



**HAL**  
open science

## On maximal massive 3D supergravity

Eric A Bergshoeff, Olaf Hohm, Jan Rosseel, Paul K Townsend

► **To cite this version:**

Eric A Bergshoeff, Olaf Hohm, Jan Rosseel, Paul K Townsend. On maximal massive 3D supergravity. *Classical and Quantum Gravity*, 2010, 27 (23), pp.235012. 10.1088/0264-9381/27/23/235012 . hal-00649073

**HAL Id: hal-00649073**

**<https://hal.science/hal-00649073>**

Submitted on 7 Dec 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On Maximal Massive 3D Supergravity

Eric A. Bergshoeff<sup>1</sup>, Olaf Hohm<sup>2</sup>, Jan Rosseel<sup>1</sup>  
and Paul K. Townsend<sup>3</sup>

<sup>1</sup> *Centre for Theoretical Physics, University of Groningen,  
Nijenborgh 4, 9747 AG Groningen, The Netherlands*  
email: E.A.Bergshoeff@rug.nl, j.rosseel@rug.nl

<sup>2</sup> *Center for Theoretical Physics, Massachusetts Institute of Technology,  
Cambridge, MA 02139, USA*  
email: ohohm@mit.edu

<sup>3</sup> *Department of Applied Mathematics and Theoretical Physics,  
Centre for Mathematical Sciences, University of Cambridge,  
Wilberforce Road, Cambridge, CB3 0WA, U.K.*  
email: P.K.Townsend@damtp.cam.ac.uk

## ABSTRACT

We construct, at the linearized level, the three-dimensional (3D)  $\mathcal{N} = 4$  supersymmetric “general massive supergravity” and the maximally supersymmetric  $\mathcal{N} = 8$  “new massive supergravity”. We also construct the maximally supersymmetric linearized  $\mathcal{N} = 7$  topologically massive supergravity, although we expect  $\mathcal{N} = 6$  to be maximal at the non-linear level.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b><math>\mathcal{N}</math>-extended Weyl and Poincaré supermultiplets</b>	<b>5</b>
2.1	$\mathcal{N} = 2$ . . . . .	7
2.2	$\mathcal{N} = 3$ . . . . .	9
2.3	$\mathcal{N} = 4$ . . . . .	9
2.4	$\mathcal{N} = 5$ . . . . .	11
2.5	$\mathcal{N} = 6$ . . . . .	11
2.6	$\mathcal{N} = 7$ . . . . .	11
2.7	$\mathcal{N} = 8$ . . . . .	12
<b>3</b>	<b><math>\mathcal{N} = 4</math> Massive Supergravities</b>	<b>12</b>
3.1	Further properties of the Weyl multiplet fields . . . . .	13
3.2	$\mathcal{N} = 4$ linearized Weyl multiplet actions . . . . .	14
3.3	$\mathcal{N} = 4$ Einstein supergravity . . . . .	16
3.4	$\mathcal{N} = 4$ massive supergravities . . . . .	19
<b>4</b>	<b><math>\mathcal{N} = 8</math> Maximal Supergravity</b>	<b>22</b>
4.1	The $\mathcal{N} = 8$ conformal multiplet . . . . .	23
4.2	$\mathcal{N} = 8$ conformal higher-derivative actions . . . . .	24
4.3	$\mathcal{N} = 8$ Einstein supergravity . . . . .	25
4.4	$\mathcal{N} = 8$ new massive supergravity . . . . .	25
<b>5</b>	<b>Linearized <math>\mathcal{N} = 7</math> TMG</b>	<b>27</b>
<b>6</b>	<b>Conclusions and Outlook</b>	<b>29</b>

# 1 Introduction

The “on-shell”  $\mathcal{N}$ -extended one-particle supermultiplets available for massless particles of four-dimensional (4D) field theories are well-known. Generically, these supermultiplets will appear in CPT-dual pairs but there are some special supermultiplets that are CPT self-dual; these are unique for a given choice of maximal spin, and have the property that  $\mathcal{N}$  is maximal for that spin. The corresponding field theories generally have improved ultra-violet (UV) behaviour. For example,  $\mathcal{N} = 4$  is maximal for maximum spin 1, and there is a unique  $\mathcal{N} = 4$  self-dual supermultiplet, which is realized by the UV-finite  $\mathcal{N} = 4$  super-Yang-Mills (SYM) theory. Similarly,  $\mathcal{N} = 8$  is maximal for maximum spin 2, and there is a unique  $\mathcal{N} = 8$  self-dual supermultiplet, which is realized by  $\mathcal{N} = 8$  supergravity; the UV status of this non-renormalizable theory is still a matter of dispute but it is certainly “improved”; see [1] for a recent review.

This paper is motivated by the observation that there is a one-to-one correspondence between the massless on-shell supermultiplets of  $\mathcal{N}$ -extended 4D supersymmetry and the *massive* on-shell supermultiplets of  $\mathcal{N}$ -extended three-dimensional (3D) supersymmetry. The  $\mathcal{N}$  3D Majorana spinor supercharges  $Q^i$ ,  $i = 1, \dots, \mathcal{N}$ , obey the anticommutation relations

$$\{Q_\alpha^i, Q_\beta^j\} = 2(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{ij}, \quad (1.1)$$

where  $P_\mu$  is the three-momentum,  $\gamma^\mu$  are the 3D Dirac matrices and  $C$  is the charge conjugation matrix. Choosing a real representation of the Dirac matrices with  $C = \gamma^0$  and choosing the rest frame three-momentum  $P_\mu = (-M, 0, 0)$  for a particle of mass  $M$ , one finds that the following combinations

$$(a^i)^\dagger = \frac{1}{2} (Q_1^i + iQ_2^i), \quad a^i = \frac{1}{2} (Q_1^i - iQ_2^i), \quad (1.2)$$

obey the anticommutation relations

$$\{a^i, (a^j)^\dagger\} = M\delta^{ij}, \quad \{a^i, a^j\} = \{(a^i)^\dagger, (a^j)^\dagger\} = 0. \quad (1.3)$$

The operators  $a$  and  $a^\dagger$  can thus be seen as raising and lowering operators. Moreover, it can be checked that they increase or decrease the space-time helicity by 1/2 (see e.g. [2]). The construction of the supermultiplets is thus straightforward. Starting from a ‘Clifford vacuum’  $|\Omega\rangle_j$  with helicity  $j$  one can act  $\mathcal{N}$  times with the raising operators  $a^\dagger$ . This leads to a massive multiplet with  $2^\mathcal{N}$  states of helicities ranging from  $j$  to  $j + \mathcal{N}/2$ , as shown in Table 1 for  $j + \mathcal{N}/2 = 2$ . Only the multiplets containing the +2 helicities are shown for  $\mathcal{N} = 1, \dots, 7$ . Parity flips the helicities and hence takes these into multiplets containing a state of helicity  $-2$  but not  $+2$ . The  $\mathcal{N} = 8$  multiplet is exceptional because it contains both helicities  $+2$  and  $-2$  and is therefore “parity self-dual”. One sees that this is formally the same construction as massless particle supermultiplets of four-dimensional  $\mathcal{N}$ -extended supersymmetry<sup>1</sup>.

---

<sup>1</sup>If the anticommutator (1.1) is modified to allow additional (non-central) charges then one can find additional, parity-preserving, multiplets that do not correspond to 4D multiplets because they are acted upon by twice as many supercharges as those discussed here [3].

**Table 1:** Some 3D massive supermultiplets

helicity	+2	+3/2	+1	+1/2	0	-1/2	-1	-3/2	-2
$\mathcal{N} = 1$	1	1							
$\mathcal{N} = 2$	1	2	1						
$\mathcal{N} = 3$	1	3	3	1					
$\mathcal{N} = 4$	1	4	6	4	1				
$\mathcal{N} = 5$	1	5	10	10	5	1			
$\mathcal{N} = 6$	1	6	15	20	15	6	1		
$\mathcal{N} = 7$	1	7	21	35	35	21	7	1	
$\mathcal{N} = 8$	1	8	28	56	70	56	28	8	1

This 4D/3D correspondence holds for each value of  $\mathcal{N}$  although it should be appreciated that the 3D theory has half the total number of supersymmetries (because the minimal spinor in 3D has half as many independent components as the minimal spinor in 4D). It should also be appreciated that the one-to-one correspondence is for supermultiplets, and not free field theories<sup>2</sup>; this is because a CPT-dual pair in 4D corresponds to a pair of 3D massive supermultiplets paired by parity, but locality in 3D does not require this pairing. This means that there can be parity-violating massive 3D field theories that have no 4D analog; the simplest ( $\mathcal{N} = 0$ ) examples are “topologically massive electrodynamics” (TME) [4–6] and “topologically massive gravity” (TMG) [6, 7].

Thus we can expect to find (at least at the linearized level) parity-preserving massive 3D SYM and massive 3D supergravity theories for each of the corresponding massless 4D SYM and supergravity theories. This expectation is realized in the SYM case by the  $\mathcal{N}$ -extended 3D SYM-Higgs theories, expanded about a Higgs vacuum in which all particles are massive. This yields 3D theories of massive spin-1 particles with  $\mathcal{N} = 1, 2, 4$  supersymmetry. Parity may then be broken by the addition of a supersymmetric Chern-Simons (CS) term, but such a term exists only for  $\mathcal{N} = 1, 2, 3$ , so  $\mathcal{N} = 3$  is maximal for spin 1 if parity is violated [8]. This feature can be understood directly from the supermultiplet structure: the  $\mathcal{N} = 4$  spin-1 supermultiplet is equivalent to a parity-dual pair of  $\mathcal{N} = 3$  spin-1 supermultiplets, so any parity violation that is visible in the linearized theory will split the degeneracy of this pair, thereby breaking  $\mathcal{N} = 4$  to  $\mathcal{N} = 3$ .

In the supergravity case, the 4D/3D analogy leads us to expect parity-preserving 3D supergravity theories propagating massive graviton supermultiplets with  $\mathcal{N} =$

<sup>2</sup>Interactions must be considered on a case-by-case basis.

1, 2, 3, 4, 5, 6, 8. Until recently, it was far from obvious how such theories could be constructed, but the  $\mathcal{N} = 0$  example of “New Massive Gravity” (NMG) [9] has shown the way, because it propagates precisely massive helicity  $\pm 2$  modes. Furthermore, one can add a Lorentz Chern-Simons (LCS) term to the NMG action to obtain a parity-violating “General Massive Gravity” (GMG) that has both NMG and TMG as limiting cases [9]. The  $\mathcal{N} = 1$  supersymmetric extension of these 3D gravity theories has now been constructed [10, 11] as has the linearized  $\mathcal{N} = 2$  extension, which nicely combines NMG with the Proca action for spin 1, and combines both CS and LCS terms in a single  $\mathcal{N} = 2$  superinvariant [10]. The main purpose of this paper is to present results for the  $\mathcal{N} > 2$  massive supergravity theories.<sup>3</sup>

We shall work exclusively at the linearized level, leaving the problem of interactions to future work. Even so, there are a number of issues that we are able to address, and resolve. Firstly, we recall that NMG consists of the Einstein-Hilbert action, with the “wrong” sign, and a curvature-squared invariant constructed from the scalar  $K = G^{\mu\nu} S_{\mu\nu}$ , where  $G_{\mu\nu}$  is the Einstein tensor and  $S_{\mu\nu}$  the Schouten tensor. We therefore need an  $\mathcal{N} = 8$  extension of the Einstein-Hilbert action (the lower  $\mathcal{N}$  cases can then be obtained by truncation). This involves coupling the  $\mathcal{N} = 8$  off-shell Weyl supermultiplet [13] (which contains the graviton field subject to linearized diffeomorphisms *and* linearized Weyl rescalings) to 8 compensating  $\mathcal{N} = 8$  scalar supermultiplets (containing only scalar and spinor fields). Each off-shell  $\mathcal{N} = 8$  scalar supermultiplet has an infinite number of auxiliary fields (arising from the expansion of a field defined on  $\mathcal{N} = 8$  harmonic superspace [14]). These should be eliminated only *after* the addition of an  $\mathcal{N} = 8$  extension of the NMG K-invariant, but the compensating supermultiplets decouple from the K-superinvariant *at the linearized level* as a consequence of an “accidental” linearized superconformal invariance; this allows us to trivially eliminate all auxiliary fields of each compensating supermultiplet and work with the simpler on-shell-supersymmetric<sup>4</sup>  $\mathcal{N} = 8$  scalar supermultiplet.

Starting with the  $\mathcal{N} = 8$  super-NMG model, we might expect to be able to construct a parity-violating  $\mathcal{N} = 8$  super-GMG model by the addition of an  $\mathcal{N} = 8$  super-LCS term. Let us recall that the LCS term is, by itself, the action of 3D conformal gravity [15] and that there exists an  $\mathcal{N} = 8$  superconformal gravity [16]. However, the  $\mathcal{N} = 8$  spin-2 supermultiplet is equivalent to a degenerate parity-dual pair of  $\mathcal{N} = 7$  spin-2 supermultiplets, and this degeneracy is lifted if parity is violated. It follows that, at best,  $\mathcal{N} = 7$  is maximal for super-GMG and also for its TMG limit. In fact, for reasons that we will give later, we believe that  $\mathcal{N} = 6$  is actually maximal<sup>5</sup> for

---

<sup>3</sup>A “disjoint” family of massive 3D  $\mathcal{N} = 1$  supergravity theories was recently found in [12] by dimensional reduction of a 6D theory with curvature-squared terms. These may also have  $\mathcal{N} > 1$  extensions but we shall have nothing to say about that here.

