations: we are unable to write them down in the first place! This is the first, and fundamental difficulty.

Then we may start from the opposite end: take a pair (X, p) where X is an algebraic curve and p is an arbitrary (not necessarily Belyi) meromorphic function on X , but both defined over $\overline{\mathbb{Q}}$. After that compose p with a Belyi function f which would take all critical values of p to $\{0, 1, \infty\}$. This is exactly what we did in Example 2.9 for the Fermat curve. However, we were successful only because the critical values of p were very simple. In the generic case the degree of f becomes too big to be manageable.

The royal way consists of first preparing a Belyi function f , and only after that looking for a pair (X, p) such that the critical values of p are among the critical points of f . Let us return to the function of Example 2.3. Looking at formulas (1) and (2) or, more exactly, at the critical points of f (that is, roots of f and of $f - 1$) we may take, for example, an elliptic curve

$$y^2 = x(7x - 1)(x^2 + 4x - 1). \quad (5)$$

The projection $p : (x, y) \mapsto x$ is ramified over four points: $x = 0$ (a black vertex of degree 6), $x = 1/7$ (a white vertex of degree 1), and two roots of $x^2 + 4x - 1$

(face centers). Thus, a composition of p with f gives a Belyi function and a dessin on the curve (5).

The majority of known examples of dessins of genus $g \geq 1$ follow this way and are therefore constructed using a composition. There are, however, examples of a different nature.

Example 4.5 (Bring's curve) In 1786, Erland Bring found the change of variables which reduces an equation of fifth degree to the form

$$p(x) = x^5 + ax + b = 0. \quad (6)$$

Felix Klein [20], pursuing the study of the equations of fifth degree, was naturally led to consider the algebraic curve \mathcal{B} in the projective space \mathbb{CP}^4 with coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$ described by the following three equations:

$$\sum_{i=1}^5 x_i = 0, \quad \sum_{i=1}^5 x_i^2 = 0, \quad \sum_{i=1}^5 x_i^3 = 0. \quad (7)$$

Indeed, the elementary symmetric functions of degrees 1, 2, 3 of the roots of $p(x)$ vanish; therefore, their power sums also vanish. Klein has shown that the curve \mathcal{B} , commonly called *Bring's curve*, is of genus 4. A set of five roots of equation (6) usually gives rise to 120 points on \mathcal{B} which are different permutations of these roots. For some particular values of parameters a and b the number of such points may become smaller.

The coefficients a and b themselves are symmetric polynomials in x_i of degree 4 and 5 respectively. Since we would like to consider $(x_1 : x_2 : x_3 : x_4 : x_5)$ as projective points, the pairs (a, b) must be considered up to the equivalence relation $(a, b) \sim (\lambda^4 a, \lambda^5 b)$ for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$. For the future use, note that the discriminant of p is

$$\Delta(p) = 256 a^5 + 3125 b^4.$$

There are numerous publications dedicated to Bring's curve; the reader may consult, for example, a survey paper [35]. It turns out that the icosahedron I_4 of genus 4 discussed in Example 1.9 lives on this surface. The automorphism group of \mathcal{B} is S_5 (it acts by permutations of coordinates) while the automorphism group of I_4 is A_5 . The additional symmetry is due to the self-duality of I_4 . It is more convenient to compute a Belyi function for the hypermap $H = I_4 \cup I_4^*$ where I_4^* is dual to I_4 , like in Example 4.1.

We claim that the Belyi function of $H = I_4 \cup I_4^*$ is

$$f(x_1 : x_2 : x_3 : x_4 : x_5) = \frac{256 a^5}{256 a^5 + 3125 b^4}.$$

Proof. The numerator of f is $256 a^5$. The equation $a = 0$ of degree 4, being added to system (7), gives rise to a system of degree 24. In this way, we get 24 points on \mathcal{B} : they are the black vertices of H (i. e., vertices and face centers

of I_4). All of them are roots of f of multiplicity 5. From the point of view of equation (6), $a = 0$ means that (6) becomes equivalent to $x^5 - 1 = 0$. The 5th roots of unity may be permuted in 120 ways, but the sequences obtained from each other by cyclic permutations are proportional and therefore lead to the same projective point.

