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SOME REMARKS ON THE CLASS OF BERNSTEIN FUNCTIONS AND SOME SUB-CLASSES

SONIA FOURATI, WISSEM JEDIDI

1. INTRODUCTION

In this paper we summarize some properties that we think complementary to the book recently published of Schilling-Song-Vondracek [4].

We recall the definition of **Bernstein functions** : a function ϕ defined on $]0, +\infty[$ is called a Bernstein function ($\phi \in \mathcal{B}$) if it has the representation

$$\phi(\lambda) = q + d\lambda + \int_{(0, \infty)} (1 - e^{-\lambda x}) \pi(dx), \quad \lambda \geq 0, \quad (1.1)$$

q, d are non negative. The measure π is called the Lévy measure of ϕ . It is a measure supported by $]0, \infty[$ that integrates the function $\min(x, 1)$.

Bernstein functions are more likely called by probabilists "Laplace exponents of infinite divisible non negative sub-distributions" (or "Laplace exponents of subordinators") and the previous representation is their Lévy-Khintchine expression.

2. A UNIFIED VIEW ON SUBCLASSES OF BERNSTEIN FUNCTIONS

In the sequel all the measures will be understood on the space $]0, \infty[$ and their densities, if they have one, are according to the Lebesgue measure on $]0, \infty[$ which will be denoted by dx .

We recall the definition of the Mellin convolution of two measures ν and τ on the space $]0, \infty[$.

$$\nu \otimes \tau(A) = \int_{]0, \infty[^2} 1_A(xy) \nu(dx) \tau(dy), \quad A \text{ borelian of }]0, \infty[.$$

Notice that this integral may be infinite when ν and/or τ are not finite measures.

Theorem 2.1. *Let π be a Lévy measure.*

1) *The measure π has a non increasing density if and only if π is of the form*

$$\pi = 1_{]0, 1]}(x) dx \otimes \nu,$$

where ν is some a Lévy measure.

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2) The measure $x\pi(dx)$ has a non increasing density if and only if π is of the form

$$\pi = 1_{]0,1]}(x) \frac{dx}{x} \otimes \nu,$$

where ν is a measure which integrates the function $f_0(x) = x1_{x \in]0,1]} + \log x 1_{x > 1}$ (in particular ν is a Lévy measure).

3) The measure π has a density of the form $x^{a-1}k(x)$ with $a \in]-1, \infty[$ and k a completely monotonic function such that $\lim_{+\infty} k(x) = 0$ if and only if π has the expression

$$\pi = x^{a-1} e^{-x} dx \otimes \nu,$$

where ν is a measure which integrates the function f_a given by

$$f_a(x) := \begin{cases} x1_{x \in]0,1]} + x^{-a}1_{x > 1} & \text{if } a \in]-1, 0[, \\ x1_{x \in]0,1]} + \log x 1_{x > 1} & \text{if } a = 0, \\ x1_{x \in]0,1]} + 1_{x > 1} & \text{if } a \in]0, \infty[. \end{cases} \quad (2.1)$$

Consequently, ν is a Lévy measure in all cases. Moreover ν is any Lévy measure in case 1) and in case 3) with $a > 0$.

Proof. Notice that if μ has a density h , then $\mu \otimes \nu$ has the density, denoted by $h \otimes \nu$ and valued in $[0, +\infty[$:

$$h \otimes \nu(x) = \int_{]0, \infty[} \frac{1}{y} h\left(\frac{x}{y}\right) \nu(dy), \quad x > 0.$$

1) Using the last expression for $h(x) = u_0(x) = 1_{]0,1]}(x)$, we have

$$u_0 \otimes \nu(x) = \int_x^{+\infty} \frac{\nu(dy)}{y}.$$

Notice that any non increasing function (valued in $[0, +\infty[$) is of the form $u_0 \otimes \nu$ and conversely. Since

$$\int_0^\infty (x \wedge 1) u_0 \otimes \nu(x) = \int_0^1 \frac{x^2}{2} \frac{1}{x} \nu(dx) + \frac{1}{2} \int_1^\infty \frac{1}{y} \nu(dy) + \int_1^\infty \frac{x-1}{x} \nu(dx),$$

we deduce that the measure with density $\int_x^{+\infty} \frac{\nu(dy)}{y}$ is a Lévy measure if and only if ν integrates the function $x \wedge 1$ or, in other words, ν is a Lévy measure.

