



**HAL**  
open science

## Bounded universal functions for sequences of holomorphic self-maps of the disk

Frédéric Bayart, Pamela Gorkin, Sophie Grivaux, Raymond Mortini

► **To cite this version:**

Frédéric Bayart, Pamela Gorkin, Sophie Grivaux, Raymond Mortini. Bounded universal functions for sequences of holomorphic self-maps of the disk. *Arkiv för Matematik*, 2009, 47 (2), pp.205 - 229. 10.1007/s11512-008-0083-z . hal-00648713

**HAL Id: hal-00648713**

**<https://hal.science/hal-00648713>**

Submitted on 9 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# BOUNDED UNIVERSAL FUNCTIONS FOR SEQUENCES OF HOLOMORPHIC SELF-MAPS OF THE DISK

by

Frédéric Bayart, Pamela Gorkin, Sophie Grivaux & Raymond Mortini

---

**Abstract.** — We give several characterizations of those sequences of holomorphic self-maps  $(\phi_n)$  of the unit disk for which there exists a function  $F$  in the unit ball  $\mathcal{B} = \{f \in H^\infty : \|f\|_\infty \leq 1\}$  of  $H^\infty$  such that the orbit  $\{F \circ \phi_n : n \in \mathbb{N}\}$  is locally uniformly dense in  $\mathcal{B}$ . Such a function  $F$  is said to be a  $\mathcal{B}$ -universal function. One of our conditions is stated in terms of the hyperbolic derivatives of the functions  $\phi_n$ . As a consequence we will see that if  $\phi_n$  is the  $n$ -th iterate of a map  $\phi$  of  $\mathbb{D}$  into  $\mathbb{D}$ , then  $(\phi_n)$  admits a  $\mathcal{B}$ -universal function if and only if  $\phi$  is a parabolic or hyperbolic automorphism of  $\mathbb{D}$ . Also we will show that whenever there exists a  $\mathcal{B}$ -universal function, then this function can be chosen to be a Blaschke product.

## 1. Introduction

Let  $H^\infty$  denote the algebra of bounded analytic functions on the unit disk  $\mathbb{D}$ , and let  $H(\mathbb{D})$  denote the space of functions analytic on  $\mathbb{D}$  with the local uniform topology. In this paper we will consider sequences of holomorphic self-maps of  $\mathbb{D}$ ,  $(\phi_n)$ , and functions that are “badly behaved” with respect to these maps, the so-called *universal functions*. Our setting will be the following:  $X$  will be  $H(\mathbb{D})$  or a subset of  $H^\infty$ . For a holomorphic self-map  $\phi$  of  $\mathbb{D}$  we define the composition operator  $C_\phi$  on  $X$  by  $C_\phi(f) = f \circ \phi$ .

**Definition 1.1.** — A function  $f \in X$  is said to be  $X$ -universal for  $(\phi_n)$  (or, equivalently, we say  $(C_{\phi_n})$  admits an  $X$ -universal function  $f$ ) if the set  $\{f \circ \phi_n : n \in \mathbb{N}\}$  is locally uniformly dense in  $X$ . Also, we will call a function  $f \in H^\infty$  universal with respect to  $(\phi_n)$  if  $\{f \circ \phi_n : n \in \mathbb{N}\}$  is locally uniformly dense in  $\{g \in H^\infty : \|g\|_\infty \leq \|f\|_\infty\}$ .

In other words,  $f \in H^\infty$  is universal if  $\{f \circ \phi_n : n \in \mathbb{N}\}$  is as big as it possibly can be.

Bounded universal functions for invertible composition operators on  $H^\infty(\Omega)$ , where  $\Omega$  is a planar domain, were studied in [16, 19, 21] and, for the case in which  $\Omega \subseteq \mathbb{C}^n$ , in [1, 2, 12, 18]. In this paper, we will restrict consideration to the case in which  $\Omega = \mathbb{D}$  while our space  $X$  will vary: sometimes we will study the case in which  $X$  is the closed

---

**2000 Mathematics Subject Classification.** — Primary 47B33, Secondary 46J15, 30D50.

**Key words and phrases.** — Universal functions, composition operators, bounded analytic functions, hyperbolic derivatives.

Research supported by the RIP-program 2006 at Oberwolfach.

unit ball,  $\mathcal{B}$ , of  $H^\infty$ . Other times  $X$  will denote the set  $\mathcal{S}$  of functions in  $\mathcal{B}$  that do not vanish on  $\mathbb{D}$ . The aim of this paper is to study  $\mathcal{B}$ -universality as well as  $\mathcal{S}$ -universality for noninvertible composition operators on  $H^\infty$ .

If we let  $(p_n)$  denote a sequence of finite Blaschke products which is dense in  $\mathcal{B}$ , then it is very easy to come up with a  $\mathcal{B}$ -universal function for the self-maps  $(p_n)$ : the identity function works, for example. Similarly the atomic inner function  $S$  is  $\mathcal{S}$ -universal for  $(p_n)$  as well. The more interesting case is one in which the  $p_n$ 's do not fill up much of  $\mathcal{B}$ ; for example, a sequence  $(p_n)$  such that  $p_n \rightarrow 1$  on compacta. Thus we were led to the following question: under what conditions on the sequence of self-maps does there exist a  $\mathcal{B}$ -universal function? an  $\mathcal{S}$ -universal function? In this paper we will give a very satisfactory complete answer.

We will always consider a sequence  $(\phi_n)$  of holomorphic self-maps with  $\phi_n(0) \rightarrow 1$ . Our sequence need not be a sequence of iterates and it need not be a sequence of automorphisms. Indeed, the more interesting cases, for the purposes of the study in this paper, are those in which this is not the case. Our results can be summarized as follows. In Section 2, which was motivated by work in [4, 16, 17, 19], we show (Theorem 2.1) that the existence of a  $\mathcal{B}$ -universal function, a  $\mathcal{B}$ -universal Blaschke product, and an  $\mathcal{S}$ -universal singular inner function are equivalent. Such functions exist precisely when

$$\limsup_{n \rightarrow +\infty} \frac{|\phi_n'(0)|}{1 - |\phi_n(0)|^2} = 1.$$

In Section 3, which follows the line of thought in papers such as [5, 1, 3, 21], we show (Theorem 3.1) that these conditions are equivalent to the existence of a closed, infinite dimensional vector space, generated by Blaschke products such that every function in the vector space is universal in the sense of Definition 1.1. We conclude the paper with several related results, including a discussion of the special cases in which the sequence of self-maps is a sequence of iterates of a particular function.