The numerator of $f - 1$ is $-3125b^4$. The equation $b = 0$ of degree 5, being added to system (7), gives rise to a system of degree 30. In this way, we get 30 points on \mathcal{B} : they are the white vertices of H , and all of them are roots of $f - 1$ of multiplicity 4 (cf. Example 4.1). From the point of view of equation (6), $b = 0$ means that (6) becomes equivalent to $x^5 - x = 0$. Its five roots are 0 and the 4th roots of unity. There are 5 ways to choose which one of x_i will be equal to 0, and then permute the four remaining roots in 24 different ways, but the sequences obtained by cyclic permutations of order 4 are proportional to each other, and we get $5 \cdot (24/4) = 30$ distinct projective points.

The denominator of f is the discriminant $\Delta(p)$. It is easy to verify that if $\Delta(p) = 0$ then p has exactly one double root. Therefore, the 120 permutations of the sequence of roots of p give us only 60 distinct points on \mathcal{B} , all of them being of multiplicity 2. They correspond to the 60 faces of degree 2 in H (cf. once again Example 4.1).

We should also verify that for all other values of f , different from 0, 1, and ∞ , we get 120 distinct points on \mathcal{B} , but this verification is rather tedious and we omit it. The fact that the automorphism group of the hypermap H thus obtained is S_5 is obvious. \square

Felix Klein in his book [20] (Part II, Chapters II and III) uses another but also very beautiful construction. He subdivides each pentagonal face of I_4 in five triangles by inserting a new vertex of degree 5 inside each face and connecting it with the five vertices incident to the face (cf. 5th operation of Example 4.3). In this way he obtains a map T which has 12 vertices of degree 10 (former vertices of I_4) and 12 vertices of degree 5 (former face centers of I_4), 60 triangular faces, and 90 edges. It turns out that T is a covering of degree 3 of I_0 , the 12 ramification points being the vertices of I_0 ; the multiplicities of the points above each ramification point are $2^1 1^1$. The best way to see this covering is to turn once again to Fig. 2: it is nothing else but the projection of the surface underlying I_4 from the origin onto the sphere. Then, if we want to get a Belyi function for T , what remains is to further apply the Belyi function for I_0 which is given by the following formulas:

$$f(x) = 1728 \cdot \frac{(x^{10} - 11x^5 - 1)^5 x^5}{(x^{20} + 228x^{15} + 494x^{10} - 228x^5 + 1)^3},$$

$$f(x) - 1 = -\frac{(x^{10} + 1)^2 (x^{20} - 522x^{15} - 10006x^{10} + 522x^5 + 1)^2}{(x^{20} + 228x^{15} + 494x^{10} - 228x^5 + 1)^3}.$$

A representation of the above covering of I_0 by the triangulation of I_4 as a concrete function $h : \mathcal{B} \rightarrow \overline{\mathbb{C}}$ may be found in Klein's book. The corresponding computations are rather long, and we omit them.

Example 4.6 (Bring's cousins) Fig. 12 shows the family of five maps with the set of vertex-and-face degrees $(5^2 1^2, 2^5, 5^2 1^2)$.

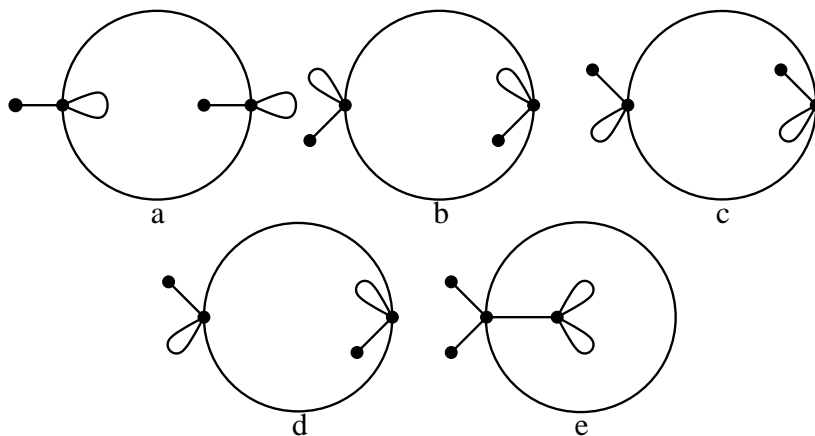


Figure 12: Maps with the set of vertex-and-face degrees $(5^2 1^2, 2^5, 5^2 1^2)$.

