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Sonia Fourati, Wissem Jedidi

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SOME REMARKS ON THE CLASS OF BERNSTEIN FUNCTIONS AND SOME
SUB-CLASSES

SONIA FOURATI, WISSEM JEDIDI

1. INTRODUCTION

In this paper we summarize some properties that we think complementary to the book recently
published of Schilling-Song-Vondracek [4].

We recall the definition of Bernstein functions: a function \( \phi \) defined on \([0, +\infty[\) is called a Bernstein function \((\phi \in \mathcal{B})\) if it has the representation

\[
\phi(\lambda) = q + d\lambda + \int_{(0,\infty)} (1-e^{-\lambda x})\pi(dx), \quad \lambda \geq 0,
\]

where \( q, d \) are non negative. The measure \( \pi \) is called the Lévy measure of \( \phi \). It is a measure supported by \([0, \infty[\) that integrates the function \( \min(x, 1) \).

Bernstein functions are more likely called by probabilists "Laplace exponents of infinite divisible non
negative sub-distributions" (or "Laplace exponents of subordinators") and the previous representation
is their Lévy-Khintchine expression.

2. A UNIFIED VIEW ON SUBCLASSES OF BERNSTEIN FUNCTIONS

In the sequel all the measures will be understood on the space \([0, \infty[\) and their densities, if they
have one, are according to the Lebesgue measure on \([0, \infty[\) which will be denoted by \( dx \).

We recall the definition of the Mellin convolution of two measures \( \nu \) and \( \tau \) on the space \([0, \infty[\).

\[
\nu \ast \tau(A) = \int_{[0,\infty]^2} 1_A(xy)\nu(dx)\tau(dy), \quad A \text{ borelian of } [0, \infty[.
\]

Notice that this integral may be infinite when \( \nu \) and/or \( \tau \) are not finite measures.

Theorem 2.1. Let \( \pi \) be a Lévy measure.

1) The measure \( \pi \) has a non increasing density if and only if \( \pi \) is of the form

\[
\pi = 1_{[0,1]}(x)dx \ast \nu,
\]

where \( \nu \) is some a Lévy measure.

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2) The measure \( x\pi(dx) \) has a non increasing density if and only if \( \pi \) is of the form

\[
\pi = 1_{[0,1]}(x) \frac{dx}{x} \oplus \nu,
\]

where \( \nu \) is a measure which integrates the function \( f_0(x) = x1_{x\in[0,1]} + \log x1_{x>1} \) (in particular \( \nu \) is a Lévy measure).

3) The measure \( \pi \) has a density of the form \( x^{a-1}k(x) \) with \( a \in ]-1,\infty[ \) and \( k \) a completely monotonic function such that \( \lim_{+\infty} k(x) = 0 \) if and only if \( \pi \) has the expression

\[
\pi = x^{a-1}e^{-x}dx \oplus \nu,
\]

where \( \nu \) is a measure which integrates the function \( f_a \) given by

\[
f_a(x) := \begin{cases} 
  x1_{x\in[0,1]} + x^{-a}1_{x>1} & \text{if } a \in ]-1,0[, \\
  x1_{x\in[0,1]} + \log x1_{x>1} & \text{if } a = 0, \\
  x1_{x\in[0,1]} + 1_{x>1} & \text{if } a \in ]0,\infty[.
\end{cases}
\]

Consequently, \( \nu \) is a Lévy measure in all cases. Moreover \( \nu \) is any Lévy measure in case 1) and in case 3) with \( a > 0 \).

