



**HAL**  
open science

# Curvature and gravity actions for matrix models: II. The case of general Poisson structures

Daniel N Blaschke, Harold Steinacker

► **To cite this version:**

Daniel N Blaschke, Harold Steinacker. Curvature and gravity actions for matrix models: II. The case of general Poisson structures. *Classical and Quantum Gravity*, 2010, 27 (23), pp.235019. 10.1088/0264-9381/27/23/235019 . hal-00649077

**HAL Id: hal-00649077**

**<https://hal.science/hal-00649077>**

Submitted on 7 Dec 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Curvature and Gravity Actions for Matrix Models II: the case of general Poisson structures

Daniel N. Blaschke\*, Harold Steinacker†

October 11, 2010

*Faculty of Physics, University of Vienna  
Boltzmannngasse 5, A-1090 Vienna (Austria)*

## Abstract

We study the geometrical meaning of higher-order terms in matrix models of Yang-Mills type in the semi-classical limit, generalizing recent results [1] to the case of 4-dimensional space-time geometries with general Poisson structure. Such terms are expected to arise e.g. upon quantization of the IKKT-type models. We identify terms which depend only on the intrinsic geometry and curvature, including modified versions of the Einstein-Hilbert action, as well as terms which depend on the extrinsic curvature. Furthermore, a mechanism is found which implies that the effective metric  $G$  on the space-time brane  $\mathcal{M} \subset \mathbb{R}^D$  “almost” coincides with the induced metric  $g$ . Deviations from  $G = g$  are suppressed, and characterized by the would-be  $U(1)$  gauge field.

---

\*daniel.blaschke@univie.ac.at

†harold.steinacker@univie.ac.at

# Contents

<b>1</b>	<b>Introduction and background</b>	<b>2</b>
<b>2</b>	<b>Matrix models and their geometry</b>	<b>3</b>
2.1	Reviewing the basic ingredients . . . . .	3
2.2	Special relations in $2n = 4$ dimensions . . . . .	5
2.3	Intrinsic curvature. . . . .	7
2.4	Cartesian tensors . . . . .	9
<b>3</b>	<b>Extensions of the matrix model action</b>	<b>11</b>
3.1	Matrix operators . . . . .	11
3.2	Potential terms $k = 0$ . . . . .	13
3.3	$\mathcal{O}(X^6)$ terms . . . . .	14
3.4	$\mathcal{O}(X^{10})$ terms . . . . .	16
<b>4</b>	<b>Gravitational action and degrees of freedom</b>	<b>18</b>
<b>5</b>	<b>Concluding remarks</b>	<b>22</b>
	<b>Appendix A</b>	<b>23</b>
	<b>Appendix B</b>	<b>24</b>

## 1 Introduction and background

This paper is a continuation of our previous work [1], where gravitational actions, in particular an analog of the Einstein-Hilbert action, were obtained from higher-order terms in matrix models of Yang-Mills type.

In this framework [2, 3, 4], space-time is realized as quantized Poisson manifold  $\mathcal{M} \subset \mathbb{R}^D$  with an induced metric  $g_{\mu\nu}$  and Poisson tensor  $\theta^{\mu\nu}$ . These structures determine an effective gravitational metric  $G^{\mu\nu} = e^{-\sigma} \theta^{\mu\mu'} \theta^{\nu\nu'} g_{\mu'\nu'}$ , to which matter couples more-or-less as in general relativity (GR). Since generic 4-dimensional geometries can be realized (at least locally) as sub-manifold  $\mathcal{M} \subset \mathbb{R}^{10}$  [5], this provides a suitable framework for a pre-geometric, “emergent” theory of gravity. As an illustration, a realization of the Schwarzschild geometry in this approach is presented in Ref. [6].

The dynamics of gravity in this framework and its relation resp. deviation from general relativity is not yet very well understood. Upon quantization, various higher-order terms are expected to arise in the matrix model, or alternatively such terms can be added by hand. In [1], we identified a matrix model action which in the semi-classical limit reduces to  $\int d^4x \sqrt{g} e^{2\sigma} R[g]$ , for the most natural case of geometries with  $G_{\mu\nu} = g_{\mu\nu}$ . However, it turns out that there are several possible matrix actions which reduce to the same semi-classical form for  $G_{\mu\nu} = g_{\mu\nu}$ . Moreover, in order to derive the equations of motion for the geometry, it is necessary to consider variations which violate this condition. In the present paper, we obtain a slightly modified action which for coinciding metrics reduces to the Einstein-Hilbert action, and which is tensorial (i.e. depends only on the intrinsic geometry of  $\mathcal{M} \subset \mathbb{R}^D$ ) for general  $G_{\mu\nu} \neq g_{\mu\nu}$ . We also identify several other terms which have an intrinsic geometrical meaning. Some of these terms depend also on the Poisson structure.

There are also “potential” terms which may set the non-commutativity (NC) scale  $e^{-\sigma}$ , as well as terms which depend on the extrinsic geometry, i.e. the embedding of  $\mathcal{M} \subset \mathbb{R}^D$ . This should be the beginning of a more systematic study.

An important issue which arises in this context is the role of the Poisson or NC structure  $\theta^{\mu\nu}$ , which in particular determines the difference  $h_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu}$ . This Poisson structure can be viewed as would-be  $U(1)$  gauge field, and is governed mainly by the “bare” Yang-Mills term in the matrix model. We show that this action suppresses  $h_{\mu\nu}$ , and singles out self-dual and anti-selfdual Poisson structures with  $G_{\mu\nu} = g_{\mu\nu}$  as vacuum solutions. In the case of Minkowski signature, this holds once a specific complexification of Poisson structures is adopted, which appears to be very natural. This is important progress in the understanding of emergent gravity in these models, and exhibits more clearly the relation with general relativity.

In the present work, we restrict ourselves essentially to the semi-classical limit of the matrix model. Of course, the main appeal for this framework compared with other descriptions of gravity is the fact that it goes beyond the classical concepts of geometry: Space-time is not put in by hand but emerges, realized as non-commutative space with an effective geometry, gauge fields, and matter. Moreover, the IKKT matrix model [7] (which is the prime candidate of this class of models with  $D = 10$ ) can alternatively be viewed as  $\mathcal{N} = 4$  supersymmetric Yang-Mills gauge theory on  $\mathbb{R}_\theta^4$ , and hence it is expected to define a good quantum theory. Therefore these models provide promising candidates for a quantum theory of fundamental interactions including gravity. Moreover, there are several intriguing hints that the role of vacuum energy in this framework may be different than in GR. Nevertheless, much more work remains to be done in order to fully understand this class of models, and we hope that the current paper provides useful results and tools for that purpose.

This paper is organized in the following way: We start by reviewing properties and important relations of the current framework of matrix models and emergent gravity in Section 2.1. This will also fix our notation for the remaining sections. We then continue Section 2 by deriving relations for the special case of a 4-dimensional embedded manifold  $\mathcal{M}^4 \subset \mathbb{R}^D$ , and discuss connections and curvature. Section 3 will be devoted to higher order extensions to Yang-Mills matrix models and their semi-classical limit, whose implications will be discussed in Section 4.

## 2 Matrix models and their geometry

We briefly collect the essential ingredients of the matrix model framework for emergent gravity, referring e.g. to the recent review [4] for more details.

### 2.1 Reviewing the basic ingredients

The starting point is given by the matrix model of Yang-Mills type,

$$S_{YM} = -\text{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}, \quad (2.1)$$

where  $\eta_{ac}$  is the (flat) metric of a  $D$  dimensional embedding space (i.e.  $a, b, c, d \in 1, \dots, D$ ). It can be purely Euclidean, or have one or more time-like directions. The “covariant coordinates”  $X^a$  (cf. [8]) are Hermitian matrices, resp. operators acting on a separable Hilbert

space  $\mathcal{H}$ . The commutator of two coordinates will be denoted as

$$[X^a, X^b] = i\theta^{ab}. \quad (2.2)$$

We are interested in configurations which can be interpreted as  $2n$  dimensional non-commutative space  $\mathcal{M}_\theta^{2n}$ , in the spirit of non-commutative geometry. Thus we consider configurations where  $2n$  of the matrices (henceforth called  $X^\mu$ ) generate a non-commutative algebra interpreted as non-commutative spaces  $\mathcal{M}_\theta^{2n}$ , and the remaining  $D - 2n$  matrices are (quantized) functions of the  $X^\mu$ , i.e. functions on  $\mathcal{M}_\theta^{2n}$ . In other words, we split<sup>1</sup> the matrices resp. coordinates as

$$X^a = (X^\mu, \phi^i), \quad \mu = 1, \dots, 2n, \quad i = 1, \dots, D - 2n, \quad (2.3)$$

so that the  $\phi^i(X) \sim \phi^i(x)$  in the semi-classical limit define an embedding of a  $2n$  dimensional submanifold

$$\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D. \quad (2.4)$$

Moreover, we can interpret<sup>2</sup>

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x) \quad (2.5)$$

in the semi-classical limit as a Poisson structure on  $\mathcal{M}^{2n}$ . Thus we are considering quantized Poisson manifolds  $(\mathcal{M}^{2n}, \theta^{\mu\nu})$ , with quantized embedding functions  $X^a$ . Throughout this paper,  $\sim$  denotes the semi-classical limit, where commutators are replaced by Poisson brackets. We will assume that  $\theta^{\mu\nu}$  is non-degenerate, so that its inverse matrix  $\theta_{\mu\nu}^{-1}$  defines a symplectic form on  $\mathcal{M}^{2n}$ . The sub-manifold  $\mathcal{M}^{2n} \subset \mathbb{R}^D$  is equipped with a non-trivial induced metric<sup>3</sup>

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j \eta_{ij}, \quad (2.6)$$

via pull-back of  $\eta_{ab}$ . Finally, we define the following quantities [13]:

$$\begin{aligned} G^{\mu\nu} &= e^{-\sigma} \theta^{\mu\rho} \theta^{\nu\sigma} g_{\rho\sigma}, & \eta &= \frac{1}{4} e^\sigma G^{\mu\nu} g_{\mu\nu}, \\ \rho &= \sqrt{\det \theta_{\mu\nu}^{-1}}, & e^{-\sigma} &= \frac{\rho}{\sqrt{\det G_{\mu\nu}}}. \end{aligned} \quad (2.7)$$

The last relation gives a unique definition for  $e^{-\sigma}$  provided  $n > 1$ , which we assume. It is easy to see that the kinetic term for scalar fields on  $\mathcal{M}^{2n}$  is governed by the effective metric  $G_{\mu\nu}(x)$ , and in fact the same metric also governs non-Abelian gauge fields and fermions in the matrix model (up to possible conformal factors), so that  $G_{\mu\nu}$  *must* be interpreted as gravitational metric. Since the embedding  $\phi^i$  is dynamical, the model describes a theory of gravity realized on dynamically determined submanifolds of  $\mathbb{R}^D$ . We also recall that

$$\text{Tr } \phi \sim \int \frac{d^{2n}x}{(2\pi)^n} \sqrt{G} e^{-\sigma} \phi(x) \quad (2.8)$$

<sup>1</sup>More generally, all of the  $X^a$  are interpreted as functions on  $\mathcal{M}_\theta^{2n}$  subject to  $D - 2n$  relations. Examples for such NC submanifolds realized by matrix models have been known for a long time, cf. [7, 9].

<sup>2</sup>In the special case where  $\theta^{\mu\nu}$  is constant, this leads to non-commutative field theories — see [10, 11] for a review of the topic. However, a dynamical commutator seems essential in the context of gravity.

<sup>3</sup>For a related discussion see e.g. [12].

in the semi-classical limit, and note the remarkable identity

$$|G_{\mu\nu}(x)| = |g_{\mu\nu}(x)|, \quad 2n=4 \quad (2.9)$$

which holds on 4-dimensional  $\mathcal{M}^4 \subset \mathbb{R}^D$ . It is also useful to define the following tensor

$$\mathcal{J}^\mu{}_\nu = e^{-\sigma/2} \theta^{\mu\mu'} g_{\mu'\nu} = -e^{\sigma/2} G^{\mu\mu'} \theta_{\mu'\nu}^{-1} \quad (2.10)$$

which satisfies

$$\begin{aligned} (\mathcal{J}^2)^\mu{}_\rho &= -G^{\mu\nu} g_{\nu\rho}, \\ \text{tr} \mathcal{J}^2 &= -4e^{-\sigma} \eta \equiv -(gG), \end{aligned} \quad (2.11)$$

where ‘tr’ denotes the trace over Lorentz indices.

In Ref. [1], we focused on the particular case of 4-dimensional geometries with

$$G^{\mu\nu} = g^{\mu\nu} \quad \rightarrow \quad \eta = e^\sigma. \quad (2.12)$$

Clearly, this defines an almost-Kähler manifold with almost-complex structure  $\mathcal{J}^2 = -1$ . For such geometries to be consistent in the case of Minkowski signature, we have to assume that  $\theta^{\mu\nu}$  has imaginary time-like components, which is natural in view of the correspondence  $X^0 \rightarrow iT$ , as discussed in [4]. It is not hard to see that this corresponds to  $\theta^{\mu\nu}$  being self-dual with respect to the metric  $g_{\mu\nu}$  (cf. Section 2.2 and Ref. [14]). Such  $\theta^{\mu\nu}$  indeed exist for generic geometries<sup>4</sup>. We then showed that the Einstein-Hilbert action can be obtained by a certain matrix action (2.46). However, variations of  $\theta^{\mu\nu}$  away from a self-dual case lead to metric variations

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \quad (2.13)$$

Therefore, in order to derive the equations of motion for both the (embedding) metric as well as the Poisson structure  $\theta^{\mu\nu}$ , it is necessary to allow at least small deviations from  $G_{\mu\nu} = g_{\mu\nu}$ . We will in fact identify a mechanism in Section 4 which generically implies  $G \approx g$  to a very good approximation, at least for geometries with mild curvature. This justifies to consider only linearized corrections in  $h_{\mu\nu}$ , and provides an important step towards clarifying the relation with general relativity.

**Notation.** We will adopt the convention that Latin matrix indices are raised and lowered with  $\eta_{ab}$  throughout this paper (resp.  $\delta_{ab}$  in the Euclidean case). As we consider deviations from the self-dual geometries introduced above, we will inevitably encounter two types of covariant derivatives: those with respect to the effective metric  $\nabla := \nabla[G]$ , and those with respect to the induced metric  $\nabla' := \nabla[g]$ . We will use this notation throughout the remainder of this paper. Furthermore, we will use the abbreviations  $(Gg) \equiv G^{\mu\nu} g_{\mu\nu}$  and  $(Gg)_\alpha^\mu \equiv G^{\mu\rho} g_{\rho\alpha}$ .

## 2.2 Special relations in $2n = 4$ dimensions

In this section we collect some basic results on the geometry of  $\mathcal{M}^4 \subset \mathbb{R}^D$  in the presence of the structures defined above. We consider the case of general metrics  $G_{\mu\nu} \neq g_{\mu\nu}$  on  $2n = 4$  dimensional manifolds where the tensor  $\mathcal{J}^\mu{}_\nu$  defined in (2.10) becomes unimodular,

---

<sup>4</sup>with suitable technical assumptions, such as global hyperbolicity or asymptotic flatness.

i.e.  $\det \mathcal{J} = 1$ . This leads to the existence of a remarkable identity which we will now derive. Consider first the Euclidean case. Since everything is formulated in a tensorial way, we can diagonalize the embedding metric at that point  $g_{\mu\nu}|_p = \delta_{\mu\nu}$ , and bring the Poisson tensor resp. the symplectic form into canonical form

$$\omega = \theta^{-1} (\alpha dx^0 dx^3 \pm \alpha^{-1} dx^1 dx^2) \quad (2.14)$$

at  $p \in \mathcal{M}$  using a suitable  $SO(4)$  rotation. This leads to

$$G^{\mu\nu} = \text{diag}(\alpha^2, \alpha^{-2}, \alpha^{-2}, \alpha^2) \quad \text{at } p \in \mathcal{M}, \quad (2.15)$$

and similarly  $\mathcal{J}^\mu{}_\nu = -\text{diag}(\alpha^2, \alpha^{-2}, \alpha^{-2}, \alpha^2)$  at  $p \in \mathcal{M}$ . In particular, it follows that

$$\frac{1}{4}(Gg) = e^{-\sigma}\eta = \frac{1}{2}(\alpha^2 + \alpha^{-2}) \geq 1. \quad (2.16)$$

Furthermore, we obtain the following characteristic equation<sup>5</sup> for  $\mathcal{J}^2$  [14]:

$$(\mathcal{J}^2)^\mu{}_\nu + 2e^{-\sigma}\eta\delta^\mu{}_\nu + (\mathcal{J}^{-2})^\mu{}_\nu = 0, \quad (2.17)$$

or equivalently

$$(GgG)^{\mu\nu} = -(\mathcal{J}^2 G)^{\mu\nu} = 2e^{-\sigma}\eta G^{\mu\nu} - g^{\mu\nu} = \frac{1}{2}(Gg)G^{\mu\nu} - g^{\mu\nu}. \quad (2.18)$$

Furthermore, observe that  $\star(dx^0 dx^3) = dx^1 dx^2$  where  $\star$  denotes the Hodge star defined by  $\varepsilon^{\mu\nu\rho\sigma}$  and  $g_{\mu\nu}$  on  $\mathcal{M}^4$ . This means that the corresponding symplectic form is (anti-) self-dual ((A)SD) if and only if

$$\star\omega = \pm\omega \quad \Leftrightarrow \quad \alpha = 1 \quad \text{resp.} \quad e^{-\sigma}\eta = 1 \quad \Leftrightarrow \quad G_{\mu\nu} = g_{\mu\nu} \quad \Leftrightarrow \quad \mathcal{J}^2 = -1, \quad (2.19)$$

in which case  $\mathcal{M}^4$  becomes an almost-Kähler manifold with almost-complex structure  $\mathcal{J}$ . These statements generalize to the case of Minkowski signature, provided we consider complexified  $\theta^{\mu\nu}$  with imaginary time-like components  $\theta^{0\nu}$ , see [4].