The cartographic group of the map a is isomorphic to A_5 (acting on 12 cosets of a cyclic subgroup C_5). An exercise to the reader is to establish that this map is the reduction of I_4 by the rotational symmetry of order 5. For the maps b and c the cartographic group is $(S_2 \wr A_6) \cap A_{12}$, of size 11 520 (the symbol “ \wr ” means wreath product); for the maps d and e the group is $\text{PSL}_2(11)$, of size 660. Accordingly, the family splits into three Galois orbits: $\{a\}$, $\{b, c\}$, and $\{d, e\}$. For the map a , which is defined over \mathbb{Q} , the Belyi function is as follows:

$$f(x) = -\frac{1}{64} \frac{(x-1)^5 (x+1)^5 (x^2 - 4x - 1)}{x^5 (x^2 + x - 1)}, \quad (8)$$

while

$$f(x) - 1 = -\frac{1}{64} \frac{(x^4 - 2x^3 - 6x^2 + 2x + 1)^2 (x^2 + 1)^2}{x^5 (x^2 + x - 1)}.$$

For the other cases the formulas are more cumbersome, and we omit them. The orbit $\{b, c\}$ is defined over the field $\mathbb{Q}(\sqrt{-15})$, and $\{d, e\}$ is defined over the field $\mathbb{Q}(\sqrt{5})$.

All the five maps are self-dual. Also, the first three, a , b , and c , are symmetric, with the automorphism group C_2 . Therefore, if we draw on the same picture the map itself together with its dual, then the automorphism group of the resulting hypermap must be an extension of order 2 of C_2 . A funny feature is that in the case of the map a we get in this way the cyclic group C_4 while in the case of b and c we get the Kleinian group V_4 .

The map d is famous, see [9]. The number field corresponding to the orbit $\{d, e\}$ must be quadratic since the orbit contains two elements; it must be *real*

quadratic since the map e is axially symmetric and therefore the complex conjugation does not send it to the map d ; but a Belyi function for d cannot be realized over a real number field since d cannot be made axially symmetric.³

In order to construct a Belyi function for d we need to further extend the field $\mathbb{Q}(\sqrt{5})$ by an imaginary quadratic irrationality; the field computed to that effect in [9] is $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$. Then replacing $\sqrt{-2}$ by $-\sqrt{-2}$ will take the map d to its mirror image d' which is isomorphic to d . In fact, the map e cannot be realized over $\mathbb{Q}(\sqrt{5})$ either since a mere replacement of $\sqrt{5}$ by $-\sqrt{5}$ can only produce a mirror symmetric map; in order to get d we must choose one of the values of the square root of -2 . Thus, the geometric figure for e obtained using a Belyi function will inevitably be bent; replacing $\sqrt{-2}$ by $-\sqrt{-2}$ we get another bent figure which is the mirror image of the previous one, and the two maps are combinatorially isomorphic to each other.

The fields $\mathbb{Q}(\sqrt{-15})$ and $\mathbb{Q}(\sqrt{5})$ are objective characteristics of the maps in question and cannot be replaced by other number fields while it is not clear if the same is true for the extension by $\sqrt{-2}$.

But all that is not our point in this example.

Let us consider coverings of degree 5 of the complex sphere ramified over four points which are two vertices of degree 1 and two face centers of degree 1 of the above maps. We take all the four permutations generating the monodromy group of these coverings to be cycles of order 5. Thus, the coverings in question are classified by the sequences of permutations (g_1, g_2, g_3, g_4) such that all g_i are cycles of order 5 and the product $g_1 g_2 g_3 g_4 = 1$. Computations with characters of the group S_5 show that there are, in total, 60 non-isomorphic sequences like that, 47 of them being asymmetric and the remaining 13 having a cyclic automorphism group C_5 .

What will be the resulting surfaces, and what will happen with the maps a, b, c, d, e after being lifted on them? It is easy to see that:

- two vertices of degree 1 become two vertices of degree 5;
- two vertices of degree 5 remain unramified and therefore repeated five times on the covering surface, thus creating 10 vertices of degree 5;
- two faces of degree 1 become two faces of degree 5;
- similar to the vertices, two faces of degree 5 on the sphere give rise to 10 faces of degree 5 on the covering surface;
- finally, no ramifications exist inside the edges; therefore, 6 edges on the sphere give rise to 30 edges on the covering surface.

The result is always the same: 12 vertices of degree 5, 30 edges, and 12 faces of degree 5, and the covering surface is therefore of genus 4.

³Before making any computations we might suppose that d and e constitute two separate orbits. In this case both of them would be defined over \mathbb{Q} ; but d cannot be realized over \mathbb{Q} for the same reason of not being invariant under the complex conjugation.