<sup>4</sup>By “on-shell-supersymmetric” we mean that the equations of motion are needed for closure of the supersymmetry algebra, due to the implicit elimination of auxiliary fields.

<sup>5</sup>This recalls the conjecture that  $\mathcal{N} = 6$  is maximal for an off-shell realization of  $\mathcal{N} = 8$  4D supergravity [17].

the non-linear TMG and generic GMG. However, we verify that there is an  $\mathcal{N} = 7$  *linearized* super-GMG theory by constructing the linearized off-shell  $\mathcal{N} = 7$  super-LCS invariant; this can be added to the  $\mathcal{N} = 8$  super-NMG action because it does not couple to the compensating supermultiplet. We also explain why the  $\mathcal{N} = 8$  super-LCS invariant cannot be similarly used to construct an  $\mathcal{N} = 8$  super-GMG theory; in brief, it is because there is no *off-shell-supersymmetric*  $\mathcal{N} = 8$  super-LCS term.

The method of construction of the linearized  $\mathcal{N} = 7, 8$  massive gravities that we have just sketched, can be applied for any  $\mathcal{N}$ . As a preliminary, we begin with a sketch of how the construction works for each value of  $\mathcal{N}$ . This involves a determination of the off-shell Weyl supermultiplet and the compensating multiplets needed for the construction of the linearized  $\mathcal{N}$ -extension of the Einstein-Hilbert term. For  $\mathcal{N} \leq 3$ , not only does a single scalar supermultiplet suffice but there is also an  $SO(\mathcal{N})$  singlet in this multiplet than can be identified with the compensating scalar for the local scale invariance of the Weyl multiplet. The situation for  $\mathcal{N} = 4$  is quite different. In this case, the scalar multiplet is the 3D version of the 4D hypermultiplet, but there are two distinct versions of it in 3D, and both are needed. Furthermore, the local scale compensator is necessarily composite because neither hypermultiplet contains a singlet of the R-symmetry group. In several respects, the  $\mathcal{N} = 4$  case is similar to the  $\mathcal{N} = 8$  case, but much simpler, so we discuss  $\mathcal{N} = 4$  in detail. This serves to illustrate features that we take over to  $\mathcal{N} = 8$ , thus obtaining the linearized  $\mathcal{N} = 8$  super-NMG theory. Finally, we construct the off-shell  $\mathcal{N} = 7$  LCS invariant and hence the maximally-supersymmetry parity-violating super-GMG and super-TMG theories, again at the linearized level. We conclude with a discussion of open problems.

## 2 $\mathcal{N}$ -extended Weyl and Poincaré supermultiplets

To construct the  $\mathcal{N}$ -extended supersymmetric LCS term, one needs only the fields of an  $\mathcal{N}$ -extended “Weyl supermultiplet”. At the linearized level, this multiplet contains the metric perturbation  $h_{\mu\nu}$  and  $\mathcal{N}$  Majorana anti-commuting vector-spinors  $\psi_\mu^i$  ( $i = 1, \dots, \mathcal{N}$ ) subject to the linearized transformations

$$\delta h_{\mu\nu} = \partial_{(\mu} v_{\nu)} + \eta_{\mu\nu} \omega, \quad \delta \psi_\mu^i = \partial_\mu \eta_Q + \gamma_\mu \epsilon_S, \quad (2.1)$$

for Minkowski 3-vector  $v$ , scalar  $\omega$  and the anticommuting Majorana spinor parameters of  $Q$ - and  $S$ -supersymmetry. At the linearized level we must distinguish between the  $Q$ -supersymmetry gauge invariances and the rigid supersymmetry that relates the various fields of the multiplet; it is only after the inclusion of interactions that these combine to become  $\mathcal{N}$  local supersymmetries. The linearized Weyl multiplet also includes  $\mathcal{N}(\mathcal{N} - 1)/2$  abelian gauge fields  $V_\mu^{ij}$  in the adjoint irrep of the  $Spin(\mathcal{N})$  R-symmetry group, and subject to the gauge transformation

$$\delta V_\mu^{ij} = \partial_\mu \Lambda^{ij}. \quad (2.2)$$

For  $\mathcal{N} \geq 3$  there are additional fields in the Weyl multiplet, and for  $\mathcal{N} \geq 6$  these include additional gauge fields, although  $\mathcal{N} = 8$  is exceptional in this respect, as we shall see.

To construct the  $\mathcal{N}$ -extended supersymmetric EH term, or  $\mathcal{N}$ -extended ‘‘Einstein supergravity’’, we need more than just the Weyl supermultiplet. We need a Poincar e supermultiplet in which the fields are subject only to the standard supergravity gauge transformations and not the additional ones of conformal supergravity. We therefore need to introduce additional degrees of freedom. One way to do this is as follows [18,19]. We start from a standard flat space action invariant under rigid superconformal transformations, for a supermultiplet (or supermultiplets) containing physical scalar and spinor fields (and possibly vector or antisymmetric tensor gauge fields); if the field supermultiplets used to construct this action are off-shell supersymmetric then there will generically be auxiliary fields too but their inclusion is optional. This action is then coupled to an off-shell  $\mathcal{N}$ -extended Weyl supermultiplet (e.g. by the Noether procedure) to give an action that is invariant under local superconformal transformations. Then one fixes the ‘unwanted’ superconformal symmetries by imposing conditions on the physical scalar and spinor fields of the multiplet(s). To do so, one needs sufficient multiplets so as to have a sufficient number of physical scalar and spinor fields, and with luck there will be none left over. In this case, the conformal coupling to the scalar fields will produce the Einstein-Hilbert term on gauge fixing and the rest of the action will be just what is required for the  $\mathcal{N}$ -extended supersymmetrization of this term. One says that the original ‘scalar’ supermultiplets are ‘compensating multiplets’. The auxiliary fields of these compensating multiplets become auxiliary fields of the final  $\mathcal{N}$ -extended Einstein supergravity action.

If we wish to construct generic higher-derivative actions in the way just described then the auxiliary fields of the compensating multiplets must be included because the starting flat space action for the conformal compensator multiplets must be a higher-derivative one in which the ‘auxiliary’ fields also propagate. The LCS term is an exception to this rule because it is superconformally invariant. In contrast, the 4th order curvature-squared term of NMG is not conformally invariant, so one really needs a full superconformal tensor calculus to construct  $\mathcal{N}$ -extended NMG. Some aspects of this calculus have been worked out for  $\mathcal{N} = 1, 2$  [20] but, even so, special ‘tricks’ were needed for the construction of the  $\mathcal{N} = 1$  supersymmetric super-NMG model [10, 11] and only linearized results have been found for  $\mathcal{N} = 2$  [10].

However, although we need an  $\mathcal{N}$ -extended superconformal tensor calculus to construct the full  $\mathcal{N}$ -extended NMG action, we do not need it to construct the quadratic approximation to this action. This is because the NMG 4th order invariant has the property (shared with an infinite series of yet higher-order invariants) that its quadratic approximation is invariant under the linearized super-Weyl gauge invariances, so compensating multiplets are not needed. Moreover, this quadratic approximation to the 4th order NMG invariant may be added to the quadratic approximation to the  $\mathcal{N}$ -extended Einstein supergravity to give the quadratic approximation to  $\mathcal{N}$ -extended

NMG. The upshot is that to construct the  $\mathcal{N}$ -extension of linearized NMG, and more generally the linearized GMG, we do not need a full superconformal tensor calculus. In particular, we need conformal compensator supermultiplets only for the construction of the  $\mathcal{N}$ -extended Einstein supergravity, and as we need that only to quadratic level, it is sufficient to consider the cubic interaction of the compensator supermultiplets to the Weyl multiplet and a few other quartic interactions that contribute to the quadratic action after fixing the superconformal gauge.

In what follows we will consider sequentially the cases from  $\mathcal{N} = 2, \dots, 8$ , presenting some details of the Weyl multiplet and determining the type and number of compensating multiplets that are needed. For the convenience of the reader we have summarized some details about this in Table 2. The  $\mathcal{N} = 2$  case has already been dealt with in some detail [10, 13] but it will serve to illustrate the issues involved.

**Table 2:** Some properties of the  $\mathcal{N}$ -extended Weyl multiplets. The fourth column indicates the number of basic compensating supermultiplets needed to obtain (Einstein, TMG or GMG) supergravity.

$\mathcal{N}$	# off-shell d.o.f.	R-symmetry	# multiplets
2	4+4	SO(2)	1
3	8+8	SO(3)	1
4	16+16	$Spin(4) \cong SU(2) \times SU(2)$	2
5	32+32	$Spin(5) \cong Sp_2$	2
6	64+64	$Spin(6) \cong SU(4)$	4
7	128+128	$Spin(7)$	8
8	128+128	SO(8)	8

## 2.1 $\mathcal{N} = 2$

The off-shell linearized Weyl multiplet has the field content

$$(h_{\mu\nu}; \psi_{\mu}^i; V_{\mu}), \quad i = 1, 2. \quad (2.3)$$

We use semicolons to separate fields of different mass dimensions, which increase by steps of 1/2. The vector field is the  $SO(2)$  gauge field. Taking all gauge invariances into account, we are left with 4 + 4 remaining off-shell field components.

To construct a Poincaré supermultiplet suitable for the construction of massive supergravities, we need to add additional degrees of freedom that will allow us to fix the scale and  $S$ -supersymmetry transformations, at least, but we may also fix the R-symmetry gauge invariance. For example, consider the  $\mathcal{N} = 2$  scalar multiplet with

4 + 4 off-shell field content  $(\varphi, \varsigma; \lambda^i; S, P)$ . After coupling to the Weyl multiplet, we may fix the scale and  $SO(2)$  gauge invariances by setting  $\varphi = 0$  and  $\varsigma = 0$ , while the  $S$ -supersymmetry gauge invariances may be fixed by setting  $\lambda^i = 0$ . This leaves the auxiliary fields  $S$  and  $P$ , which survive as auxiliary fields of the off-shell Poincaré supermultiplet, which has the 8 + 8 off-shell field content  $(h_{\mu\nu}; \psi_\mu^i; V_\mu, S, P)$ , now subject only to diffeomorphisms and local  $Q$ -supersymmetry gauge transformations. This multiplet was called the (1, 1) Poincaré supermultiplet in [13] in order to distinguish it from the (2, 0) Poincaré supermultiplet found by taking the compensator fields to belong to the 4 + 4 off-shell vector multiplet, which has the field content  $(\varphi, A_\mu; \lambda^i; D)$ . In this case, we use  $\varphi$  and  $\lambda^i$  to compensate for the scale and  $S$ -supersymmetry transformations, as before, but we leave uncompensated the local  $SO(2)$  gauge invariance. The linearized Poincaré supermultiplet now has the 8 + 8 off-shell field content  $(h_{\mu\nu}; A_\mu; \psi_\mu^i; V_\mu; D)$ . In the construction of the supersymmetric Einstein-Hilbert term the (2, 0) Poincaré supermultiplet leads to a  $VdA$  Chern-Simons type term [13], and a  $D^2$  term for the auxiliary field  $D$ .

Variant choices of conformal compensator multiplets therefore lead to variant versions of the supersymmetric EH action; in this case the (1, 1) or (2, 0) versions. Whatever the choice, one can add to this action any off-shell supersymmetric  $\mathcal{N} = 2$  superconformal action, such as LCS. Also, at the linearized level, we can add the quadratic approximation to the  $\mathcal{N} = 2$  supersymmetric 4th order invariant of NMG, thereby constructing the  $\mathcal{N} = 2$  extension of linearized GMG. This was done for the (1, 1) case in [10]. The generic GMG model propagates one multiplet of helicities  $(2, 3/2, 1)$  with mass  $m_+$  and another multiplet of helicities  $(-1, -3/2, -2)$  with mass  $m_-$ . The same computations may be done for the (2, 0) case, but with more difficulty because the zero Weyl weight of the vector field  $A$  means that the starting flat space action cannot be quadratic. Nevertheless, it can be done and the final spectrum is the same but the spin 1 modes are propagated differently. In the (1, 1) case the action for the spin 1 modes is the Proca action for  $V$  with a CS term<sup>6</sup>. In the (2, 0) case one finds that the action for the spin 1 modes has the Lagrangian

$$L_1 = \frac{1}{2}\tilde{F}^2 + A^\mu\tilde{G}_\mu + \frac{1}{2\mu}V^\mu\tilde{G}_\mu + \frac{1}{2m^2}\tilde{G}^2 \quad (2.4)$$

where

$$\tilde{F}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu A_\rho, \quad \tilde{G}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu V_\rho. \quad (2.5)$$

One may verify that this propagates two spin 1 modes of helicities  $\pm 1$  and masses  $m_\pm$  given by

$$m^2 = m_+m_- \quad \mu = m^2/(m_- - m_+), \quad (2.6)$$

exactly as for the (1, 1) case.

---

<sup>6</sup>But here we use a different notation: what was called  $A$  in [10] is  $V$  here.