Furthermore, we also note the following useful identity

$$\partial_\alpha(\rho\theta^{\mu\alpha}) = 0 \quad (2.20)$$

which holds in any coordinates, and follows from the Jacobi identity. On  $2n = 4$ -dimensional branes, it implies

$$\begin{aligned} 0 &= \partial_\alpha(e^{-\sigma}\sqrt{|g|}\theta^{\mu\alpha}) = \sqrt{|g|}\nabla'_\alpha(e^{-\sigma}\theta^{\mu\alpha}) \\ &= \partial_\alpha(e^{-\sigma}\sqrt{|G|}\theta^{\mu\alpha}) = \sqrt{|G|}\nabla'_\alpha(e^{-\sigma}\theta^{\mu\alpha}) \end{aligned} \quad (2.21)$$

using  $|g| = |G|$ . Note furthermore that

$$\begin{aligned} G^{\mu\alpha}\nabla'_\alpha\theta_{\mu\nu}^{-1} &= \nabla'_\alpha(G^{\mu\alpha}\theta_{\mu\nu}^{-1}) - \theta_{\mu\nu}^{-1}\nabla'_\alpha G^{\mu\alpha} \\ &= -\nabla'_\alpha(e^{-\sigma}\theta^{\mu\alpha}g_{\mu\nu}) - \theta_{\mu\nu}^{-1}\nabla'_\alpha G^{\mu\alpha} \\ &= -\theta_{\mu\nu}^{-1}\nabla'_\alpha G^{\mu\alpha} \end{aligned} \quad (2.22)$$

using the basic identity (2.21).

---

<sup>5</sup>If we would consider real  $\theta^{\mu\nu}$  in the Minkowski case, this relation would be replaced by  $\mathcal{J}^2 + 2e^{-\sigma}\eta - \mathcal{J}^{-2} = 0$ .

**Determinants.** Consider the scalar function

$$\det \mathcal{J} = e^{-n\sigma} \det(\theta^{\mu\nu}) \det(g_{\mu\nu}) \quad (2.23)$$

which satisfies  $\det \mathcal{J} = 1$  in  $2n = 4$  dimensions. In that case, it follows that

$$\begin{aligned} \partial_\alpha e^{2\sigma} &= \partial_\alpha \det(\theta^{\mu\eta} g_{\eta\nu}) = e^{2\sigma} g^{\mu\sigma} \theta_{\sigma\nu}^{-1} \partial_\alpha (\theta^{\nu\eta} g_{\eta\mu}) \\ &= e^{2\sigma} (\theta_{\eta\nu}^{-1} \partial_\alpha \theta^{\nu\eta} + g^{\mu\eta} \partial_\alpha g_{\eta\mu}) . \end{aligned} \quad (2.24)$$

We can replace  $\partial_\alpha$  with any covariant derivative operator  $\nabla_\alpha$  in this formula. In particular, for  $\nabla' = \nabla[g]$  we obtain

$$\partial_\alpha e^{2\sigma} = e^{2\sigma} \theta_{\eta\nu}^{-1} \nabla'_\alpha \theta^{\nu\eta} . \quad (2.25)$$

Similarly, using  $\mathcal{J}^\mu{}_\nu = -e^{\sigma/2} G^{\mu\eta} \theta_{\eta\nu}^{-1}$  we get

$$\partial_\alpha e^{-2\sigma} = e^{-2\sigma} \theta^{\nu\eta} \nabla_\alpha \theta_{\eta\nu}^{-1} , \quad (2.26)$$

so for  $2n = 4$  we have

$$2\partial_\alpha \sigma = \theta_{\eta\nu}^{-1} \nabla'_\alpha \theta^{\nu\eta} = \theta_{\eta\nu}^{-1} \nabla_\alpha \theta^{\nu\eta} . \quad (2.27)$$

Since  $\det(G^{\mu\eta} g_{\eta\nu}) = 1$  in  $2n = 4$  dimensions, a similar argument yields

$$\begin{aligned} 0 &= \partial_\alpha \det(G^{\mu\eta} g_{\eta\nu}) = g^{\mu\sigma} G_{\sigma\nu} \partial_\alpha (G^{\nu\eta} g_{\eta\mu}) \\ &= G_{\eta\nu} \partial_\alpha G^{\nu\eta} + g^{\mu\eta} \partial_\alpha g_{\eta\mu} , \end{aligned} \quad (2.28)$$

and likewise for any covariant derivatives. This implies

$$g^{\mu\eta} \nabla_\alpha g_{\eta\mu} = 0 = G_{\eta\nu} \nabla'_\alpha G^{\nu\eta} . \quad (2.29)$$

In the computations of the subsequent sections, we will make use of the important relations (2.18), (2.21), (2.22) and (2.29) in many places.

### 2.3 Intrinsic curvature.

Since we consider general geometries  $G_{\mu\nu} \neq g_{\mu\nu}$  in this paper, we will inevitably encounter the tensor

$$C_{\alpha;\mu\nu} := \partial_\alpha x^a \nabla_\mu \partial_\nu x_a = \frac{1}{2} (\nabla_\mu g_{\nu\alpha} + \nabla_\nu g_{\mu\alpha} - \nabla_\alpha g_{\mu\nu}) , \quad (2.30)$$

in subsequent computations. Contracting this tensor with  $G^{\mu\nu}$ , one derives

$$\partial_\alpha x^a \square_G x_a = \nabla_\mu (G^{\mu\nu} g_{\nu\alpha}) - 2\partial_\alpha (e^{-\sigma} \eta) = \nabla^\nu g_{\nu\alpha} - \frac{1}{2} \partial_\alpha (gG) , \quad (2.31a)$$

$$\stackrel{2n=4}{=} -G_{\alpha\nu} \nabla_\mu g^{\mu\nu} . \quad (2.31b)$$

$$\partial_\alpha x^a \nabla_\mu \partial^\alpha x_a = \frac{1}{2} \partial_\mu (Gg) , \quad (2.31c)$$

where the 4D identity (2.18) is used in (2.31b) and “l.h.s.  $\stackrel{2n=4}{=}$  r.h.s.” denotes equality iff  $2n = 4$ .

Keeping these relations in mind, we now derive the curvature tensor with respect to the metrics  $G_{\mu\nu}$  and  $g_{\mu\nu}$ : For a general embedding  $\mathcal{M} \subset \mathbb{R}^D$  with Cartesian embedding functions  $x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$ , consider the expression

$$\begin{aligned}
& \nabla_\sigma \nabla_\mu x^a \nabla_\rho \nabla_\nu x_a - \nabla_\sigma \nabla_\nu x^a \nabla_\mu \nabla_\rho x_a \\
= & \nabla_\sigma (\nabla_\mu x^a \nabla_\rho \nabla_\nu x_a) - \nabla_\mu x^a \nabla_\sigma \nabla_\rho \nabla_\nu x_a - \nabla_\rho (\nabla_\sigma \nabla_\nu x^a \nabla_\mu x_a) + \nabla_\rho \nabla_\sigma \nabla_\nu x^a \nabla_\mu x_a \\
= & \nabla_\sigma C_{\mu;\rho\nu} - \nabla_\rho C_{\mu;\sigma\nu} + [\nabla_\rho, \nabla_\sigma] \nabla_\nu x^a \nabla_\mu x_a \\
= & \nabla_\sigma C_{\mu;\rho\nu} - \nabla_\rho C_{\mu;\sigma\nu} + (Gg)^\eta{}_\mu R_{\rho\sigma\nu\eta}[G].
\end{aligned} \tag{2.32}$$

Unless stated otherwise, we will always understand  $R_{\rho\sigma\nu\eta} \equiv R_{\rho\sigma\nu\eta}[G]$  throughout this paper. All the terms in (2.32) are tensorial, and we obtain

$$(Gg)^\eta{}_\mu R_{\rho\sigma\nu\eta}[G] = \nabla_\sigma \nabla_\mu x^a \nabla_\rho \nabla_\nu x_a - \nabla_\sigma \nabla_\nu x^a \nabla_\mu \nabla_\rho x_a - \nabla_\sigma C_{\mu;\rho\nu} + \nabla_\rho C_{\mu;\sigma\nu}. \tag{2.33}$$

Repeating this calculation with  $\nabla$  replaced by the covariant derivative with respect to the induced metric  $\nabla[g] = \nabla'$ , we recover the Gauss-Codazzi theorem due to  $\nabla'_\mu x^a \nabla'_\rho \nabla'_\nu x_a = 0$ :

$$R_{\rho\sigma\nu\mu}[g] = g_{\mu\tau} R[g]_{\rho\sigma\nu}{}^\tau = \nabla'_\sigma \nabla'_\mu x^a \nabla'_\rho \nabla'_\nu x_a - \nabla'_\sigma \nabla'_\nu x^a \nabla'_\mu \nabla'_\rho x_a. \tag{2.34}$$

For the self-dual case  $C_{\mu;\rho\nu} = \nabla_\mu x^a \nabla_\rho \nabla_\nu x_a = 0$ , and both curvature tensors (2.33) and (2.34) coincide.

**Relating  $R[g]$  and  $R[G]$ .** The covariant derivatives  $\nabla_\mu$  and  $\nabla'_\mu$  are related via the tensors  $C_{\alpha;\mu\nu}$  as follows:

$$\nabla'_\mu V_\nu = \nabla_\mu V_\nu - C_{\alpha;\mu\nu} g^{\alpha\beta} V_\beta = \nabla_\mu V_\nu + \tilde{C}_{\alpha;\mu\nu} G^{\alpha\beta} V_\beta, \tag{2.35}$$

for some vector  $V_\nu$ , and where  $\tilde{C}_{\alpha;\mu\nu}$  is defined by replacing  $g$  with  $G$  (and hence  $\nabla$  with  $\nabla'$ ) in (2.30). This implies

$$\begin{aligned}
g^{\alpha\beta} C_{\alpha;\mu\nu} &= \frac{1}{2} g^{\alpha\beta} (\nabla_\mu g_{\nu\alpha} + \nabla_\nu g_{\mu\alpha} - \nabla_\alpha g_{\mu\nu}) \\
&= -G^{\alpha\beta} \tilde{C}_{\alpha;\mu\nu} = -\frac{1}{2} G^{\alpha\beta} (\nabla'_\mu G_{\nu\alpha} + \nabla'_\nu G_{\mu\alpha} - \nabla'_\alpha G_{\mu\nu}),
\end{aligned} \tag{2.36}$$

which has a number of useful consequences:

$$\begin{aligned}
g^{\alpha\mu} C_{\alpha;\mu\nu} &= \frac{1}{2} g^{\alpha\mu} \nabla_\nu g_{\mu\alpha} = -G^{\alpha\mu} \tilde{C}_{\alpha;\mu\nu} = -\frac{1}{2} G^{\alpha\mu} \nabla'_\nu G_{\mu\alpha} = 0, \\
g^{\alpha\beta} g^{\mu\nu} C_{\alpha;\mu\nu} &\stackrel{2n=4}{=} g^{\alpha\beta} g^{\mu\nu} \nabla_\mu g_{\nu\alpha} = -\nabla_\mu g^{\mu\beta} \\
&= -G^{\alpha\beta} g^{\mu\nu} \nabla'_\mu G_{\nu\alpha} + \frac{1}{2} G^{\alpha\beta} \partial_\alpha (g^{\mu\nu} G_{\mu\nu}), \\
g^{\alpha\beta} G^{\mu\nu} C_{\alpha;\mu\nu} &= g^{\alpha\beta} G^{\mu\nu} \nabla_\mu g_{\nu\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\alpha (Gg) \\
&\stackrel{2n=4}{=} -G^{\alpha\beta} G^{\mu\nu} \nabla'_\mu G_{\nu\alpha} = \nabla'_\mu G^{\mu\beta},
\end{aligned} \tag{2.37}$$

where we have used (2.29). Furthermore, we may define projectors on the tangential resp. normal bundle of  $\mathcal{M} \subset \mathbb{R}^D$  as

$$\mathcal{P}_T^{ab} = g^{\mu\nu} \partial_\mu x^a \partial_\nu x^b, \quad \mathcal{P}_N^{ab} = \eta^{ab} - \mathcal{P}_T^{ab}. \tag{2.38}$$

Hence, by the very definition of the covariant derivative associated to  $g_{\mu\nu}$ , we have

$$\begin{aligned}\nabla'_\sigma \nabla'_\nu x^a &= \nabla_\sigma \nabla_\nu x^a - g^{\alpha\beta} C_{\beta;\sigma\nu} \partial_\alpha x^a \\ &= \nabla_\sigma \nabla_\nu x^a - g^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b \nabla_\sigma \nabla_\nu x_b \\ &= \mathcal{P}_N^{ab} \nabla_\sigma \nabla_\nu x_b.\end{aligned}\tag{2.39}$$

This allows to relate the curvature tensors<sup>6</sup> associated to  $G_{\mu\nu}$  resp.  $g_{\mu\nu}$ :

$$\begin{aligned}R_{\rho\sigma\nu\mu}[g] &= \nabla'_\sigma \nabla'_\mu x^a \nabla'_\rho \nabla'_\nu x_a - \nabla'_\sigma \nabla'_\nu x^a \nabla'_\mu \nabla'_\rho x_a \\ &= \mathcal{P}_N^{ab} \nabla_\sigma \nabla_\mu x_a \nabla_\rho \nabla_\nu x_b - \mathcal{P}_N^{ab} \nabla_\sigma \nabla_\nu x_a \nabla_\mu \nabla_\rho x_b \\ &= (Gg)_\mu^\eta R_{\rho\sigma\nu\eta}[G] + \nabla_\sigma C_{\mu;\rho\nu} - \nabla_\rho C_{\mu;\sigma\nu} - C_{\alpha;\sigma\mu} C_{\beta;\rho\nu} g^{\alpha\beta} + C_{\alpha;\sigma\nu} C_{\beta;\mu\rho} g^{\alpha\beta}, \\ R_{\rho\nu}[g] &= R_{\rho\nu}[G] + g^{\sigma\mu} \nabla_\sigma C_{\mu;\rho\nu} - g^{\sigma\mu} \nabla_\rho C_{\mu;\sigma\nu} - g^{\sigma\mu} C_{\alpha;\sigma\mu} C_{\beta;\rho\nu} g^{\alpha\beta} + g^{\sigma\mu} C_{\alpha;\sigma\nu} C_{\beta;\mu\rho} g^{\alpha\beta},\end{aligned}\tag{2.40}$$

using (2.38) and (2.32). The last terms can be evaluated using

$$g^{\alpha\beta} C_{\alpha;\sigma\nu} C_{\beta;\mu\rho} g^{\rho\nu} g^{\sigma\mu} = -\frac{3}{4} g^{\rho\nu} \nabla_\nu g^{\beta\mu} \nabla_\rho g_{\mu\beta} - \frac{1}{2} g_{\rho\mu} \nabla_\beta g^{\rho\nu} \nabla_\nu g^{\beta\mu},\tag{2.41a}$$

$$g^{\alpha\beta} C_{\alpha;\sigma\mu} g^{\sigma\nu} C_{\beta;\rho\nu} g^{\rho\nu} \stackrel{2n=4}{=} g_{\beta\nu} \nabla_\alpha g^{\alpha\beta} \nabla_\rho g^{\rho\nu},\tag{2.41b}$$

$$\begin{aligned}G^{\alpha\beta} C_{\alpha;\sigma\nu} C_{\beta;\mu\rho} G^{\rho\nu} G^{\sigma\mu} &\stackrel{2n=4}{=} 4\partial_\nu(e^{-\sigma}\eta)\partial^\nu(e^{-\sigma}\eta) + 2\partial_\alpha(e^{-\sigma}\eta)\nabla_\mu g^{\mu\alpha} \\ &\quad - \frac{3}{4} \nabla_\nu g^{\beta\mu} \nabla^\nu g_{\mu\beta} - \frac{1}{2} G_{\mu\beta} \nabla_\alpha g^{\mu\rho} \nabla_\rho g^{\alpha\beta},\end{aligned}\tag{2.41c}$$

$$\begin{aligned}g^{\sigma\mu} \nabla_\sigma C_{\mu;\rho\nu} - g^{\sigma\mu} \nabla_\rho C_{\mu;\sigma\nu} &= \frac{1}{2} g^{\sigma\mu} \nabla_\sigma (\nabla_\rho g_{\mu\nu} + \nabla_\nu g_{\rho\mu} - \nabla_\mu g_{\rho\nu}) - \frac{1}{2} g^{\sigma\mu} \nabla_\rho \nabla_\nu g_{\sigma\mu} \\ &= \frac{1}{2} \left( -\nabla_\rho \nabla^\mu h_{\mu\nu} - R_{\rho\beta}[g] h^{\beta\alpha} g_{\alpha\nu} + (\rho \leftrightarrow \nu) \right) \\ &\quad + \frac{1}{2} \square_g h_{\rho\nu} + R_{\alpha\rho\beta\nu}[g] h^{\alpha\beta} + \mathcal{O}(h^2),\end{aligned}\tag{2.41d}$$

as derived in Appendix A. Hence to leading order in  $h_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu}$ , we have

$$\begin{aligned}R_{\rho\nu}[g] &= R_{\rho\nu}[G] - \frac{1}{2} \left( \nabla_\rho \nabla^\mu h_{\mu\nu} + R_{\rho\beta}[g] h^{\beta\alpha} g_{\alpha\nu} + (\rho \leftrightarrow \nu) \right) + \frac{1}{2} \square_g h_{\rho\nu} + R_{\alpha\rho\beta\nu}[g] h^{\alpha\beta} \\ &\quad + \mathcal{O}(h^2), \\ R[g] &= R_{\rho\nu}[G] g^{\rho\nu} - \nabla^\nu \nabla^\mu h_{\mu\nu} + \mathcal{O}(h^2), \\ R[G] &= R_{\rho\nu}[g] G^{\rho\nu} + \nabla^\nu \nabla^\mu h_{\mu\nu} + \mathcal{O}(h^2).\end{aligned}\tag{2.42}$$

## 2.4 Cartesian tensors

Now consider the following expressions, which play an important role in the following:

$$\begin{aligned}H^{ab} &= \frac{1}{2} [[X^a, X^c], [X^b, X_c]]_+ \sim -e^\sigma G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b, \\ H &= H^{ab} \eta_{ab} = [X^c, X^d] [X_c, X_d] \sim -e^\sigma G^{\mu\nu} g_{\mu\nu} = -4\eta(x).\end{aligned}\tag{2.43}$$

---

<sup>6</sup>cp. also [15].