2) Using the expression of $h \otimes \nu$ with $h(x) = u_1(x) = \frac{1}{x} 1_{]0,1]}(x)$, we have :

$$u_1 \otimes \nu(x) = \frac{1}{x} \int_x^\infty \nu(dy) = \frac{\nu(]x, +\infty[)}{x}.$$

Notice that any function π , valued in $[0, +\infty[$, such that $x\pi(x)$ is non increasing is of the form $u_1 \otimes \nu$ and conversely. After that, note that

$$\int_0^\infty (x \wedge 1) u_1 \otimes \nu(x) dx = \int_0^1 x \nu(dx) + \nu(]1, +\infty[) + \int_1^{+\infty} \log x \nu(dx).$$

Thus, the measure with density $u_1 \otimes \nu(x)$ is a Lévy measure if and only if $\nu(dx)$ integrates $f_0(x) = x1_{x < 1} + \log x 1_{x > 1}$.

3) Let $a \in]-1, \infty[$, k a complete monotone function such that $k(+\infty) = 0$ and $\pi(dx) = x^{a-1} k(x) 1_{]0, \infty[} dx$. By Bernstein theorem, k is the Laplace transform of some measure μ on $[0, \infty[$, $k(x) = \int_{]0, \infty[} e^{-xt} \mu(dt)$. Since $k(+\infty) = \mu(\{0\})$, the measure μ does not charge 0. Then, we can build the measure $\Delta(du) = \mu(du)u^{-a}$ and write

$$k(x) := \int_{]0, \infty[} e^{-xu} u^a \Delta(du), \quad x > 0. \quad (2.2)$$

Let the function

$$h_a(u) := \int_0^\infty (x \wedge u) x^{a-1} e^{-x} dx, \quad u > 0.$$

By Fubini theorem, we have

$$\int_{]0, \infty[} (x \wedge 1) \pi(dx) = \int_{]0, \infty[} (x \wedge 1) x^{a-1} k(x) dx = \int_{]0, \infty[} h_a(u) \frac{\Delta(du)}{u}.$$

We will find the necessary and sufficient conditions on Δ so that the last integral is finite.

First, notice that $h_a(u) \nearrow \Gamma(a+1)$ when $u \rightarrow +\infty$ and then h_a is bounded for any $a > -1$. Second, elementary computations give the following behavior of h_a at the neighborhood of 0 ,

$$\begin{aligned} \lim_{0+} \frac{h_a(u)}{u} &= \Gamma(a) && \text{if } a > 0; \\ 0 < \liminf_{0+} \frac{h_a(u) - u}{u |\log u|} &\leq \limsup_{0+} \frac{h_a(u) - u}{u |\log u|} < \infty, && \text{if } a = 0; \\ \lim_{0+} \frac{h_a(u)}{u^{1-|a|}} &= \frac{1}{|a|} + \frac{1}{1-|a|}, && \text{if } a \in]-1, 0[. \end{aligned}$$

and then π is a Lévy measure iff $\int_1^{+\infty} \frac{\Delta(du)}{u} du < +\infty$ and

$$\Delta([0, 1]) < +\infty, \quad \text{if } a > 0, \quad \int_{(0,1]} |\log u| \Delta(du) < \infty, \quad \text{if } a = 0, \quad \int_{(0,1]} \frac{\Delta(du)}{u^{|a|}} < \infty, \quad \text{if } -1 < a < 0.$$