## 2. Universality in the ball of $H^\infty$

**2.1. The main result.** — In [19] Heins showed that there exists a sequence of automorphisms of  $\mathbb{D}$  which admits a  $\mathcal{B}$ -universal function. The main result in [16] shows that this  $\mathcal{B}$ -universal function can be chosen to be a Blaschke product whose zeroes belong to a given sequence of points  $(z_n)$  of  $\mathbb{D}$  going to the boundary, and indicates how the construction of the Blaschke product proceeds. In [17], the paper focuses on the construction of an  $\mathcal{S}$ -universal singular inner function for a sequence of automorphisms. The proof of Theorem 2.1 below is quite different. First, we do not assume the self-maps are automorphisms; thus the proof requires the development of new techniques. Second, we show the existence of “special” universal functions (Blaschke products, singular inner functions), but our proof is not constructive. This simplifies, in many ways, much of the work. Here is the main result of this section:

**Theorem 2.1.** — *Let  $(\phi_n)$  be a sequence of holomorphic self-maps of the unit disk  $\mathbb{D}$  such that  $\phi_n(0) \rightarrow 1$  as  $n \rightarrow +\infty$ . The following conditions are equivalent:*

1. the sequence  $(C_{\phi_n})$  admits a  $\mathcal{B}$ -universal function
2. the sequence  $(C_{\phi_n})$  admits a  $\mathcal{B}$ -universal Blaschke product
3. the sequence  $(C_{\phi_n})$  admits an  $\mathcal{S}$ -universal singular inner function
4. the set  $\mathcal{B}_m = \{u \circ \phi_n ; u \in \mathcal{B}, n \geq m\}$  is locally uniformly dense in  $\mathcal{B}$  for every  $m \geq 1$
5.  $\limsup_{n \rightarrow +\infty} \frac{|\phi'_n(0)|}{1 - |\phi_n(0)|^2} = 1$ .

Before going on with the proof of Theorem 2.1, we point out that Theorem 2.1 also holds under the weaker assumption that  $|\phi_n(0)| \rightarrow 1$  as  $n \rightarrow +\infty$ . The necessary modifications of the proof below will be left to the reader.

**2.2. Proof of (2)  $\Leftrightarrow$  (4).** — The key to the proof of Theorem 2.1 is the equivalence between assertions (2) and (4), which is a “technical” equivalent formulation of  $C_{\phi_n}$ -universality. Our proofs will repeatedly use the following elementary fact:

**Fact 2.2.** — *If a sequence of holomorphic functions on a domain is bounded by 1 and converges to 1 at some point of the domain, then the sequence converges to 1 uniformly on compact subsets of the domain.*

The direction (2)  $\Rightarrow$  (4) in Theorem 2.1 is clear since there exists a sequence  $(n_k)$  and a (universal) function  $f \in \mathcal{B}$  such that  $f \circ \phi_{n_k} \rightarrow z$  on compacta, so that  $u \circ f \circ \phi_{n_k} \rightarrow u$  for every  $u \in \mathcal{B}$ . The converse direction (4)  $\Rightarrow$  (2) relies on the following two lemmas:

**Lemma 2.3.** — *The set of finite Blaschke products  $B$  with  $B(1) = 1$  is dense in  $\mathcal{B}$  for the topology of local uniform convergence.*

*Proof.* — By Carathéodory’s theorem (see [11, p. 6]) the set of finite Blaschke products is dense in  $\mathcal{B}$ . Thus, for  $f \in \mathcal{B}$  and  $K \subseteq \mathbb{D}$  compact, there exists a finite Blaschke product  $B$  such that  $|f - B| < \varepsilon/2$  on  $K$ . Use [14, Lemma 2.10] to obtain an automorphism  $b$  of  $\mathbb{D}$  with  $b(1) = \overline{B(1)}$  and  $|b - 1| < \varepsilon/2$  on  $K$ . The function  $bB$  is a finite Blaschke product and  $|Bb - f| < \varepsilon$  on  $K$ , which proves our claim.  $\square$

The second lemma establishes the existence of certain “anti-peak” functions.

**Lemma 2.4.** — *Let  $K \subseteq \mathbb{D}$  be a compact subset of  $\mathbb{D}$  and  $\varepsilon > 0$ . There exists a function  $\gamma \in A(\mathbb{D})$  such that  $\gamma(1) = 0$ ,  $\|\gamma - 1\|_K < \varepsilon$  and  $\|\gamma\|_\infty < 1$ .*

*Proof.* — For a positive integer  $n$ , define the function  $\gamma_n$  by

$$\gamma_n(z) = \left(1 - \frac{\varepsilon}{2}\right) \left[1 - \left(\frac{1+z}{2}\right)^n\right] \text{ for } z \in \mathbb{D}.$$

Then if  $n$  is large enough,  $\gamma_n$  satisfies all the required conditions.  $\square$

Under assumption (4) of Theorem 2.1 above, we prove the following proposition, which is the main ingredient in the proof of Theorem 2.1:

**Proposition 2.5.** — *Let  $(\phi_n)$  be a sequence of holomorphic self-maps of  $\mathbb{D}$  such that  $\phi_n(0) \rightarrow 1$  and for any  $m \geq 1$ , the set  $\mathcal{B}_m$  is dense in  $\mathcal{B}$ . Then for any  $m_0 \geq 1$ , any  $\varepsilon > 0$ , any function  $f$  in  $\mathcal{B}$  and any  $K \subseteq \mathbb{D}$  compact, there exist a finite Blaschke product  $B$  and an integer  $m \geq m_0$  such that*

- (a)  $B(1) = 1$
- (b)  $\|B - 1\|_K < \varepsilon$
- (c)  $\|B \circ \phi_m - f\|_K < \varepsilon$ .