The matrix “energy-momentum tensor” is then defined by [13]

$$T^{ab} = H^{ab} - \frac{1}{4}\eta^{ab}H \quad \sim \quad \eta\eta^{ab} - e^\sigma G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b. \quad (2.44)$$

It is instructive to consider the projectors defined in Eqn. (2.38) acting on these expressions in the semi-classical limit, i.e.  $(\mathcal{P}_T H)^{ab} \sim H^{ab}$  and  $(\mathcal{P}_N T)^{ab} \sim \eta\mathcal{P}_N^{ab}$ . In the special case of  $g_{\mu\nu} = G_{\mu\nu}$ , the semi-classical limit of the energy-momentum tensor becomes truly related to the projectors:

$$T^{ab} \sim e^\sigma \mathcal{P}_N^{ab}, \quad \text{and} \quad H^{ab} \sim -e^\sigma \mathcal{P}_T^{ab}. \quad (2.45)$$

Moreover, then

$$T^{ab} \square X_a \square X_b - \frac{1}{2} T^{ab} \square H_{ab} \sim e^{3\sigma} R, \quad (2.46)$$

as shown<sup>7</sup> in [1]. However, there are several similar matrix actions which for  $g_{\mu\nu} = G_{\mu\nu}$  reduce to the same semi-classical form. It turns out that for general  $g_{\mu\nu} \neq G_{\mu\nu}$ , which we study in the present paper, the left-hand side of (2.46) is no longer intrinsic, i.e. it depends also on the embedding  $\mathcal{M} \subset \mathbb{R}^D$ . This makes the derivation of the equations of motion more difficult. However, we will identify a slightly modified matrix action which is intrinsic for general geometries in the semi-classical limit.

Before we continue, let us add a brief remark concerning  $H^{ab}$  in  $2n = 4$  dimensions: The 4D identity (2.18) implies

$$(H^3)^{ad} - \frac{1}{2} H(H^2)^{ad} + e^{2\sigma} H^{ad} \sim -e^{2\sigma} (Gg)^\mu_\rho (e^\sigma (GgG)^{\rho\nu} - 2\eta G^{\rho\nu} + e^\sigma g^{\rho\nu}) \partial_\mu x^a \partial_\nu x^d \underset{2n=4}{\sim} 0. \quad (2.47)$$

This means that  $e^{-\sigma} H^{ab}$  has 3 eigenvalues  $\{0, \alpha^2, \alpha^{-2}\}$  with  $e^{-\sigma} \eta = \frac{1}{2}(\alpha^2 + \alpha^{-2})$  and  $H \sim -4\eta$  (cf. Section 2.2 and Ref. [14]). Hence the last relation essentially characterizes the 4-dimensional nature of  $\mathcal{M}^4$ , and it also encodes the reality structure of  $\theta^{\mu\nu}$  at the matrix level because it is non-linear.

**Semi-classical limit of the tangential conservation law.** The following useful results for various Poisson brackets are essentially obtained in [14]: Since  $H^{ab}$  is a scalar field on  $\mathcal{M} \subset \mathbb{R}^D$ , we have<sup>8</sup>

$$\begin{aligned} \{x_a, H^{ab}\} &= -\theta^{\mu\nu} \partial_\mu x_a \nabla_\nu (e^\sigma G^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b) \\ &= -e^\sigma G^{\alpha\beta} \left( \partial_\nu \sigma g_{\mu\alpha} \theta^{\mu\nu} \partial_\beta x^b + \theta^{\mu\nu} \nabla_\nu g_{\mu\alpha} \partial_\beta x^b + g_{\mu\alpha} \theta^{\mu\nu} \nabla_\nu \partial_\beta x^b \right) \\ &= -G^{\alpha\beta} \theta^{\mu\nu} \nabla_\nu (e^\sigma g_{\mu\alpha}) \partial_\beta x^b. \end{aligned} \quad (2.48)$$

This is again tensorial, and can be written in a number of different ways:

$$\begin{aligned} \{x_a, H^{ab}\} &= -e^\sigma G^{\alpha\beta} \nabla_\nu (\theta^{\mu\nu} g_{\mu\alpha}) \partial_\beta x^b \\ &= -e^\sigma G^{\alpha\beta} \nabla^\mu (e^\sigma \theta_{\mu\alpha}^{-1}) \partial_\beta x^b \\ &= (\partial_\alpha \eta - e^\sigma \nabla^\rho g_{\rho\alpha} - 2\eta \partial_\alpha \sigma) \theta^{\beta\alpha} \partial_\beta x^b \\ &= (e^\sigma \square_G x^a \partial_\alpha x^a + \partial_\alpha \eta) \theta^{\alpha\beta} \partial_\beta x^b \end{aligned} \quad (2.49)$$

<sup>7</sup>The derivation given in [1] for  $\int d^4x \sqrt{g} e^{2\sigma} R$  also applies without the integral resp. trace.

<sup>8</sup>Notice, that we use the same symbols  $H^{ab}$  and  $T^{ab}$  for their respective semi-classical limits whenever it is clear from context what is meant.

using the identity (2.21) and

$$\theta^{\nu\mu}\partial_\mu\eta = e^\sigma\nabla_\mu(G^{\mu\mu'}g_{\mu'\nu'}\theta^{\nu'\nu}) + e^\sigma\theta^{\nu\alpha}\nabla^\rho g_{\rho\alpha} + 2\eta\theta^{\nu\mu}\partial_\mu\sigma \quad (2.50)$$

which follows from the Jacobi identity [14]. Together with (2.31a), we obtain

$$\{x_a, T^{ab}\} = e^\sigma (\square_G x_a \partial_\alpha x^a) \theta^{\alpha\beta} \partial_\beta x^b \quad (2.51)$$

which also follows directly from the matrix identity (2.53). For Yang-Mills matrix models, the tangential conservation law  $[X_a, T^{ab}] = 0$  holds in fact at the matrix level [13] as a consequence of the symmetry  $X^a \rightarrow X^a + c^a \mathbf{1}$ . However, higher order terms in the matrix model as considered below may modify this relation. Note also that for 4-dimensional branes, (2.31b) implies

$$\{x_a, T^{ab}\} \stackrel{2n=4}{=} -e^\sigma \nabla_\mu g^{\mu\nu} G_{\nu\alpha} \theta^{\alpha\beta} \partial_\beta x^b, \quad (2.52)$$

so that the tangential conservation law is equivalent to  $\nabla_\mu g^{\mu\nu} = 0$ .

**Exact matrix identities.** The above semi-classical conservation law (2.51) can also be obtained from the following matrix identities:

$$\begin{aligned} [X_a, H^{ab}] &= \frac{1}{2} \left( [\square X_c, [X^b, X^c]]_+ + \frac{1}{2} [X^b, H] \right), \\ [X_a, T^{ab}] &= \frac{1}{2} [\square X_c, [X^b, X^c]]_+. \end{aligned} \quad (2.53)$$

### 3 Extensions of the matrix model action

We now want to consider more general terms in the matrix model, which in general have the form

$$S_P[X] = \text{Tr}(X^{a_1} \dots X^{a_\ell}) P_{a_1 \dots a_\ell}, \quad (3.1)$$

where  $P_{a_1 \dots a_\ell}$  is an invariant tensor of  $SO(D)$  (resp.  $SO(1, D-1)$  etc. in the case of Minkowski signature). Imposing also translational invariance  $X^a \rightarrow X^a + c^a \mathbf{1}$ , only terms built out of commutators are admissible. We will organize such polynomial terms in the matrix model according to the power  $\ell$  of matrices  $X^a$ , as well as the number  $d$  of commutators. It is clear that translational invariance implies  $d \geq \ell/2$ , and that  $k = d - \ell/2$  corresponds to the number of derivatives of geometrical tensors such as  $\theta^{\mu\nu}$  in the semi-classical limit. It is thus natural to consider an expansion in  $k$  as well as  $\ell$ .

#### 3.1 Matrix operators

Before diving into the possible extensions to the matrix model action, we collect some basic “building blocks” for which we derive the following semi-classical results:

**Lemma 1** *For any matrices  $\Phi \sim \phi(x)$ ,  $\Psi \sim \psi(x)$ , we have*

$$\eta^{ab} [X_a, \Phi] [X_b, \Psi] \sim -e^\sigma G^{\mu\nu} \partial_\mu \phi \partial_\nu \psi, \quad (3.2a)$$

$$\square \Phi \equiv [X^a, [X_a, \Phi]] \sim -\{x^b, \{x^c, \phi\}\} \eta_{bc} = -e^\sigma \square_G \phi, \quad (3.2b)$$

$$H^{ab}[X_a, \Phi][X_b, \Psi] \sim e^{2\sigma}(GgG)^{\mu\nu} \partial_\mu \phi \partial_\nu \psi, \quad (3.2c)$$

$$\begin{aligned} H^{ab}[X_a, [X_b, \Phi]] &\sim e^{2\sigma}(GgG)^{\beta\eta} \nabla_\beta \partial_\eta \phi + e^\sigma \partial_\beta e^\sigma (GgG)^{\eta\beta} \partial_\eta \phi \\ &\quad + \frac{1}{4} e^{2\sigma} (\partial_\rho (Gg) - (Gg) \partial_\rho \sigma) G^{\eta\rho} \partial_\eta \phi \\ &\stackrel{g=G}{\sim} e^{2\sigma} \square_G \phi. \end{aligned} \quad (3.2d)$$

In particular, for  $2n = 4$ -dimensional branes, we have

$$(H^{ab} - \frac{1}{2} H \eta^{ab})[X_a, \Phi][X_b, \Psi] \sim -e^{2\sigma} g^{\mu\nu} \partial_\mu \phi \partial_\nu \psi, \quad (3.2e)$$

$$[X_a, (H^{ab} - \frac{1}{2} H \eta^{ab})[X_b, \Phi]] \sim -e^{2\sigma} (\square_g \phi + g^{\mu\nu} \partial_\mu \sigma \partial_\nu \phi). \quad (3.2f)$$

**Proof.** Relations (3.2a) and (3.2b) are by now well-known [14], and (3.2c) can be computed straightforwardly as

$$\begin{aligned} H^{ab}[X_a, \Phi][X_b, \Psi] &\sim e^\sigma G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \theta^{\alpha\beta} \partial_\alpha x_a \partial_\beta \phi \theta^{\alpha'\beta'} \partial_{\alpha'} x_b \partial_{\beta'} \psi \\ &= e^{2\sigma} (GgG)^{\mu\nu} \partial_\mu \phi \partial_\nu \psi. \end{aligned} \quad (3.3)$$

Now (3.2d) can be shown either by a direct computation which is given in Appendix B.1, or more elegantly by considering the following bilinear form

$$\text{Tr} \left( \Phi_1 H^{ab} [X_a, [X_b, \Phi_2]] \right) = \text{Tr} \left( -[X_a, H^{ab}] [X_b, \Phi_2] \Phi_1 - H^{ab} [X_b, \Phi_2] [X_a, \Phi_1] \right) \quad (3.4)$$

for any matrices  $\Phi_i \sim \phi_i(x)$ . The first term vanishes for self-dual  $\theta$  (up to  $\mathcal{O}(\hbar^2)$ , resp. is easy to evaluate), and reads

$$\begin{aligned} \text{Tr}([X_a, H^{ab}] [X_b, \Phi_2] \Phi_1) &\sim - \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} e^{-\sigma} \phi_1 (e^\sigma \square_G x^c \partial_\alpha x_c + \partial_\alpha \eta) \theta^{\alpha\beta} \partial_\beta x^b \theta^{\mu\nu} \partial_\mu x_b \partial_\nu \phi_2 \\ &= \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} e^\sigma \phi_1 \left( \nabla^\beta g_{\beta\alpha} - \frac{1}{4} \partial_\alpha (gG) + \frac{1}{4} \partial_\alpha \sigma (gG) \right) G^{\alpha\nu} \partial_\nu \phi_2, \end{aligned} \quad (3.5)$$

using (2.49) and (2.31a). The second term of (3.4) can be computed using (3.2c) yielding

$$\begin{aligned} \text{Tr}(H^{ab} [X_b, \Phi_2] [X_a, \Phi_1]) &\sim \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} e^\sigma (GgG)^{\mu\nu} \partial_\mu \phi_2 \partial_\nu \phi_1 \\ &= - \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} \phi_1 (e^\sigma \nabla_\nu \sigma (GgG)^{\mu\nu} \partial_\mu \phi_2 + e^\sigma \nabla^\rho g_{\rho\eta} G^{\mu\eta} \partial_\mu \phi_2 + e^\sigma (GgG)^{\mu\nu} \nabla_\nu \partial_\mu \phi_2). \end{aligned} \quad (3.6)$$

Hence

$$\begin{aligned} \text{Tr}(\Phi_1 H^{ab} [X_a, [X_b, \Phi_2]]) &= \text{Tr} \left( -[X_a, H^{ab}] [X_b, \Phi_2] \Phi_1 - H^{ab} [X_b, \Phi_2] [X_a, \Phi_1] \right) \\ &\sim \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} e^\sigma \phi_1 \left( \frac{1}{4} (\partial_\alpha (gG) - \partial_\alpha \sigma (gG)) G^{\alpha\nu} \partial_\nu \phi_2 \right. \\ &\quad \left. + \nabla_\nu \sigma (GgG)^{\mu\nu} \partial_\mu \phi_2 + (GgG)^{\mu\nu} \nabla_\nu \partial_\mu \phi_2 \right), \end{aligned} \quad (3.7)$$

which implies (3.2d) since  $\phi_1$  is arbitrary. Further simplification of this formula can be achieved in  $2n = 4$  dimensions, where (3.2e) follows directly from (3.2c) using the 4D identity (2.18). Hence in particular

$$\begin{aligned} (2\pi)^2 \text{Tr}(\Phi_2[X_a, (H^{ab} - \frac{1}{2}H\eta^{ab})[X_b, \Phi_1]]) &= -(2\pi)^2 \text{Tr}((H^{ab} - \frac{1}{2}H\eta^{ab})[X_a, \Phi_2][X_b, \Phi_1]) \\ &\sim \int d^4x \sqrt{g} e^\sigma g^{\mu\nu} \partial_\mu \phi_2 \partial_\nu \phi_1 \\ &= - \int d^4x \sqrt{g} g^{\mu\nu} \phi_2 \nabla'_\mu (e^\sigma \partial_\nu \phi_1), \end{aligned} \quad (3.8)$$

which for arbitrary  $\phi_2$  implies (3.2f).  $\blacksquare$

Finally, we also note the following identity which will be useful below:

$$\begin{aligned} H^{ab}[X_a, [X_b, \Phi]] &= [X_a, H^{ab}[X_b, \Phi]] - [X_a, H^{ab}][X_b, \Phi] \\ &\sim e^\sigma (2\eta \square_G \phi - e^\sigma g^{\mu\nu} \nabla_\mu \partial_\nu \phi + 2G^{\mu\nu} \partial_\nu \eta \partial_\mu \phi - \nabla_\nu (e^\sigma g^{\mu\nu}) \partial_\mu \phi) \\ &\quad - e^\sigma (e^\sigma \square_G x^c \partial_\alpha x_c + \partial_\alpha \eta) G^{\alpha\nu} \partial_\nu \phi \\ &= e^{2\sigma} (2e^{-\sigma} \eta \square_G \phi - g^{\mu\nu} \nabla_\nu \partial_\mu \phi + (e^{-\sigma} G^{\mu\nu} \partial_\nu \eta - g^{\mu\nu} \nabla_\nu \sigma) \partial_\mu \phi). \end{aligned} \quad (3.9)$$

### 3.2 Potential terms $k = 0$

For  $k = 0$ , consider first the following terms

$$\text{Tr}\left(-\frac{1}{4}H\right)^\ell \sim \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{-\sigma} \eta^\ell, \quad \text{for } \ell \in \mathbb{N}. \quad (3.10)$$

For  $\ell = 1$ , we recover the basic Yang-Mills matrix model

$$S_{YM} = -\frac{1}{4} \text{Tr}H \sim \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{-\sigma} \eta. \quad (3.11)$$

Now recall that (2.16)

$$e^{-\sigma} \eta = \frac{1}{2}(\alpha^2 + \alpha^{-2}) \geq 1, \quad (3.12)$$

which assumes its minimum  $e^{-\sigma} \eta = 1$  if and only if  $\alpha = \pm 1$ , i.e. for  $g_{\mu\nu} = G_{\mu\nu}$ . This means that for fixed embedding, the minimum of the action  $S_{YM}$  is achieved<sup>9</sup> if  $\alpha = \pm 1$ , i.e. if  $\theta^{\mu\nu}$  is self-dual w.r.t.  $g_{\mu\nu}$ . Curvature terms as discussed below may lead to small deviations from self-duality,

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad (3.13)$$

however the potential is expected to dominate as long as the curvature is “small”. This is an important mechanism, which justifies to focus on geometries where  $G_{\mu\nu} \approx g_{\mu\nu}$ . The deviations from (anti-)self-duality will be studied in more detail in Section 4; e.g. it will also be shown that  $e^{-\sigma} \eta = 1 + \mathcal{O}(h^2)$ .