Notice that in each case $\Delta([0, 1]) < +\infty$ and then $\frac{\Delta(du)}{u}$ is a Lévy measure. Define the measure ν as the image of $\Delta(du)$ induced by the function $u \mapsto 1/u$, it is also a Lévy measure. Notice that properties of the measure Δ are equivalent to the fact that the function f_a in the statement of the theorem is integrable for ν . In order to conclude, write

$$x^{a-1} k(x) = x^{a-1} \int_{]0, \infty[} e^{-xu} u^a \Delta(du) = \int_{]0, \infty[} \left(\frac{x}{y}\right)^{a-1} e^{-\frac{x}{y}} \frac{\nu(dy)}{y} = [y^{a-1} e^{-y} \otimes \nu](x).$$

□

Remark 2.2. Bernstein functions ϕ associated by (1.1) to Lévy measures π of theorem 2.1, are particular in the context of subordinators and infinitely divisible random variables :

1) A Bernstein function whose Lévy measure is of type (1) is called Jurek Bernstein functions. The Jurek class is also characterized by $\phi \geq 0$ and $(\lambda\phi)' \in \mathcal{B}$.

2) A Bernstein function whose Lévy measure is of type (2) is called self-decomposable Bernstein functions, we denote \mathcal{SDB} their set. It is easy to check (see theorem 2.6 ch. VI [5], for instance), that

$$\phi \in \mathcal{SDB} \iff \phi(0) \geq 0 \quad \text{and} \quad \lambda \mapsto \lambda\phi'(\lambda) \in \mathcal{B}.$$

The infinite divisible positive variables X associated to the SDB -functions ϕ ($\mathbb{E}(e^{-\lambda X}) = e^{-\phi(\lambda)}$) are the self-decomposable r.v. X , that is the random variable for which there exists a family of positive r.v. $(Y_c)_{0 < c < 1}$, each Y_c is independent from X and satisfy $X \stackrel{d}{=} cX + Y_c$.

3) In the book of Schilling-Song-Vondracek the set \mathcal{CB} ([4] p. 49) of Complete Bernstein functions corresponds to the Bernstein functions appearing in part (3) of the above theorem with parameter $a = 1$. The \mathcal{TB} ([4] p. 73) of Thorin Bernstein functions are the ones that with parameter $a = 0$. We will talk again in the next section about CB functions. The class \mathcal{TB} corresponds to the Laplace exponents of the generalized Gamma distributions, shortly GGC, introduced by Bondesson [1], [2] and the GGC subordinators studied by James, Roynette and Yor [3]. For more developments on \mathcal{TB} , see [4]. Part 3) of the last theorem suggests a generalization of these two notions for any parameter $a > -1$.

Notice that $\mathcal{CB}_a \subset \mathcal{CB}_b$ for every $a \leq b$, $\mathcal{TB} \subset SDB$ and that $\mathcal{TB} \subset \mathcal{CB}_a \subset \mathcal{CB}$ for every $0 \leq a \leq 1$.

The simplest functions in \mathcal{CB}_a are given when taking the complete monotonic functions k of the form $k(x) = e^{-bx}$, which is the Laplace transform of δ_b , the Dirac mass at point b . Then the associated Bernstein function is

$$\phi_a(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) x^{a-1} e^{-bx} dx = \begin{cases} \Gamma(a) \left(\frac{1}{b^a} - \frac{1}{(b+\lambda)^a} \right) & \text{if } a \neq 0 \\ \log(1 + \frac{\lambda}{b}) & \text{if } a = 0. \end{cases} \quad (2.3)$$

Notice that for $a \in]-1, 0[$, these functions are those associated to the so-called tempered stable processes of index $\alpha = -a$ and for $a = 0$, it is the normalized Gamma process. As settled in the next theorem, any \mathcal{CB}_a function is a conic combinations of these simple ones. This result is a straightforward consequence of theorem 2.1:

Theorem 2.3. representation of \mathcal{CB}_a -functions. *Let $a > -1$ and ϕ_a defined in (2.3). A function ϕ belongs to \mathcal{CB}_a if and only if its a Mellin convolution of ϕ_a with some measure. Namely, for $a \neq 0$*

$$\phi(\lambda) = q + d\lambda + \int_{]0, \infty[} \left(1 - \frac{t^a}{(t + \lambda)^a} \right) \nu(dt),$$

where $q, d \geq 0$, ν is a measure that integrates the function $f_a(t)$ given by (2.1). In this case, the Lévy measure associated to ϕ is $\pi(dx) = x^{a-1} \int_0^{+\infty} e^{-xt} t^a \nu(dt)$.