Proof. Notice that if \( \mu \) has a density \( h \), then \( \mu \oplus \nu \) has the density, denoted by \( h \oplus \nu \) and valued in \( [0,+\infty] \):

\[
h \oplus \nu(x) = \int_{[0,\infty[} \frac{1}{y}h\left(\frac{x}{y}\right)\nu(dy), \quad x > 0.
\]

1) Using the last expression for \( h(x) = u_0(x) = 1_{[0,1]}(x) \), we have

\[
u \oplus \nu(x) = \int_x^{+\infty} \nu(dy) \frac{1}{y}.
\]

Notice that any non increasing function (valued in \( [0,+\infty] \)) is of the form \( u_0 \oplus \nu \) and conversely. Since

\[
\int_0^{+\infty} (x \wedge 1)u_0 \oplus \nu(x) = \int_0^{1} \frac{1}{2} x \nu(dx) + \frac{1}{2} \int_1^{+\infty} \nu(dy) + \int_1^{+\infty} \frac{x-1}{x} \nu(dx),
\]

we deduce that the measure with density \( \int_x^{+\infty} \nu(dy) \frac{1}{y} \) is a Lévy measure if and only if \( \nu \) integrates the function \( x \wedge 1 \) or, in other words, \( \nu \) is a Lévy measure.

2) Using the expression of \( h \oplus \nu \) with \( h(x) = u_1(x) = \frac{1}{x}1_{[0,1]}(x) \), we have:

\[
u \oplus u_1 = \frac{1}{x} \int_x^{+\infty} \nu(dy) = \frac{\nu([x, +\infty[)}{x}.
\]

Notice that any function \( \pi \), valued in \( [0,+\infty] \), such that \( x\pi(x) \) is non increasing is of the form \( u_1 \oplus \nu \) and conversely. After that, note that

\[
\int_0^{+\infty} (x \wedge 1)u_1 \oplus \nu(x)dx = \int_0^{1} x \nu(dx) + \nu([1, +\infty[) + \int_1^{+\infty} \log x \nu(dx).
\]

Thus, the measure with density \( u_1 \oplus \nu(x) \) is a Lévy measure if and only if \( \nu(dx) \) integrates \( f_0(x) = x1_{x<1} + \log x1_{x>1} \).
3) Let $a \in ]-1, +\infty[$, $k$ a complete monotone function such that $k(+\infty) = \pi(dx) = x^{a-1} k(x) 1_{[0,\infty]}(dx)$.

By Bernstein theorem, $k$ is the Laplace transform of some measure $\mu$ on $[0, \infty[$, $k(x) = \int_{[0,\infty]} e^{-xt} \mu(dt)$.

Since $k(+\infty) = \mu(\{0\})$, the measure $\mu$ does not charge $0$. Then, we can build the measure $\Delta(du) = \mu(du) u^{-a}$ and write

$$k(x) := \int_{[0,\infty]} e^{-xu} u^a \Delta(du), \quad x > 0.$$  \hspace*{1cm} (2.2)

Let the function

$$h_a(u) := \int_{[0,\infty]} (x \wedge u) x^{a-1} e^{-x} dx, \quad u > 0.$$  

By Fubini theorem, we have

$$\int_{[0,\infty]} (x \wedge 1) \pi(dx) = \int_{[0,\infty]} (x \wedge 1) x^{a-1} k(x) dx = \int_{[0,\infty]} h_a(u) \frac{\Delta(du)}{u}.$$ 

We will find the necessary and sufficient conditions on $\Delta$ so that the last integral is finite.

First, notice that $h_a(u) \not\nearrow \Gamma(a + 1)$ when $u \to +\infty$ and then $h_a$ is bounded for any $a > -1$. Second, elementary computations give the following behavior of $h_a$ at the neighborhood of $0$

$$\lim_{0+} \frac{h_a(u)}{u} = \Gamma(a) \quad \text{if} \quad a > 0;$$

$$0 < \liminf_{0+} \frac{h_a(u) - u}{u |\log u|} \leq \limsup_{0+} \frac{h_a(u) - u}{u |\log u|} < \infty \quad \text{if} \quad a = 0;$$

$$\lim_{0+} \frac{h_a(u)}{u^{1-|a|}} = \frac{1}{|a|} + \frac{1}{1-|a|}, \quad \text{if} \quad a \in ]-1, 0[.$$  

and then $\pi$ is a Lévy measure iff $\int_1^{+\infty} \frac{\Delta(du)}{u} du < +\infty$ and

$$\Delta([0,1]) < +\infty, \quad \text{if} \quad a > 0, \quad \int_{[0,1]} |\log u| \Delta(du) < \infty, \quad \text{if} \quad a = 0, \quad \int_{[0,1]} \frac{\Delta(du)}{u^{|a|}} < \infty, \quad \text{if} \quad -1 < a < 0.$$  