*Proof.* — Let  $K$ ,  $\varepsilon > 0$  and  $f$  be given as above. It is always possible to assume that  $\|f\| \leq 1 - \eta < 1$  for some  $\eta > 0$ . By Lemma 2.2, since  $\phi_n(0) \rightarrow 1$  we know that  $(\phi_n)$  converges uniformly to 1 on  $K$ . Let  $\gamma$  be the anti-peak function given by Lemma 2.4. There exists  $\alpha > 0$  such that  $|z - 1| < \alpha$  implies  $|\gamma(z)| < \min(\varepsilon, \eta)$ . We fix a peak function  $\psi$  such that  $\psi(1) = 1$ ,  $\|\psi\|_K < \varepsilon$ , and for  $|z - 1| > \alpha$ ,  $|\psi(z)| < 1 - \|\gamma\|_\infty$  (for instance, we can take for  $\psi$  a sufficiently large power of  $z \mapsto (1 + z)/2$ ). For this peak function  $\psi$ , there exists a number  $\beta$  with  $0 < \beta < \alpha$  such that for  $|z - 1| < \beta$ , we have  $|\psi(z) - 1| < \varepsilon$ . Finally since  $(\phi_n)$  converges to 1 uniformly on  $K$ , we can choose an integer  $m_1 \geq m_0$  such that for every  $m \geq m_1$ ,  $\phi_m(K)$  is contained in the disk  $\{z \in \mathbb{D} : |z - 1| < \beta\}$ .

We now apply our assumption to find an integer  $m \geq m_1$  and a function  $u \in \mathcal{B}$  such that

$$\|u \circ \phi_m - f\|_K < \varepsilon.$$

Since  $f$  has norm less than  $1 - \eta$ , modifying  $u$  if necessary we can assume that  $\|u\|_\infty \leq 1 - \eta$ . Let

$$h = (1 - \varepsilon)\gamma + u\psi.$$

We claim that the function  $h$  solves the problem, except that it is not Blaschke product and  $h(1)$  is not equal to 1. Once we have established this claim, we will modify  $h$  into a Blaschke product which will truly solve the problem.

If  $z \in \phi_m(K)$ , we have

$$|h(z) - u(z)| \leq (1 - \varepsilon)|\gamma(z)| + |\psi(z) - 1| \leq \varepsilon(1 - \varepsilon) + \varepsilon,$$

where the last inequality uses the fact that  $\phi_m(K) \subseteq \{z \in \mathbb{D} : |z - 1| < \beta\}$ . Thus,  $h \circ \phi_m$  is close to  $f$  on  $K$ .

If  $z$  is in  $K$ , then

$$|h(z) - 1| \leq \varepsilon + |\gamma(z) - 1| + \varepsilon \leq 3\varepsilon.$$

Now observe that  $|h(z)| \leq 1$  for any  $z \in \mathbb{D}$ . Indeed if  $|z - 1| > \alpha$  this follows from the property of  $\psi$ , whereas for  $|z - 1| < \alpha$  this is a consequence of the values of  $|\gamma(z)|$  and  $\|u\|_\infty$ .

To obtain the required Blaschke product, apply Lemma 2.3 to replace the function  $h$  by a Blaschke product  $B$  with  $B(1) = 1$  such that the required uniform approximations of (b) and (c) on  $K$  hold.  $\square$

*Proof of (4)  $\Rightarrow$  (2) in Theorem 2.1.* — This is a direct consequence of Proposition 2.5. Indeed, let  $(f_l)_{l \geq 1}$  be a dense sequence of elements of  $\mathbb{D}$ , and  $K_l = \overline{\mathbb{D}}(0, 1 - 2^{-l})$  (for instance) be an exhaustive sequence of compact subsets of  $\mathbb{D}$ . We claim the existence of finite Blaschke products  $(B_l)_{l \geq 1}$  and of a sequence  $(n_l)_{l \geq 1}$  such that for every  $l \geq 1$ ,

$$(2.1) \quad B_l(1) = 1$$

$$(2.2) \quad \|B_l - 1\|_{K_l} < 2^{-l}$$

and

$$(2.3) \quad \left\| C_{\phi_{n_l}}(B_1 \dots B_k) - f_l \right\|_{K_l} < 2^{-l} \text{ for every } k \geq l.$$

The existence of a universal Blaschke product follows from properties (2.1), (2.2) and (2.3): it suffices to set

$$B = \prod_{l \geq 1} B_l$$

(which is a convergent Blaschke product by (2.3)). By (2.3),  $\|C_{\phi_{n_l}}(B) - f_l\|_{K_l} < 2^{-l}$  for every  $l \geq 1$ , and we are done.

The construction of the  $B_l$ 's and the  $n_l$ 's is done by induction on  $l$  using Proposition 2.5: the case  $l = 1$  is nothing more than Proposition 2.5. If the construction has been carried out until Step  $l - 1$ , we choose  $B_l$  and  $n_l$  large enough by Proposition 2.5 so that

1.  $B_l(1) = 1$
2.  $\|B_l - 1\|_K$  is very small where  $K = K_l \cup \bigcup_{j \leq l-1} \phi_{n_j}(K_l)$
3.  $\|C_{\phi_{n_j}}(B_l) - 1\|_{K_j}$  is so small as to ensure that

$$\left\| C_{\phi_{n_j}}(B_1 \dots B_{l-1}) C_{\phi_{n_j}}(B_l) - f_j \right\|_{K_j} < 2^{-j}$$

for every  $j \leq l - 1$

4.  $\left\| C_{\phi_{n_l}}(B_1 \dots B_{l-1}) - 1 \right\|_{K_l} < 2^{-(l+1)}$
5.  $\left\| C_{\phi_{n_l}}(B_l) - f_l \right\|_{K_l} < 2^{-(l+1)}$ .

This yields

$$\left\| C_{\phi_{n_l}}(B_1 \dots B_l) - f_l \right\|_{K_l} < 2^{-l}$$

which terminates the proof of Proposition 2.5.  $\square$

We are now ready for the rest of the proof of Theorem 2.1.

**2.3. Proof of (1)  $\Rightarrow$  (5).** — By assumption (1), there exists a function  $f \in \mathcal{B}$  and an increasing sequence  $(n_k)$  such that  $f \circ \phi_{n_k} \rightarrow z$  uniformly on compacta as  $n_k \rightarrow +\infty$ . In particular  $|(f \circ \phi_{n_k})'(0)| = |\phi'_{n_k}(0)| \cdot |f'(\phi_{n_k}(0))| \rightarrow 1$ , i.e.

$$\frac{|\phi'_{n_k}(0)|}{1 - |\phi_{n_k}(0)|^2} \cdot (1 - |\phi_{n_k}(0)|^2) |f'(\phi_{n_k}(0))| \rightarrow 1.$$

By the Schwarz Lemma  $(1 - |\phi_{n_k}(0)|^2) |f'(\phi_{n_k}(0))| \leq 1$  for every  $n_k$ . Since we also have

$$\frac{|\phi'_{n_k}(0)|}{1 - |\phi_{n_k}(0)|^2} \leq 1$$

for all  $n_k$ 's by the Schwarz-Pick inequalities, it follows that

$$\frac{|\phi'_{n_k}(0)|}{1 - |\phi_{n_k}(0)|^2} \rightarrow 1,$$

which proves (5).