When $a = 0$, we have seen that we meet the well known Thorin class, then we do not recall the facts about that.

3. NEW INJECTIVE MAPPINGS FROM \mathcal{B} TO \mathcal{CB}

We recall that a \mathcal{CB} function is a Bernstein function whose Lévy measure has a density which is a completely monotonic function. We recall the connection between \mathcal{CB} functions; ϕ is a \mathcal{CB} function if and only if it admits the representation

$$\phi(\lambda) = q + a\lambda + \int_{]0, +\infty[} \frac{\lambda}{\lambda + x} \nu(dx) \quad (3.1)$$

where $q, a \geq 0$ and ν is a measure that integrates $\inf(1, 1/x)$.

Another characterization of \mathcal{CB} functions is given by theorem 6.2 of [4] and is known as the **Pick-Nevanlinna characterization of \mathcal{CB} functions** : *Let ϕ a non negative continuous function on $[0, +\infty[$ is a \mathcal{CB} function if and only if it has an analytic continuation on $\mathbf{C} \setminus]-\infty, 0]$ such that $\Im(\phi(z)) \geq 0$ for all z s.t. $\Im(z) > 0$.*

In the next theorem, we will state a representation similar to (3.1) which is valid for any function ϕ in class \mathcal{B} .

First, notice that a function ϕ in \mathcal{B} has an analytic continuation on the half plane $\{z, \Re(z) > 0\}$ which can be extended by continuity to the closed half plane $\{z, \Re(z) \geq 0\}$. We still denote ϕ this continuous continuation.

Theorem 3.1. *Let $\phi \in \mathcal{B}$, then, for all $\lambda \geq 0$,*

$$\phi(\lambda) = a\lambda + \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + u^2} \Re[\phi(iu)] du.$$

Proof. We suppose without loss of generality that $q = a = 0$. Recall $(C_t)_{t \geq 0}$ is a standard Cauchy process. Since $\phi(ix) = \int_{(0, \infty)} (1 - e^{-ixs}) \pi(ds)$, $x \in \mathbb{R}$, then we can write for all $\lambda > 0$:

$$\begin{aligned} \phi(\lambda) &= \int_{(0, \infty)} (1 - e^{-\lambda s}) \pi(ds) \\ &= \int_{(0, \infty)} \mathbb{E}[1 - e^{-isC_\lambda}] \pi(ds) \\ &= \int_{(0, \infty)} \left(\int_{\mathbb{R}} (1 - e^{-ius}) \frac{\lambda}{\pi(u^2 + \lambda^2)} du \right) \pi(ds) \\ &= \int_{\mathbb{R}} \frac{\lambda}{\pi(u^2 + \lambda^2)} \phi(iu) du \\ &= \int_0^\infty \frac{\lambda}{\pi(u^2 + \lambda^2)} (\phi(iu) + \phi(-iu)) du \\ &= \int_0^\infty \frac{\lambda}{\pi(u^2 + \lambda^2)} 2\Re(\phi(iu)) du \end{aligned}$$

□

Notice that $v(u) := \frac{2}{\pi} \Re(\phi(iu))$ is $[0, +\infty[$ -valued, negative definite function (see definition 4.3 and theorem 4.6 of [4]) and the identity of the preceding lemma proves that the function $\frac{v(u)}{u^2}$ is integrable at infinity. Denote by \mathcal{CB}^- the class of functions

$$\mathcal{CB}^- := \left\{ \lambda \mapsto \psi(\lambda) = q + d\lambda + \int_0^{+\infty} \frac{\lambda}{\lambda + u^2} v(u) du, \quad q, d \geq 0, v : [0, \infty[\rightarrow [0, \infty[\text{ negative definite function} \right\}$$

It is obvious that \mathcal{CB}^- is a (strict) subclass of \mathcal{CB} .