Notice that in each case $\Delta([0,1]) < +\infty$ and then $\Delta(du)$ is a Lévy measure. Define the measure $\nu$ as the image of $\Delta(du)$ induced by the function $u \mapsto 1/u$, it is also a Lévy measure. Notice that properties of the measure $\Delta$ are equivalent to the fact that the function $f_a$ in the statement of the theorem is integrable for $\nu$. In order to conclude, write

$$x^{a-1} k(x) = x^{a-1} \int_{[0,\infty]} e^{-xu} u^a \Delta(du) = \int_{[0,\infty]} \left( \frac{x}{y} \right)^{a-1} e^{-x/y} \nu(dy) \frac{y^{a-1} e^{-y}}{y} = [y^{a-1} e^{-y} \otimes \nu] (x).$$

\[\Box\]

**Remark 2.2.** Bernstein functions $\phi$ associated by (1.1) to Lévy measures $\pi$ of theorem 2.1, are particular in the context of subordinators and infinitely divisible random variables:

1) A Bernstein function whose Lévy measure is of type (1) is called Jurek Bernstein functions. The Jurek class is also characterized by $\phi \geq 0$ and $(\lambda \phi)' \in \mathcal{B}$.

2) A Bernstein function whose Lévy measure is of type (2) is called self-decomposable Bernstein functions, we denote $SDB$ their set. It is easy to check (see theorem 2.6 ch. VI [5], for instance), that

$$\phi \in SDB \iff \phi(0) \geq 0 \quad \text{and} \quad \lambda \mapsto \lambda \phi' (\lambda) \in \mathcal{B}.$$
The infinite divisible positive variables $X$ associated to the $SDB$-functions $\phi$ ( $\mathbb{E}(e^{-\lambda X}) = e^{-\phi(\lambda)}$ ) are the self-decomposable r.v. $X$, that is the random variable for which there exists a family of positive r.v. $(Y_e)_{0 < e < 1}$, each $Y_e$ is independent from $X$ and satisfy $X = cX + Y_e$.

3) In the book of Schilling-Song-Vondracek the set $CB$ ([4] p. 49) of Complete Bernstein functions corresponds to the Bernstein functions appearing in part (3) of the above theorem with parameter $a = 1$. The $TB$ ([4] p. 73) of Thorin Bernstein functions are the ones that with parameter $a = 0$. We will talk again in the next section about $CB$ functions. The class $TB$ corresponds to the Laplace exponents of the generalized Gamma distributions, shortly GGC, introduced by Bondesson [1], [2] and the GGC subordinators studied by James, Roynette and Yor [3]. For more developments on $TB$, see [4]. Part 3) of the last theorem suggests a generalization of these two notions for any parameter $a > -1$.

Notice that $CB_a \subset CB_b$ for every $a \leq b$, $TB \subset SDB$ and that $TB \subset CB_a \subset CB$ for every $0 \leq a \leq 1$.

The simplest functions in $CB_a$ are given when taking the complete monotonic functions $k$ of the form $k(x) = e^{-bx}$, which is the Laplace transform of $\delta_b$, the Dirac mass at point $b$. Then the associated Bernstein function is

$$\phi_a(\lambda) = \int_0^{\infty} (1 - e^{-\lambda x}) x^{a-1} e^{-bx} dx = \begin{cases} \Gamma(a) \left( b^{-a} - \frac{1}{(b+\lambda)^{a}} \right) & \text{if } a \neq 0 \\ \log(1 + \frac{\lambda}{b}) & \text{if } a = 0. \end{cases}$$ (2.3)

Notice that for $a \in \] -1, 0[$, these functions are those associated to the so-called tempered stable processes of index $\alpha = -a$ and for $a = 0$, it is the normalized Gamma process. As settled in the next theorem, any $CB_a$ function is a conic combinations of these simple ones. This result is a straightforward consequence of theorem 2.1:

**Theorem 2.3. representation of $CB_a$-functions.** Let $a > -1$ and $\phi_a$ defined in (2.3). A function $\phi$ belongs to $CB_a$ if and only if its a Mellin convolution of $\phi_a$ with some measure. Namely, for $a \neq 0$

$$\phi(\lambda) = q + d\lambda + \int_{[0,\infty]} \left( 1 - \frac{t^a}{(t+\lambda)^a} \right) \nu(dt),$$

where $q, d \geq 0$, $\nu$ is a measure that integrates the function $f_a(t)$ given by (2.1). In this case, the Lévy measure associated to $\phi$ is $\pi(dx) = x^{a-1} \int_0^{+\infty} e^{-xt} t^{a} \nu(dt)$.

When $a = 0$, we have seen that we meet the well known Thorin class, then we do not recall the facts about that.

3. NEW INJECTIVE MAPPINGS FROM $B$ TO $CB$

We recall that a $CB$ function is a Bernstein function whose Lévy measure has a density which is a completely monotonic function. We recall the connection between $CB$ functions; $\phi$ is a $CB$ function if and only if it admits the representation

$$\phi(\lambda) = q + a\lambda + \int_{[0,\infty]} \frac{\lambda}{\lambda + x} \nu(dx)$$ (3.1)

where $q, a \geq 0$ and $\nu$ is a measure that integrates $\inf(1, 1/x)$.

Another characterization of $CB$ functions is given by theorem 6.2 of [4] and is known as the Pick-Nevanlinna characterization of $CB$ functions: Let $\phi$ a non negative continuous function on $[0, +\infty]$ is a $CB$ function if and only if it has an analytic continuation on $\mathbb{C} \setminus -\infty, 0]$ such that $\Im(\phi(z)) \geq 0$ for all $z$ s.t. $\Im(z) > 0$. 


In the next theorem, we will state a representation similar to (3.1) which is valid for any function \( \phi \) in class \( \mathcal{B} \).

First, notice that a function \( \phi \in \mathcal{B} \) has an analytic continuation on the half plane \( \{ z, \Re(z) > 0 \} \) which can be extended by continuity to the closed half plane \( \{ z, \Re(z) \geq 0 \} \). We still denote \( \phi \) this continuous continuation.

**Theorem 3.1.** Let \( \phi \in \mathcal{B} \), then, for all \( \lambda \geq 0 \),

\[
\phi(\lambda) = a\lambda + \frac{2}{\pi} \int_{0}^{\infty} \frac{\lambda}{\lambda^2 + u^2} \Re[\phi(iu)] \, du.
\]

*Proof.* We suppose without loss of generality that \( q = a = 0 \). Recall \((C_t)_{t \geq 0}\) is a standard Cauchy process. Since \( \phi(ix) = \int_{\mathbb{R}} (1 - e^{-ixs}) \pi(ds), x \in \mathbb{R} \), then we can write for all \( \lambda > 0 \):

\[
\phi(\lambda) = \int_{\mathbb{R}} \frac{\lambda}{\pi(u^2 + \lambda^2)} \phi(iu) \, du.
\]

Notice that \( v(u) := \frac{2}{\pi} \Re(\phi(iu)) \) is \([0, +\infty)\)-valued, negative definite function (see definition 4.3 and theorem 4.6 of [4]) and the identity of the preceding lemma proves that the function \( v(u) \) is integrable at infinity. Denote by \( \mathcal{CB}^- \) the class of functions

\[
\mathcal{CB}^- := \{ \lambda \mapsto \psi(t) = q + d\lambda + \int_{0}^{\infty} \frac{\lambda}{\lambda + u^2} v(u) \, du, \, q, \, d \geq 0, \, v : [0, \infty[ \rightarrow [0, \infty[ \text{ negative definite function} \}
\]

It is obvious that \( \mathcal{CB}^- \) is a (strict) subclass of \( \mathcal{CB} \).