We cannot guarantee that all the $5 \cdot 60 = 300$ maps thus obtained are different since some of them may turn out to be isomorphic. But it is clear that there are plenty of them. Among all these maps, only I_4 is regular: the list of all regular maps of genera up to 15 may be found in [4], and there is only one of them having our set of vertex-and-face degrees. However, not being regular, the other maps are “symmetric” in our generalized meaning of the word: they are compositions. We don’t know if there also exist maps with the same set of vertex-and-face degrees which cannot be represented as compositions.

To give an example, let us consider the dessin of the map a . Its vertices of degree 1 are the roots of the polynomial $x^2 - 4x - 1$, and the centers of the faces of degree 1 are the roots of $x^2 + x - 1$ (cf. equation (8)). Then consider the curve \mathcal{A} defined by the following equation:

$$y^5 = \frac{x^2 - 4x - 1}{x^2 + x - 1} \quad (9)$$

(note that y^5 has only two multiple roots, namely, $y = 0$ and $y = \infty$). We claim that \mathcal{A} and \mathcal{B} are not isomorphic as algebraic curves. Indeed, there are two disjoint classes of algebraic curves of genus $g \geq 2$: hyperelliptic curves, and curves of canonical type. A curve of genus g and of degree $2g - 2$, embedded without degeneracies in the projective space of dimension $g - 1$, is of canonical type (see [10], Chapter 2, § 3, Section “Canonical curves”). This is exactly the case for the curve \mathcal{B} : the space $\mathbb{CP}^3 \subset \mathbb{CP}^4$ is the hyperplane of the equation $\sum_{i=1}^5 x_i = 0$, sf. (7). On the other hand, the curve \mathcal{A} is hyperelliptic: the birational substitutions

$$y = s, \quad t = 4xy^5 + 2y^5 - 4x + 8, \quad x = \frac{1}{4} \cdot \frac{t - 2s^5 - 8}{s^5 - 1}$$

transform \mathcal{A} into the curve

$$t^2 = 20s^{10} + 80.$$

Since the dessin entirely determines the complex structure of the curve on which it is drawn, the lifting of the map a on the curve \mathcal{A} via the covering (9) cannot be isomorphic to I_4 which lives on \mathcal{B} .

This is a very curious example: we have proved the non-isomorphism of two maps without even explicitly constructing one of them, the proof being based on some facts from algebraic geometry. I don’t know of any other example of this kind.

To the best of my knowledge, the existence of the above huge family of *uniform* maps (“uniform” means that all vertices have the same degree and all faces have the same degree) has so far remained unnoticed. It would be interesting to classify them, as well as the corresponding hyperbolic tessellations.

4.5 Concluding remarks

There are other important applications of the composition, but it would be difficult to make their overview here since it would need quite a lot of preparatory material. Therefore, we just mention them briefly.

The problem of topological classification of polynomials has remained open for almost 140 years. A huge amount of experimental data was obtained by the author (see Sec. 5.4 of [22]). There are a couple of well understood exceptional phenomena; putting them aside, all known cases of non-uniqueness of equivalence classes of polynomials with the same ramification data are due to compositions.

The so-called polynomial moment problem is closely related to the classical Poincaré center-focus problem. Its aim is to describe polynomials orthogonal to all powers of a given polynomial. The problem was recently solved, see [27]. It turns out that all solutions are described as sums of compositions.

The problem of non-uniqueness of decomposition of polynomials was solved by Ritt [31] in 1922; its analogue for Laurent polynomials was solved by Pakovich [26] in 2007. The problem for general rational functions still remains open.

Last but not least, compositions of polynomials and rational functions constitute the main subject of complex dynamical systems. Belyi functions that send the set $\{0, 1, \infty\}$ to itself can be composed: the result is again a Belyi function. But this is only a particular case of the so-called *postcritically finite* dynamical systems for which an orbit of every critical point is finite; see [29]. First of all, they are also rigid and therefore defined over $\overline{\mathbb{Q}}$. Next, the towers of monodromy groups play an important role. But, contrary to the Belyi case, not every such dynamical system can be realized by a rational function: there are so-called Thurston's obstructions to that. A wonderful new world!

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Figure 13: Marble mosaic floor in St. Mark's basilica, Venice; attributed to Paolo Ucello (around 1430). Source: Weisstein, Eric W. "Small Stellated Dodecahedron." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/SmallStellatedDodecahedron.html>