## 2.2 $\mathcal{N} = 3$

The off-shell  $\mathcal{N} = 3$  linearized Weyl multiplet has  $8 + 8$  components and the field content

$$(h_{\mu\nu}; \psi_{\mu}^i; V_{\mu}^i; \chi), \quad i = 1, 2, 3. \quad (2.7)$$

The minimal  $\mathcal{N} = 3$  scalar multiplet has the physical (i.e. non-auxiliary) field content  $(\varphi, \varphi^i; \lambda, \lambda^i)$ ; i.e. both scalars and spinors are in the  $1 \oplus 3$  representation of the  $SO(3)$   $R$ -symmetry group. There is an off-shell version of this multiplet with a finite number of auxiliary fields (a 3D version of the relaxed hypermultiplet [21]) but it is not needed for present purposes. The  $1 \oplus 3$  scalars are precisely those needed to compensate for the one scale and three  $SO(3)$  gauge invariances. The fermion triplet  $\lambda^i$  compensates for the three  $S$ -supersymmetries. This leaves the singlet spinor  $\lambda$ . In the context of the  $\mathcal{N} = 3$  supersymmetrization of the Einstein-Hilbert action, the pair of spinors  $(\chi, \lambda)$  is auxiliary. As the (non-gauge) vectors  $V_{\mu}^{ij}$  are also auxiliary in this context, only  $h_{\mu\nu}$  and  $\psi_{\mu}^i$  remain as the 'physical' fields, which actually do not propagate modes.

In the context of  $\mathcal{N} = 3$  super-TMG there is a  $\bar{\chi}\chi$  term associated with the Lorentz-Chern-Simons term and this means that  $\chi$  is no longer a Lagrange multiplier for the constraint  $\lambda = 0$ . The combined effect of the ' $\chi\lambda$ ' terms is to propagate one spin-1/2 mode. Thus, the physical field content in the context of TMG is  $(h_{\mu\nu}; \psi_{\mu}^i, \lambda; V_{\mu}^i; \chi)$  subject only to linearized diffeomorphism and local  $Q$ -supersymmetry transformations. Note that the effect of the compensating multiplet is no longer merely to reduce the gauge invariances of the Weyl multiplet fields, as is the case for  $\mathcal{N} = 2$ , but also to provide an additional spinor field. The physical field content for NMG is the same as for TMG but we expect the higher-order interactions to lead to a second massive particle supermultiplet of opposite helicities in exactly the way spelled out for  $\mathcal{N} = 2$  in [10].

As for  $\mathcal{N} = 2$ , we could consider how things change if we use vector multiplets in place of scalar multiplets, but this is already much more complicated for  $\mathcal{N} = 3$  so we shall henceforth restrict the discussion to scalar supermultiplets.

## 2.3 $\mathcal{N} = 4$

The off-shell  $\mathcal{N} = 4$  Weyl multiplet, with  $16+16$  components, has the field content

$$(h_{\mu\nu}; \psi_{\mu}^i; V_{\mu}^{ij}, E; \chi^i; D), \quad i = 1, 2, 3, 4. \quad (2.8)$$

The  $R$ -symmetry group is now reducible:  $Spin(4) \cong SU(2) \times SU(2)$ . As a consequence, the vector fields may be written as

$$V^{ij} = V_{+}^{ij} + V_{-}^{ij}, \quad \frac{1}{2}\varepsilon^{ijkl}V_{\pm}^{kl} = \pm V_{\pm}^{ij}. \quad (2.9)$$

This sum of self-dual and anti-self-dual terms corresponds to the direct sum representation  $(3, 1) \oplus (1, 3)$ . This sum is of course unchanged under a switch in the duality

assignments of  $V_{\pm}$ , which amounts to an exchange of the two  $SU(2)$  factors of  $Spin(4)$  by an outer  $\mathbb{Z}_2$  automorphism. However, the scalars  $E, D$  and the spinors  $\chi^i$  are *odd* under this  $\mathbb{Z}_2$  exchange<sup>7</sup>.

The minimal scalar  $\mathcal{N} = 4$  supermultiplet is the 3D analog of the 4D,  $\mathcal{N} = 2$  hypermultiplet with  $4 + 4$  physical degrees of freedom. However, in 3D there are two distinct versions of the hypermultiplet. The hypermultiplet scalars transform as a complex doublet of one or the other of the  $SU(2)$  factors of the R-symmetry group. Whichever  $SU(2)$  factor we choose, the spinor fields will transform as a complex doublet of the other  $SU(2)$  factor. In other words, we have a hypermultiplet with physical fields  $(\varphi^\alpha, \lambda^\alpha)$  or one with physical fields  $(\varphi^{\dot{\alpha}}, \lambda^\alpha)$ , where  $\alpha = 1, 2$  and  $\dot{\alpha} = 1, 2$  are spinor indices for the two  $SU(2)$  factors of  $Spin(4)$ . The  $SU(2)$  currents that may be constructed from these scalars will be either self-dual or anti-self-dual, according to which of the two versions of the hypermultiplet we choose. Without loss of generality, we may assume that the scalar fields  $\varphi^\alpha$  yield a self-dual current  $J_+^{ij}$  and that the scalar fields  $\varphi^{\dot{\alpha}}$  yield an anti-self-dual current  $J_-^{ij}$ .

Both types of hypermultiplet are needed for conformal compensation, because we have to compensate for both  $SU(2)$  gauge invariances. Another difference from the  $\mathcal{N} = 3$  case is that any  $R$ -symmetry singlet constructed from the scalar fields of the compensating multiplets must be at least quadratic in the scalar fields. The scalar bilinears

$$\varphi_+^2 \equiv \varphi^\alpha \varphi_\alpha + \varphi^{\dot{\alpha}} \varphi_{\dot{\alpha}}, \quad \varphi_-^2 \equiv \varphi^\alpha \varphi_\alpha - \varphi^{\dot{\alpha}} \varphi_{\dot{\alpha}} \quad (2.10)$$

are respectively even and odd under the  $\mathbb{Z}_2$  exchange. The former acts as a compensator for the dilatations. After fixing the dilatation gauge by imposing  $\varphi_+^2 = 1$ , one finds that the  $Spin(4)$  currents take the form

$$J_\mu^{ij}{}_{\pm} = \partial_\mu \varphi_\pm^{ij} + \dots, \quad (2.11)$$

where  $\varphi_\pm^{ij}$  are, collectively,  $Spin(4)$  Stueckelberg scalars that may be used to compensate for the local  $Spin(4)$  gauge invariance.

In contrast to  $\mathcal{N} = 3$ , the conformal compensation mechanism is closer to the non-linear Higgs mechanism than it is to the linear Stueckelberg mechanism, because  $\varphi_-^2$  can be identified as the field of a residual Higgs boson that survives the ‘spontaneous’ breaking of symmetries implied by the conformal gauge condition  $\varphi_+^2 = 1$ . However, this Higgs boson field is set to zero by the  $D$  field equation of the Weyl multiplet. The details of the construction will be discussed in the next section.

---

<sup>7</sup>It was suggested in [13] that there should exist an 8+8 Weyl multiplet that contains only  $V_+$  or  $V_-$ , and without the higher dimension fields ( $E; \chi^i; D$ ), but it does not appear to be possible to close the supersymmetry algebra on this smaller set.

## 2.4 $\mathcal{N} = 5$

The off-shell linearized Weyl multiplet for  $\mathcal{N} = 5$  has  $32 + 32$  components and the field content

$$(h_{\mu\nu}; \psi_{\mu}^i; V_{\mu}^{ij}, E^i; \chi^{ij}, \chi; D^i), \quad i = 1, 2, 3, 4, 5. \quad (2.12)$$

Fields with multiple indices are antisymmetric in these indices. Thus, the vectors are in the adjoint **10** irrep of  $Spin(5) \cong Sp_2$ , as are the spinors  $\chi^{ij}$ .

The minimal scalar supermultiplet now has  $8 + 8$  physical fields, in the complex 4 representation of  $Sp_2$ . Let  $(\varphi^{\alpha}, \lambda^{\alpha})$  ( $\alpha = 1, 2, 3, 4$ ) be these physical fields. As for  $\mathcal{N} = 4$ , we need *two* scalar supermultiplets,  $\varphi_1$  and  $\varphi_2$ . The 16 scalar fields in these two multiplets may be traded for scalar bilinears in the  $\mathbf{1} \oplus \mathbf{5}$  representations of  $Sp_2$ , and currents in the adjoint **10** of  $Sp_2$ . The singlet bilinear is  $\Omega_{\alpha\beta}\varphi_1^{\alpha}\varphi_2^{\beta}$ , which we set to unity to fix the local scale invariance. The currents then become derivatives of a **10** of Stueckelberg scalars, to linear order, which may be set to zero to compensate for the local  $Sp_2$  invariance. The 5-plet of scalar bilinears is set to zero by the  $D^i$  field equation.

## 2.5 $\mathcal{N} = 6$

The off-shell linearized Weyl multiplet now has  $64 + 64$  components, and the field content

$$(h_{\mu\nu}; \psi_{\mu}^i; V_{\mu}^{ij}, V_{\mu}, E^{ij}; \chi^{ijk}, \chi^i; D^{ij}), \quad i = 1, 2, 3, 4, 5, 6. \quad (2.13)$$

Fields with multiple indices are again antisymmetric in these indices. The  $R$ -symmetry group is  $Spin(6) \cong SU(4)$ , but the gauge symmetry is enhanced to  $U(4)$ , because of the presence of the additional gauge field  $V_{\mu}$ .

The minimal scalar multiplet again has  $8+8$  physical fields  $(\varphi^{\alpha}, \chi^{\alpha})$  in the (complex) **4** of  $SU(4)$ . We can choose a gauge given by the sum of squares of all the scalars equal to one to fix the dilatations. There will be 15 additional constraints coming from the  $D$ -field equation. The  $U(4)$  currents then become to linearized order the derivatives of 16 Stueckelberg scalars, which we set to zero to fix the  $U(4)$  invariance. We therefore need a total of 32 scalar fields and hence *four* scalar multiplets.

## 2.6 $\mathcal{N} = 7$

The off-shell linearized Weyl multiplet has  $128 + 128$  components, and the field content

$$(h_{\mu\nu}; \psi_{\mu}^i, \psi_{\mu}; V_{\mu}^{ij}, V_{\mu}^i, E^{ijk}; \chi^{ijk}, \chi^{ij}; D^{ijk}), \quad i = 1, 2, 3, 4, 5, 6, 7. \quad (2.14)$$

Fields with multiple indices are again antisymmetric in these indices. The  $R$ -symmetry group is  $Spin(7)$  but the presence of 7 additional vector fields implies an enhancement of the gauge symmetry. We also have an additional Rarita-Schwinger field implying an enhanced local supersymmetry at the non-linear level, but we avoid the associated difficulties here by restricting to the linear theory.

The minimal scalar multiplet has  $8 + 8$  components in the spinor representation of  $\text{Spin}(7)$ . The  $D$ -fields now impose 35 constraints on scalar bilinears. In addition, we need  $1 + 21 + 7 = 29$  scalars to compensate for the scale and gauge invariances. In total we need 64 scalars and hence *eight* compensating scalar multiplets.

## 2.7 $\mathcal{N} = 8$

The off-shell Weyl multiplet, again with  $128+128$  components, has the field content

$$(h_{\mu\nu} ; \psi_{\mu}^i ; V_{\mu}^{ij}, E^{ijkl} ; \chi^{ijk} ; D^{ijkl}), \quad i = 1, 2, 3, 4, 5, 6, 7, 8. \quad (2.15)$$

Fields with multiple indices are again antisymmetric in these indices, and  $E^{ijkl}$  and  $D^{ijkl}$  have 'opposite  $SO(8)$  dualities': they are, respectively, self-dual and anti-selfdual, or vice-versa. As there are two choices of duality assignments, there are two (equivalent) versions of the  $\mathcal{N} = 8$  Weyl multiplet.

The minimal scalar multiplet has  $8 + 8$  physical components and also comes in two versions. One version has the scalars in the spinor representation and the fermions in the conjugate spinor representation of  $\text{Spin}(8)$  and vice versa for the other version. These are the only possibilities consistent with the supersymmetry parameter being a vector of  $SO(8)$ . However, only one of the two versions of the scalar multiplet can be consistently coupled to a given version of the Weyl multiplet, and hence all compensating supermultiplets must be of the same type. The  $D$ -field will impose 35 constraints on scalar bilinears and we need  $1 + 28$  scalars to compensate for the scale and  $SO(8)$  gauge invariances. We therefore need a total of 64 scalars and hence *eight* compensating scalar multiplets.