**Theorem 3.2.** 1) If \( \phi \) is in \( \mathcal{B} \), then the function \( \lambda \mapsto \sqrt{\lambda} \phi(\sqrt{\lambda}) \) is in \( \mathcal{CB}^- \), more precisely,

\[
\sqrt{\lambda} \phi(\sqrt{\lambda}) = a\lambda + \int_{0}^{\infty} \frac{\lambda}{\lambda + u^2} v(u) \, du
\]

with \( v(u) = \frac{2}{\pi} \Re(\phi(iu)) \).

2) Conversely, any function in \( \mathcal{CB}^- \) is of the form \( \lambda \mapsto \sqrt{\lambda} \phi(\sqrt{\lambda}) \) where \( \phi \) is in \( \mathcal{B} \).

*Proof.* Part 1) is given by theorem 3.1. In order to show part 2), suppose \( v \) is a (non negative) real valued function, and a negative definite function. Then, according to the Lévy-Khintchine theorem it is the Lévy Fourier exponent of an infinitely divisible distribution and it is symmetric because \( v \) is real valued. Necessarily, it is of the form \( \Re(\phi(iu)) \) where \( \phi(iu) \) is a Lévy Fourier exponent of a spectrally
positive indefinitely divisible distribution. Now, the integrability of \( \frac{v(u)}{u^2} \) at infinity implies that the quadratic component is null and that the Lévy measure integrates \( x \wedge 1 \) (this is easily seen by the Lévy-Khintchine formula, and can also be found in Vigon thesis [6]). Thus \( \phi(iu) \) can be chosen to be the Lévy exponent of a subordinator without a drift. Then, part 1) can be applied to finish the proof.

**□**

**Corollary 3.3.** Any Bernstein function leaves globally invariant the cônes \( \{re^{i\pi \alpha}; \rho \geq 0, \alpha \in [-\theta, +\theta]\} \) for every \( \theta \in [0, \frac{\pi}{2}] \).

**Proof.** The image of a point \( \lambda \) on the half line \( \{re^{i\pi \theta}; \rho \geq 0\} \) by the function \( \lambda \mapsto \sqrt{\lambda} \) (for any \( u > 0 \), is in the cône \( \{re^{i\pi \alpha}; \rho \geq 0, \alpha \in [-\theta, +\theta]\} \). This cône is convex and closed, thus, any conic combination of this function, say the integral

\[
\frac{2}{\pi} \int_{0}^{\infty} \frac{\lambda}{\lambda^2 + u^2} \Re(\phi(iu)) \, du,
\]

is in the same cône. **□**

**Remark 3.4.** This property has to be approached to the much stronger property fulfilled by \( \mathcal{CB} \) function: A \( \mathcal{CB} \) function has an analytic continuation on \( \mathbb{C} \setminus (-\infty, 0] \) and this continuation leaves globally invariant the cônes \( \{re^{i\pi \alpha}; \rho \geq 0, \alpha \in [0, \theta]\} \) for any \( \theta \in [0, 1[ \). Moreover this property fully characterizes the \( \mathcal{CB} \) functions (the Pick-Nevenlinna characterization given above is equivalent to this property for \( \theta = 1 \)) which is not the case for the property of \( \mathcal{B} \)-functions given in this corollary.

In the two following results, we give some extension of theorem 3.2 by replacing the function \( \lambda \mapsto \sqrt{\lambda} \) by other functions.

**Corollary 3.5.** Let \( \phi \) a \( \mathcal{B} \)-function.

1) For any function \( \psi \) such that \( \lambda \mapsto \psi(\lambda^2) \) is in \( \mathcal{B} \), the function \( \lambda \mapsto \sqrt{\lambda} \phi(\psi(\lambda)) \) is in \( \mathcal{CB}^- \).

2) For any function \( \psi \) such that \( \psi^2 \) is in \( \mathcal{CB} \), the functions and \( \lambda \mapsto \psi(\lambda)\phi(\psi(\lambda)) \) and \( \sqrt{\lambda} \phi(\psi(\lambda)) \) are in \( \mathcal{CB} \).