Theorem 3.2. *1) If ϕ is in \mathcal{B} , then the function $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ is in \mathcal{CB}^- , more precisely,*

$$\sqrt{\lambda}\phi(\sqrt{\lambda}) = a\lambda + \int_0^\infty \frac{\lambda}{\lambda + u^2} v(u) du$$

with $v(u) = \frac{2}{\pi} \Re(\phi(iu))$.

2) Conversely, any function in \mathcal{CB}^- is of the form $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ where ϕ is in \mathcal{B} .

Proof. Part 1) is given by theorem 3.1. In order to show part 2), suppose v is a (non negative) real valued function, and a negative definite function. Then, according to the Lévy-Khintchine theorem it is the Lévy Fourier exponent of an infinitely divisible distribution and it is symmetric because v is real valued. Necessarily, it is of the form $\Re(\phi(iu))$ where $\phi(iu)$ is a Lévy Fourier exponent of a spectrally

positive indefinitely divisible distribution. Now, the integrability of $\frac{v(u)}{u^2}$ at infinity implies that the quadratic component is null and that the Lévy measure integrates $x \wedge 1$ (this is easily seen by the Lévy-Khintchine formula, and can also be found in Vigon thesis [6]). Thus $\phi(iu)$ can be chosen to be the Lévy exponent of a subordinator without a drift. Then, part 1) can be applied to finish the proof. \square

Corollary 3.3. *Any Bernstein function leaves globally invariant the cônes $\{\rho e^{i\pi\alpha}; \rho \geq 0, \alpha \in [-\theta, +\theta]\}$ for every $\theta \in [0, \frac{\pi}{2}]$.*

Proof. The image of a point λ on the half line $\{\rho e^{i\pi\theta}; \rho \geq 0\}$ by the function $\lambda \mapsto \frac{\lambda}{\lambda^2 + u^2}$ (for any $u > 0$), is in the cône $\{\rho e^{i\pi\alpha}; \rho \geq 0, \alpha \in [-\theta, +\theta]\}$. This cône is convex and closed, thus, any conic combination of this function, say the integral

$$\frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + u^2} \Re(\phi(iu)) du,$$

is in the same cône. \square

Remark 3.4. *This property has to be approached to the much stronger property fulfilled by CB function : A CB function has an analytic continuation on $\mathbf{C} \setminus]-\infty, 0]$ and this continuation leaves globally invariant the cônes $\{\rho e^{i\pi\alpha}; \rho \geq 0, \alpha \in [0, \theta]\}$ for any $\theta \in [0, 1[$. Moreover this property fully characterizes the CB functions (the Pick-Nevenlinna characterization given above is equivalent to this property for $\theta = 1$) which is not the case for the property of B- functions given in this corollary.*

In the two following results, we give some extension of theorem 3.2 by replacing the function $\lambda \mapsto \sqrt{\lambda}$ by other functions.

Corollary 3.5. *Let ϕ a B-function.*

- 1) *For any function ψ such that $\lambda \mapsto \psi(\lambda^2)$ is in \mathcal{B} , the function $\lambda \mapsto \sqrt{\lambda} \phi(\psi(\lambda))$ is in \mathcal{CB}^- .*
- 2) *For any function ψ such that ψ^2 is in \mathcal{CB} , the functions and $\lambda \mapsto \psi(\lambda) \phi(\psi(\lambda))$ and $\frac{\lambda}{\psi(\lambda)} \phi(\psi(\lambda))$ are in \mathcal{CB} .*

Proof. We notice easily that we can suppose $q = a = 0$ without loss of generality and we do so.