## 3 $\mathcal{N} = 4$ Massive Supergravities

In this section we present details of the construction of linearized 3D  $\mathcal{N} = 4$  supergravities with various higher-derivative interactions using the superconformal tensor calculus. In particular, we obtain the  $\mathcal{N} = 4$  extension of linearized GMG and its limits. Essential to all these constructions is the  $16 + 16$  component linearized Weyl multiplet of (2.8). The linearized supersymmetry transformation rules (in the same conventions as in [10]) are

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}^i \gamma_{(\mu} \psi_{\nu)}^i, \\ \delta \psi_{\mu}^i &= -\frac{1}{4} \gamma^{\rho\sigma} \partial_{\rho} h_{\mu\sigma} \epsilon^i - V_{\mu}^{ij} \epsilon^j, \\ \delta V_{\mu}^{ij} &= \frac{1}{2} \bar{\epsilon}^{[i} \phi_{\mu}^{j]} + \varepsilon^{ijkl} \bar{\epsilon}^k \gamma_{\mu} \chi^l, \\ \delta E &= \frac{1}{4} \bar{\epsilon}^i \chi^i, \\ \delta \chi^i &= \frac{1}{8} \varepsilon^{ijkl} \gamma^{\mu} \epsilon^l F_{\mu(\text{lin})}^{jk} + \gamma^{\mu} \partial_{\mu} E \epsilon^i + D \epsilon^i, \\ \delta D &= \frac{1}{4} \bar{\epsilon}^i \gamma^{\mu} \partial_{\mu} \chi^i. \end{aligned} \quad (3.1)$$

Here  $F_{(\text{lin})}^{\mu ij}$  and the (dependent)  $S$ -supersymmetry gauge field  $\phi_\mu^i$  are given by

$$F_{(\text{lin})}^{\mu ij} = \epsilon^{\mu\nu\rho} \partial_\nu V_\rho^{ij}, \quad \phi_\mu^i = \gamma_\nu \gamma_\mu \mathcal{R}_{(\text{lin})}^{\nu i}, \quad \mathcal{R}_{(\text{lin})}^{\mu i} = \epsilon^{\mu\nu\rho} \partial_\nu \psi_\rho^i. \quad (3.2)$$

We furthermore insist on gauge invariance with respect to the linearized gauge transformations of (2.1) and (2.2). The supersymmetry algebra now closes in the sense that the commutator of two supersymmetries on any field, with parameters  $\epsilon_1$  and  $\epsilon_2$ , gives a translation plus (field-dependent) gauge transformations (represented by the dots):

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \frac{1}{2} (\bar{\epsilon}_2^i \gamma^\mu \epsilon_1^i) \partial_\mu + \dots. \quad (3.3)$$

### 3.1 Further properties of the Weyl multiplet fields

The transformation laws given above for the independent fields of the Weyl multiplet imply the following supersymmetry transformation of the dependent  $S$ -supersymmetry gauge field:

$$\delta \phi_\mu^i = \gamma^\nu \epsilon^i S_{\mu\nu}^{(\text{lin})} - \gamma_\nu \gamma_\mu \epsilon^j F_{(\text{lin})}^{\nu ij}, \quad (3.4)$$

where  $S_{\mu\nu}$  is the linearized 3D Schouten tensor

$$S_{\mu\nu}^{(\text{lin})} = R_{\mu\nu}^{(\text{lin})} - \frac{1}{4} \eta_{\mu\nu} R^{(\text{lin})}, \quad R^{(\text{lin})} \equiv \eta^{\mu\nu} R_{\mu\nu}^{(\text{lin})}. \quad (3.5)$$

Using the expression

$$R_{\mu\nu}^{(\text{lin})} = -\frac{1}{2} [\square h_{\mu\nu} - 2\partial_{(\mu} h_{\nu)} + \partial_\mu \partial_\nu h], \quad h_\mu \equiv \eta^{\nu\rho} \partial_\rho h_{\mu\nu}, \quad h = \eta^{\mu\nu} h_{\mu\nu}, \quad (3.6)$$

one can verify that the linearized Weyl transformation of the linearized Schouten tensor takes the following simple form

$$\delta_\omega S_{\mu\nu}^{(\text{lin})} = -\frac{1}{2} \partial_\mu \partial_\nu \omega. \quad (3.7)$$

It follows that the linearized Cotton tensor, defined as

$$C_{\mu\nu}^{(\text{lin})} = \epsilon_\mu^{\tau\rho} \partial_\tau S_{\rho\nu}^{(\text{lin})}, \quad (3.8)$$

is linearized Weyl invariant; this is a consequence of the Weyl invariance of the non-linear Cotton tensor, which is the 3D analog of the 4D Weyl tensor. The linearized Cotton tensor, which is parity odd, satisfies the identities

$$\partial^\mu C_{\mu\nu}^{(\text{lin})} \equiv 0, \quad C_{\mu\nu}^{(\text{lin})} \equiv C_{\nu\mu}^{(\text{lin})}, \quad \eta^{\mu\nu} C_{\mu\nu}^{(\text{lin})} \equiv 0, \quad (3.9)$$

which are also consequences of similar identities satisfied by the full Cotton tensor.

The superpartner of the Cotton tensor is the Cottino tensor. The linearized  $\mathcal{N} = 4$  Cottino tensor is

$$C_{(\text{lin})}^{\mu i} = \gamma^\nu \partial_\nu \mathcal{R}_{(\text{lin})}^{\mu i} + \epsilon^{\mu\nu\rho} \partial_\nu \mathcal{R}_{\rho(\text{lin})}^i. \quad (3.10)$$

It satisfies the identities

$$\partial_\mu \mathcal{C}_{(\text{lin})}^{\mu i} = 0, \quad \gamma_\mu \mathcal{C}_{(\text{lin})}^{\mu i} = 0. \quad (3.11)$$

The linearized Cotton, Cottino and SO(4) curvature tensors have the following supersymmetry transformations:

$$\begin{aligned} \delta C_{\mu\nu}^{(\text{lin})} &= -\frac{1}{4} \bar{\epsilon}^i \gamma_{(\mu}{}^\rho \partial_\rho \mathcal{C}_{\nu)}^{i \text{lin}}, \\ \delta \mathcal{C}_{(\text{lin})}^{\mu i} &= \gamma_\nu \epsilon^i C_{(\text{lin})}^{\mu\nu} + \epsilon^{\mu\nu\rho} \gamma_\sigma \gamma_\nu \epsilon^j \partial_\rho F_{(\text{lin})}^{\sigma ij}, \\ \delta F_{(\text{lin})}^{\mu ij} &= \frac{1}{2} \bar{\epsilon}^{[i} \mathcal{C}_{(\text{lin})}^{\mu j]} + \bar{\epsilon}^k \epsilon^{ijkl} \gamma^{\mu\rho} \partial_\rho \chi^l. \end{aligned} \quad (3.12)$$

These transformation rules define an  $\mathcal{N} = 4$  conformal field-strength multiplet with components

$$\{C_{\mu\nu}^{(\text{lin})}, \mathcal{C}_{(\text{lin})}^{\mu i}, F_{(\text{lin})}^{\mu ij}, \chi^i, D, E\}. \quad (3.13)$$

With the above definitions in hand we may construct various conformal higher-derivative actions for the conformal multiplet. Below we give a few examples of such actions where the leading term, bilinear in the graviton field, has 3,4,5 and 6 derivatives.

### 3.2 $\mathcal{N} = 4$ linearized Weyl multiplet actions

Let us now consider invariants that may be constructed from the Weyl multiplet fields alone, at least in the quadratic approximation.

#### (1) $\mathcal{N} = 4$ Supersymmetric LCS

One may verify that the following action containing a linearized Lorentz Chern-Simons term is supersymmetric,

$$S_3^{\mathcal{N}=4} = \int d^3x \left\{ h^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{\psi}_\mu^i \mathcal{C}_{(\text{lin})}^{\mu i} - 2V_\mu^{ij} F_{(\text{lin})}^{\mu ij} + 16\bar{\chi}^i \chi^i - 128ED \right\}. \quad (3.14)$$

As this action is the quadratic approximation to an  $\mathcal{N} = 4$  superconformal extension of the LCS term, no compensating fields are needed to construct the non-linear invariant.

#### (2) $\mathcal{N} = 4$ Supersymmetric NMG invariant

Using (3.12), one may verify invariance of the following action:

$$\begin{aligned} S_4^{\mathcal{N}=4} = \int d^3x \left\{ -\frac{1}{2} \epsilon^{\mu\tau\rho} h_\mu{}^\nu \partial_\tau C_{\rho\nu}^{(\text{lin})} - \frac{1}{2} \bar{\psi}_\mu^i \not{\partial} \mathcal{C}^{i\mu}{}_{(\text{lin})} + F_{\mu(\text{lin})}^{ij} F_{(\text{lin})}^{\mu ij} \right. \\ \left. + 32E \square E - 8\bar{\chi}^i \not{\partial} \chi^i + 32D^2 \right\}. \end{aligned} \quad (3.15)$$

The leading term is just the linearization of the fourth-order  $K$  invariant of NMG [9]. To see this we note first that  $K \equiv R^{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2 = G^{\mu\nu} S_{\mu\nu}$ . A convenient form for the linearized 3D Einstein tensor is

$$G_{(\text{lin})}^{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\tau\rho} \epsilon^{\nu\eta\sigma} \partial_\tau \partial_\eta h_{\rho\sigma}. \quad (3.16)$$

Using this, one may show that

$$G_{(lin)}^{\mu\nu} S_{\mu\nu}^{(lin)} = -\frac{1}{2} \varepsilon^{\mu\tau\rho} h_{\mu}{}^{\nu} \partial_{\tau} C_{\rho\nu}^{(lin)} + \text{total derivative}. \quad (3.17)$$

The expression on the right hand side makes manifest the linearized Weyl invariance.

(3)  $\mathcal{N} = 4$  *Supersymmetric Ricci times Cotton*

One can also construct a fifth-order parity-odd action that starts with the product of a Ricci tensor with a Cotton tensor:

$$S_5^{\mathcal{N}=4} = \int d^3x \left\{ R_{(lin)}^{\mu\nu} C_{\mu\nu}^{(lin)} + \bar{C}_{\mu i}^{(lin)} C_{(lin)}^{\mu i} + \epsilon^{\mu\nu\rho} F_{\mu}^{(lin)ij} \partial_{\nu} F_{\rho}^{(lin)ij} \right. \\ \left. + 16\bar{\chi}^i \square \chi^i - 128E \square D \right\}. \quad (3.18)$$

(4)  $\mathcal{N} = 4$  *Supersymmetric Cotton Tensor Squared*

It is not difficult to construct linearized higher-derivative supersymmetric actions by using the following observation. Schematically the supersymmetric action (3.14) consists of terms that are all of the form

$$\text{Weyl} \times \text{Weyl}^c, \quad (3.19)$$

where Weyl indicates a field from the Weyl multiplet and  $\text{Weyl}^c$  denotes a field from the conformal field-strength multiplet (3.13). The fact that the supersymmetry transformations are linearized and thus global has the following immediate consequence: Given a set of fields  $\{\text{Weyl}\}$  that transforms as (3.1), the set of fields

$$\{\text{Weyl}^{(n)}\} = \{\square^n \text{Weyl}\} \quad (3.20)$$

transforms in the same way. This means that the action that is obtained from (3.15) by replacing in each term the first factor Weyl by  $\text{Weyl}^{(n)}$ , such that we obtain

$$\text{Weyl}^{(n)} \times \text{Weyl}^c, \quad (3.21)$$

also defines a linearized supersymmetric invariant. The case  $n = 1$  is particularly interesting since one can rewrite, using conformal gauge transformations, the action such that each term is bilinear in the fields of the conformal field-strength Weyl multiplet. This leads to the supersymmetric Cotton tensor squared action:

$$S_6^{\mathcal{N}=4} = \int d^3x \left\{ C_{(lin)}^{\mu\nu} C_{\mu\nu}^{(lin)} - \frac{1}{4} \bar{C}_{\mu}^{i(lin)} \not{\partial} C_{(lin)}^{\mu i} + F_{\mu(lin)}^{ij} \square F_{(lin)}^{\mu ij} \right. \\ \left. + 32E \square^2 E - 8\bar{\chi}^i \square \not{\partial} \chi^i + 32D \square D \right\}. \quad (3.22)$$

This concludes our discussion of linearized higher-derivative actions that can be constructed from the fields of the Weyl multiplet alone.

### 3.3 $\mathcal{N} = 4$ Einstein supergravity

To obtain the  $\mathcal{N} = 4$  supersymmetric Einstein action we need two compensating hypermultiplets, one of each type, for reasons explained in the previous section. Following the superconformal approach we couple these compensating hypermultiplets to the Weyl multiplet and take suitable gauge choices for the superfluous (super-) conformal symmetries. This general procedure simplifies due to the fact that we only consider the linearized version of  $\mathcal{N} = 4$  Einstein supergravity.