Thus assuming  $G \approx g$ , the above potential terms for  $\ell > 1$  amount to

$$\text{Tr}\left(-\frac{1}{4}H\right)^\ell \sim \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{(\ell-1)\sigma} (e^{-\sigma} \eta)^\ell \stackrel{g \approx G}{\approx} \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{(\ell-1)\sigma}. \quad (3.14)$$

<sup>9</sup>This is certainly true in the Euclidean case, and in the Minkowski case provided we adopt complexified  $\theta^{\mu\nu}$  as discussed in Section 2.2 and Ref. [14].

Then these terms essentially determine a potential

$$S_{\text{pot}} = \sum_{\ell} a_{\ell} \text{Tr} H^{\ell} \stackrel{g \approx G}{\approx} \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} V(\sigma), \quad (3.15)$$

for  $e^{\sigma}$ . This is very interesting: if  $V(\sigma)$  has a non-trivial minimum, it will dynamically determine the vacuum expectation value of  $e^{\sigma}$  and hence the scale of non-commutativity. Thus  $e^{\sigma}$  will be essentially constant, simplifying considerably some of the considerations below. This is also important in order to preserve the equivalence principle, at least approximately, because the effective metric for fermions and scalars a priori differ by a conformal factor  $\sim e^{\sigma/3}$  [16, 17]. There are other terms with  $k = 0$  of type  $\text{Tr}(H^{ab}H_{bc}H^{ca})$  etc. For  $g \approx G$ , they essentially reduce to the same potential terms as above due to the projector property  $4H^{ab}\eta_{bb'}H^{b'c} = HH^{ac}$  which holds for  $g_{\mu\nu} = G_{\mu\nu}$ , assuming  $2n = 4$ . However this type of terms also depends on the dimension of  $\mathcal{M} \subset \mathbb{R}^D$ , and might help to single out 4-dimensional branes. This should be investigated elsewhere. (In fact,  $g_{\mu\nu} = G_{\mu\nu}$  is only possible for  $2n = 4$ , which alone would single out 4-dimensional branes.)

We can summarize these observations as follows: In the case of near-flat geometries the potential terms with  $k = 0$  are expected to dominate, leading to  $g_{\mu\nu} \approx G_{\mu\nu}$  and  $e^{\sigma} \approx \text{const}$ . Additional terms with  $k > 0$  involving more commutators typically correspond to curvature contributions as shown below, and may lead to small deviations from  $g = G$ . In fact, it turns out that  $\sigma = \text{const}$ . is incompatible with self-dual  $\theta^{\mu\nu}$  resp.  $g = G$  for general geometries<sup>10</sup>. Nevertheless, the presence of a potential  $V(\sigma)$  should ensure that  $\sigma$  is constant to a very good approximation, even in the presence of curvature. This is important because  $e^{\sigma}$  determines e.g. the gauge coupling constant. It also suggests that the symplectic structure obtained in [6] based on self-duality will be modified near the horizon, such that  $e^{\sigma} \approx \text{const}$ . is preserved. This should be studied in more detail elsewhere.

### 3.3 $\mathcal{O}(X^6)$ terms

For the sake of systematics we start our discussion of  $k > 0$  terms with the  $\mathcal{O}(X^6)$ , although the  $\mathcal{O}(X^{10})$  turn out to be much more appealing. As shown in [1], there are only two independent terms of order  $X^6$ , given by

$$S_6 = \text{Tr} \left( \alpha \square X^a \square X_a + \frac{\beta}{2} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] \right). \quad (3.16)$$

In the general case  $g_{\mu\nu} \neq G_{\mu\nu}$ , it seems that the easiest way to evaluate them is in terms of  $R[g]$  (also allowing us to compare with the one-loop results in [17]). We start our derivation by considering

$$\begin{aligned} \square X^a &\sim -\theta^{\mu\nu} \partial_{\mu} x^b \nabla'_{\nu} \left( \theta^{\alpha\beta} \partial_{\alpha} x_b \partial_{\beta} x^a \right) = - \left( e^{\sigma} G^{\nu\beta} \nabla'_{\nu} \partial_{\beta} x^a + e^{\sigma} \theta_{\alpha\rho}^{-1} G^{\nu\rho} \nabla'_{\nu} \theta^{\alpha\beta} \partial_{\beta} x^a \right) \\ &= -e^{\sigma} \left( G^{\mu\nu} \nabla'_{\mu} \nabla'_{\nu} x^a - G^{\mu\rho} \nabla'_{\nu} x^a \theta^{\sigma\nu} \nabla'_{\mu} \theta_{\sigma\rho}^{-1} \right), \end{aligned} \quad (3.17)$$

since  $\partial_{\mu} x^b \nabla'_{\nu} \partial_{\alpha} x_b = 0$ . It follows that

$$\begin{aligned} \square X^a \square X_a &\sim e^{2\sigma} \left( G^{\mu\nu} G^{\alpha\beta} \nabla'_{\mu} \nabla'_{\nu} x^a \nabla'_{\alpha} \nabla'_{\beta} x_a + e^{\sigma} G^{\mu\rho} G^{\alpha\tau} G^{\sigma\epsilon} \nabla'_{\mu} \theta_{\sigma\rho}^{-1} \nabla'_{\alpha} \theta_{\epsilon\tau}^{-1} \right) \\ &= e^{\sigma} \left( e^{\sigma} G^{\mu\nu} G^{\alpha\beta} \nabla'_{\mu} \nabla'_{\nu} x^a \nabla'_{\alpha} \nabla'_{\beta} x_a + g_{\rho\tau} \nabla'_{\mu} G^{\mu\rho} \nabla'_{\alpha} G^{\alpha\tau} \right), \end{aligned} \quad (3.18)$$

<sup>10</sup>For example, such a self-dual  $\theta^{\mu\nu}$  was determined for the Schwarzschild geometry in [6], and it turns out that  $e^{\sigma} \neq \text{const}$ .

using (2.22) for the second term (which is manifestly order  $\mathcal{O}(h^2)$ ). The second part of  $S_6$  is derived in Appendix B.2, and using the 4D identity (2.18) we find

$$\begin{aligned}
S_6 \sim & (\alpha + \beta) \int \frac{d^4x}{(2\pi)^2} \sqrt{g} e^\sigma G^{\mu\nu} G^{\rho\sigma} \nabla'_\mu \nabla'_\nu x^a \nabla'_\rho \nabla'_\sigma x_a \\
& + \beta \int \frac{d^4x}{(2\pi)^2} \sqrt{g} e^\sigma \left[ 2e^{-\sigma} \theta^{\mu\nu} \hat{\theta}^{\rho\sigma} R[g]_{\mu\rho\nu\sigma} - G^{\mu\nu} G^{\rho\sigma} R[g]_{\mu\rho\nu\sigma} - (GgG)^{\mu\nu} R[g]_{\mu\nu} \right. \\
& \quad \left. + \left( \frac{3}{4} (Gg) G^{\mu\nu} - g^{\mu\nu} \right) \partial_\mu \sigma \partial_\nu \sigma \right] \\
& + \alpha \int \frac{d^4x}{(2\pi)^2} \sqrt{g} \nabla'_\mu G^{\mu\rho} \nabla'_\alpha (Gg)_\rho^\alpha \\
& + \beta \int \frac{d^4x}{(2\pi)^2} \sqrt{g} e^\sigma \left[ \frac{1}{2} \left( \frac{3}{2} G^{\mu\nu} \nabla'_\mu (Gg) - \nabla'_\mu G^{\mu\nu} (Gg) \right) \partial_\nu \sigma + g^{\nu\rho} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} \partial_\mu \sigma \right. \\
& \quad - \nabla'_\mu (Gg)_\sigma^\nu G^{\rho\sigma} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} - \nabla'_\mu (Gg)_\sigma^\rho \theta_{\rho\tau}^{-1} \nabla'_\nu G^{\tau\mu} \theta^{\nu\sigma} \\
& \quad \left. + \frac{1}{2} e^\sigma G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \theta_{\nu\rho'}^{-1} \theta_{\sigma\rho}^{-1} \nabla'_\tau G^{\nu\sigma} \right] \tag{3.19}
\end{aligned}$$

where

$$\hat{\theta}^{\mu\nu} := G^{\mu\mu'} g_{\mu'\eta} \theta^{\eta\nu} \tag{3.20}$$

is an anti-symmetric tensor. This is manifestly tensorial for  $\alpha = -\beta$ . Using Eqn. (2.39) the first line of  $S_6$  in the semi-classical limit Eqn. (3.19) can also be written as

$$\frac{\alpha + \beta}{(2\pi)^2} \int d^4x \sqrt{g} e^\sigma \mathcal{P}_N^{ab} \square_{Gx_a} \square_{Gx_b}. \tag{3.21}$$

The action (3.19) simplifies considerably in the self-dual case  $g_{\mu\nu} = G_{\mu\nu}$ , reducing to the one previously computed in Ref. [1]. Furthermore, the terms surviving that limit are of the same type as those induced at one loop when coupling fermions to the matrix model, as was found in Ref. [17]. The leading order deviations from the self-dual case may be studied by expanding the above action around  $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ : To order  $\mathcal{O}(h)$  the action  $S_6$  semi-classically reads

$$\begin{aligned}
S_6 \sim & \frac{\alpha + \beta}{(2\pi)^2} \int d^4x \sqrt{g} e^\sigma (\square_g x^a - 2h^{\mu\nu} \nabla'_\mu \nabla'_\nu x^a) \square_g x_a \\
& + \frac{\beta}{(2\pi)^2} \int d^4x \sqrt{g} e^\sigma \left[ 2e^{-\sigma} \theta^{\mu\nu} (\theta^{\rho\sigma} - h^{\rho\alpha} g_{\alpha\beta} \theta^{\beta\sigma}) R[g]_{\mu\rho\nu\sigma} - 2R[g] + 4h^{\mu\nu} R[g]_{\mu\nu} \right. \\
& \quad + 2\partial_\mu \sigma \partial^\mu \sigma - 3h^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + 2\nabla'_\mu h^{\mu\nu} \partial_\nu \sigma - \frac{3}{4} \partial^\nu (hg) \partial_\nu \sigma \\
& \quad \left. + g^{\nu\rho} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} \partial_\mu \sigma + \nabla'_\mu h^{\nu\rho} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} \right] \\
& + \mathcal{O}(h^2). \tag{3.22}
\end{aligned}$$

As explained in Section 4,  $h_{\mu\nu}$  can be parametrized in terms of the deviation of the symplectic structure around its self-dual version, i.e.  $\theta_{\mu\nu}^{-1} = \bar{\theta}_{\mu\nu}^{-1} + F_{\mu\nu}$  where  $\bar{\theta}_{\mu\nu}^{-1}$  is self-dual with respect

to  $g_{\mu\nu}$ . Then the above action can be simplified further by considering terms only up to order  $\mathcal{O}(F)$ . This implies that  $(hg) = \mathcal{O}(F^2)$  can be dropped, and  $g^{\nu\rho}\nabla'_\nu\theta_{\tau\rho}^{-1} = g^{\nu\rho}\nabla'_\nu F_{\tau\rho} = \mathcal{O}(\partial h)$ . The same type of matrix model terms have also been considered on 2-dimensional branes in [18], where  $S_6$  for  $\alpha + \beta = 0$  reduces essentially to an integral over the Ricci scalar.

We also note that

$$R_{\mu\nu\rho\sigma}[g]\theta^{\mu\nu}\theta^{\rho\sigma} = W_{\mu\nu\rho\sigma}[g]\theta^{\mu\nu}\theta^{\rho\sigma} - 2e^\sigma R_{\mu\rho}[g]G^{\mu\rho} + \frac{1}{3}e^\sigma R[g](Gg) \quad (3.23)$$

where [15]

$$W_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\sigma\nu} - g_{\mu\sigma}R_{\rho\nu} - g_{\nu\rho}R_{\sigma\mu} + g_{\nu\sigma}R_{\rho\mu}) + \frac{1}{6}(Rg_{\mu\rho}g_{\sigma\nu} - Rg_{\mu\sigma}g_{\rho\nu}) \quad (3.24)$$

is the Weyl tensor on the 4 dimensional submanifold  $\mathcal{M}^4$ . In the case of (anti-)self-dual  $\theta$  we have  $g = G$ , and

$$R_{\mu\nu\rho\sigma}[g]\theta^{\mu\nu}\theta^{\rho\sigma} = W_{\mu\nu\rho\sigma}[g]\theta^{\mu\nu}\theta^{\rho\sigma} - \frac{2}{3}e^\sigma R[g]. \quad (3.25)$$

This is interesting for the following reason: As discussed below, it may be appropriate to average over the moduli space of Poisson structures  $\theta^{\mu\nu}$ , which essentially consists of (anti-)self-dual 2-forms with fixed determinant. This averaging over the asymptotic orientations leads to  $W_{\mu\nu\rho\sigma}[g]\langle\theta^{\mu\nu}\theta^{\rho\sigma}\rangle = 0$  since  $\langle\theta^{\mu\nu}\theta^{\rho\sigma}\rangle$  is Lorentz-invariant for (A)SD  $\theta$ , so that the term  $R_{\mu\nu\rho\sigma}[g]\theta^{\mu\nu}\theta^{\rho\sigma}$  essentially reduces to the Ricci scalar.

### 3.4 $\mathcal{O}(X^{10})$ terms

We now consider  $\mathcal{O}(X^{10})$  terms with  $k > 0$  (i.e. ignoring contributions to the potential as discussed above in Section 3.2). We are especially interested in a combination of terms which semi-classically more or less leads to the Einstein-Hilbert action. For  $g_{\mu\nu} \neq G_{\mu\nu}$ , the answer is not as simple as Eqn. (2.46) derived in Ref. [1]. As a starting point, we hence consider the term  $H^{ab}\square H_{ab}$  which previously has been shown to be the ‘‘central piece’’ leading to the Ricci-scalar in the semi-classical limit (i.e. the additional matrix terms were needed to make it intrinsic). The corresponding derivation is given in Appendix B.3. It reveals that the following combination of terms depend only on the intrinsic geometry through  $G_{\mu\nu}$ ,  $g_{\mu\nu}$  and  $e^\sigma$ , independent of the embedding  $\mathcal{M}^4 \subset \mathbb{R}^D$ :

$$\begin{aligned} & H^{ab}\square H_{ab} + 2\square X^c H^{ab}[X_a, [X_b, X_c]] \\ & \sim -e^{3\sigma}\left((GgGg)e^{-\sigma}\square_G e^\sigma + 8e^{-2\sigma}(\partial_\alpha\eta\partial^\alpha\eta - \eta\partial_\alpha\eta\partial^\alpha\sigma) - \frac{3}{2}\nabla_\nu g^{\mu\beta}\nabla^\nu g_{\mu\beta} + 2\nabla^\mu g_{\mu\beta}\nabla_\alpha g^{\alpha\beta}\right. \\ & \quad + (Gg)R_{\mu\eta}[G](GgG)^{\mu\eta} - 2R[G] - 2\nabla_{\mu'}(G^{\mu\mu'}g_{\mu\beta}\nabla_\alpha g^{\alpha\beta}) - G_{\mu\beta}\nabla_\alpha g^{\mu\rho}\nabla_\rho g^{\alpha\beta} \\ & \quad \left. + 2\nabla_\beta g^{\alpha\beta}e^{-\sigma}\partial_\alpha\eta - 4e^{-\sigma}\eta\nabla_\mu g^{\mu\alpha}\partial_\alpha\sigma + 2g^{\mu\nu}G_{\mu\alpha}\nabla_\beta g^{\alpha\beta}\partial_\nu\sigma\right). \end{aligned} \quad (3.26)$$

The second term in the first line is needed in order to cancel extrinsic terms, and in the self-dual limit it semi-classically coincides with its counter part of Ref. [1] (resp. the first term of Eqn. (2.46)).

In order to make the following results more transparent, we keep only terms of order  $\mathcal{O}(h)$  and drop higher-order terms in  $h$ . This is justified by the observation in Section 3.2

that the Yang-Mills action  $S_{YM}$  is quadratic in  $h$ , and therefore suppresses the deviation from self-duality. Then the above result yields

$$\begin{aligned} & \text{Tr}(H^{ab}\square H_{ab} + 2\square X^c H^{ab}[X_a, [X_b, X_c]]) \\ & \sim - \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{2\sigma} \left( 4R_{\mu\eta}[G](GgG)^{\mu\eta} - 2R[G] + 4e^{-\sigma}\square_G e^\sigma + 4\nabla_\beta g^{\alpha\beta} \partial_\alpha \sigma \right) + \mathcal{O}(\partial h^2). \end{aligned} \quad (3.27)$$

Using the intrinsic terms (3.30a), (3.30b), we also obtain the following forms

$$\begin{aligned} S_R & := \text{Tr}\left(\frac{1}{2}T^{ab}\square H_{ab} + \square X^c H^{ab}[X_a, [X_b, X_c]]\right) \\ & \sim - \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{2\sigma} \left( 3R[G] - 2R_{\mu\eta}[G]g^{\mu\eta} + 2\nabla_\beta g^{\alpha\beta} \partial_\alpha \sigma \right) + \mathcal{O}(\partial h^2), \end{aligned} \quad (3.28a)$$

$$\begin{aligned} \tilde{S}_R & := \text{Tr}\left(\frac{1}{2}H^{ab}(\square H_{ab} - [X^a, [X^b, H]]) + \square X^c H^{ab}[X_a, [X_b, X_c]]\right) \\ & \sim - \int \frac{d^4x}{(2\pi)^2} \sqrt{G} e^{2\sigma} \left( 3R[G] - 2R_{\mu\eta}[G]g^{\mu\eta} \right) + \mathcal{O}(\partial h^2), \end{aligned} \quad (3.28b)$$

noting that  $\eta = e^\sigma + \mathcal{O}(h^2)$  as well as

$$\begin{aligned} 2R_{\mu\eta}[G](GgG)^{\mu\eta} - R[G] & = (4e^{-\sigma}\eta - 1)R[G] - 2R_{\mu\eta}[G]g^{\mu\eta} \\ & = 3R[G] - 2R_{\mu\eta}[G]g^{\mu\eta} + \mathcal{O}(h^2 p^2). \end{aligned} \quad (3.29)$$

Here  $p^2$  stands for the curvature scale of the gravitational field  $R[G]$ , and we will assume that  $\mathcal{O}(h^2 p^2) = \mathcal{O}(\partial h^2)$ . For  $G = g$ , we recover the result obtained in [1], and the ‘‘local’’ formula (2.46) follows from (3.26).