**2.4. Proof of (5)  $\Rightarrow$  (2).** — Passing to a subsequence we can assume that  $\phi_n(0)$  is a thin sequence such that  $\phi_n(0) \rightarrow 1$  and

$$\lim_{n \rightarrow \infty} \frac{|\phi'_n(0)|}{1 - |\phi_n(0)|^2} = 1.$$

Recall that a sequence of distinct points  $(z_n)_{n \geq 1}$  in  $\mathbb{D}$  is said to be *thin* if

$$\lim_{n \rightarrow \infty} \prod_{\substack{j=1 \\ j \neq n}}^{\infty} \rho(z_j, z_n) = 1,$$

where  $\rho$  denotes the pseudo-hyperbolic distance on  $\mathbb{D}$ . Let  $B$  denote the interpolating Blaschke product corresponding to the sequence  $(\phi_n(0))$ . Note that the assumption that  $(\phi_n(0))$  is thin means that

$$\lim_{n \rightarrow \infty} (1 - |\phi_n(0)|^2) |B'(\phi_n(0))| = 1.$$

Since  $(B \circ \phi_n)'(0) = B'(\phi_n(0)) \phi'_n(0)$ , we obtain that

$$(B \circ \phi_n)'(0) = B'(\phi_n(0)) (1 - |\phi_n(0)|^2) \frac{\phi'_n(0)}{1 - |\phi_n(0)|^2}.$$

Now, using the fact that  $B$  is thin and our hypothesis, we get

$$|(B \circ \phi_n)'(0)| = |B'(\phi_n(0))| (1 - |\phi_n(0)|^2) \frac{|\phi'_n(0)|}{1 - |\phi_n(0)|^2} \rightarrow 1.$$

Since  $(B \circ \phi_n)$  is a bounded family of analytic functions a normal families argument implies that there is a subsequence  $(B \circ \phi_{n_j})$  and an analytic function  $f$  of norm at most one such that  $B \circ \phi_{n_j} \rightarrow f$ . But  $(B \circ \phi_{n_j})(0) = 0$  for every  $n_j$  and  $|(B \circ \phi_{n_j})'(0)| \rightarrow 1$ , so  $f(0) = 0$  and  $|f'(0)| = 1$ . By Schwarz's lemma,  $f(z) = \lambda z$  for some  $|\lambda| = 1$ . Writing  $u = \bar{\lambda} B$  we see that  $u \circ \phi_{n_j} \rightarrow z$  uniformly on compacta. Hence assumption (4) of Theorem 2.1 is satisfied, which proves assertion (2). Since (2)  $\Rightarrow$  (1) is obvious, we have proved the equivalence between assertions (1), (2) and (5) of Theorem 2.1.

It now remains to deal with the case of universal singular inner functions.

**2.5. Proof of (2)  $\Rightarrow$  (3).** — Consider the atomic singular inner function

$$S(z) = \exp\left(\frac{z+1}{z-1}\right).$$

By our assumption we can choose a Blaschke product  $B$  that is universal with respect to  $(\phi_n)$ . Now  $S \circ B$  is a singular inner function. If  $f \in \mathcal{S}$ , let  $h$  be a solution of the equation  $\log f = -\left(\frac{1+h}{1-h}\right)$  with  $h \in \mathcal{B}$ . Note that  $f = S \circ h$ . Since  $B$  is universal for  $(\phi_n)$  we can choose  $(\phi_{n_k})$  such that  $B \circ \phi_{n_k} \rightarrow h$ . Then  $(S \circ B) \circ \phi_{n_k} \rightarrow S \circ h = f$ , and thus  $S \circ B$  is  $\mathcal{S}$ -universal for  $(C_{\phi_n})$ .

**2.6. Proof of (3)  $\Rightarrow$  (5).** — Our study here will be aided by a result that is related to a conjecture of Krzyz. Writing a function  $f$  of  $\mathcal{S}$  as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the Krzyz conjecture is concerned with estimating the size of  $a_n$ . He conjectured that  $\max_n |a_n| = 2/e$  and this occurs if and only if  $f(z)$  is a rotation of the function  $\exp(\frac{z^n+1}{z^n-1})$ . Showing that  $\max |a_1| = 2/e$  is, however, not difficult (see, for example [22]). For the reader's convenience we present a proof here.

**Lemma 2.6.** —  $\sup\{|f'(0)| : f \in \mathcal{S}\} = 2/e$ .

*Proof.* — Let  $f \in \mathcal{S}$ . Without loss of generality we can assume that  $f(0) > 0$  and that the function  $f$  is not constant. Choose the principal branch of the logarithm and let  $g(z) = -\log f(z)$  with  $g(0) > 0$ . Then  $g$  is an holomorphic function on  $\mathbb{D}$  with positive real part. We have

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{it} + z}{re^{it} - z} \operatorname{Re} g(re^{it}) dt + i \operatorname{Im} g(0).$$

Hence

$$g'(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{2re^{it}}{(re^{it})^2} \operatorname{Re} g(re^{it}) dt.$$

Consequently, for all  $r \in ]0, 1[$

$$|g'(0)| \leq \frac{2}{r} \int_0^{2\pi} \operatorname{Re} g(re^{it}) \frac{dt}{2\pi} = \frac{2}{r} \operatorname{Re} g(0).$$

Thus  $|g'(0)| \leq 2|g(0)|$ , from which we conclude that

$$|f'(0)| = |f(0)| |g'(0)| \leq 2|f(0)| |g(0)| = -2f(0) \log f(0).$$

Since the maximum of the function  $-x \log x$  on the interval  $[0, 1]$  is equal to  $1/e$ , we see that  $|f'(0)| \leq 2/e$ .  $\square$