The hypermultiplet with physical fields  $(\varphi^{\dot{\alpha}}, \lambda_{\alpha})$  has the supersymmetry transformations

$$\delta\varphi^{\dot{\alpha}} = \bar{\epsilon}^i \lambda_{\beta} \bar{\sigma}^{i\dot{\alpha}\beta}, \quad \delta\lambda_{\alpha} = \frac{1}{4} \gamma^{\mu} \partial_{\mu} \varphi^{\dot{\beta}} \epsilon^i \sigma_{\alpha\dot{\beta}}^i, \quad (3.23)$$

where  $\sigma^i = (\mathbb{1}, -i\sigma^a)$  and  $\bar{\sigma}^i = (\mathbb{1}, i\sigma^a)$  (with  $\sigma^a$  the usual Pauli matrices). Similarly, the hypermultiplet with physical fields  $(\varphi_{\alpha}, \lambda^{\dot{\alpha}})$  has the supersymmetry transformations

$$\delta\varphi_{\alpha} = \bar{\epsilon}^i \lambda^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^i, \quad \delta\lambda^{\dot{\alpha}} = \frac{1}{4} \gamma^{\mu} \partial_{\mu} \varphi_{\alpha} \epsilon^i \bar{\sigma}^{i\dot{\alpha}\alpha}. \quad (3.24)$$

These transformations leave invariant the following action with  $8 + 8$  on-shell degrees of freedom:

$$S = \int d^3x \left( -\partial^{\mu} \varphi^{\dot{\alpha}} \partial_{\mu} \varphi_{\dot{\alpha}} - \partial^{\mu} \varphi_{\alpha} \partial_{\mu} \varphi^{\alpha} - 4\bar{\lambda}^{\beta} \gamma^{\mu} \partial_{\mu} \lambda_{\beta} - 4\bar{\lambda}_{\dot{\beta}} \gamma^{\mu} \partial_{\mu} \lambda^{\dot{\beta}} \right), \quad (3.25)$$

where

$$\varphi_{\dot{\alpha}} = (\varphi^{\dot{\alpha}})^*, \quad \varphi^{\alpha} = (\varphi_{\alpha})^*, \quad \bar{\lambda}^{\beta} = i(\lambda_{\beta})^{\dagger} \gamma^0, \quad \bar{\lambda}_{\dot{\alpha}} = i(\lambda^{\dot{\alpha}})^{\dagger} \gamma^0. \quad (3.26)$$

After coupling to conformal supergravity, the action (3.25) gets extended by many terms which are schematically of the form

$$(\text{matter})^2 \times \{1 + (\text{Weyl}) + (\text{Weyl})^2\}, \quad (3.27)$$

where the terms in brackets denote factors that are independent, linear or bilinear in the fields of the conformal supergravity multiplet. In particular, the  $(\text{matter})^2 \times (\text{Weyl})^2$  terms contain a term proportional to  $R\varphi^2$ , and this gives rise to the Einstein-Hilbert term on fixing the conformal gauge. Similarly, there will be a term  $\bar{\psi}\phi\varphi^2$  from which the gravitino kinetic term follows, after fixing the conformal gauge, on substitution for  $\phi_{\mu}^i$ .

It turns out that many terms in the action are irrelevant for the determination of the final quadratic expression. To keep things simple we only present those terms in the action that contribute to the final answer. These terms, collectively denoted by  $\mathcal{L}_{\text{quadr.}}^{\mathcal{N}=4}$ , are given by

$$\begin{aligned} \mathcal{L}_{\text{quadr.}}^{\mathcal{N}=4} &= -8D^{\mu} \varphi^{\dot{\alpha}} D_{\mu} \varphi_{\dot{\alpha}} - 8D^{\mu} \varphi_{\alpha} D_{\mu} \varphi^{\alpha} - 32\bar{\lambda}^{\beta} \gamma^{\mu} D_{\mu} \lambda_{\beta} - 32\bar{\lambda}_{\dot{\beta}} \gamma^{\mu} D_{\mu} \lambda^{\dot{\beta}} \\ &- R\varphi_+^2 - \frac{1}{2} \bar{\psi}_{\mu}^k \gamma^{\mu\nu} \phi_{\nu}^k \varphi_+^2 - 64D\varphi_-^2 - 32E^2\varphi_+^2 \\ &+ 16\bar{\chi}^i (\bar{\sigma}^{i\dot{\alpha}\alpha} \lambda_{\alpha} \varphi_{\dot{\alpha}} - \sigma_{\alpha\dot{\alpha}}^i \lambda^{\dot{\alpha}} \varphi^{\alpha}), \end{aligned} \quad (3.28)$$

where  $\varphi_+^2$  and  $\varphi_-^2$  are defined in (2.10).

Note that the  $R\varphi^2$  term is precisely such that the scalar wave equation is conformally coupled to the background metric. Note also that the action is invariant under the exchange of the two SU(2) factors of the  $R$ -symmetry group because the  $D$ ,  $E$  and  $\chi^i$  fields are odd under this exchange. The covariant derivatives  $D_\mu$  are covariant with respect to Lorentz and SO(4) rotations only. Defining

$$V_\mu^{ij} = (\sigma^{ij})_\alpha{}^\beta V_\mu{}^\alpha{}_\beta + (\bar{\sigma}^{ij})_{\dot{\alpha}}{}^{\dot{\beta}} V_\mu{}^{\dot{\alpha}}{}_{\dot{\beta}}, \quad (3.29)$$

the covariant derivative  $D_\mu\phi_\alpha$  is given by

$$D_\mu\varphi_\alpha = \partial_\mu\varphi_\alpha - V_{\mu\alpha}{}^\beta\varphi_\beta. \quad (3.30)$$

In principle, all coefficients in the action (3.28) follow from the supersymmetry of the full non-linear action. In practice, it is much easier to take the action (3.28) only as a guideline to obtain the final quadratic expression. The coefficients in this final expression are easily determined by requiring (linearized) supersymmetry.

After coupling to conformal supergravity the fields of the two hypermultiplets transform under local dilatations and  $S$ -supersymmetry transformations as follows:

$$\delta\varphi^{\dot{\alpha}} = \omega\varphi^{\dot{\alpha}}, \quad \delta\lambda_\alpha = \sigma_{\alpha\dot{\beta}}^i\varphi^{\dot{\beta}}\eta^i, \quad (3.31)$$

$$\delta\varphi_\alpha = \omega\varphi_\alpha, \quad \delta\lambda^{\dot{\alpha}} = \bar{\sigma}^{i\dot{\alpha}\beta}\varphi_\beta\eta^i. \quad (3.32)$$

To fix the dilatations and  $S$ -supersymmetry transformations we impose the gauge choices

$$\varphi_+^2 = 1, \quad \bar{\sigma}^{i\dot{\alpha}\alpha}\lambda_\alpha\varphi_{\dot{\alpha}} + \sigma_{\alpha\dot{\alpha}}^i\lambda^{\dot{\alpha}}\varphi^\alpha = 0, \quad (3.33)$$

respectively. These gauge choices can be used in (3.28). The  $D$  and  $\chi$  fields are Lagrange multipliers that lead to a single bosonic constraint

$$\Phi \equiv \varphi_-^2 = 0 \quad (3.34)$$

and 4 fermionic constraints

$$\psi^i \equiv \bar{\sigma}^{i\dot{\alpha}\alpha}\lambda_\alpha\varphi_{\dot{\alpha}} - \sigma_{\alpha\dot{\alpha}}^i\lambda^{\dot{\alpha}}\varphi^\alpha = 0, \quad (3.35)$$

respectively. Since we are going to add further higher-derivative terms to the action in the next subsection we leave these Lagrange multipliers in the action.

Using the gauge choices (3.33), the field equation of the 6 vector fields allows for these vector fields to be solved in terms of the scalars:

$$V_\mu^{ij} = (\sigma^{ij})_\alpha{}^\beta\varphi^\alpha\partial_\mu\varphi_\beta + (\bar{\sigma}^{ij})_{\dot{\alpha}}{}^{\dot{\beta}}\varphi^{\dot{\alpha}}\partial_\mu\varphi_{\dot{\beta}}. \quad (3.36)$$

We next fix the local SO(4) by imposing the following 6 conditions:

$$\varphi^\alpha\partial_\mu\varphi_\beta + \text{h.c.} = 0, \quad \varphi^{\dot{\alpha}}\partial_\mu\varphi_{\dot{\beta}} + \text{h.c.} = 0, \quad (3.37)$$

which implies  $V_\mu^{ij} = 0$ .

We note that for pure Einstein supergravity (no further higher-derivative terms added to the action) the 8+8 matter fields of the compensating hypermultiplets are all fixed by the constraints and gauge choices, as the following counting shows:

$$\begin{aligned} \text{bosons} &: 8 - 1 \text{ (D-gauge)} - 6 \text{ (SO(4)-gauge)} - 1 \text{ (constraint)} = 0, \\ \text{fermions} &: 8 - 4 \text{ (S-gauge)} - 4 \text{ (constraints)} = 0. \end{aligned} \quad (3.38)$$

Using the gauge choices (3.33) and (3.37), one can show that it is possible to rewrite the kinetic terms of the compensating matter scalars and fermions, appearing in the conformal action, as standard kinetic terms for  $\Phi$  and  $\psi^i$  only. One thus obtains the following linearized  $\mathcal{N} = 4$  supersymmetric Einstein action

$$\begin{aligned} S_2^{\mathcal{N}=4} = \int d^3x \left\{ \frac{1}{2} h^{\mu\nu} G_{\mu\nu}^{(\text{lin})} + \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^i \partial_\nu \psi_\rho^i - V_\mu^{ij} V_{\mu ij} - 32E^2 \right. \\ \left. - 8\bar{\psi}\not{\partial}\psi + 16\bar{\psi}\chi - 64D\Phi + 32\Phi\Box\Phi \right\}. \end{aligned} \quad (3.39)$$

Using the fact that the gauge-fixing requires a compensating  $S$ -transformation and SO(4) transformation with parameters

$$\eta^i = \frac{1}{2} \gamma^\rho V_\rho^{ij} \epsilon^j, \quad \Lambda^{ij} = -\bar{\epsilon}^{[i} \psi^{j]}, \quad (3.40)$$

respectively, we find that the action (3.39) is invariant under the following supersymmetry rules

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}^i \gamma_{(\mu} \psi_{\nu)}^i, \\ \delta \psi_\mu^i &= -\frac{1}{4} \gamma^{\rho\sigma} \partial_\rho h_{\mu\sigma} \epsilon^i - V_\mu^{ij} \epsilon^j + \frac{1}{2} \gamma_\mu \gamma^\rho V_\rho^{ij} \epsilon^j, \\ \delta V_\mu^{ij} &= \frac{1}{2} \bar{\epsilon}^{[i} \phi_\mu^{j]} + \epsilon^{ijkl} \bar{\epsilon}^k \gamma_\mu \chi^l - \epsilon^{ijkl} \bar{\epsilon}^k \partial_\mu \psi^l, \quad \delta E = \frac{1}{4} \bar{\epsilon}^i \chi^i, \\ \delta \chi^i &= \frac{1}{8} \epsilon^{ijkl} \gamma^\mu F_{\mu(\text{lin})}^{jk} \epsilon^l + \gamma^\mu (\partial_\mu E) \epsilon^i + D \epsilon^i, \quad \delta D = \frac{1}{4} \bar{\epsilon}^i \not{\partial} \chi^i, \\ \delta \psi^i &= \frac{1}{8} \epsilon^{ijkl} \gamma^\mu V_\mu^{jk} \epsilon^l + E \epsilon^i + \gamma^\mu (\partial_\mu \Phi) \epsilon^i, \quad \delta \Phi = \frac{1}{4} \bar{\epsilon}^i \psi^i. \end{aligned} \quad (3.41)$$

Note that the action (3.39) does not describe any massive degrees of freedom. Several fields could be integrated out to leave us with only the Einstein term and the kinetic term of the gravitino. We do not do this because the equations of motion of these fields change as soon as we add one of the superconformal higher-derivative actions constructed in the previous subsection.

### 3.4 $\mathcal{N} = 4$ massive supergravities

In this subsection we will consider the sum of the supersymmetric Einstein action and the conformal higher-derivative actions constructed in subsection 3.2 with 3,4 and 5 derivatives thereby introducing three mass parameters  $\mu$ ,  $m$  and  $M$ , respectively. This leads to the following action that is left invariant under the same transformation rules (3.41) derived in the previous subsection:

$$\begin{aligned}
S^{\mathcal{N}=4} = & \int d^3x \left\{ \frac{1}{2} h^{\mu\nu} G_{\mu\nu}^{(\text{lin})} + \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^i \partial_\nu \psi_\rho^i - V_\mu^{ij} V_{\mu ij} - 32E^2 \right. \\
& \left. - 8\bar{\psi} \not{\partial} \psi + 16\bar{\psi} \chi - 64D\Phi + 32\Phi \square \Phi \right\} \\
& + \frac{1}{\mu} \left\{ h^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{\psi}_\mu^i C_{(\text{lin})}^{\mu i} - 2V_\mu^{ij} F_{(\text{lin})}^{\mu ij} + 16\bar{\chi}^i \chi^i - 128ED \right\} \\
& + \frac{1}{m^2} \left\{ -\frac{1}{2} \epsilon^{\mu\tau\rho} h_\mu{}^\nu \partial_\tau C_{\rho\nu}^{(\text{lin})} - \frac{1}{2} \bar{\psi}_\mu^i \not{\partial} C^{i\mu}{}_{(\text{lin})} + F_{\mu(\text{lin})}^{ij} F_{(\text{lin})}^{\mu ij} + \right. \\
& \left. + 32E \square E - 8\bar{\chi}^i \not{\partial} \chi^i + 32D^2 \right\} \\
& + \frac{1}{M^3} \left\{ R_{(\text{lin})}^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{C}_{\mu i}^{(\text{lin})} C_{(\text{lin})}^{\mu i} + \epsilon^{\mu\nu\rho} F_{\mu}^{(\text{lin}) ij} \partial_\nu F_{\rho}^{(\text{lin}) ij} \right. \\
& \left. + 16\bar{\chi}^i \square \chi^i - 128E \square D \right\}. \tag{3.42}
\end{aligned}$$

To analyze which massive supermultiplets are propagating for the different values of the mass parameters, it is enough to consider the scalar equations of motion:

$$\begin{aligned}
E + \frac{2}{\mu} D - \frac{1}{m^2} \square E + \frac{2}{M^3} \square D &= 0, \\
\Phi + \frac{2}{\mu} E - \frac{1}{m^2} D + \frac{2}{M^3} \square E &= 0, \\
\square \Phi - D &= 0,
\end{aligned} \tag{3.43}$$

and the fermionic equations of motion:

$$\begin{aligned}
\not{\partial} \psi^i &= \chi^i, \\
\psi^i + \frac{2}{\mu} \chi^i - \frac{1}{m^2} \not{\partial} \chi^i + \frac{2}{M^3} \square \chi^i &= 0.
\end{aligned} \tag{3.44}$$

We may now analyze the following models

- (1) *Supersymmetric Einstein*:  $\mu, m, M \rightarrow \infty$

We find

$$E = D = \Phi = \psi^i = \chi^i = 0, \quad (3.45)$$

and hence there are no propagating multiplets.