**Additional  $\mathcal{O}(X^{10})$  terms.** Consider the following terms, whose semi-classical limit is obtained easily from our previous results (2.43), (2.49) and (2.31b):

$$\begin{aligned} [X^a, H_{ab}][X^b, H] & \sim 4e^\sigma (e^\sigma \nabla_\alpha g^{\beta\alpha} \partial_\beta \eta - G^{\alpha\beta} \partial_\alpha \eta \partial_\beta \eta) \\ & = 4e^{3\sigma} (\nabla_\alpha h^{\beta\alpha} \partial_\beta \sigma - G^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma) + \mathcal{O}(h^2), \end{aligned} \quad (3.30a)$$

$$\begin{aligned} [X^a, H][X_a, H] & \sim -16e^\sigma G^{\mu\nu} \partial_\mu \eta \partial_\nu \eta \\ & = -16e^{3\sigma} G^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \mathcal{O}(h^2). \end{aligned} \quad (3.30b)$$

There are additional  $\mathcal{O}(X^{10})$  terms which are of order  $\mathcal{O}(h^2)$ , which we will not discuss in this paper. These include

$$H^{ab}\square X^a \square X^b = -e^{3\sigma} G_{\mu\nu} \nabla_\alpha g^{\alpha\mu} \nabla_\beta g^{\beta\nu} = \mathcal{O}(\partial h^2), \quad (3.31a)$$

$$\begin{aligned} [X_a, H^{ab}][X^c, H_{bc}] & \sim -e^{2\sigma} (\square_G x^a \partial_\alpha x_a + e^{-\sigma} \partial_\alpha \eta) \theta^{\alpha\beta} \partial_\beta x^b (\square_G x^c \partial_\delta x_c + e^{-\sigma} \partial_\delta \eta) \theta^{\delta\gamma} \partial_\gamma x_b \\ & = e^{3\sigma} \left( -G_{\alpha\beta} \nabla_\gamma g^{\alpha\gamma} \nabla_\kappa g^{\beta\kappa} + 2e^{-\sigma} \nabla_\alpha g^{\beta\alpha} \partial_\beta \eta - e^{-2\sigma} G^{\alpha\beta} \partial_\alpha \eta \partial_\beta \eta \right) \\ & = \mathcal{O}(\partial h^2). \end{aligned} \quad (3.31b)$$

The trace of the last term can in fact be written in a number of different ways,

$$\begin{aligned} \text{Tr}([X^a, H^{ab}][X^c, H^{bc}]) & = -\text{Tr}(H^{ab}[X^a, [X^c, H^{bc}]]) \\ & = \text{Tr}(H^{ab}[X^c, [H^{bc}, X^a]]) + \text{Tr}(H^{ab}[H^{bc}, [X^a, X^c]]) \\ & = \text{Tr}([X^c, H^{ab}][X^a, H^{bc}]) + \text{Tr}([H^{ab}, H^{bc}][X^a, X^c]). \end{aligned} \quad (3.32)$$

**Extrinsic terms.** The  $\mathcal{O}(X^{10})$  terms above have been tailored to be tensorial, i.e. such that they only depend on the intrinsic geometry of  $\mathcal{M}$  in the semi-classical limit. There are of course also terms which depend on the “extrinsic curvature” i.e. on the embedding of  $\mathcal{M} \subset \mathbb{R}^D$ . The prototype of such a term is given by

$$\square X^a \square X_a \sim e^{2\sigma} \square_G x^a \square_G x_a, \quad (3.33)$$

cf. (3.16), or similarly

$$\begin{aligned} \text{Tr} \square X_a [X^b, (H_{bc} - \frac{1}{2} \eta_{bc} H) [X^c, X_a]] &\sim \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} e^\sigma \square_G x^a (\square_g x_a + g^{\mu\nu} \partial_\mu \sigma \partial_\nu x_a) \\ &= \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} e^\sigma (\square_G x^a \square_g x_a - g^{\mu\nu} G_{\nu\alpha} \nabla_\beta g^{\alpha\beta} \partial_\mu \sigma), \\ H \square X^a \square X^b &\sim -4e^{2\sigma} \eta \square_G x^a \square_G x_a. \end{aligned} \quad (3.34)$$

For  $g_{\mu\nu} \sim G_{\mu\nu}$ , these terms essentially coincide, and single out harmonic embeddings  $\square_G x^a = 0$  as vacuum geometries. In general, such terms should be expected to arise upon quantization, and their physical significance must be investigated. It seems plausible that they become important at cosmological scales where the intrinsic curvature is small, leading to long-distance modifications of gravity somewhat along the lines of the “harmonic” solutions given in [19, 20]. Such long-distance modifications are very interesting in view of the major puzzles in cosmology, notably in the context of dark energy and dark matter.

On the other hand, the term  $\square_G x^a \square_G x_a$  might also serve as a UV cutoff for perturbation theory, since it behaves as  $(p^2)^2$  on  $\mathbb{R}_\theta^4$ , where  $p$  denotes the momentum scale.

## 4 Gravitational action and degrees of freedom

Now consider the matrix model action combining (2.1) with curvature terms such as (3.28), which in the semi-classical limit become

$$\begin{aligned} \tilde{S}_R &\sim - \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} e^{2\sigma} (3R[G] - 2R_{\mu\eta} [G] g^{\mu\eta}) + \mathcal{O}(\partial h^2) \\ &= - \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} e^{2\sigma} (R[g] - 3R^{\mu\nu} [g] h_{\mu\nu} + \nabla^\nu \nabla'^\mu h_{\mu\nu}) + \mathcal{O}(\partial h^2), \end{aligned} \quad (4.1a)$$

$$\begin{aligned} S_{\text{simple}} &= \tilde{S}_R + \frac{1}{2} \text{Tr} [X^a, T_{ab}] [X^b, H] \\ &\sim - \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} e^{2\sigma} (R[g] - 3R^{\mu\nu} [g] h_{\mu\nu}) + \mathcal{O}(\partial h^2), \end{aligned} \quad (4.1b)$$

using (2.42) where

$$G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \quad (4.2)$$

and therefore  $G^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$ . The term  $\nabla^\nu \nabla'^\mu h_{\mu\nu}$  can be eliminated by subtracting suitable terms of type (3.30a), (3.30b) from the action. We will therefore drop it and consider  $S_{\text{simple}}$  in order to simplify the presentation. For the same reason the possible additional contributions from  $S_6$  (3.22) will also be omitted here. We will furthermore drop all terms of order  $\mathcal{O}(\partial h^2)$ , however we keep the  $\mathcal{O}(h^2)$  e.g. in the Yang-Mills terms and the potential terms, which are expected to be important for weak gravity. This will be justified below, and ensures a well-defined and compact moduli space of vacuum solutions for  $\theta^{\mu\nu}$ .

Because these actions are tensorial (i.e. independent of the embedding  $\mathcal{M}^4 \subset \mathbb{R}^D$ ), the semi-classical equations of motion are obtained simply by varying the independent geometrical degrees of freedom encoded in  $g_{\mu\nu}$  and  $\theta^{\mu\nu}$ . To understand these degrees of freedom, note that in a given “coordinate patch”, the embedding metric  $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j \eta_{ij}$  is determined by the scalar fields  $\phi^i(x)$ . The Poisson tensor  $\theta^{\mu\nu}$  can be parametrized as

$$\theta_{\mu\nu}^{-1} = \bar{\theta}_{\mu\nu}^{-1} + F_{\mu\nu} \quad (4.3)$$

where  $\bar{\theta}_{\mu\nu}^{-1}$  is self-dual<sup>11</sup> with respect to  $g_{\mu\nu}$ , and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Thus the independent degrees of freedom are given by the embedding  $\phi^i$  and  $F_{\mu\nu}$  resp.  $A_\mu$ .

In principle, one could now derive the equations of motion resulting from (4.1) as well as from the other possible terms such as  $S_6$ , Eqn. (3.22). This is straightforward as long as only “intrinsic” terms are considered, which depend on  $g_{\mu\nu}$  and  $\theta^{\mu\nu}$ . The variation of the fundamental degrees of freedom can be separated into variations  $\delta_\phi$  of the embedding leading to

$$\delta_\phi g_{\mu\nu} = \delta\phi^i \phi^j \eta_{ij} + \phi^i \delta\phi^j \eta_{ij}, \quad (4.4)$$

and the variation  $\delta_A$  of the Poisson tensor given by

$$\delta_A F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu. \quad (4.5)$$

We postpone this straightforward but tedious task to future work, and only draw some generic and qualitative conclusions below. In the presence of terms which also depend on the embedding resp. extrinsic curvature such as  $\square_G x^a \square_G x^a$ , the action would lead to higher-order equations of motion in the embedding  $\phi^i$ . In particular, this leads to the “harmonic branch” as discussed in [20], whose physical relevance requires further study. It may suffice here to say that such extrinsic terms may lead to very interesting cosmological solutions [19], while the viability for solar system gravity is not clear.

**Yang-Mills action and vacuum configurations for  $\theta^{\mu\nu}$ .** We can gain some important insights even without deriving equations of motion. Let us expand the Yang-Mills term to  $\mathcal{O}(F^2)$ , but keep only  $\mathcal{O}(\partial F)$  resp.  $\mathcal{O}(\partial h)$  in the curvature terms due to the explicit gravitational momentum scale. This gives

$$\begin{aligned} \theta^{\mu\nu} &= ((1 + \bar{\theta}F)^{-1} \bar{\theta})^{\mu\nu} \\ &= (\bar{\theta} - \bar{\theta}F\bar{\theta} + \bar{\theta}F\bar{\theta}F\bar{\theta})^{\mu\nu} + \mathcal{O}(F^3) \\ &= \bar{\theta}^{\mu\nu} + \bar{\theta}^{\mu\mu'} \bar{\theta}^{\nu\nu'} F_{\mu'\nu'} + (\bar{\theta}F\bar{\theta}F\bar{\theta})^{\mu\nu} + \mathcal{O}(F^3), \end{aligned} \quad (4.6a)$$

$$h_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu} = -e^{\bar{\sigma}} (\bar{\theta}^{-1} g F)_{\mu\nu} - e^{\bar{\sigma}} (F g \bar{\theta}^{-1})_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\bar{\theta}F) + \mathcal{O}(F^2), \quad (4.6b)$$

$$g_{\mu\nu} h^{\mu\nu} = 0 + \mathcal{O}(F^2), \quad (4.6c)$$

$$h_{\mu\nu} h^{\mu\nu} = 2(\bar{\theta}F\bar{\theta}F) - 2e^{\bar{\sigma}} (F g F g) - (\bar{\theta}F)(\bar{\theta}F) + \mathcal{O}(F^3), \quad (4.6d)$$

$$\begin{aligned} e^{-\sigma} &= e^{-\bar{\sigma}} \det(1 + \bar{\theta}F)^{1/2} \\ &= e^{-\bar{\sigma}} \left( 1 + \frac{1}{2} (\bar{\theta}F) + \frac{1}{8} (\bar{\theta}F)(\bar{\theta}F) - \frac{1}{4} (\bar{\theta}F\bar{\theta}F) + \mathcal{O}(F^3) \right), \end{aligned} \quad (4.6e)$$

$$(\theta g \theta g) = -4e^{\bar{\sigma}} + 2e^{\bar{\sigma}} (\bar{\theta}F) + e^{2\bar{\sigma}} (g F g F) - 2e^{\bar{\sigma}} (\bar{\theta}F\bar{\theta}F) + \mathcal{O}(F^3), \quad (4.6f)$$

---

<sup>11</sup>One could equally well consider the case of small perturbations around anti-self-dual  $\bar{\theta}_{\mu\nu}^{-1}$ .

$$\begin{aligned}
\frac{1}{4}(Gg) &= -\frac{1}{4}e^{-\sigma}(\theta g\theta g) = 1 + \frac{1}{4}(\bar{\theta}F\bar{\theta}F) - \frac{1}{8}(\bar{\theta}F)(\bar{\theta}F) - \frac{1}{4}e^{\bar{\sigma}}(gFgF) + \mathcal{O}(F^3) \\
&= 1 + \frac{1}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(F^3).
\end{aligned} \tag{4.6g}$$

Here we use a condensed notation where neighbouring indices are contracted and  $()$  denotes a trace (e.g.  $\bar{\theta}F \equiv \bar{\theta}^{\mu\nu}F_{\nu\eta}$  and  $(\bar{\theta}F) \equiv \bar{\theta}^{\mu\nu}F_{\nu\mu}$ ), as well as

$$e^{\bar{\sigma}}\delta_{\nu}^{\mu} = -(\bar{\theta}g\bar{\theta}g)_{\nu}^{\mu}, \quad \nabla^{\mu}\bar{\theta}_{\mu\nu}^{-1} = 0. \tag{4.7}$$

The relation (4.6c) is in fact a consequence of  $|G| = |g|$  in 4 dimensions, (4.7) holds for any self-dual  $\bar{\theta}_{\mu\nu}^{-1}$ , and  $\bar{\sigma}$  is defined through  $\bar{\theta}^{\mu\nu}$ , so that  $\bar{\mathcal{J}}_{\nu}^{\mu}$  defines an almost-complex structure. We will assume  $h_{\mu\nu}$  to be small, and accordingly we will drop all terms of order  $\mathcal{O}(\partial F^2)$ .

The r.h.s. of (4.6g) acquires a geometric meaning due to the relation

$$\frac{1}{8}(F\bar{\theta})(F\bar{\theta}) - \frac{1}{4}(F\bar{\theta}F\bar{\theta}) = \text{Pfaff}(F_{\mu\nu})\text{Pfaff}(\bar{\theta}^{\mu\nu}), \tag{4.8}$$

(cf. [2]) where

$$\text{Pfaff}(\bar{\theta}^{\mu\nu}) = \frac{1}{8}\epsilon_{\mu\nu\rho\eta}\bar{\theta}^{\mu\nu}\bar{\theta}^{\rho\eta} = \frac{1}{4}\frac{1}{\sqrt{|g|}}\bar{\theta}^{\mu\nu}(\star_g\bar{\theta})^{\mu'\nu'}g_{\mu\mu'}g_{\nu\nu'} = \pm\sqrt{|\bar{\theta}^{\mu\nu}|}. \tag{4.9}$$

Note that  $\text{Pfaff}(\bar{\theta})$  is positive (negative) for (anti-) self-dual  $\bar{\theta}^{\mu\nu}$ . Then the Yang-Mills matrix model action (2.1) in the semi-classical limit becomes<sup>12</sup>

$$\begin{aligned}
S_{YM} &\sim \int \frac{d^4x}{(2\pi)^2}\sqrt{g}e^{-\sigma}\eta = \int \frac{d^4x}{(2\pi)^2}\sqrt{g}\left(1 + \frac{1}{8}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(F^3)\right) \\
&= \int \frac{d^4x}{(2\pi)^2}\sqrt{g}\left(1 + \frac{1}{4}e^{\bar{\sigma}}F_{\mu\nu}F_{\mu'\nu'}g^{\mu\mu'}g^{\nu\nu'} - \text{Pfaff}(F_{\mu\nu})\text{Pfaff}(\bar{\theta}^{\mu\nu}) + \mathcal{O}(F^3)\right) \\
&= \int \frac{d^4x}{(2\pi)^2}\sqrt{g}\left(1 + \frac{1}{8}e^{\bar{\sigma}}(F \mp \star_g F)_{\mu\nu}(F \mp \star_g F)_{\mu'\nu'}g^{\mu\mu'}g^{\nu\nu'} + \mathcal{O}(F^3)\right),
\end{aligned} \tag{4.10}$$

where  $\star_g$  denotes the Hodge star w.r.t.  $g_{\mu\nu}$ , and  $\mp$  is minus for self-dual  $\bar{\theta}^{\mu\nu}$  and vice versa.

Recalling that any 2-form can be decomposed into self-dual (SD) and anti-selfdual (ASD) components, we arrive at an important result: ASD fluctuations  $F_{\mu\nu}$  around a SD background  $\bar{\theta}_{\mu\nu}^{-1}$  give a *positive* contribution to  $S_{YM}$  and are hence suppressed, consistent with (2.16). On the other hand, the SD part of  $F_{\mu\nu}$  does *not* contribute to  $S_{YM}$  but determines the “dilaton field”  $e^{\sigma}$ . Conversely, SD fluctuations around an ASD background are suppressed by  $S_{YM}$ , while  $e^{\sigma}$  encodes ASD fluctuations. This justifies to focus on geometries with  $G_{\mu\nu} \approx g_{\mu\nu}$ , and makes clear that it is the embedding rather than the  $\theta_{\mu\nu}^{-1}$  which plays the central role for the emergent gravity<sup>13</sup>.

In particular, it follows that the moduli space of vacuum configurations of  $S_{YM}$  (for fixed embedding) consists of 2 disjoint components  $\bar{\Sigma} = \bar{\Sigma}^+ \cup \bar{\Sigma}^-$  given by the space of (A)SD

<sup>12</sup>It is interesting to compare this with the action for non-Abelian field strength [14], which has a somewhat similar structure. The Abelian case has also been considered by A. Schenkel (unpublished).