Let us now go back to the proof of the implication (3)  $\Rightarrow$  (5). The idea is the same as in the proof of (1)  $\Rightarrow$  (5). By assumption, there exists  $f \in \mathcal{S}$  such that

$$f \circ \phi_{n_k} \rightarrow S \text{ uniformly on compacta.}$$

Thus

$$|(f \circ \phi_{n_k})'(0)| = |f'(\phi_{n_k}(0)) \phi'_{n_k}(0)| \rightarrow \left| S(z) \left( \frac{-2}{(1-z)^2} \right) \right|_{z=0} = 2/e,$$

which can be rewritten as

$$(2.4) \quad (1 - |\phi_{n_k}(0)|^2) |f'(\phi_{n_k}(0))| \frac{|\phi'_{n_k}(0)|}{1 - |\phi_{n_k}(0)|^2} \rightarrow 2/e.$$

By Lemma 2.6, if  $F \in \mathcal{S}$ , then  $|F'(0)| \leq 2/e$ . This implies that for  $a \in \mathbb{D}$  we have

$$|(F \circ \tau_a)'(0)| = |F'(a)(1 - |a|^2)| \leq 2/e.$$

Thus, letting  $F = f$  and  $a = \phi_{n_k}(0)$  we get that for every  $n_k$

$$|f'(\phi_{n_k}(0))| (1 - |\phi_{n_k}(0)|^2) \leq 2/e.$$



Since the quantity  $\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}$  in (2.4) above is less than or equal to 1 for all  $n_k$ 's, we must have

$$\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2} \rightarrow 1.$$

**2.7. Some consequences and remarks.** — Our first corollary is:

**Corollary 2.7.** — *Let  $(\phi_n)$  be a sequence of self-maps of  $\mathbb{D}$  with  $\phi_n(0) \rightarrow 1$ . Then*

$$\limsup_{n \rightarrow +\infty} \frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2} = 1 \text{ if and only if } \limsup_{n \rightarrow +\infty} \frac{(1-|a|^2)|\phi'_n(a)|}{1-|\phi_n(a)|^2} = 1$$

for every  $a \in \mathbb{D}$ .

*Proof.* — One direction is clear, so suppose that  $\limsup_{n \rightarrow +\infty} \frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2} = 1$ . By Theorem 2.1 there exists an  $\mathcal{S}$ -universal singular inner function  $u$  for  $(C_{\phi_n})$ . Let  $a \in \mathbb{D}$  and consider  $(\phi_n \circ \tau_a)$ . If  $f \in \mathcal{S}$ , then  $f \circ \tau_a^{-1} \in \mathcal{S}$  and, therefore,  $u \circ \phi_{n_k} \rightarrow f \circ \tau_a^{-1}$  for some subsequence  $(\phi_{n_k})$ . Thus  $u \circ (\phi_{n_k} \circ \tau_a) \rightarrow f$  and  $(\phi_{n_k} \circ \tau_a)$  has an  $\mathcal{S}$ -universal singular inner function too. Again applying Theorem 2.1 we get

$$\frac{|(\phi_{n_l} \circ \tau_a)'(0)|}{1-|\phi_{n_l}(\tau_a(0))|^2} = \frac{(1-|a|^2)|\phi'_{n_l}(a)|}{1-|\phi_{n_l}(a)|^2} \rightarrow 1,$$

thus completing the proof of the corollary.  $\square$

The next corollary states that whenever  $(C_{\phi_n})$  admits a universal Blaschke product, it admits a dense set of universal Blaschke products:

**Corollary 2.8.** — *Let  $(\phi_n)$  be a sequence of self-maps with  $\phi_n(0) \rightarrow 1$ . Suppose that  $(C_{\phi_n})$  admits a  $\mathcal{B}$ -universal function. Then the set of  $\mathcal{B}$ -universal Blaschke products for  $(C_{\phi_n})$  is dense in  $\mathcal{B}$ .*

*Proof.* — By Theorem 2.1,  $(C_{\phi_n})$  admits a universal Blaschke product  $B$ . We can write  $B = \prod_{j=1}^{\infty} L_j$ , where the  $L_j$ 's are automorphisms of  $\mathbb{D}$ . Let  $(b_n)$  be a sequence of finite Blaschke products that are dense in  $\mathcal{B}$  and satisfy  $b_n(1) = 1$  (see Lemma 2.3). Consider the tails  $B_n = \prod_{j=n}^{\infty} L_j$ . Then  $\{b_n B_n : n \in \mathbb{N}\}$  is dense in  $\mathcal{B}$  and each member of this set is  $\mathcal{B}$ -universal for  $(C_{\phi_n})$ .  $\square$

**Remark 2.9.** — Suppose that  $B_1$  and  $B_2$  are  $\mathcal{B}$ -universal for  $(C_{\phi_n})$ . Then  $B_1 \circ B_2$  is  $\mathcal{B}$ -universal, too. To see this, note that since  $B_2$  is universal, for every  $m$  there exists a subsequence  $(\phi_{n_j(m)})$  such that  $B_2 \circ \phi_{n_j(m)} \rightarrow \phi_m$ . So  $B_1 \circ \phi_m = \lim B_1 \circ (B_2 \circ \phi_{n_j})$  and therefore the orbit of  $B_1$  under  $C_{\phi_n}$  is contained in the closure of the orbit of  $B_1 \circ B_2$ . Since the former is dense in  $\mathcal{B}$ , the latter must be too. Thus  $B_1 \circ B_2$  is  $\mathcal{B}$ -universal. However, the situation for  $\mathcal{S}$ -universality is quite different. In fact, if we let  $(p_n)$  be a sequence of finite Blaschke products that is dense in  $\mathcal{B}$ , then the atomic inner function  $S$  is  $\mathcal{S}$ -universal for  $(C_{p_n})$ , but  $S \circ S$  is not. To see this, note that if  $S \circ S$  were universal, there would exist a subsequence of  $(p_n)$  such that  $S \circ S \circ p_{n_k} \rightarrow S$ . Taking logarithms we see that in a sufficiently small compact neighborhood  $K$  of the origin

$$\frac{1 + S \circ p_{n_k}}{1 - S \circ p_{n_k}} \rightarrow \frac{1 + z}{1 - z}.$$

Hence  $S \circ p_{n_k}$  tends uniformly to  $z$  on  $K$ , which contradicts Hurwitz's lemma.

### 3. Infinite dimensional spaces of universal functions

**3.1. The main result.** — In this section we follow the line of thought of papers such as [1, 3, 5, 21] and study how large the subspaces of universal functions can be. We say that a uniformly closed subspace  $V$  of  $H(\mathbb{D})$  is topologically generated by the set  $E$  if the linear span of  $E$  is uniformly dense in  $V$ .