(2) *Supersymmetric TMG*:  $m, M \rightarrow \infty$

We find one independent propagating scalar that satisfies

$$\square\Phi - \frac{\mu^2}{4}\Phi = 0, \quad (3.46)$$

and hence there is one propagating massive spin 2 supermultiplet. For the fermions we find

$$\not{\partial}\psi^i = \chi^i, \quad \psi^i = -\frac{2}{\mu}\chi^i, \quad (3.47)$$

and hence

$$\not{\partial}\psi^i = -\frac{\mu}{2}\psi^i. \quad (3.48)$$

Applying an extra  $\not{\partial}$  leads to

$$\square\psi^i - \frac{\mu^2}{4}\psi = 0, \quad (3.49)$$

which is in agreement with the  $\mathcal{N} = 4$  spin 2 supermultiplet content displayed in Table 1.

(3) *Supersymmetric NMG*:  $\mu, M \rightarrow \infty$

We find two independent scalars with

$$\square D - m^2 D = 0, \quad \square E - m^2 E = 0, \quad (3.50)$$

and hence there are two propagating  $\mathcal{N} = 4$  massive multiplets, of the same mass but with opposite helicity. For the fermions we find:

$$\not{\partial}\psi^i = \chi^i, \quad \not{\partial}\chi^i = m^2\psi^i. \quad (3.51)$$

Upon diagonalization of these equations one infers that their mass eigenvalues are  $\pm m$ , which implies that they describe opposite helicities, as required. By applying  $\not{\partial}$ , one finds

$$\square\chi^i - m^2\chi^i = 0, \quad \square\psi^i - m^2\psi^i = 0, \quad (3.52)$$

which is again in agreement with the scalar analysis.

(4) *Supersymmetric GMG*:  $M \rightarrow \infty$

We now have

$$\begin{aligned}
E + \frac{2}{\mu}D - \frac{1}{m^2}\square E &= 0, \\
\Phi + \frac{2}{\mu}E - \frac{1}{m^2}D &= 0, \\
\square\Phi - D &= 0.
\end{aligned} \tag{3.53}$$

Eliminating the scalar  $\Phi$  and replacing  $E \rightarrow 1/\mu E$  we obtain

$$\begin{aligned}
\square D - \left(m^2 + \frac{4m^4}{\mu^2}\right)D - \frac{2m^4}{\mu^2}E &= 0, \\
\square E - 2m^2D - m^2E &= 0.
\end{aligned} \tag{3.54}$$

After a diagonalization we find two propagating multiplets with different masses :

$$m_{\pm}^2 = \frac{m^2 \left(2m^2 + \mu^2 \pm 2\sqrt{m^2(m^2 + \mu^2)}\right)}{\mu^2}, \tag{3.55}$$

such that<sup>8</sup>

$$m_+m_- = m^2, \quad m_- - m_+ = 2\frac{m^2}{\mu}. \tag{3.56}$$

The corresponding eigenscalars are  $D + b_{\pm}E$ , where

$$b_{\pm} = \frac{-m^2 \pm \sqrt{m^4 + m^2\mu^2}}{\mu^2}. \tag{3.57}$$

For the fermions we find

$$\begin{aligned}
\rlap{-}\partial\psi^i &= \chi^i, \\
\rlap{-}\partial\chi^i &= m^2\psi^i + \frac{2m^2}{\mu}\chi^i.
\end{aligned} \tag{3.58}$$

By applying  $\rlap{-}\partial$  and similar manipulations as done for the scalars, one arrives at

$$\begin{aligned}
\square\psi^i - 2m^2\chi^i - m^2\psi^i &= 0, \\
\square\chi^i - \left(m^2 + \frac{4m^4}{\mu^2}\right)\chi^i - \frac{2m^4}{\mu^2}\psi^i &= 0.
\end{aligned} \tag{3.59}$$

This is the same system of equations as encountered in the scalar case, with  $D \rightarrow \chi^i$ ,  $E \rightarrow \psi^i$ . The diagonalisation procedure and mass<sup>2</sup> eigenvalues are thus the same as in the scalar case.

(5) *Supersymmetric NTMG*:  $M \rightarrow \infty$ ,  $\mu, m \rightarrow 0$  such that  $m^2/\mu = \text{const}$ .

---

<sup>8</sup>We have fixed the sign of  $m_- - m_+$  in accordance with [9]. To compare, use that  $\tilde{\mu} = 2m^2/\mu^2$ .

We now have one propagating scalar,

$$\square E - \frac{4m^4}{\mu^2} E = 0, \quad (3.60)$$

and hence one propagating multiplet. Similarly, we find for the fermions:

$$\not{\partial}\psi^i = \chi^i, \quad \not{\partial}\chi^i = \frac{2m^2}{\mu}\chi^i, \quad (3.61)$$

from which one derives

$$\square\chi^i - \frac{4m^4}{\mu^2}\chi^i = 0, \quad (3.62)$$

again in agreement with the scalars.

(6) *Supersymmetric ENMG*:  $m^2 \rightarrow \infty, \mu, M \rightarrow 0$  such that  $\mu/M^3 = \text{const.}$

This is the case of ‘Extended New Massive Gravity’ (ENMG) discussed in the third reference of [9]. This case leads to the scalar equations

$$\frac{1}{\mu}D + \frac{1}{M^3}\square D = 0, \quad \frac{1}{\mu}E + \frac{1}{M^3}\square E = 0. \quad (3.63)$$

Since in the action we have

$$E\square D \sim A\square A - B\square B, \quad E = A + B, D = A - B, \quad (3.64)$$

we end up with one physical and one ghost multiplet, each with  $(\text{mass})^2 = M^3/\mu$ .

For the fermions we find

$$\frac{1}{\mu}\chi^i + \frac{1}{M^3}\square\chi^i = 0, \quad (3.65)$$

which also implies one physical and one ghost multiplet.

We have summarized the analysis of the different models in Table 2.

## 4 $\mathcal{N} = 8$ Maximal Supergravity

In this section we consider the case of maximal supersymmetry. Since the application of the superconformal tensor calculus has been explained in the previous section we will be brief whenever there is overlap with the  $\mathcal{N} = 4$  case.

**Table 3:** Some 3D higher-derivative sugra models. The second column indicates the number of derivatives in the different terms for the spin 2 field. The 3rd and 4th column indicate the massive spin 2 multiplets described by the corresponding supergravity model. The last 2 columns indicate the masses of these  $\mathcal{N} = 4$  multiplets, respectively. The boldface number in the lowest row indicates a ghost multiplet.

sugra model	action	$\mathcal{N} = 4$	opposite helicity	(mass) <sup>2</sup>	(mass) <sup>2</sup>
Einstein	2	–	–	–	–
TMG	2+3	1	–	$\mu^2$	–
NMG	2+4	1	1	$m^2$	$m^2$
GMG	2+3+4	1	1	$m_+^2$	$m_-^2$
NTMG	3+4	1	–	$m^4/\mu^2$	–
ENMG	3+5	1	<b>1</b>	$M^3/\mu$	$M^3/\mu$

#### 4.1 The $\mathcal{N} = 8$ conformal multiplet

The  $D = 3$ ,  $\mathcal{N} = 8$  linearized conformal supergravity multiplet has 128+128 field degrees of freedom and contains the following components:

$$\{h_{\mu\nu}, \psi_\mu^i, V_\mu^{ij}, E^{ijkl}, \chi^{ijk}, D^{ijkl}\}, \quad (4.1)$$

with  $h_{\mu\nu} = h_{\nu\mu}$  the linearized graviton,  $\psi_\mu^i$  ( $i = 1, \dots, 8$ ) the 8 gravitini and  $V_\mu^{ij} = -V_\mu^{ji}$  the SO(8) R-symmetry gauge field. The matter fields  $E^{ijkl}$  and  $D^{ijkl}$  (of different mass dimension) are in the antisymmetric selfdual  $\mathbf{35}^+$  and anti-selfdual  $\mathbf{35}^-$  representation of SO(8), respectively, while the fermions  $\chi^{ijk}$  form an antisymmetric  $\mathbf{56}$ -plet of SO(8).

The linearized supersymmetry transformations are given by [13] (using the same conventions as in [10])

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}^i \gamma_{(\mu} \psi_{\nu)}^i, \\ \delta \psi_\mu^i &= -\frac{1}{4} \gamma^{\rho\sigma} \partial_\rho h_{\mu\sigma} \epsilon^i - V_\mu^{ij} \epsilon^j, \\ \delta V_\mu^{ij} &= \frac{1}{2} \bar{\epsilon}^{[i} \phi_\mu^{j]} + \bar{\epsilon}^k \gamma_\mu \chi^{ijk}, \\ \delta E^{ijkl} &= -\bar{\epsilon}^{[i} \chi^{jkl]} + \text{dual}, \\ \delta \chi^{ijk} &= -\frac{3}{4} \gamma^\mu F_{\mu(\text{lin})}^{[ij} \epsilon^{k]} + \gamma^\mu (\partial_\mu E^{ijkl}) \epsilon^l + D^{ijkl} \epsilon^l, \\ \delta D^{ijkl} &= -\bar{\epsilon}^{[i} \not{\partial} \chi^{jkl]} - \text{dual}. \end{aligned} \quad (4.2)$$

Here  $F_{(\text{lin})}^{\mu ij}$  and  $\phi_\mu^i$  are defined in an analogous manner as in (3.2) for  $\mathcal{N} = 4$ . We insist on gauge invariance with respect to the analogue of the linear gauge transformations

of (2.1). Finally, the supersymmetry algebra closes in the same way as in the  $\mathcal{N} = 4$  case.

## 4.2 $\mathcal{N} = 8$ conformal higher-derivative actions

Using the same definitions of the Cotton and Cottino tensor as in the  $\mathcal{N} = 4$  case we may construct the following parity-even conformal higher-derivative actions for the  $\mathcal{N} = 8$  conformal multiplet.

### (1) $\mathcal{N} = 8$ supersymmetric $R^2$

There exists a supersymmetric action of the  $\mathcal{N} = 8$  conformal multiplet that starts with the linearized version of the  $R_{\mu\nu}^2 - \frac{3}{8}R^2$  term of NMG [9]. More precisely, the transformation rules (4.2) leave the following action invariant:

$$S_4^{\mathcal{N}=8} = \int d^3x \left\{ -\frac{1}{2}\epsilon^{\mu\tau\rho}h_\mu{}^\nu\partial_\tau C_{\rho\nu}^{(\text{lin})} - \frac{1}{2}\bar{\psi}_\mu^i\partial\mathcal{C}^{\mu(\text{lin})i} + F_{\mu(\text{lin})}^{ij}F_{(\text{lin})}^{\mu ij} + \frac{2}{3}E^{ijkl}\square E_{ijkl} - \frac{4}{3}\bar{\chi}^{ijk}\not{\partial}\chi^{ijk} + \frac{2}{3}D^{ijkl}D^{ijkl} \right\}. \quad (4.3)$$

Under the rigid supersymmetry rules (4.2) the Cotton, Cottino and SO(8) curvature tensors transform as follows

$$\begin{aligned} \delta C_{\mu\nu}^{(\text{lin})} &= -\frac{1}{4}\bar{\epsilon}^i\gamma_{(\mu}{}^\rho\partial_\rho C_{\nu)}^{i(\text{lin})}, \\ \delta \mathcal{C}_{(\text{lin})}^{\mu i} &= \gamma_\nu\epsilon^i C_{(\text{lin})}^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\sigma\gamma_\nu\epsilon^j\partial_\rho F_{(\text{lin})}^{\sigma ij}, \\ \delta F_{(\text{lin})}^{\mu ij} &= \frac{1}{2}\bar{\epsilon}^{[i}C_{(\text{lin})}^{\mu j]} + \bar{\epsilon}^k\gamma^{\mu\rho}\partial_\rho\chi^{ijk}. \end{aligned} \quad (4.4)$$

These transformation rules define a  $\mathcal{N} = 8$  conjugated Weyl multiplet with components

$$\{C_{\mu\nu}^{(\text{lin})}, \mathcal{C}_{(\text{lin})}^{\mu i}, F_{(\text{lin})}^{\mu ij}, \chi^{ijk}, D^{ijkl}, E^{ijkl}\}. \quad (4.5)$$

### (2) $\mathcal{N} = 8$ supersymmetric Cotton tensor squared

Following the same procedure as in the  $\mathcal{N} = 4$  case, making use of the conjugated Weyl multiplet (4.5), we obtain the following supersymmetric Cotton tensor squared action:

$$S_6^{\mathcal{N}=8} = \int d^3x \left\{ C_{(\text{lin})}^{\mu\nu}C_{\mu\nu}^{(\text{lin})} - \frac{1}{4}\bar{\mathcal{C}}_\mu^{i(\text{lin})}\not{\partial}\mathcal{C}_{(\text{lin})}^{\mu i} + F_{\mu(\text{lin})}^{ij}\square F_{(\text{lin})}^{\mu ij} + \frac{2}{3}E^{ijkl}\square^2 E_{ijkl} - \frac{4}{3}\bar{\chi}^{ijk}\square\not{\partial}\chi^{ijk} + \frac{2}{3}D^{ijkl}\square D^{ijkl} \right\}. \quad (4.6)$$

This concludes our discussion of the (parity-even) conformal higher-derivative actions. In the next subsection we consider the case of (non-conformal)  $\mathcal{N} = 8$  Einstein supergravity.