**Proof.** We notice easily that we can suppose \( q = a = 0 \) without loss of generality and we do so.

1) Suppose \( \lambda \mapsto \psi(\lambda^2) \) is in \( \mathcal{B} \), then by composition, \( \phi \circ \psi(\lambda^2) \) is a Bernstein function. Theorem 3.1 says that it admits the representation

\[
\phi(\psi(\lambda^2)) = \int_{0}^{\infty} \frac{\lambda}{\lambda^2 + u^2} v(u) \, du
\]

with \( v \) a completely negative function. By change of variables \( \lambda \to \sqrt{\lambda} \), we then obtain :

\[
\phi(\psi(\lambda)) = \int_{0}^{\infty} \frac{\sqrt{\lambda}}{\lambda + u^2} v(u) \, du
\]

and

\[
\sqrt{\lambda} \phi(\psi(\lambda)) = \int_{0}^{+\infty} \frac{\lambda}{\lambda + u^2} v(u) \, du.
\]

Thus the function \( \sqrt{\lambda} \phi(\psi(\lambda)) \) is in \( \mathcal{CB}^- \).
2) Compose the CB function $\psi^2(\lambda)$ with the CB function $\sqrt[2]{\phi}(\sqrt[2]{\lambda})$ and get the new CB function $\psi \circ \phi$ (see corollary 7.6 of [4]). This gives the first part of this assertion. Now, take the representation of $\phi$ of theorem 3.1, and compose with the function $\psi$, to get

$$\phi(\psi(\lambda)) = \int_0^\infty \frac{\psi(\lambda)}{\psi^2(\lambda) + u^2} v(u) du,$$

(3.2) with $v(u) = \frac{2}{\pi}(\Re(\phi(iu))).$

Multiply by $\frac{\lambda}{\psi(\lambda)}$ and obtain

$$\frac{\lambda}{\psi(\lambda)} \phi(\psi(\lambda)) = \int_0^\infty \frac{\lambda}{\psi^2(\lambda) + u^2} v(u) du.$$  

Since $\psi^2$ is in CB, then, so are $\lambda \mapsto \psi^2(\lambda) + u^2$ and $\frac{\lambda}{\psi^2(\lambda) + u^2}$ is CB for every $u$ (see proposition 7.1 of [4]). Thus, the previous integral is a a CB-function as a conic combination of CB-functions.

Notice that if $f$ is a B-function, then $f(\sqrt{\lambda})$ satisfies property 1). If further $f$ is a CB-function then $f(\sqrt{\lambda})$ and $\sqrt{F(\lambda)}$ both satisfy property 2).

We summarize now the properties that can be settled when composing a Bernstein function $\phi$ by a power function $\psi(\lambda) = \lambda^\alpha$, $\alpha \in [0,1]$.

**Proposition 3.6.** Let $\phi$ a function in $\mathcal{B}$,

1) for all $\alpha \in [0,1]$ the function $\lambda \mapsto \lambda^{1-\alpha} \phi(\lambda^\alpha)$ is in $\mathcal{B}$.

2) If $\alpha \in [0, \frac{1}{2}]$ and $\gamma \in [\alpha, 1 - \alpha]$ then $\lambda \mapsto \lambda^\gamma \phi(\lambda^\alpha)$ is in $\mathcal{B}$. Moreover, if $\gamma = \frac{1}{2}$ it is in $\mathcal{B}^-$.

**Proof.**

1) Let $\alpha \in [0,1]$. Denote $f_\alpha$ the density of a normalized $\alpha$-stable distribution:

$$\int_0^{+\infty} e^{-\lambda x} f_\alpha(x) \, dx = e^{-\lambda^\alpha}.$$  

Then, for any $t > 0$,

$$e^{-t \lambda^\alpha} = t^{-1/\alpha} \int_0^{\infty} e^{-\lambda x} f_\alpha(x t^{-1/\alpha}) \, dx. \quad (3.2)$$

Let $\phi$ in $\mathcal{B}$ with Lévy-Khintchine representation:

$$\phi(\lambda) = a + b \lambda + \int_0^{+\infty} (1 - e^{-\lambda x}) \pi(dx) = a + b \lambda + \lambda \int_0^{+\infty} e^{-\lambda t} \pi(t) \, dt,$$

where $\pi$ is the Lévy measure and $\pi$ its tail function $\pi(t) = \pi([t, +\infty[)$. We have

$$\lambda^{1-\alpha} \phi(\lambda^\alpha) = a + b \lambda + \lambda \int_0^{+\infty} e^{-\lambda^\alpha t} \pi(t) \, dt.$$  