1) Suppose $\lambda \mapsto \psi(\lambda^2)$ is in \mathcal{B} , then by composition, $\phi \circ \psi(\lambda^2)$ is a Bernstein function. Theorem 3.1 says that it admits the representation

$$\phi(\psi(\lambda^2)) = \int_0^\infty \frac{\lambda}{\lambda^2 + u^2} v(u) du$$

with v a completely negative function. By change of variables $\lambda \rightarrow \sqrt{\lambda}$, we then obtain :

$$\phi(\psi(\lambda)) = \int_0^\infty \frac{\sqrt{\lambda}}{\lambda + u^2} v(u) du$$

and

$$\sqrt{\lambda} \phi(\psi(\lambda)) = \int_0^{+\infty} \frac{\lambda}{\lambda + u^2} v(u) du.$$

Thus the function $\sqrt{\lambda} \phi(\psi(\lambda))$ is in \mathcal{CB}^- .

2) Compose the \mathcal{CB} function $\psi^2(\lambda)$ with the \mathcal{CB} function $\sqrt{\lambda}\phi(\sqrt{\lambda})$ and get the new \mathcal{CB} function $\psi \circ \phi \circ \psi$ (see corollary 7.6 of [4]). This gives the first part of this assertion. Now, take the representation of ϕ of theorem 3.1, and compose with the function ψ , to get

$$\phi(\psi(\lambda)) = \int_0^\infty \frac{\psi(\lambda)}{\psi^2(\lambda) + u^2} v(u) du, \quad \left(\text{with } v(u) = \frac{2}{\pi} (\Re(\phi(iu))) \right).$$

Multiply by $\frac{\lambda}{\psi(\lambda)}$ and obtain

$$\frac{\lambda}{\psi(\lambda)} \phi(\psi(\lambda)) = \int_0^\infty \frac{\lambda}{\psi^2(\lambda) + u^2} v(u) du.$$

Since ψ^2 is in \mathcal{CB} , then, so are $\lambda \mapsto \psi^2(\lambda) + u^2$ and $\frac{\lambda}{\psi^2(\lambda) + u^2}$ is \mathcal{CB} for every u (see proposition 7.1 of [4]). Thus, the previous integral is a \mathcal{CB} -function as a conic combination of \mathcal{CB} -functions. \square

Notice that if f is a \mathcal{B} -function, then $f(\sqrt{\lambda})$ satisfies property 1). If further f is a \mathcal{CB} -function then $f(\sqrt{\lambda})$ and $\sqrt{f(\lambda)}$ both satisfy property 2).

We summarize now the properties that can be settled when composing a Bernstein function ϕ by a power function $\psi(\lambda) = \lambda^\alpha$, $\alpha \in]0, 1[$.

Proposition 3.6. *Let ϕ a function in \mathcal{B} ,*

- 1) *for all $\alpha \in]0, 1[$ the function $\lambda \mapsto \lambda^{1-\alpha}\phi(\lambda^\alpha)$ is in \mathcal{B} .*
- 2) *If $\alpha \in]0, \frac{1}{2}[$ and $\gamma \in]\alpha, 1 - \alpha[$ then $\lambda \mapsto \lambda^\gamma\phi(\lambda^\alpha)$ is in \mathcal{CB} . Moreover, if $\gamma = \frac{1}{2}$ it is in \mathcal{CB}^- .*

Proof. 1) Let $\alpha \in [0, 1[$ Denote f_α the density of a normalized α -stable distribution :

$$\int_0^{+\infty} e^{-\lambda x} f_\alpha(x) dx = e^{-\lambda^\alpha}.$$

Then, for any $t > 0$,

$$e^{-t\lambda^\alpha} = t^{-1/\alpha} \int_0^\infty e^{-\lambda x} f_\alpha(xt^{-1/\alpha}) dx. \quad (3.2)$$