<sup>13</sup>It is nevertheless interesting to recall that this subject was sparked by the observation that the  $U(1)$  “would-be” gauge modes acquire a geometrical meaning through  $G^{\mu\nu}$ , leading to  $h_{\mu\nu}$  which do give Ricci-flat fluctuations around flat backgrounds [2, 21]. This gauge sector is given a central role in [22]. The ultimate physical relevance of these  $U(1)$  modes is still to be understood.

symplectic structures  $\bar{\theta}_{\mu\nu}^{-1}$  w.r.t.  $g_{\mu\nu}$ , and  $S_{YM}$  provides a positive definite action which suppresses fluctuations away from  $\bar{\Sigma}$ . These sectors  $\bar{\Sigma}^\pm$  are disconnected, and characterized by the sign of  $\text{Pfaff}(\bar{\theta}^{\mu\nu})$ . Now observe that  $e^\sigma$  defines a scalar function on  $\bar{\Sigma}$  (4.6e) which measures the “strength” of  $\theta^{\mu\nu}$ , i.e. the non-commutativity scale. Hence a potential  $V(e^\sigma)$  as in (3.15) with a non-trivial minimum,

$$\begin{aligned} V(e^\sigma) &= V_0 + \frac{1}{2}M^2(e^\sigma - x_0)^2 + \dots \\ &= V_0 + \frac{1}{2}M^2\left(e^{\bar{\sigma}} - x_0 - \frac{1}{2}e^{\bar{\sigma}}(\bar{\theta}F)\right)^2 + \dots, \end{aligned} \quad (4.11)$$

where  $V_0$ ,  $M$  and  $x_0$  are constants, will set the NC scale resp. the vacuum scale  $e^\sigma \approx \text{const.}$  Then  $\Sigma$  becomes compact, e.g.  $\Sigma^\pm \cong S^2$  in the near-flat case. On the other hand, terms in the gravitational action such as  $R_{\mu\nu}h^{\mu\nu}$  may lead to small deviations from (anti-)self-duality. Moreover,  $e^\sigma = \text{const.}$  may not be compatible with (A)SD  $\theta_{\mu\nu}^{-1}$  in the presence of curvature, cf. [6]. Then (4.11) suggests  $(\bar{\theta}F) \approx 2(1 - e^{-\bar{\sigma}}x_0) \neq 0$  if  $M$  is large, with  $F \rightarrow 0$  as  $x \rightarrow \infty$ . Therefore the physical moduli space  $\Sigma = \Sigma^+ \cup \Sigma^-$  of vacua will consist of symplectic forms  $\theta_{\mu\nu}^{-1} = \bar{\theta}_{\mu\nu}^{-1} + F_{\mu\nu}$  which are small deformations of (A)SD fields, characterized (in the asymptotically flat case) by the asymptotic orientation of  $\bar{\theta}_{\mu\nu}^{-1}$ .

If the function  $V(e^\sigma)$  has flat directions, then one can pick a vacuum with arbitrary scale  $e^{\bar{\sigma}}$ . The kinetic term  $\partial^\mu\sigma\partial_\mu\sigma$  would still suppress variations of  $\sigma$ .

We conclude that the above type of action represents a well-defined variational problem for the geometry, and leads to metrics with  $g_{\mu\nu} \approx G_{\mu\nu}$  as well as  $e^\sigma \approx \text{const.}$  Note that although we focused on the case of Euclidean signature, the steps go through in the Minkowski case provided one adopts complexified  $\theta^{\mu\nu}$  as discussed above, which do admit (anti-)self-dual configurations  $\star_g\theta = \pm i\theta$ . This provides an important simplification and progress for the analysis of the emergent gravity theory.

**Further perspectives and physical implications.** One obvious class of vacuum solutions of (3.28b) and (2.1) is given by Ricci-flat spaces along with an (A)SD  $\theta^{\mu\nu}$  (hence  $h_{\mu\nu} = 0$ ) such that  $e^\sigma = \text{const.}$  The problem is that in general, Ricci-flat spaces may not admit such (A)SD  $\theta^{\mu\nu}$  such that  $e^\sigma = \text{const.}$  This is illustrated in [6] where a self-dual  $\bar{\theta}_{\mu\nu}^{-1}$  was found with  $e^\sigma \neq \text{const.}$

The above analysis suggests the following strategy to find solutions for the coupled system  $(g_{\mu\nu}, \theta^{\mu\nu})$ : for a given metric  $g_{\mu\nu}$ , compute first a self-dual symplectic form  $\bar{\theta}_{\mu\nu}^{-1}$ ; this will lead to some  $e^{\bar{\sigma}}$  which in general is not constant. Then  $F_{\mu\nu}$  resp.  $h_{\mu\nu}$  should be determined through the full equations of motion, which will take the form of modified inhomogeneous Maxwell equations, schematically

$$g^{\mu\mu'}\nabla^\nu(e^{\bar{\sigma}}F_{\mu'\nu}) = J^\mu. \quad (4.12)$$

Here  $J^\mu$  will depend on  $\partial_\nu\tilde{V}(e^{\bar{\sigma}})$  and  $(\bar{\theta}F)$ , and may include matter contributions which turn out to act as dipole sources [4]. In the presence of a suitable potential  $V(e^\sigma)$  and/or a kinetic term  $\partial^\mu\sigma\partial_\mu\sigma$ , this will lead to  $e^\sigma \approx \text{const.}$  Since the gauge coupling as well as the NC scale depends on  $e^\sigma$ , this is probably essential to meet precision tests of general relativity and the time-independence of the fine structure constant.

The example of the Schwarzschild geometry [6] indicates a certain tension between the requirements  $e^\sigma = \text{const.}$  and  $g_{\mu\nu} = G_{\mu\nu}$ , since  $\theta^{\mu\nu}$  is determined by solving Maxwell-like equations with non-trivial boundary conditions. This would presumably be acceptable if

$h_{\mu\nu} = \mathcal{O}(R)$  for asymptotically flat 4-dimensional geometry, where  $R$  denotes the scale of the gravitational curvature. In that case, the additional terms in the gravitational action such as  $h^{\mu\nu}R_{\mu\nu} = \mathcal{O}(R^2)$  are suppressed at least in the case of weak gravity, leading to nearly-Ricci-flat spaces  $R_{\mu\nu} \approx 0$  as (vacuum) solutions in agreement with general relativity. However, this has not been shown at this point.

Even if the equations governing  $\theta^{\mu\nu}$  are so rigid that  $h_{\mu\nu}$  cannot be neglected, one might still effectively recover an (almost)-constant  $e^\sigma$  along with (almost)-ASD  $\theta_{\mu\nu}^{-1}$  e.g. by considering branes with compact extra dimensions, such as  $\mathcal{M}^4 \times S^2 \subset \mathbb{R}^{10}$ . This is very natural also to obtain non-Abelian gauge groups as required for particle physics (cf. [23]), and will be studied elsewhere in more detail.

There is another interesting point which should be kept in mind. Once a solution for  $\theta^{\mu\nu}$  is found, the quantization of the theory requires to integrate over the fluctuations in  $F_{\mu\nu}$  (recall that this would-be  $U(1)$  gauge field couples only to the gravitational sector). However, there is in fact a moduli space  $\Sigma$  of solutions  $\theta^{\mu\nu}$ , corresponding to different asymptotic orientations of  $\theta^{\mu\nu}$  (this is obvious in the flat case). The question then arises whether one should also integrate over this moduli space<sup>14</sup>. In particular, this would amount to an integration over all configurations corresponding to different asymptotics of  $\theta^{\mu\nu}$  related by Lorentz rotations. The Lorentz-violating term  $W\theta\theta$  (3.25) would then disappear from the action. This issue boils down to the question whether or not there really is a non-trivial VEV  $\langle\theta^{\mu\nu}\rangle$ , spontaneously breaking Lorentz invariance. Note that this is not essential for the mechanism of gravity presented here, which works also (and in fact simplifies) under weaker assumptions such as  $\langle\theta^{\mu\nu}\rangle = 0$  but  $\langle\theta^{\mu\nu}\theta^{\mu'\nu'}\rangle \neq 0$ .

Finally, we should perhaps comment on the cosmological constant problem, which in the present setting amounts to explaining why  $V' = 0$  implies  $V \approx 0$ , i.e. that  $V \approx 0$  at its minimum (cp. (4.11)). At this stage (in the ‘‘Einstein branch’’ [4]) this problem may appear to be similar as in standard GR, but again there are additional ingredients such as extrinsic curvature, compact extra dimensions, an additional (harmonic) branch of solutions, etc. which may shed new light on this problem.

## 5 Concluding remarks

The results of this paper represent a further step in the long-term project of studying the effective gravity theory emergent from matrix models of Yang-Mills type. One important new insight is that the ‘‘bare’’ Yang-Mills term defines a positive-definite action for  $h_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu}$ , which implies that the effective metric approximately coincides with the induced (embedding) metric. Furthermore, we studied the geometrical meaning of higher-order terms in the matrix model for general backgrounds, identifying in particular an action which is very similar to the Einstein-Hilbert action, taking into account  $G_{\mu\nu} \approx g_{\mu\nu}$  and  $e^\sigma \approx \text{const}$ . Such terms are expected at the level of the quantum effective action, or alternatively they can be added to the action by hand. These results are very welcome in the quest for a realistic theory of (quantum) gravity.

We also identified some specific issues and potential problems in clarifying the physical viability and the relation with general relativity. One issue is a certain ‘‘tension’’ between self-dual  $\theta^{\mu\nu}$  and  $e^\sigma \approx \text{const}$ , which both seem natural and desirable in view of the above results. Once this is understood, one can proceed to reliably analyze the equations for the

<sup>14</sup>See also [10] for a related discussion in the context of non-commutative field theory.

embedding resp. for the effective metric, which then describes gravity and its deviation from GR.

The bottom line is that the model defines a highly non-trivial coupled system for the embedding  $g_{\mu\nu}$  and the Poisson structure  $\theta^{\mu\nu}$ , and contains some (quantum) theory of gravity. This complexity is of course essential for any serious candidate for a realistic theory, but makes the identification of the “relevant” configurations and solutions non-trivial. An additional complication is that quantum effects must be taken into account, e.g. through higher-order terms as discussed here. Furthermore, the case of compact extra dimensions and the implications of non-trivial extrinsic terms such as  $\square_G x^a \square_G x^a$  must be studied systematically. Clearly much more work is needed before the physical viability of these models can be reliably addressed. On the other hand, the models are sufficiently clear-cut such that their physical content can finally be understood.

## Acknowledgements

This work was supported by the “Fonds zur Förderung der Wissenschaftlichen Forschung” (FWF) under contract P21610-N16. H.S. is grateful for hospitality at the AEI Golm, and useful discussions with J. Arnlind, J. Hoppe, D. Oriti, M. Sivakumar, A. Schenkel, P. Schupp, R. Szabo and S. Theisen are acknowledged.

## Appendix A: Derivation of (2.41c) and (2.41d)

Consider

$$\begin{aligned}
& G^{\alpha\beta} C_{\alpha;\sigma\nu} C_{\beta;\mu\rho} G^{\rho\nu} G^{\sigma\mu} \\
&= \frac{1}{4} G^{\alpha\beta} (\nabla_\nu g_{\sigma\alpha} + \nabla_\sigma g_{\nu\alpha} - \nabla_\alpha g_{\sigma\nu}) (\nabla_\mu g_{\rho\beta} + \nabla_\rho g_{\mu\beta} - \nabla_\beta g_{\rho\mu}) G^{\rho\nu} G^{\sigma\mu} \\
&= \frac{1}{2} G^{\alpha\beta} \left( \nabla_\nu g_{\sigma\alpha} G^{\sigma\mu} - \frac{1}{2} \nabla_\alpha g_{\sigma\nu} G^{\sigma\mu} \right) (\nabla_\mu g_{\rho\beta} + \nabla_\rho g_{\mu\beta} - \nabla_\beta g_{\rho\mu}) G^{\rho\nu} \\
&= \frac{1}{2} \left( \nabla_\nu (2e^{-\sigma} \eta G^{\beta\mu} - g^{\beta\mu}) - \frac{1}{2} G^{\alpha\beta} \nabla_\alpha g_{\sigma\nu} G^{\sigma\mu} \right) G^{\rho\nu} (\nabla_\mu g_{\rho\beta} + \nabla_\rho g_{\mu\beta} - \nabla_\beta g_{\rho\mu}) \\
&= \left( G^{\beta\mu} \partial_\nu (e^{-\sigma} \eta) - \frac{1}{2} \nabla_\nu g^{\beta\mu} \right) \nabla^\nu g_{\mu\beta} - G^{\alpha\beta} \nabla_\alpha \left( e^{-\sigma} \eta G^{\mu\rho} - \frac{1}{2} g^{\mu\rho} \right) \left( \nabla_\rho g_{\mu\beta} - \frac{1}{2} \nabla_\beta g_{\rho\mu} \right) \\
&= \frac{3}{2} \partial_\nu (e^{-\sigma} \eta) \partial^\nu (Gg) - \frac{3}{4} \nabla_\nu g^{\beta\mu} \nabla^\nu g_{\mu\beta} - \partial_\alpha (e^{-\sigma} \eta) \nabla_\mu (2e^{-\sigma} \eta G^{\mu\alpha} - g^{\mu\alpha}) \\
&\quad + \frac{1}{2} \nabla_\alpha g^{\mu\rho} \nabla_\rho (2e^{-\sigma} \eta \delta_\mu^\alpha - G_{\mu\beta} g^{\alpha\beta}) \\
&= 4\partial_\nu (e^{-\sigma} \eta) \partial^\nu (e^{-\sigma} \eta) - \frac{3}{4} \nabla_\nu g^{\beta\mu} \nabla^\nu g_{\mu\beta} + 2\partial_\alpha (e^{-\sigma} \eta) \nabla_\mu g^{\mu\alpha} - \frac{1}{2} G_{\mu\beta} \nabla_\alpha g^{\mu\rho} \nabla_\rho g^{\alpha\beta} \quad (\text{A.1})
\end{aligned}$$

assuming  $2n = 4$ , where we have used (2.18).

The relation (2.41d) can be seen as follows:

$$\begin{aligned}
& g^{\sigma\mu} \nabla_\sigma C_{\mu;\rho\nu} - g^{\sigma\mu} \nabla_\rho C_{\mu;\sigma\nu} \\
&= \frac{1}{2} g^{\sigma\mu} \nabla_\sigma (\nabla_\rho g_{\mu\nu} + \nabla_\nu g_{\rho\mu} - \nabla_\mu g_{\rho\nu}) - \frac{1}{2} g^{\sigma\mu} \nabla_\rho \nabla_\nu g_{\sigma\mu}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}g^{\sigma\mu} \left( (\nabla_\rho \nabla_\sigma g_{\mu\nu} + \nabla_\nu \nabla_\sigma g_{\rho\mu} - \nabla_\sigma \nabla_\mu g_{\rho\nu}) - \nabla_\rho \nabla_\nu g_{\sigma\mu} \right. \\
&\quad \left. + (R_{\sigma\rho\mu}{}^\alpha g_{\alpha\nu} + R_{\sigma\rho\nu}{}^\alpha g_{\mu\alpha}) + (R_{\sigma\nu\rho}{}^\alpha g_{\alpha\mu} + R_{\sigma\nu\mu}{}^\alpha g_{\rho\alpha}) \right) \\
&= \frac{1}{2}g^{\sigma\mu} \left( \nabla_\rho \nabla_\sigma g_{\mu\nu} + \nabla_\nu \nabla_\sigma g_{\rho\mu} - \nabla_\sigma \nabla_\mu g_{\rho\nu} - \nabla_\rho \nabla_\nu g_{\sigma\mu} \right) \\
&\quad + \frac{1}{2} \left( g^{\sigma\mu} R_{\sigma\rho\mu\beta} [G] (Gg)_\nu^\beta + g^{\sigma\mu} R_{\sigma\nu\mu\beta} [G] (Gg)_\rho^\beta - 2R_{\alpha\rho\beta\nu} [G] G^{\alpha\beta} \right) \\
&= \frac{1}{2} \left( -\nabla_\rho \nabla^\mu h_{\mu\nu} - \nabla_\nu \nabla^\mu h_{\rho\mu} + \square_g h_{\rho\nu} + g^{\sigma\mu} \nabla_\rho \nabla_\nu h_{\sigma\mu} \right) \\
&\quad + \frac{1}{2} \left( -R_{\rho\beta} [g] h^{\beta\alpha} g_{\alpha\nu} - R_{\nu\beta} [g] h^{\beta\alpha} g_{\alpha\rho} + 2R_{\alpha\rho\beta\nu} [g] h^{\alpha\beta} \right) + \mathcal{O}(h^2). \tag{A.2}
\end{aligned}$$

Now (2.41d) follows noting that  $g^{\rho\nu} \nabla \nabla h_{\rho\nu} = 0 + \mathcal{O}(h^2)$  due to (4.6).