### 4.3 $\mathcal{N} = 8$ Einstein supergravity

To obtain supersymmetric Einstein we introduce 8 scalar multiplets and couple them to conformal supergravity. The supersymmetry transformation rules of 8 scalar multiplets with components  $(\phi^{\alpha A}, \chi_{\dot{\alpha}}^A)$  ( $A = 1, \dots, 8$ ) are given by

$$\begin{aligned}\delta\varphi^{\alpha A} &= \bar{\epsilon}^i(\gamma^i)^{\alpha\dot{\beta}}\lambda_{\dot{\beta}}^A, \\ \delta\lambda_{\dot{\alpha}}^A &= \frac{1}{4}\gamma^\mu(\partial_\mu\varphi^{\alpha A})\tilde{\gamma}_{\dot{\alpha}\alpha}^i\epsilon^i,\end{aligned}\tag{4.7}$$

where  $(i, \alpha, \dot{\alpha}) = (1, \dots, 8)$  denote the 8-dimensional  $\mathbf{v}, \mathbf{s}$  and  $\mathbf{c}$  representations of SO(8), respectively. For the SO(8) Dirac matrices we use the notation of [22].

The transformation rules (4.7) leave the following action invariant

$$S = \int d^3x \left\{ \partial_\mu\varphi^{\alpha A}\partial^\mu\varphi_{\dot{\alpha}}^A + 4\bar{\lambda}^{\dot{\alpha}A}\not{\partial}\lambda_{\dot{\alpha}}^A \right\}.\tag{4.8}$$

We next couple the above action to  $\mathcal{N} = 8$  conformal supergravity and follow precisely the same steps as in the  $\mathcal{N} = 4$  case. From now on we identify  $A = \alpha$  and write  $\varphi^{\alpha B} = \varphi^{\alpha\beta}$ . This leads to the following Einstein supergravity action

$$\begin{aligned}S_2^{\mathcal{N}=8} &= \int d^3x \left\{ \frac{1}{2}h^{\mu\nu}G_{\mu\nu}^{(\text{lin})} + \epsilon^{\mu\nu\rho}\bar{\psi}_\mu^i\partial_\nu\psi_\rho^i - V_\mu^{ij}V_{\mu ij} - \frac{2}{3}E^{ijkl}E_{ijkl} \right. \\ &\quad \left. - \frac{4}{3}\bar{\psi}^{ijk}\not{\partial}\psi^{ijk} + \frac{8}{3}\bar{\psi}^{ijk}\chi_{ijk} - \frac{4}{3}D^{ijkl}\Phi_{ijkl} + \frac{2}{3}\Phi^{ijkl}\square\Phi_{ijkl} \right\}.\end{aligned}\tag{4.9}$$

We find that this action is invariant under the following supersymmetry rules:

$$\begin{aligned}\delta h_{\mu\nu} &= \bar{\epsilon}^i\gamma_{(\mu}\psi_{\nu)}^i, \\ \delta\psi_\mu^i &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon^i - V_\mu^{ij}\epsilon^j + \frac{1}{2}\gamma_\mu\gamma^\rho V_\rho^{ij}\epsilon^j, \\ \delta V_\mu^{ij} &= \frac{1}{2}\bar{\epsilon}^{[i}\phi_{\mu}^{j]} + \bar{\epsilon}^k\gamma_\mu\chi^{ijk} - \bar{\epsilon}^k\partial_\mu\psi^{ijk}, \\ \delta E^{ijkl} &= -\bar{\epsilon}^{[i}\chi^{jkl]} + \text{dual}, \\ \delta\chi^{ijk} &= -\frac{3}{4}\gamma^\mu F_\mu^{[ij}\epsilon^{k]} + \gamma^\mu(\partial_\mu E^{ijkl})\epsilon^l + D^{ijkl}\epsilon^l, \\ \delta D^{ijkl} &= -\bar{\epsilon}^{[i}\not{\partial}\chi^{jkl]} - \text{dual}, \\ \delta\psi^{ijk} &= -\frac{3}{4}\gamma^\mu V_\mu^{[ij}\epsilon^{k]} + E^{ijkl}\epsilon^l + \gamma^\mu(\partial_\mu\Phi^{ijkl})\epsilon^l, \\ \delta\Phi^{ijkl} &= -\bar{\epsilon}^{[i}\psi^{jkl]} - \text{dual}.\end{aligned}\tag{4.10}$$

### 4.4 $\mathcal{N} = 8$ new massive supergravity

To obtain  $\mathcal{N} = 8$  new massive supergravity we add the supersymmetric  $R^2$  action (4.3), which we give a coefficient  $1/m^2$ , to the supersymmetric Einstein action. Since

the conformal multiplet is off-shell we retain supersymmetry. The combined action is given by

$$\begin{aligned}
S_{\text{NMG}}^{\mathcal{N}=8} = & \int d^3x \left\{ \frac{1}{2} h^{\mu\nu} G_{\mu\nu}^{(\text{lin})} + \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^i \partial_\nu \psi_\rho^i - V_\mu^{ij} V_{\mu ij} - \frac{2}{3} E^{ijkl} E_{ijkl} \right. \\
& - \frac{4}{3} \bar{\psi}^{ijk} \not{\partial} \psi^{ijk} + \frac{8}{3} \bar{\psi}^{ijk} \chi_{ijk} - \frac{4}{3} D^{ijkl} \Phi_{ijkl} + \frac{2}{3} \Phi^{ijkl} \square \Phi_{ijkl} \left. \right\} \\
& + \frac{1}{m^2} \left\{ -\frac{1}{2} \epsilon^{\mu\tau\rho} h_\mu^\nu \partial_\tau C_{\rho\nu}^{(\text{lin})} - \frac{1}{2} \bar{\psi}_\mu^i \not{\partial} \mathcal{C}^{\mu(\text{lin})i} + F_{\mu(\text{lin})}^{ij} F_{(\text{lin})}^{\mu ij} \right. \\
& \left. - \frac{4}{3} \bar{\chi}^{ijk} \not{\partial} \chi^{ijk} + \frac{2}{3} E^{ijkl} \square E_{ijkl} + \frac{2}{3} D^{ijkl} D_{ijkl} \right\}. \tag{4.11}
\end{aligned}$$

It is invariant under the same supersymmetry transformations (4.10). The only thing that changes with respect to the pure Einstein supergravity action is that the equations of motion of the conformal fields  $E, D, V$  and  $\chi$ , receive  $1/m^2$  corrections which leads to propagating massive degrees of freedom. To be precise, the corrected equations of motion read:

$$E_{ijkl} - \frac{1}{m^2} \square E_{ijkl} = 0, \tag{4.12}$$

$$V_{\mu ij} - \frac{1}{m^2} \partial^\lambda F_{\lambda\mu ij}^{(\text{lin})}(V) = 0, \tag{4.13}$$

$$\Phi^{ijkl} - \frac{1}{m^2} D^{ijkl} = 0, \tag{4.14}$$

$$\psi^{ijk} - \frac{1}{m^2} \not{\partial} \chi^{ijk} = 0. \tag{4.15}$$

The first equation shows that the  $E$  scalars describe 35 massive helicity 0 d.o.f. The third equation, together with the uncorrected equation for  $\Phi^{ijkl}$ ,

$$\square \Phi^{ijkl} - D^{ijkl} = 0, \tag{4.16}$$

can be used to show that the  $D$  scalars satisfy

$$D^{ijkl} - \frac{1}{m^2} \square D^{ijkl} = 0, \tag{4.17}$$

and hence describe another 35 helicity d.o.f. of the same mass  $m$ . From the second equation it follows that the vector fields describe 28 helicity  $\pm 1$  states:

$$V_\mu^{ij} - \frac{1}{m^2} \square V_\mu^{ij} = 0. \tag{4.18}$$

Concerning the helicity  $\pm 1/2$  degrees of freedom we end up with the following two equations of motion:

$$\not{\partial} \chi^{ijk} - m^2 \psi^{ijk} = 0, \quad \not{\partial} \psi^{ijk} - \chi^{ijk} = 0. \tag{4.19}$$

By taking sums and differences we see that this system describes  $56 + 1/2$  helicity states and  $56 - 1/2$  helicity states of the same mass  $m$ , i.e.

$$\chi^{ijk} - \frac{1}{m^2} \square \chi^{ijk} = 0, \quad \psi^{ijk} - \frac{1}{m^2} \square \psi^{ijk} = 0. \quad (4.20)$$

Adding up all massive degrees of freedom, including the helicity  $\pm 2$  and  $\pm 3/2$  states, we precisely obtain the content of the  $\mathcal{N} = 8$  massive super multiplet given in Table 1, as it should be.

## 5 Linearized $\mathcal{N} = 7$ TMG

In this section we study maximal supersymmetry for parity-odd actions. We first consider the Lorentz Chern-Simons (LCS) term. It turns out that there does not exist a  $\mathcal{N} = 8$  supersymmetric LCS action for the *full* conformal multiplet. The reason for this is that such an action would require a term of the form  $E^{ijkl} D^{ijkl}$  but, due to the opposite dualities of the  $E$  and  $D$  fields, such a term does not exist. The best one can do is write down the following so-called ‘‘pseudo-action’’ assuming that  $E^{ijkl}$  and  $D^{ijkl}$  are not (anti-)selfdual (the Cotton tensor  $C$  and the Cottino tenor  $\mathcal{C}$  are defined below):

$$S_3^{\mathcal{N}=8} = \int d^3x \left\{ h^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{\psi}_\mu \mathcal{C}_{(\text{lin})}^{\mu i} - 2V_\mu^{ij} F_{(\text{lin})}^{\mu ij} + \frac{8}{3} \bar{\chi}^{ijk} \chi^{ijk} \right\}. \quad (5.1)$$

This action is invariant under supersymmetry up to terms proportional to  $E^{ijkl}$  or  $D^{ijkl}$ . Indeed, up to a total derivative, the variation of the above action is given by

$$\delta S_3^{\mathcal{N}=8} = \int d^3x \left\{ \frac{16}{3} \bar{\chi}^{ijk} \gamma^\mu \epsilon^l \partial_\mu E^{ijkl} + \frac{16}{3} \bar{\chi}^{ijk} \epsilon^l D^{ijkl} \right\}. \quad (5.2)$$

The (anti-)selfduality of the  $E$  and  $D$  fields is only imposed at the level of the equations of motion. Note that setting  $E^{ijkl} = D^{ijkl} = 0$  in the conformal multiplet leads to equations of motion for the remaining fields that can be integrated to an action which is precisely the above action [16]. We have not been able to write down a similar pseudo-action for the combined Einstein-Chern-Simons system. This is only possible if one can impose (anti-) selfduality in the equations of motion for  $D^{ijkl}$  and  $E^{ijkl}$ . This seems unlikely since we expect an equation of motion of the form  $E^{ijkl} = \frac{1}{\mu} D^{ijkl} + \dots$  where  $\mu$  is a mass parameter. Such an equation obviously is inconsistent with the duality properties of the  $E$  and  $D$  fields. The non-existence of such a  $\mathcal{N} = 8$  topologically massive supergravity theory can also be anticipated from the fact that such a theory would lead to a  $\mathcal{N} = 8$  supersymmetric massive supermultiplet with broken parity. According to Table 1 such a multiplet does not exist.