**Theorem 3.1.** — *Let  $(\phi_n)$  be a sequence of self-maps of  $\mathbb{D}$ . The following are equivalent.*

1. *the sequence  $(C_{\phi_n})$  admits a  $\mathcal{B}$ -universal function*
2. *the set  $\mathcal{B}_q^p = \{(u_1 \circ \phi_n, \dots, u_p \circ \phi_n) : n \geq q, u_i \in \mathcal{B}\}$  is dense in  $\mathcal{B}^p$  for any integers  $p, q \geq 1$*
3. *there exists a uniformly closed infinite dimensional vector subspace  $V$  of  $H^\infty$ , topologically generated by Blaschke products and linearly isometric to  $\ell^1$ , with the property that every function in  $V$  is universal with respect to  $(C_{\phi_n})$ .*

The proof of this theorem requires Lemmas 2.3 and 2.4 of the previous section as well as Proposition 3.2 below, which is the “multidimensional version” of Proposition 2.5. The proof is exactly the same, so we omit it.

**Proposition 3.2.** — *Let  $(\phi_n)$  be a sequence of holomorphic self-maps of  $\mathbb{D}$  such that  $\phi_n(0) \rightarrow 1$  and for any  $p, q \geq 1$ , the set  $\mathcal{B}_q^p = \{(u_1 \circ \phi_n, \dots, u_p \circ \phi_n) : n \geq q, u_i \in \mathcal{B}\}$  is dense in  $\mathcal{B}^p$ . Then, for any  $p \geq 1$ , any  $m_0 \geq 1$ , any  $\varepsilon > 0$ , any functions  $(f_1, \dots, f_p)$  in  $\mathcal{B}^p$  and any  $K \subseteq \mathbb{D}$  compact, there exist finite Blaschke products  $B_1, \dots, B_p$  and an integer  $m \geq m_0$  such that*

$$\|B_j \circ \phi_m - f_j\|_K < \varepsilon, \|B_j - 1\|_K < \varepsilon \text{ and } B_j(1) = 1 \text{ for all } j \in \{1, \dots, p\}.$$

The proof of Theorem 3.1 is very similar to the one that appears in [3]. A special case of the theorem was proved using maximal ideal space techniques in [21]. Another useful lemma is:

**Lemma 3.3.** — *Let  $(\phi_n)$  be a sequence of holomorphic self-maps of  $\mathbb{D}$ . Then the uniform limit of universal functions in  $H^\infty$  is a universal function with respect to  $(C_{\phi_n})$ .*

*Proof.* — Let  $f_n$  be a sequence of universal functions in  $H^\infty$  that converge uniformly to  $f$  on  $\mathbb{D}$  and let  $K$  be a compact subset of  $\mathbb{D}$ . Let  $g \in H^\infty$  with  $\|g\| \leq \|f\|$ . Since  $\left\| \frac{\|f_n\|_\infty}{\|f\|_\infty} g \right\|_\infty \leq \|f_n\|_\infty$  and  $f_n$  is universal, there exists a subsequence  $(\phi_{j_k(n)})$  of  $(\phi_j)$  (depending on  $n$ ), such that  $f_n \circ \phi_{j_k(n)} \rightarrow \frac{\|f_n\|_\infty}{\|f\|_\infty} g$  locally uniformly in  $\mathbb{D}$ . So

$$\|f \circ \phi_{j_k(n)} - g\|_K \leq \|f \circ \phi_{j_k(n)} - f_n \circ \phi_{j_k(n)}\|_\infty + \left\| f_n \circ \phi_{j_k(n)} - g \frac{\|f_n\|_\infty}{\|f\|_\infty} \right\|_K + \left\| \frac{\|f_n\|_\infty}{\|f\|_\infty} g - g \right\|_\infty,$$

from which the result follows.  $\square$

Here is now the proof of Theorem 3.1.

**3.2. Proof of (1)  $\Rightarrow$  (2).** — For  $k = 1, \dots, p$ , let  $(B_m^k)_{m \geq 1}$  be  $p$  sequences of finite Blaschke products, each of them being locally uniformly dense in  $\mathcal{B}$ . By assumption (1), there exists a universal function  $u \in \mathcal{B}$  and a sequence  $(n_j)$  of integers such that  $u \circ \phi_{n_j} \rightarrow z$ . If we let  $u_m^k := B_m^k \circ u$ , then  $u_m^k \circ \phi_{n_j} = B_m^k \circ (u \circ \phi_{n_j}) \rightarrow B_m^k$  uniformly on compacta as  $j \rightarrow \infty$ . Now  $u_m^k \in \mathcal{B}$  for each  $m$  and  $k = 1, \dots, p$  and therefore

$$\{(u_1 \circ \phi_n, \dots, u_p \circ \phi_n) : n \geq q, u_i \in \mathcal{B}\}$$

is dense in  $\mathcal{B}^p$  for every  $q \in \mathbb{N}$ . This yields (2).

Since it is clear that (3)  $\Rightarrow$  (1), our proof will be complete if we show that (2)  $\Rightarrow$  (3). Our proof follows closely the proof of Theorem 1 in [3], with minor modifications so as to adapt the proof in [3] to the situation we discuss here.

**3.3. Proof of (2)  $\Rightarrow$  (3).** — Let  $(\alpha_n)$  be a countable dense sequence of points on the unit circle with  $\alpha_1 = 1$  and let  $(h_n)_{n \geq 1}$  be a sequence in  $\mathcal{B}$  which is locally uniformly dense. There is no loss of generality in assuming that  $h_n$  is not identically equal to 1. Write  $\mathbb{D}$  as  $\mathbb{D} = \bigcup_{n=1}^{\infty} K_n$  where  $K_n$  is compact and  $K_n \subseteq K_{n+1}^\circ$  for every  $n$ .