## Appendix B: Semi-classical results for matrix model extensions

### B.1 Derivation of (3.2d)

To see (3.2d), consider

$$\begin{aligned}
H^{ab}[X_a, [X_b, \Phi]] &\sim e^\sigma G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \theta^{\alpha\beta} \partial_\alpha x_a \partial_\beta (\theta^{\rho\eta} \partial_\rho x_b \partial_\eta \phi) \\
&= e^\sigma (Gg)^\nu{}_\alpha \theta^{\alpha\beta} \left( \theta^{\rho\eta} g_{\nu\rho} \partial_\beta \partial_\eta \phi + \partial_\beta (\theta^{\rho\eta} g_{\rho\nu}) \partial_\eta \phi - \partial_\beta \partial_\nu x^b \theta^{\rho\eta} \partial_\rho x_b \partial_\eta \phi \right) \\
&= e^{2\sigma} (GgG)^{\beta\eta} \partial_\beta \partial_\eta \phi + e^\sigma \hat{\theta}^{\nu\beta} \partial_\beta (e^\sigma G^{\eta\rho} \theta_{\rho\nu}^{-1}) \partial_\eta \phi \\
&= e^{2\sigma} (GgG)^{\beta\eta} \partial_\beta \partial_\eta \phi + e^\sigma \partial_\beta e^\sigma (GgG)^{\eta\beta} \partial_\eta \phi \\
&\quad + e^{2\sigma} (Gg)^\beta{}_\rho \partial_\beta G^{\eta\rho} \partial_\eta \phi - \frac{1}{2} e^{2\sigma} \hat{\theta}^{\nu\beta} \partial_\rho \theta_{\nu\beta}^{-1} G^{\eta\rho} \partial_\eta \phi \tag{B.1}
\end{aligned}$$

using the fact that  $\hat{\theta}^{\mu\nu}$  is anti-symmetric, and

$$\begin{aligned}
\hat{\theta}^{\nu\beta} \partial_\beta \theta_{\rho\nu}^{-1} &= -\hat{\theta}^{\nu\beta} \partial_\rho \theta_{\nu\beta}^{-1} - \hat{\theta}^{\nu\beta} \partial_\nu \theta_{\beta\rho}^{-1} \\
2\hat{\theta}^{\nu\beta} \partial_\beta \theta_{\rho\nu}^{-1} &= -\hat{\theta}^{\nu\beta} \partial_\rho \theta_{\nu\beta}^{-1}. \tag{B.2}
\end{aligned}$$

On the other hand, consider

$$\begin{aligned}
(GgG)^{\mu\nu} \Gamma_{\mu\nu}^\alpha [G] &= \frac{1}{2} (GgG)^{\mu\nu} (\partial_\mu G_{\nu\beta} + \partial_\nu G_{\mu\beta} - \partial_\beta G_{\mu\nu}) G^{\alpha\beta} \\
&= -(Gg)^\mu{}_\beta \partial_\mu G^{\alpha\beta} - \frac{1}{2} (GgG)^{\mu\nu} \partial_\beta G_{\mu\nu} G^{\alpha\beta} \\
&= -(Gg)^\mu{}_\beta \partial_\mu G^{\alpha\beta} + \frac{1}{2} (G^{\mu\nu} \partial_\beta g_{\mu\nu} + 2\hat{\theta}^{\alpha\beta} \partial_\mu \theta_{\alpha\beta}^{-1} - (Gg) \partial_\beta \sigma) G^{\alpha\beta} \\
&= -(Gg)^\mu{}_\beta \partial_\mu G^{\alpha\beta} + \frac{1}{2} \left( \frac{1}{2} \partial_\beta (Gg) + \hat{\theta}^{\alpha\beta} \partial_\mu \theta_{\alpha\beta}^{-1} - \frac{1}{2} (Gg) \partial_\beta \sigma \right) G^{\alpha\beta}
\end{aligned}$$

using

$$\frac{1}{2} \partial_\mu (Gg) = G^{\alpha\beta} \partial_\mu g_{\alpha\beta} + \hat{\theta}^{\alpha\beta} \partial_\mu \theta_{\alpha\beta}^{-1} - \frac{1}{2} (Gg) \partial_\mu \sigma. \tag{B.3}$$

Therefore we get

$$\begin{aligned}
H^{ab}[X_a, [X_b, \phi]] &\sim e^{2\sigma} (GgG)^{\beta\eta} \partial_\beta \partial_\eta \phi + e^\sigma \partial_\beta e^\sigma (GgG)^{\eta\beta} \partial_\eta \phi \\
&\quad - e^{2\sigma} ((GgG)^{\mu\nu} \Gamma_{\mu\nu}^\eta - \frac{1}{4} \partial_\rho (Gg) G^{\eta\rho} + \frac{1}{4} (Gg) \partial_\rho \sigma G^{\eta\rho}) \partial_\eta \phi \\
&= e^{2\sigma} (GgG)^{\beta\eta} \nabla_\beta \partial_\eta \phi + e^\sigma \partial_\beta e^\sigma (GgG)^{\eta\beta} \partial_\eta \phi \\
&\quad + \frac{1}{4} e^{2\sigma} (\partial_\rho (Gg) - (Gg) \partial_\rho \sigma) G^{\eta\rho} \partial_\eta \phi,
\end{aligned} \tag{B.4}$$

which is indeed tensorial.

## B.2 Derivation of (3.19)

We use the (constant) background metric  $\eta_{ab}$  to pull down Latin indices, i.e.  $x_a \equiv x^b \eta_{ab}$ , and consider first

$$\begin{aligned}
\frac{1}{2} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] &\sim \frac{1}{2} e^\sigma G^{\nu\sigma} \nabla'_\nu \left( \theta^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b \right) \nabla'_\sigma (\theta^{\tau\epsilon} \partial_\tau x_a \partial_\epsilon x_b) \\
&= e^\sigma G^{\nu\sigma} \left( e^\sigma G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + \frac{1}{2} g_{\alpha\tau} g_{\beta\epsilon} \nabla'_\nu \theta^{\alpha\beta} \nabla'_\sigma \theta^{\tau\epsilon} \right).
\end{aligned} \tag{B.5}$$

From the Jacobi identity

$$\theta^{\mu\alpha} \nabla'_\alpha \theta^{\nu\sigma} + \theta^{\nu\alpha} \nabla'_\alpha \theta^{\sigma\mu} + \theta^{\sigma\alpha} \nabla'_\alpha \theta^{\mu\nu} = 0, \tag{B.6}$$

it follows that

$$\nabla'_\rho \theta^{\mu\nu} = (\theta^{\mu\alpha} \theta^{\nu\sigma} - \theta^{\mu\sigma} \theta^{\nu\alpha}) \nabla'_\alpha \theta_{\rho\sigma}^{-1}, \tag{B.7}$$

which enables us to simplify the second term of (B.5) further:

$$\begin{aligned}
\frac{1}{2} g_{\alpha\tau} g_{\beta\epsilon} \nabla'_\nu \theta^{\alpha\beta} \nabla'_\sigma \theta^{\tau\epsilon} &= g_{\alpha\tau} g_{\beta\epsilon} \nabla'_\nu \theta^{\alpha\beta} \theta^{\tau\mu} \theta^{\epsilon\rho} \nabla'_\mu \theta_{\sigma\rho}^{-1} \\
&= \theta^{\epsilon\rho} \nabla'_\nu (e^\sigma (Gg)_\epsilon^\mu) \nabla'_\mu \theta_{\sigma\rho}^{-1} + e^\sigma (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu} \nabla'_\mu \theta_{\sigma\rho}^{-1} \\
&= e^\sigma \left( \partial_\nu \sigma \frac{1}{2} \hat{\theta}^{\mu\rho} \nabla'_\sigma \theta_{\mu\rho}^{-1} + (\nabla'_\nu (Gg)_\tau^\mu \theta^{\tau\rho} + (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu}) \nabla'_\mu \theta_{\sigma\rho}^{-1} \right),
\end{aligned} \tag{B.8}$$

where  $\hat{\theta}^{\mu\nu} := (Gg)_\epsilon^\mu \theta^{\epsilon\nu}$ . Hence,

$$\begin{aligned}
&\frac{(2\pi)^2}{2} \text{Tr}([X^c, [X^a, X^b]][X_c, [X_a, X_b]]) \\
&\sim \int d^4x \sqrt{g} e^\sigma G^{\nu\sigma} \left( G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + \partial_\nu \sigma \frac{1}{2} \hat{\theta}^{\mu\rho} \nabla'_\sigma \theta_{\mu\rho}^{-1} \right. \\
&\quad \left. + (\nabla'_\nu (Gg)_\tau^\mu \theta^{\tau\rho} + (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu}) \nabla'_\mu \theta_{\sigma\rho}^{-1} \right) \\
&= \int d^4x \sqrt{g} e^\sigma \left( G^{\nu\sigma} G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + G^{\nu\sigma} \partial_\nu \sigma \frac{1}{2} \hat{\theta}^{\mu\rho} \nabla'_\sigma \theta_{\mu\rho}^{-1} \right. \\
&\quad \left. + G^{\nu\sigma} \nabla'_\nu (Gg)_\tau^\mu \theta^{\tau\rho} \nabla'_\mu \theta_{\sigma\rho}^{-1} - (\partial_\mu \sigma G^{\nu\sigma} + \nabla'_\mu G^{\nu\sigma}) (Gg)_\tau^\rho \theta_{\sigma\rho}^{-1} \nabla'_\nu \theta^{\tau\mu} \right. \\
&\quad \left. - G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \nabla'_\mu (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu} - G^{\nu\sigma} (Gg)_\tau^\rho \theta_{\sigma\rho}^{-1} \nabla'_\mu \nabla'_\nu \theta^{\tau\mu} \right)
\end{aligned}$$

$$\begin{aligned}
&= \int d^4x \sqrt{g} e^\sigma \left( G^{\nu\sigma} G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + G^{\nu\sigma} \partial_\nu \sigma e^{-\sigma} \nabla'_\sigma \eta \right. \\
&\quad - G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \nabla'_\nu (Gg)_\tau^\mu \nabla'_\mu \theta^{\tau\rho} + (\partial_\mu \sigma G^{\nu\sigma} + \nabla'_\mu G^{\nu\sigma}) (Gg)_\sigma^\rho \theta_{\tau\rho}^{-1} \nabla'_\nu \theta^{\tau\mu} \\
&\quad \left. - G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \nabla'_\mu (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu} - (GgG)^{\nu\rho} \theta_{\rho\tau}^{-1} ([\nabla'_\mu, \nabla'_\nu] \theta^{\tau\mu} + \nabla'_\nu (\partial_\mu \sigma \theta^{\tau\mu})) \right) \\
&= \int d^4x \sqrt{g} e^\sigma \left( G^{\nu\sigma} G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + \frac{1}{4} G^{\nu\mu} \partial_\nu \sigma (\nabla'_\mu (Gg) + (Gg) \partial_\mu \sigma) \right. \\
&\quad - G^{\nu\sigma} \theta_{\sigma\rho}^{-1} (\nabla'_\nu (Gg)_\tau^\mu \nabla'_\mu \theta^{\tau\rho} + \nabla'_\mu (Gg)_\tau^\nu \nabla'_\nu \theta^{\tau\mu}) \\
&\quad + \partial_\mu \sigma (GgG)^{\nu\rho} \theta_{\tau\rho}^{-1} \nabla'_\nu \theta^{\tau\mu} + \nabla'_\mu (Gg)_\alpha^\nu G^{\rho\alpha} \theta_{\tau\rho}^{-1} \nabla'_\nu \theta^{\tau\mu} \\
&\quad - (GgG)^{\nu\rho} \theta_{\rho\tau}^{-1} (R[g]_{\nu\mu\eta}{}^\tau \theta^{\eta\mu} + R[g]_{\nu\eta}{}^{\tau\eta}) \\
&\quad \left. - (GgG)^{\nu\rho} (\nabla'_\nu \partial_\rho \sigma + \partial_\mu \sigma \theta_{\rho\tau}^{-1} \nabla'_\nu \theta^{\tau\mu}) \right) \\
&= \int d^4x \sqrt{g} e^\sigma \left( G^{\nu\sigma} G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + \frac{1}{4} G^{\nu\mu} \partial_\nu \sigma (\nabla'_\mu (Gg) + (Gg) \partial_\mu \sigma) \right. \\
&\quad - \nabla'_\mu (Gg)_\sigma^\nu \nabla'_\nu \theta_{\tau\rho}^{-1} (G^{\mu\rho} \theta^{\sigma\tau} + G^{\rho\sigma} \theta^{\tau\mu}) - G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \nabla'_\mu (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu} \\
&\quad - 2\partial_\mu \sigma (GgG)^{\nu\rho} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} + \nabla'_\nu (GgG)^{\nu\rho} \partial_\rho \sigma + (GgG)^{\nu\rho} \partial_\nu \sigma \partial_\rho \sigma \\
&\quad \left. + e^{-\sigma} \hat{\theta}^{\nu\beta} R[g]_{\nu\mu\eta\beta} \theta^{\eta\mu} - (GgG)^{\nu\rho} R[g]_{\nu\rho} \right) \\
&= \int d^4x \sqrt{g} e^\sigma \left( G^{\nu\sigma} G^{\alpha\tau} \nabla'_\nu \nabla'_\alpha x^a \nabla'_\sigma \nabla'_\tau x_a + \frac{1}{4} G^{\nu\mu} \partial_\nu \sigma (\partial_\mu (Gg) + (Gg) \partial_\mu \sigma) \right. \\
&\quad - \nabla'_\mu (Gg)_\sigma^\nu G^{\rho\sigma} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} - \nabla'_\mu G^{\rho\rho'} \theta_{\rho'\tau}^{-1} \nabla'_\nu G^{\tau\mu} \theta^{\nu\sigma} g_{\sigma\rho} \\
&\quad + \frac{1}{2} e^\sigma G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \theta_{\nu\rho'}^{-1} \theta_{\sigma\rho}^{-1} \nabla'_\tau G^{\nu\sigma} - 2(GgG + \frac{1}{2}g)^{\nu\rho} \nabla'_\nu \theta_{\tau\rho}^{-1} \theta^{\tau\mu} \partial_\mu \sigma + \nabla'_\nu (GgG)^{\nu\rho} \partial_\rho \sigma \\
&\quad \left. + (GgG)^{\nu\rho} \partial_\nu \sigma \partial_\rho \sigma + e^{-\sigma} \hat{\theta}^{\nu\beta} R[g]_{\nu\mu\eta\beta} \theta^{\eta\mu} - (GgG)^{\nu\rho} R[g]_{\nu\rho} \right) \tag{B.9}
\end{aligned}$$

using (2.18), (2.21), (2.22), and the identities  $\nabla'_\sigma \eta = \frac{1}{2}(g\theta g)_{\mu\nu} \nabla'_\sigma \theta^{\mu\nu}$  and  $(Gg)_\alpha^\mu \theta_{\beta\mu}^{-1} = -(Gg)_\beta^\mu \theta_{\alpha\mu}^{-1}$  as well as

$$\begin{aligned}
G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \nabla'_\mu (Gg)_\tau^\rho \nabla'_\nu \theta^{\tau\mu} &= -\nabla'_\mu G^{\rho\rho'} \nabla'_\nu (e^\sigma \theta_{\rho'\tau}^{-1} G^{\tau\mu}) G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \\
&= e^\sigma \nabla'_\mu G^{\rho\rho'} \nabla'_{\rho'\tau} \theta_{\tau\nu}^{-1} G^{\tau\mu} G^{\nu\sigma} \theta_{\sigma\rho}^{-1} + e^\sigma G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \nabla'_\tau \theta_{\nu\rho'}^{-1} G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \\
&\quad - \nabla'_\mu G^{\rho\rho'} \theta_{\rho'\tau}^{-1} \nabla'_\nu (e^\sigma G^{\tau\mu}) G^{\nu\sigma} \theta_{\sigma\rho}^{-1} \\
&= -\nabla'_\mu (G^{\rho\rho'} g_{\sigma\rho}) \nabla'_{\rho'\tau} \theta_{\tau\nu}^{-1} G^{\tau\mu} \theta^{\nu\sigma} + e^\sigma \frac{1}{2} G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \nabla'_\tau (\theta_{\nu\rho'}^{-1} G^{\nu\sigma} \theta_{\sigma\rho}^{-1}) \\
&\quad - \nabla'_\mu G^{\rho\rho'} \theta_{\rho'\tau}^{-1} \nabla'_\nu (e^\sigma G^{\tau\mu}) G^{\nu\sigma} \theta_{\sigma\rho}^{-1} - e^\sigma \frac{1}{2} G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \theta_{\nu\rho'}^{-1} \theta_{\sigma\rho}^{-1} \nabla'_\tau G^{\nu\sigma} \\
&= \nabla'_\mu (Gg)_\sigma^\nu \nabla'_\nu \theta_{\tau\rho}^{-1} G^{\rho\mu} \theta^{\tau\sigma} - \frac{1}{2} G^{\tau\mu} \partial_\mu (Gg) \partial_\tau \sigma - e^\sigma \nabla'_\mu G^{\rho\rho'} G^{\tau\mu} \theta_{\rho'\tau}^{-1} \theta_{\sigma\rho}^{-1} G^{\nu\sigma} \partial_\nu \sigma \\
&\quad - e^\sigma \nabla'_\mu G^{\rho\rho'} \theta_{\rho'\tau}^{-1} \nabla'_\nu G^{\tau\mu} G^{\nu\sigma} \theta_{\sigma\rho}^{-1} - e^\sigma \frac{1}{2} G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \theta_{\nu\rho'}^{-1} \theta_{\sigma\rho}^{-1} \nabla'_\tau G^{\nu\sigma} \\
&= \nabla'_\mu (Gg)_\sigma^\nu \nabla'_\nu \theta_{\tau\rho}^{-1} G^{\rho\mu} \theta^{\tau\sigma} - g^{\mu\sigma'} \nabla'_\mu \theta_{\sigma\sigma'}^{-1} \theta^{\nu\sigma} \partial_\nu \sigma \\
&\quad - e^\sigma \nabla'_\mu G^{\rho\rho'} \theta_{\rho'\tau}^{-1} \nabla'_\nu G^{\tau\mu} G^{\nu\sigma} \theta_{\sigma\rho}^{-1} - e^\sigma \frac{1}{2} G^{\tau\mu} \nabla'_\mu G^{\rho\rho'} \theta_{\nu\rho'}^{-1} \theta_{\sigma\rho}^{-1} \nabla'_\tau G^{\nu\sigma} \tag{B.10}
\end{aligned}$$

where the last step follows from

$$e^\sigma \nabla'_\mu G^{\rho\rho'} G^{\tau\mu} \theta_{\rho'\tau}^{-1} \theta_{\sigma\rho}^{-1} G^{\nu\sigma} \partial_\nu \sigma = e^{-\sigma} \nabla'_\mu G^{\rho\rho'} (g_{\rho'\tau} \theta^{\tau\mu}) (g_{\sigma\rho} \theta^{\nu\sigma}) \partial_\nu \sigma$$

$$\begin{aligned}
&= e^{-\sigma} \nabla'_\mu \left( \frac{1}{2} (Gg) g_{\tau\sigma} - G_{\tau\sigma} \right) \theta^{\tau\mu} \theta^{\nu\sigma} \partial_\nu \sigma \\
&= -\frac{1}{2} G^{\mu\nu} \partial_\mu (Gg) \partial_\nu \sigma + g^{\mu\sigma'} \nabla'_\mu \theta_{\sigma\sigma'}^{-1} \theta^{\nu\sigma} \partial_\nu \sigma
\end{aligned}$$

using the 4D identity (2.18), since

$$\begin{aligned}
e^{-\sigma} \nabla'_\mu G_{\tau\sigma} \theta^{\tau\mu} \theta^{\nu\sigma} \partial_\nu \sigma &= e^{-\sigma} \nabla'_\mu (e^\sigma \theta_{\tau\tau}^{-1} \theta_{\sigma\sigma'}^{-1} g^{\tau'\sigma'}) \theta^{\tau\mu} \theta^{\nu\sigma} \partial_\nu \sigma \\
&= \nabla'_\mu \theta_{\tau\tau}^{-1} g^{\tau'\nu} \theta^{\tau\mu} \partial_\nu \sigma - g^{\mu\sigma'} \nabla'_\mu \theta_{\sigma\sigma'}^{-1} \theta^{\nu\sigma} \partial_\nu \sigma - g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \\
&= -\theta_{\tau\tau}^{-1} g^{\tau'\nu} \nabla'_\mu \theta^{\tau\mu} \partial_\nu \sigma - g^{\mu\sigma'} \nabla'_\mu \theta_{\sigma\sigma'}^{-1} \theta^{\nu\sigma} \partial_\nu \sigma - g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \\
&= -g^{\mu\sigma'} \nabla'_\mu \theta_{\sigma\sigma'}^{-1} \theta^{\nu\sigma} \partial_\nu \sigma
\end{aligned} \tag{B.11}$$

due to (2.21). Together with the definition of the curvature tensor with respect to the induced metric (2.34) we obtain (3.19).