Although a supersymmetric  $\mathcal{N} = 8$  LCS action does not exist, one can construct a supersymmetric  $\mathcal{N} = 7$  version. In order to do this, one decomposes the  $R$ -symmetry index  $i$  as  $i = \{I, 8\}$ , where  $I = 1, \dots, 7$ . Performing this decomposition, one arrives

at the following fields:

$$\begin{aligned}
h_{\mu\nu}, \quad \psi_\mu^I, \quad \psi_\mu \equiv \psi_\mu^8, \quad V_\mu^{IJ}, \quad V_\mu^I \equiv V_\mu^{I8}, \quad \chi^{IJK}, \quad \chi^{IJ} \equiv \chi^{IJ8}, \\
E^{IJKL}, \quad E^{IJK} \equiv E^{IJK8}, \quad D^{IJKL}, \quad D^{IJK} \equiv D^{IJK8}.
\end{aligned} \tag{5.3}$$

The transformation rules can be found from the  $\mathcal{N} = 8$  ones, by making the above decomposition and by putting  $\epsilon^8 = 0$ . In this way one finds

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar{\epsilon}^I \gamma_{(\mu} \psi_{\nu)}^I, \\
\delta \psi_\mu^I &= -\frac{1}{4} \gamma^{\rho\sigma} \partial_\rho h_{\mu\sigma} \epsilon^I - V_\mu^{IJ} \epsilon^J, \\
\delta \psi_\mu &= V_\mu^I \epsilon^I, \\
\delta V_\mu^{IJ} &= \frac{1}{2} \bar{\epsilon}^{[I} \phi_{\mu}^{J]} + \bar{\epsilon}^K \gamma_\mu \chi^{IJK}, \\
\delta V_\mu^I &= \frac{1}{4} \bar{\epsilon}^I \phi_\mu - \bar{\epsilon}^J \gamma_\mu \chi^{IJ}, \\
\delta \chi^{IJK} &= -\frac{3}{4} \gamma^\mu F_\mu^{[IJ} \epsilon^{K]} + \gamma^\mu \partial_\mu E^{IJKL} \epsilon^L + D^{IJKL} \epsilon^L, \\
\delta E^{IJKL} &= -\bar{\epsilon}^{[I} \chi^{JKL]} - \frac{1}{8} \epsilon^{IJKLMNO} \bar{\epsilon}^M \chi^{NO}, \\
\delta D^{IJKL} &= -\bar{\epsilon}^{[I} \phi \chi^{JKL]} + \frac{1}{8} \epsilon^{IJKLMNO} \bar{\epsilon}^M \phi \chi^{NO},
\end{aligned} \tag{5.4}$$

where  $\phi_\mu = \phi_\mu^8$ . The (anti-)self duality conditions for  $E^{IJKL}$  and  $D^{IJKL}$  no longer hold but are instead replaced by

$$\begin{aligned}
E^{IJKL} &= \frac{1}{3!} \epsilon^{IJKLMNO} E^{MNO}, \\
D^{IJKL} &= -\frac{1}{3!} \epsilon^{IJKLMNO} D^{MNO}.
\end{aligned} \tag{5.5}$$

(One could as well remove  $E^{IJK}$  and  $D^{IJK}$  using these rules.) The combination  $E^{IJKL} D^{IJKL}$  is thus no longer zero and can be added to the action. One obtains that the following action is invariant

$$\begin{aligned}
S_3^{\mathcal{N}=7} &= \int d^3x \left\{ h^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{\psi}_\mu^I \mathcal{C}_{(\text{lin})}^{\mu I} - 2V_\mu^{IJ} F_{(\text{lin})}^{\mu IJ} + \frac{8}{3} \bar{\chi}^{IJK} \chi^{IJK} \right. \\
&\quad \left. - \frac{16}{3} E^{IJKL} D^{IJKL} - 8 \bar{\chi}^{IJ} \chi^{IJ} + 4V_\mu^I F_{(\text{lin})}^{\mu I} - \bar{\psi}_\mu \mathcal{C}_{(\text{lin})}^\mu \right\},
\end{aligned} \tag{5.6}$$

where the definitions of  $F_{(\text{lin})}^{\mu I}$  and  $\mathcal{C}_{(\text{lin})}^\mu$  are the usual definitions of the (dual) field strength of  $V_\mu^I$  and Cottino tensor of  $\psi_\mu$  respectively.

Adding the  $\mathcal{N} = 7$  supersymmetric LCS action to the  $\mathcal{N} = 8$  supersymmetric Einstein action we obtain the action of  $\mathcal{N} = 7$  topologically massive supergravity:

$$\begin{aligned}
S_{\text{TMG}}^{\mathcal{N}=7} = & \int d^3x \left\{ \frac{1}{2} h^{\mu\nu} G_{\mu\nu}^{(\text{lin})} + \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^i \partial_\nu \psi_\rho^i - V_\mu^{ij} V_{\mu ij} - \frac{2}{3} E^{ijkl} E_{ijkl} \right. \\
& - \frac{4}{3} \bar{\psi}^{ijk} \not{\partial} \psi^{ijk} + \frac{8}{3} \bar{\psi}^{ijk} \chi_{ijk} - \frac{4}{3} D^{ijkl} \Phi_{ijkl} + \frac{2}{3} \Phi^{ijkl} \square \Phi_{ijkl} \left. \right\} \\
& + \frac{1}{\mu} \left\{ h^{\mu\nu} C_{\mu\nu}^{(\text{lin})} + \bar{\psi}_\mu^I C_{(\text{lin})}^{\mu I} - 2V_\mu^{IJ} F_{(\text{lin})}^{\mu IJ} + \frac{8}{3} \bar{\chi}^{IJK} \chi^{IJK} \right. \\
& \left. - \frac{16}{3} E^{IJKL} D^{IJKL} - 8\bar{\chi}^{IJ} \chi^{IJ} + 4V_\mu^I F_{(\text{lin})}^{\mu I} - \bar{\psi}_\mu C_{(\text{lin})}^\mu \right\}. \tag{5.7}
\end{aligned}$$

One may verify that the equations of motion lead precisely to a  $\mathcal{N} = 7$  spin 2 massive supermultiplet, see Table 1. The analysis is similar to the  $\mathcal{N} = 4$  case treated in the section 3.

## 6 Conclusions and Outlook

The correspondence between massless supermultiplets of 4D supersymmetry and massive supermultiplets of 3D supersymmetry, and the existence of  $\mathcal{N}$ -extended 4D supergravity theories for  $\mathcal{N} = 1, 2, 3, 4, 5, 6$  and  $\mathcal{N} = 8$  suggests the existence of analogous parity-preserving 3D massive supergravity theories, with  $\mathcal{N}$  now counting the number of 3D two-component Majorana spinor supercharges. In particular, the analogy suggests the existence of an  $\mathcal{N} = 8$  maximally supersymmetric extension of “new massive gravity” [9]. We also expect additional parity-violating supergravity theories, such as “topologically massive supergravity” [7, 23] but representation theory only allows  $\mathcal{N} \leq 7$  in this case.

The general massive  $\mathcal{N} = 1$  3D supergravity was constructed in [10, 11], as was one version of the linearized  $\mathcal{N} = 2$  3D supergravity. A simplifying feature of these unitary higher-derivative models is that the higher-derivative terms have a quadratic approximation (in an expansion about the Minkowski vacuum) that is invariant under linearized superconformal gauge invariances. As a consequence, the superconformal compensating multiplets needed for the construction of generic higher-derivative invariants are needed, *at the linearized level*, only for the construction of the supersymmetric extension of the Einstein-Hilbert term. This means that we do not need a full superconformal tensor calculus to construct the linearized theories, which is fortunate because this has not been worked out for  $\mathcal{N} \geq 3$  and must involve an infinite number of auxiliary fields for  $\mathcal{N} \geq 5$ .

We have discussed the general picture of compensating fields for all  $\mathcal{N}$ . In particular  $\mathcal{N} = 3$  is similar to the  $\mathcal{N} = 2$  case discussed in [10], in that we only need one compensating multiplet which is a Stueckelberg multiplet in the sense that its coupling to the Weyl multiplet is bilinear. For  $\mathcal{N} \geq 4$ , we need multiplet copies of the compensating multiplet, and the Weyl multiplet couples to bilinears of them. The  $R$ -symmetry

group gets spontaneously broken when we fix the scale transformations and the corresponding Goldstone bosons are the Stueckelberg fields. We have presented the details of how things work out for  $\mathcal{N} = 4$ , thereby constructing the general linearized massive 3D  $\mathcal{N} = 4$  supergravity, as well as a number of (non-unitary) linearized models with yet higher derivative terms.

Maximally supersymmetric supergravity models are of particular interest, and we have constructed the linearized maximally-supersymmetric  $\mathcal{N} = 8$  “new massive supergravity”. The inclusion of the parity-violating Lorentz-Chern-Simons term necessarily breaks  $\mathcal{N} = 8$  supersymmetry, so that  $\mathcal{N} = 7$  is maximal for models such as topologically massive gravity and we have constructed a linearized  $\mathcal{N} = 7$  topologically massive supergravity action. However, it is not clear that  $\mathcal{N} = 7$  is realizable beyond the linear level, because there are eight Rarita-Schwinger fields in the  $\mathcal{N} = 7$  multiplet; we think it likely that  $\mathcal{N} = 6$  is maximal for TMG.

The extension of our results to the full non-linear level is a challenging task that we leave to the future. Given the striking properties of  $\mathcal{N} = 8$  supergravity in four dimensions, one may hope that the 3D “new massive”  $\mathcal{N} = 8$  supergravity will have similar nice properties. For example, a cosmological extension of the  $\mathcal{N} = 8$  massive gravity might allow an AdS vacuum preserving all 16 supersymmetries, in which case it might have a holographic dual 2D conformal field theory with maximal  $(4, 4)$  supersymmetry.

## Acknowledgements

We thank Paul Howe for helpful correspondence. The work of OH is supported by the DFG – The German Science Foundation and in part by funds provided by the U.S. Department of Energy (DoE) under the cooperative research agreement DE-FG02-05ER41360. PKT thanks the EPSRC for financial support.

## References

- [1] L. J. Dixon, “Ultraviolet Behavior of  $N=8$  Supergravity,” arXiv:1005.2703 [hep-th].
- [2] O. Hohm and J. Louis, “Spontaneous  $N = 2 \rightarrow N = 1$  supergravity breaking in three dimensions,” *Class. Quant. Grav.* **21** (2004) 4607 [arXiv:hep-th/0403128].
- [3] E. A. Bergshoeff and O. Hohm, “A Topologically Massive Gauge Theory with 32 Supercharges,” *Phys. Rev. D* **78** (2008) 125017 [arXiv:0810.0377 [hep-th]].
- [4] W. Siegel, “Unextended Superfields In Extended Supersymmetry”, *Nucl. Phys. B* **156** (1979) 135.
- [5] J. F. Schonfeld, “A Mass Term For Three-Dimensional Gauge Fields”, *Nucl. Phys. B* **185** (1981) 157.

- [6] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” *Annals Phys.* **140** (1982) 372.
- [7] S. Deser, R. Jackiw and S. Templeton, “Three-Dimensional Massive Gauge Theories”, *Phys. Rev. Lett.* **48** (1982) 975.
- [8] H. C. Kao and K. M. Lee, “Selfdual Chern-Simons systems with an N=3 extended supersymmetry,” *Phys. Rev. D* **46** (1992) 4691 [arXiv:hep-th/9205115].
- [9] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimensions,” *Phys. Rev. Lett.* **102** (2009) 201301 [arXiv:0901.1766 [hep-th]]; “More on Massive 3D Gravity,” *Phys. Rev. D* **79** (2009) 124042 [arXiv:0905.1259 [hep-th]]; “On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories,” *Annals Phys.* **325** (2010) 1118 [arXiv:0911.3061 [hep-th]]. “On massive gravitons in 2+1 dimensions,” arXiv:0912.2944 [hep-th].
- [10] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, “Massive 3D Supergravity,” arXiv:0907.4658 [hep-th].
- [11] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, “More on Massive 3D Supergravity,” arXiv:1005.3952 [hep-th].
- [12] H. Lu, C. N. Pope and E. Sezgin, “Massive Three-Dimensional Supergravity From  $R + R^2$  Action in Six Dimensions,” arXiv:1007.0173 [hep-th].
- [13] P. S. Howe, J. M. Izquierdo, G. Papadopoulos and P. K. Townsend, “New supergravities with central charges and Killing spinors in (2+1)-dimensions,” *Nucl. Phys. B* **467** (1996) 183 [arXiv:hep-th/9505032].
- [14] P. S. Howe, “On harmonic superspace,” arXiv:hep-th/9812133.
- [15] P. van Nieuwenhuizen, “D = 3 Conformal Supergravity And Chern-Simons Terms,” *Phys. Rev. D* **32** (1985) 872.
- [16] U. Gran and B. E. W. Nilsson, “Three-dimensional N=8 superconformal gravity and its coupling to BLG M2-branes,” *JHEP* **0903** (2009) 074 [arXiv:0809.4478 [hep-th]].
- [17] P. S. Howe and K. S. Stelle, “Supersymmetry counterterms revisited,” *Phys. Lett. B* **554** (2003) 190 [arXiv:hep-th/0211279].
- [18] W. Siegel, “A Polynomial Action For A Massive, Selfinteracting Chiral Superfield Coupled To Supergravity,”
- [19] M. Kaku and P. K. Townsend, “Poincare Supergravity As Broken Superconformal Gravity,” *Phys. Lett. B* **76** (1978) 54.

- [20] T. Uematsu, “Structure Of N=1 Conformal And Poincare Supergravity In (1+1)-Dimensions And (2+1)-Dimensions,” *Z. Phys. C* **29** (1985) 143; “Constraints And Actions In Two-Dimensional And Three-Dimensional N=1 Conformal Supergravity,” *Z. Phys. C* **32** (1986) 33.
- [21] P. S. Howe, K. S. Stelle and P. K. Townsend, “The Relaxed Hypermultiplet: An Unconstrained N=2 Superfield Theory,” *Nucl. Phys. B* **214** (1983) 519.
- [22] M. B. Green and J. H. Schwarz, “Superstring Interactions,” *Nucl. Phys. B* **218** (1983) 43.
- [23] S. Deser and J. H. Kay, “Topologically Massive Supergravity,” *Phys. Lett. B* **120** (1983) 97.