Consider the infinite matrix  $A$  defined by

$$\begin{pmatrix} \alpha_1 h_1 & \alpha_1 h_2 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_2 h_2 & \alpha_1 h_3 & \alpha_1 h_3 & \alpha_1 h_3 & \dots \\ 1 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_1 h_3 & \alpha_1 h_3 & \alpha_1 h_3 & \dots \\ 1 & 1 & 1 & 1 & 1 & \alpha_1 h_3 & \alpha_2 h_3 & \alpha_3 h_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where the second block (that is, the 2nd through 5th column) contains all possible combinations of  $(\alpha_1, \alpha_2)$  multiplied by  $h_2$  (there are four possibilities,  $(\alpha_i h_2, \alpha_j h_2)$  where  $i = 1, 2$  and  $j = 1, 2$ , listed as the first two entries in columns 2, 3, 4, and 5), the third block (6th through 32nd column) contains all possible combinations of  $(\alpha_1, \alpha_2, \alpha_3)$  multiplied by the function  $h_3$ , and so on. Then each element of the matrix  $A$  is of the form  $\alpha_{s(i,j)} h_{t(j)}$ , where  $s(i, j)$  is determined by both the column and row of the entry and  $t(j)$  is determined solely by the column. Let  $M(j)$  denote the number of terms in column  $j$  that are different from 1, that is that are written in the form  $\alpha_{s(i,j)} h_{t(j)}$ .

By induction on  $j$ , we will choose finite Blaschke products  $B_{i,j}$  and a sequence of integers  $(m_j)$  such that

$$(3.1) \quad \forall i \leq M(j), \|B_{i,j} \circ \phi_{m_j} - \alpha_{s(i,j)} h_{t(j)}\|_{K_j} < \frac{1}{2^j};$$

$$(3.2) \quad B_{i,j} = 1 \text{ for } i > M(j);$$

$$(3.3) \quad \forall k < j, \forall i, |B_{i,k} \circ \phi_{m_j}(0) - 1| < \frac{1}{2^j};$$

$$(3.4) \quad \forall k < j, \forall i, |B_{i,j} \circ \phi_{m_k}(0) - 1| < \frac{1}{2^j};$$

$$(3.5) \quad \forall i \leq M(j), \|B_{i,j} - 1\|_{K_j} < \frac{1}{2^j};$$

$$(3.6) \quad \forall i, B_{i,j}(1) = 1.$$

We note that the first step of the induction follows from Proposition 3.2.

Now we assume that the construction has been carried out until step  $j - 1$  and show how to complete step  $j$ . Using the continuity at the point 1 of the (finite) set of functions  $(B_{i,k})_{i \leq M(k), k < j}$ , we choose an integer  $m'$  such that, for every  $k < j$ , for every  $i \geq 1$  and for every  $m \geq m'$ ,  $|B_{i,k} \circ \phi_m(0) - 1| < 1/2^j$ . Now let  $K = K_j \cup \bigcup_{k < j} \{\phi_{m_k}(0)\}$ . The functions  $B_{i,j}$  are automatically given by Proposition 3.2.

We define a function  $B_i$  on  $\mathbb{D}$  by

$$B_i = \prod_{j=1}^{\infty} B_{i,j}$$

and establish the following claim:

**Claim 3.4.** — For each  $i$ , the function  $B_i$  is a Blaschke product and

$$\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}(0) \rightarrow 1 \text{ as } k \rightarrow \infty.$$

Consequently,  $|B_i \circ \phi_{m_k} - B_{i,k} \circ \phi_{m_k}| \rightarrow 0$  uniformly on compacta as  $k \rightarrow \infty$ .

By (3.5), the infinite product  $\prod_j B_{i,j}$  converges uniformly to  $B_i$  on compact subsets of  $\mathbb{D}$ , which proves the first part of the claim. Note that

$$|B_i \circ \phi_{m_k} - B_{i,k} \circ \phi_{m_k}| = \left| \left(1 - \prod_{j \neq k} B_{i,j} \circ \phi_{m_k}\right) B_{i,k} \circ \phi_{m_k} \right|.$$

Thus, if we can show that  $\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}(0) \rightarrow 1$  and combine this with the fact that  $\|\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}\| \leq 1$  and Lemma 2.2, we will be able to conclude that  $\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}$  converges to 1 uniformly on compact sets. This, in turn, will imply that

$$|B_i \circ \phi_{m_k} - B_{i,k} \circ \phi_{m_k}| \rightarrow 0 \text{ uniformly on compacta,}$$

completing the proof of the claim.

*Proof of Claim 3.4.* — We turn to showing that  $\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}(0) \rightarrow 1$  uniformly on compacta. Recall that for complex numbers  $z_j$ ,  $j = 1, \dots, m$  with  $|z_j| \leq 1$ , we have

$$\left|1 - \prod_{j=1}^m z_j\right| \leq \sum_{j=1}^m |1 - z_j|.$$

We will use this inequality to establish the rest of the claim as follows:

$$\begin{aligned} \left|1 - \prod_{j \neq k} B_{i,j} \circ \phi_{m_k}(0)\right| &\leq \sum_{j \neq k} |1 - B_{i,j} \circ \phi_{m_k}(0)| \\ &\leq \sum_{j=k+1}^{\infty} |1 - B_{i,j} \circ \phi_{m_k}(0)| + \sum_{j=1}^{k-1} |1 - B_{i,j} \circ \phi_{m_k}(0)| \\ &\leq \sum_{j=k+1}^{\infty} \frac{1}{2^j} + \sum_{j=1}^{k-1} \frac{1}{2^k}, \end{aligned}$$

where the first sum was estimated using (3.4) and the second one using (3.3). This completes the proof that  $\prod_{j \neq k} B_{i,j} \circ \phi_{m_k}(0)$  converges to 1, and therefore the proof of the claim.  $\square$

Let the vector space  $V$  be the uniform closure of the vector space generated by the functions  $B_j$ ,  $j = 1, \dots, +\infty$ . We must show that if  $f \in V$ , then  $f$  is universal for  $\{g \in H^\infty : \|g\| \leq \|f\|\}$ . Since this is clear for  $f = 0$ , we can assume  $f$  is not identically equal to 0, and if we divide  $f$  by its norm, we see that we can reduce our study to functions of norm 1. By Lemma 3.3 the uniform limit of universal functions is universal, so it is enough to consider functions that are in the span of the  $B_j$ 's. Given these considerations, we now choose  $f \in V$  with  $\|f\| = 1$  and  $f = \sum_{j=1}^n \lambda_j B_j$ . Let  $h \in H^\infty(U)$  with  $\|h\| \leq 1$ . Now let  $\mu_j = \frac{\overline{\lambda_j}}{|\lambda_j|}$  (where  $\mu_j = 1$  if  $\lambda_j = 0$ ). Let  $K_r$  be one of our compact sets. Choose a column,  $k$ , in the matrix such that the function  $h_{t(k)}$  appearing in that column satisfies  $\|h - h_{t(k)}\|_{K_r} < \epsilon / (n \sum_{j=1}^n |\lambda_j|)$  and the column has the  $n$  permuted values  $\alpha_{s(i,k)}$ , for  $i = 1, \dots, n$  in the correct order so that

$$\max_{i=1, \dots, n} |\alpha_{s(i,k)} - \mu_i| < \epsilon / (n \sum_{j=1}^n |\lambda_j|).$$