It is enough to prove that the integral $\int_0^{\infty} e^{-\lambda^\alpha t} \pi(t) \, dt$ is the Laplace transform of a non increasing function. For this purpose, we use (3.2) and write

$$\int_0^{\infty} e^{-\lambda^\alpha t} \pi(t) \, dt = \int_0^{\infty} t^{-1/\alpha} \left( \int_0^{\infty} e^{-\lambda z} f_\alpha(z t^{-1/\alpha}) \, dz \right) \pi(t) \, dt.$$
By Fubini’s theorem, we obtain that the last integral is the Laplace transform at point \( \lambda \) of the function
\[
Q(z) := \int_0^\infty f_\alpha(z t^{-\frac{1}{\alpha}}) \pi(t) t^{-\frac{1}{\alpha}} dt = \int_0^{+\infty} \pi(z^\alpha x^{-\alpha}) f_\alpha(x) \frac{dx}{x}.
\]
The second identity comes from the change of variables \( t = \frac{z^\alpha}{x} \), \( x = z t^{-\frac{1}{\alpha}} \). Since the functions \( z \mapsto \pi(z^\alpha x^{-\alpha}) \) are non increasing for all \( x > 0 \), we deduce that \( Q \) is non-increasing.

Now, for \( \alpha \leq \frac{1}{2} \), the function \( \psi(\lambda) = \lambda^\alpha \) satisfies the properties of corollary 3.5. Part 2) yields that the function \( \lambda \mapsto \lambda^{1-\alpha} \phi(\lambda^\alpha) \) is in \( CB \), and part 1), that \( \sqrt{\lambda} \phi(\lambda^\alpha) \) is in \( CB^- \).

2) If \( \alpha \leq \frac{1}{2} \), then the function \( \psi(\lambda) = \lambda^\alpha \) satisfy the part 1) of corollary 3.5 and we get that \( \sqrt{\lambda} \phi(\lambda^\alpha) \) is in \( CB^- \).

Let now \( \gamma \) and \( \alpha \) s.t. \( 0 < 2\alpha \leq \gamma + \alpha \leq 1 \) and take the representation 3.1 of \( \phi \) in order to obtain,
\[
\lambda^\gamma \phi(\lambda^\alpha) = a \lambda^\gamma + \int_0^\infty \frac{\lambda^\gamma + \alpha}{\lambda^\gamma + \alpha + u^2} v(u) du.
\]
Since \( 0 < 2\alpha \leq \gamma + \alpha \leq 1 \), one obtains that the function \( \lambda \mapsto \frac{\lambda^\gamma + \alpha}{\lambda^\gamma + \alpha + u^2} \) leaves the half plane \( \{ \Re(\lambda) > 0 \} \) globally invariant. Thus, it is a \( CB \) function for every \( u > 0 \). By an argument of conic combination, this property remains true for the previous integral and we can conclude that the function \( \lambda \mapsto \lambda^\gamma \phi(\lambda^\alpha) \) is \( CB \).

\[
\square
\]

4. Three classes of related Bernstein functions

We will discuss about the classes of functions of the form \( \lambda \mapsto \phi(\lambda^\alpha) \), and of the form \( \lambda \mapsto \phi^\alpha(\lambda) \) and of the form \( \lambda \mapsto \int_0^\infty (\frac{\lambda}{\lambda + x})^\alpha \) for any real parameter \( \alpha \). This part will be on line later.

References


Sonia Fourati, LMI, INSA Rouen, LPMA Paris 6.

E-mail address: sonia.fourati@upmc.fr

Wissem Jedidi, Department of Mathematics, Faculty of Sciences of Tunis, Campus Universitaire, 1060 Tunis, Tunisia

E-mail address: wissem.jedidi@fst.rnu.tn