### B.3 Derivation of $\mathcal{O}(X^{10})$ terms

Consider first

$$\begin{aligned}
H^{ab} \square H_{ab} &\sim -e^{2\sigma} G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \square_G (e^\sigma G^{\mu'\nu'} \partial_{\mu'} x_a \partial_{\nu'} x_b) \\
&= -e^{2\sigma} G^{\mu\nu} \partial_\mu x^a \partial_\nu x^b \left( \square_G e^\sigma G^{\mu'\nu'} \partial_{\mu'} x_a \partial_{\nu'} x_b + 2e^\sigma G^{\mu'\nu'} \square_G \partial_{\mu'} x_a \partial_{\nu'} x_b \right. \\
&\quad \left. + 4\partial^\alpha e^\sigma G^{\mu'\nu'} \nabla_\alpha \partial_{\mu'} x_a \partial_{\nu'} x_b + 2e^\sigma G^{\mu'\nu'} \nabla_\alpha \partial_{\mu'} x_a \nabla^\alpha \partial_{\nu'} x_b \right) \\
&= -e^{3\sigma} \left( (GgGg) e^{-\sigma} \square_G e^\sigma + 2G^{\mu\nu} G^{\mu'\nu'} G^{\alpha\beta} C_{\mu;\alpha\mu'} C_{\nu;\beta\nu'} \right. \\
&\quad \left. + 2(GgG)^{\mu\mu'} (\partial_\mu x^a \nabla_{\mu'} \square_G x_a + R_{\mu\eta} (Gg)_{\mu'}^\eta + 2C_{\mu;\alpha\mu'} \partial^\alpha \sigma) \right) \\
&= -e^{3\sigma} \left( (GgGg) e^{-\sigma} \square_G e^\sigma + 2G^{\mu\nu} G^{\mu'\nu'} G^{\alpha\beta} C_{\mu;\alpha\mu'} C_{\nu;\beta\nu'} \right. \\
&\quad \left. + 2(GgG)^{\mu\mu'} (2C_{\mu;\alpha\mu'} \partial^\alpha \sigma - G_{\mu\beta} \nabla_{\mu'} \nabla_\alpha g^{\alpha\beta} - \nabla_{\mu'} \partial_\mu x^a \square_G x_a + (Gg)_{\mu'}^\eta R_{\mu\eta} [G]) \right)
\end{aligned} \tag{B.12}$$

using (2.31b). The second term is elaborated in (2.41c), and using the 4D identity (2.18), (2.37) and (2.29) we obtain

$$\begin{aligned}
H^{ab} \square H_{ab} &\sim -e^{3\sigma} \left( (GgGg) e^{-\sigma} \square_G e^\sigma + \frac{1}{2} \partial_\nu (Gg) \partial^\nu (Gg) + \partial_\alpha (Gg) \nabla_\mu g^{\mu\alpha} - \frac{3}{2} \nabla_\nu g^{\beta\mu} \nabla^\nu g_{\mu\beta} \right. \\
&\quad \left. + ((Gg) G^{\mu\mu'} - 2g^{\mu\mu'}) (2C_{\mu;\alpha\mu'} \partial^\alpha \sigma - \nabla_{\mu'} \partial_\mu x^a \square_G x_a + R_{\mu\eta} (Gg)_{\mu'}^\eta) \right. \\
&\quad \left. - 2G^{\mu\mu'} g_{\mu\beta} \nabla_{\mu'} \nabla_\alpha g^{\alpha\beta} - G_{\mu\beta} \nabla_\alpha g^{\mu\rho} \nabla_\rho g^{\alpha\beta} \right) \\
&= -e^{3\sigma} \left( (GgGg) e^{-\sigma} \square_G e^\sigma + \frac{1}{2} \partial_\nu (Gg) \partial^\nu (Gg) + \partial_\alpha (Gg) \nabla_\mu g^{\mu\alpha} - \frac{3}{2} \nabla_\nu g^{\beta\mu} \nabla^\nu g_{\mu\beta} \right. \\
&\quad \left. - G_{\mu\beta} \nabla_\alpha g^{\mu\rho} \nabla_\rho g^{\alpha\beta} + (Gg) \left( \partial_\alpha (Gg) \partial^\alpha \sigma - \square_G x^a \square_G x_a + R_{\mu\eta} [G] (GgG)^{\mu\eta} \right) \right. \\
&\quad \left. - 2G^{\mu\mu'} g_{\mu\beta} \nabla_{\mu'} \nabla_\alpha g^{\alpha\beta} + 2g^{\mu\mu'} \nabla_{\mu'} \partial_\mu x^a \square_G x_a - 2R[G] \right).
\end{aligned} \tag{B.13}$$

Note that there are two terms  $g^{\mu\mu'} \nabla_{\mu'} \partial_\mu x^a \square_G x_a$  and  $\square_G x^a \square_G x_a$ , which are not tensorial but depend on the embedding of  $\mathcal{M}^4 \subset \mathbb{R}^D$ . They coincide in the self-dual case where

$g_{\mu\nu} = G_{\mu\nu}$ , but in general they are independent. In order to obtain tensorial expressions, we must cancel these terms. This can be achieved using (3.9):

$$\begin{aligned}
& H^{ab}\square H_{ab} + 2\square X^c H^{ab}[X_a, [X_b, X_c]] \\
& \sim -e^{3\sigma} \left( (GgGg)e^{-\sigma}\square_G e^\sigma + \frac{1}{2}\partial_\nu(Gg)\partial^\nu(Gg) + \partial_\alpha(Gg)\nabla_\mu g^{\mu\alpha} - \frac{3}{2}\nabla_\nu g^{\beta\mu}\nabla^\nu g_{\mu\beta} - 2R[G] \right. \\
& \quad \left. - G_{\mu\beta}\nabla_\alpha g^{\mu\rho}\nabla_\rho g^{\alpha\beta} + (Gg)\left(\partial_\alpha(Gg)\partial^\alpha\sigma + R_{\mu\eta}[G](GgG)^{\mu\eta}\right) - 2G^{\mu\mu'}g_{\mu\beta}\nabla_{\mu'}\nabla_\alpha g^{\alpha\beta} \right) \\
& \quad - 2e^{3\sigma}\square_G x^c\partial_\mu x^c(e^{-\sigma}G^{\mu\nu}\partial_\nu\eta - g^{\mu\nu}\nabla_\nu\sigma) \\
& = -e^{3\sigma} \left( (GgGg)e^{-\sigma}\square_G e^\sigma + \frac{1}{2}\partial_\nu(Gg)\partial^\nu(Gg) + \partial_\alpha(Gg)\nabla_\mu g^{\mu\alpha} - \frac{3}{2}\nabla_\nu g^{\mu\beta}\nabla^\nu g_{\mu\beta} \right. \\
& \quad \left. + (Gg)R_{\mu\eta}[G](GgG)^{\mu\eta} - 2R[G] - 2\nabla_{\mu'}(G^{\mu\mu'}g_{\mu\beta}\nabla_\alpha g^{\alpha\beta}) + 2\nabla^\mu g_{\mu\beta}\nabla_\alpha g^{\alpha\beta} \right. \\
& \quad \left. - G_{\mu\beta}\nabla_\alpha g^{\mu\rho}\nabla_\rho g^{\alpha\beta} + (Gg)\partial_\alpha(Gg)\partial^\alpha\sigma + 2\nabla_\beta g^{\alpha\beta}e^{-\sigma}\partial_\alpha\eta - 2g^{\mu\nu}G_{\mu\alpha}\nabla_\beta g^{\alpha\beta}\partial_\nu\sigma \right), \quad (\text{B.14})
\end{aligned}$$

where we also used (2.31b). This is manifestly tensorial, and can be rewritten in various ways. Under the integral, (B.14) can be simplified further using

$$\int d^4x\sqrt{G}e^{2\sigma}\nabla_\nu(G^{\mu\nu}g_{\mu\beta}\nabla_\alpha g^{\alpha\beta}) = \int d^4x\sqrt{G}e^{2\sigma}\left(2g^{\mu\nu}G_{\nu\eta}\nabla_\alpha g^{\alpha\eta}\partial_\mu\sigma - 4e^{-\sigma}\eta\nabla_\alpha g^{\alpha\beta}\partial_\nu\sigma\right),$$

so that

$$\begin{aligned}
& (2\pi)^2\text{Tr}(H^{ab}\square H_{ab} + 2\square X^c H^{ab}[X_a, [X_b, X_c]]) \\
& \sim -\int d^4x\sqrt{G}e^{2\sigma}\left((GgGg)e^{-\sigma}\square_G e^\sigma + \frac{1}{2}\partial_\nu(Gg)\partial^\nu(Gg) + \partial_\alpha(Gg)\nabla_\mu g^{\mu\alpha} - \frac{3}{2}\nabla_\nu g^{\mu\beta}\nabla^\nu g_{\mu\beta} \right. \\
& \quad \left. + (Gg)R_{\mu\eta}[G](GgG)^{\mu\eta} - 2R[G] + 8e^{-\sigma}\eta\nabla_\alpha g^{\alpha\beta}\partial_\nu\sigma + 2\nabla^\mu g_{\mu\beta}\nabla_\alpha g^{\alpha\beta} \right. \\
& \quad \left. - G_{\mu\beta}\nabla_\alpha g^{\mu\rho}\nabla_\rho g^{\alpha\beta} + (Gg)\partial_\alpha(Gg)\partial^\alpha\sigma + 2\nabla_\beta g^{\alpha\beta}e^{-\sigma}\partial_\alpha\eta - 6g^{\mu\nu}G_{\mu\alpha}\nabla_\beta g^{\alpha\beta}\partial_\nu\sigma\right) \\
& = -\int d^4x\sqrt{G}e^{2\sigma}\left(4e^{-\sigma}\square_G e^\sigma + 4R_{\mu\eta}[G](GgG)^{\mu\eta} - 2R[G] + 4\nabla_\alpha g^{\alpha\beta}\partial_\nu\sigma + \mathcal{O}(h^2)\right), \quad (\text{B.15})
\end{aligned}$$

noting that  $(Gg) = 4 + \mathcal{O}(h^2)$  due to (4.6c),  $(GgGg) = \frac{1}{2}(Gg)(Gg) - 4$  and  $\eta = e^\sigma + \mathcal{O}(h^2)$ .

## References

- [1] D. N. Blaschke and H. Steinacker, ‘Curvature and Gravity Actions for Matrix Models’, *Class. Quant. Grav.* **27** (2010) 165010, [arXiv:1003.4132].
- [2] H. Steinacker, ‘Emergent Gravity from Noncommutative Gauge Theory’, *JHEP* **12** (2007) 049, [arXiv:0708.2426].
- [3] H. Grosse, H. Steinacker and M. Wohlgenannt, ‘Emergent Gravity, Matrix Models and UV/IR Mixing’, *JHEP* **04** (2008) 023, [arXiv:0802.0973].
- [4] H. Steinacker, ‘Emergent Geometry and Gravity from Matrix Models: An Introduction’, *Class. Quant. Grav.* **27** (2010) 133001, [arXiv:1003.4134].
- [5] A. Friedman, ‘Local isometric embedding of Riemannian manifolds with indefinite metric’, *J. Math. Mech.* **10** (1961) 625;  
C. Clarke, ‘On the Global Isometric Embedding of Pseudo-Riemannian Manifolds’, *Proc. Royal Soc. London* **A314** (1970) 417.

- [6] D. N. Blaschke and H. Steinacker, ‘Schwarzschild Geometry Emerging from Matrix Models’, *Class. Quantum Grav.* **27** (2010) 185020, [arXiv:1005.0499].
- [7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, ‘A large-N reduced model as superstring’, *Nucl. Phys.* **B498** (1997) 467–491, [arXiv:hep-th/9612115].
- [8] J. Madore, S. Schraml, P. Schupp and J. Wess, ‘Gauge theory on noncommutative spaces’, *Eur. Phys. J.* **C16** (2000) 161–167, [arXiv:hep-th/0001203].
- [9] T. Banks, N. Seiberg and S. H. Shenker, ‘Branes from matrices’, *Nucl. Phys.* **B490** (1997) 91–106, [arXiv:hep-th/9612157];  
I. Chepelev, Y. Makeenko and K. Zarembo, ‘Properties of D-branes in matrix model of IIB superstring’, *Phys. Lett.* **B400** (1997) 43–51, [arXiv:hep-th/9701151];  
J. Hoppe, ‘Some classical solutions of membrane matrix model equations’, [arXiv:hep-th/9702169].
- [10] S. Doplicher, K. Fredenhagen and J. E. Roberts, ‘The Quantum structure of space-time at the Planck scale and quantum fields’, *Commun. Math. Phys.* **172** (1995) 187–220, [arXiv:hep-th/0303037].
- [11] M. R. Douglas and N. A. Nekrasov, ‘Noncommutative field theory’, *Rev. Mod. Phys.* **73** (2001) 977–1029, [arXiv:hep-th/0106048];  
R. J. Szabo, ‘Quantum Field Theory on Noncommutative Spaces’, *Phys. Rept.* **378** (2003) 207–299, [arXiv:hep-th/0109162];  
V. Rivasseau, ‘Non-commutative renormalization,’ in *Quantum Spaces — Poincaré Seminar 2007*, B. Duplantier and V. Rivasseau eds., Birkhäuser Verlag, [arXiv:0705.0705].
- [12] S. A. Paston and V. A. Franke, ‘Canonical formulation of the embedded theory of gravity equivalent to Einstein’s General Relativity’, *Theor. Math. Phys.* **153** (2007) 1581–1595, [arXiv:0711.0576].
- [13] H. Steinacker, ‘Emergent Gravity and Noncommutative Branes from Yang-Mills Matrix Models’, *Nucl. Phys.* **B810** (2009) 1–39, [arXiv:0806.2032].
- [14] H. Steinacker, ‘Covariant Field Equations, Gauge Fields and Conservation Laws from Yang-Mills Matrix Models’, *JHEP* **02** (2009) 044, [arXiv:0812.3761].
- [15] R. M. Wald, *General Relativity*, Chicago: University Press, 1984.
- [16] D. Klammer and H. Steinacker, ‘Fermions and Emergent Noncommutative Gravity’, *JHEP* **08** (2008) 074, [arXiv:0805.1157].
- [17] D. Klammer and H. Steinacker, ‘Fermions and noncommutative emergent gravity II: Curved branes in extra dimensions’, *JHEP* **02** (2010) 074, [arXiv:0909.5298].
- [18] J. Arnlind, J. Hoppe and G. Huisken, ‘Discrete curvature and the Gauss-Bonnet theorem’, [arXiv:1001.2223];  
J. Arnlind, J. Hoppe and G. Huisken, ‘On the classical geometry of embedded surfaces in terms of Poisson brackets’, [arXiv:1001.1604]
- [19] D. Klammer and H. Steinacker, ‘Cosmological solutions of emergent noncommutative gravity’, *Phys. Rev. Lett.* **102** (2009) 221301, [arXiv:0903.0986].
- [20] H. Steinacker, ‘On the Newtonian limit of emergent NC gravity and long-distance corrections’, *JHEP* **12** (2009) 024, [arXiv:0909.4621].
- [21] V. O. Rivelles, ‘Noncommutative field theories and gravity’, *Phys. Lett.* **B558** (2003) 191–196, [arXiv:hep-th/0212262].
- [22] H. S. Yang, ‘Emergent Geometry and Quantum Gravity’, *Mod. Phys. Lett.* **A25** (2010) 2381, [arXiv:1007.1795];  
H. S. Yang, ‘On The Correspondence Between Noncommutative Field Theory And Gravity’, *Mod. Phys. Lett.* **A22** (2007) 1119, [arXiv:hep-th/0612231].
- [23] A. Chatzistavrakidis, H. Steinacker and G. Zoupanos, ‘Orbifolds, fuzzy spheres and chiral fermions’, *JHEP* **05** (2010) 100, [arXiv:1002.2606];

H. Grosse, F. Lizzi and H. Steinacker, ‘Noncommutative gauge theory and symmetry breaking in matrix models’, *Phys. Rev.* **D81** (2010) 085034, [[arXiv:1001.2703](#)].