Let ϕ in \mathcal{B} with Lévy-Khintchine representation:

$$\phi(\lambda) = a + b\lambda + \int_0^{+\infty} (1 - e^{-\lambda x}) \pi(dx) = a + b\lambda + \lambda \int_0^{+\infty} e^{-\lambda t} \bar{\pi}(t) dt,$$

where π is the Lévy measure and $\bar{\pi}$ its tail function $\bar{\pi}(t) = \pi(]t, +\infty[)$. We have

$$\lambda^{1-\alpha}\phi(\lambda^\alpha) = a + b\lambda + \lambda \int_0^{+\infty} e^{-\lambda^\alpha t} \bar{\pi}(t) dt.$$

It is enough to prove that the integral $\int_0^\infty e^{-\lambda^\alpha t} \bar{\pi}(t) dt$ is the Laplace transform of a non increasing function. For this purpose, we use (3.2) and write

$$\int_0^\infty e^{-\lambda^\alpha t} \bar{\pi}(t) dt = \int_0^\infty t^{-1/\alpha} \left(\int_0^\infty e^{-\lambda z} f_\alpha(zt^{-1/\alpha}) dz \right) \bar{\pi}(t) dt.$$

By Fubini's theorem, we obtain that the last integral is the Laplace transform at point λ of the function

$$z \mapsto Q(z) := \int_0^{\infty} f_{\alpha}(zt^{-\frac{1}{\alpha}})\bar{\pi}(t)t^{-\frac{1}{\alpha}} dt = \int_0^{+\infty} \bar{\pi}(z^{\alpha}x^{-\alpha})f_{\alpha}(x)\frac{dx}{x}.$$

The second identity comes from the change of variables $t = \frac{z^{\alpha}}{x^{\alpha}}$, $x = zt^{-\frac{1}{\alpha}}$. Since the functions $z \mapsto \bar{\pi}(z^{\alpha}x^{-\alpha})$ are non increasing for all $x > 0$, we deduce that Q is non-increasing.

Now, for $\alpha \leq \frac{1}{2}$, the function $\psi(\lambda) = \lambda^{\alpha}$ satisfies the properties of corollary 3.5. Part 2) yields that the function $\lambda \mapsto \lambda^{1-\alpha}\phi(\lambda^{\alpha})$ is in \mathcal{CB} , and part 1), that $\sqrt{\lambda}\phi(\lambda^{\alpha})$ is in \mathcal{CB}^{-} .

2) If $\alpha \leq \frac{1}{2}$, then the function $\psi(\lambda) = \lambda^{\alpha}$ satisfy the part 1) of corollary 3.5 and we get that $\sqrt{\lambda}\phi(\lambda^{\alpha})$ is in \mathcal{CB}^{-} .

Let now γ and α s.t. $0 < 2\alpha \leq \gamma + \alpha \leq 1$ and take the representation 3.1 of ϕ in order to obtain,

$$\lambda^{\gamma}\phi(\lambda^{\alpha}) = a\lambda^{\gamma+\alpha} + \int_0^{\infty} \frac{\lambda^{\gamma+\alpha}}{\lambda^{2\alpha} + u^2} v(u) du \quad v(u) = \frac{2}{\pi} \Re(\phi(iu))$$

Since $0 < 2\alpha \leq \gamma + \alpha \leq 1$, one obtains that the function $\lambda \mapsto \frac{\lambda^{\gamma+\alpha}}{\lambda^{2\alpha} + u^2}$ leaves the half plane $\{\Im(\lambda) > 0\}$ globally invariant. Thus, it is a \mathcal{CB} function for every $u > 0$. By an argument of conic combination, this property remains true for the previous integral and we can conclude that the function $\lambda \mapsto \lambda^{\gamma}\phi(\lambda^{\alpha})$ is \mathcal{CB} .

□

4. THREE CLASSES OF RELATED BERNSTEIN FUNCTIONS

We will discuss about the classes of functions of the form $\lambda \mapsto \phi(\lambda^{\alpha})$, and of the form $\lambda \mapsto \phi^{\alpha}(\lambda)$ and of the form $\lambda \mapsto \int_0^{\infty} (\frac{\lambda}{\lambda+x})^{\alpha}$ for any real parameter α . This part will be on line later.

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