Then using Claim 3.4 and property (3.1) on  $K_r$  for sufficiently large  $k$  we get

$$f \circ \phi_{m_k} = \sum_{i=1}^n \lambda_i (B_i \circ \phi_{m_k}) \approx \sum_{i=1}^n \lambda_i B_{i,k} \circ \phi_{m_k} \approx \sum_{i=1}^n \lambda_i (\alpha_{s(i,k)} h_{t(k)}) \quad \text{on } K_r.$$

So

$$\begin{aligned} \left\| f \circ \phi_{m_k} - \sum_{j=1}^n |\lambda_j| h \right\|_{K_r} &\approx \left\| \sum_{j=1}^n \lambda_j \alpha_{s(j,k)} h_{t(k)} - \sum_{j=1}^n |\lambda_j| h \right\|_{K_r} \\ &= \left\| \sum_{j=1}^n \lambda_j \mu_j h_{t(k)} + \sum_{j=1}^n \lambda_j (\alpha_{s(j,k)} - \mu_j) h_{t(k)} - \sum_{j=1}^n |\lambda_j| h \right\|_{K_r} \\ &\leq \left\| \sum_{j=1}^n |\lambda_j| (h_{t(k)} - h) \right\|_{K_r} + \epsilon < 2\epsilon. \end{aligned}$$

Now we can choose any function in  $\mathcal{B}$  in place of  $h$ , for instance the constant function 1. For any  $\delta > 0$ , we find that there exists an integer  $m'_k$  such that

$$\left| f \circ \phi_{m'_k}(0) - \sum_{j=1}^n |\lambda_j| \right| < \delta.$$

This yields  $\sum_{j=1}^n |\lambda_j| \leq \|f\|_\infty + \delta$ , and since  $\delta$  is arbitrary, we get  $\sum_{j=1}^n |\lambda_j| = 1 = \|f\|_\infty$ . Thus,  $\|f \circ \phi_{m_k} - h\|_{K_r} < 2\epsilon$ , and  $f$  is universal. By homogeneity, our vector space is infinite dimensional, isometric to  $\ell^1$ , and each function  $f \in V$  can be written as  $f = \sum_{j=1}^\infty a_j B_j$  where the sequence  $(a_j)$  belongs to  $\ell^1$ . This finishes the proof of Theorem 3.1.

If we choose the Blaschke generators more carefully, we obtain the following proposition:

**Proposition 3.5.** — *Let  $(\phi_n)$  be a sequence of self-maps of  $\mathbb{D}$  with  $\phi_n(0) \rightarrow 1$ . Suppose  $(C_{\phi_n})$  admits a  $\mathcal{B}$ -universal function  $f$ . Then there exists a subspace  $V$  of  $H(\mathbb{D})$  that is dense in the local uniform topology, closed in the sup-norm topology, and such that every element of  $V$  is bounded and universal for  $(C_{\phi_n})$ .*

*Proof.* — We begin by applying Theorem 3.1 to obtain a uniformly closed vector space  $V$  generated by  $\mathcal{B}$ -universal Blaschke products  $B_i$ . Then we modify the  $B_i$ 's: Let  $(p_i)$  denote the sequence of monomials

$$(p_i) = (\mathbf{1}, 1, z, \mathbf{1}, \mathbf{z}, \mathbf{z}^2, 1, z, z^2, z^3, \mathbf{1}, \mathbf{z}, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4, 1, z, \dots)$$

and let  $C_i = p_i B_i$ . Since  $p_i(1) = 1$  for every  $i$ , each one of these new functions is  $\mathcal{B}$ -universal again for  $(C_{\phi_n})$ . As in the proof of Theorem 3.1 above, every function in the uniformly closed vector space  $W$  generated by the  $C_i$  is universal. We claim that  $B_i \rightarrow 1$  locally uniformly. From this claim it will follow that the closure of  $W$  in the local uniform topology contains the polynomials and therefore is dense. Thus, once we establish our claim, Proposition 3.5 will be proved.

So let  $K \subseteq \mathbb{D}$  be a compact subset of  $\mathbb{D}$ . Choose an index  $i$ . Returning to the notation of the previous theorem (recall that  $M(j)$  denotes the number of terms in column  $j$  that are different from 1), let  $j_i$  denote that smallest index for which  $M(j_i) \geq i$  for the first time. Choose  $i_0$  so that  $K \subseteq K_{i_0}$  and  $2^{-j_i} < \varepsilon/2$  whenever  $j_i \geq M(j_i) \geq i \geq i_0$ . On  $K$ , by (3.2) and (3.5), we have

$$|B_i - 1| \leq \sum_{j=1}^{j_i-1} |B_{ij} - 1| + \sum_{j=j_i}^{\infty} |B_{ij} - 1| \leq 0 + \sum_{j=j_i}^{\infty} 2^{-j} < \varepsilon.$$

This proves the claim and completes the proof of the proposition.  $\square$

**Remark 3.6.** — Suppose that  $E$  is a subspace of  $H(\mathbb{D})$  which is closed for the topology of uniform convergence on compact sets. If  $E$  is infinite-dimensional, then  $E$  contains a function which is unbounded on  $\mathbb{D}$ . Indeed, the contrary would imply that  $E$  is a closed subspace of  $H^2(\mathbb{D})$  consisting of bounded functions, which is impossible.

## 4. Applications, examples and remarks.

**4.1. Sequences of iterates.** — Here is our first application of Theorem 2.1. We let  $\phi$  be a self-map of  $\mathbb{D}$  and  $\phi_{[n]}$  denote the  $n$ -th iterate of  $\phi$ .

**Corollary 4.1.** — *Let  $\phi$  be a holomorphic self-map of  $\mathbb{D}$ . Then the sequence of iterates  $(C_{\phi_{[n]}})$  of  $C_\phi$  admits a universal function if and only if  $\phi$  is a parabolic or hyperbolic automorphism.*

*Proof.* — If  $\phi$  is an automorphism, then  $\frac{|\phi'(0)|}{1-|\phi(0)|^2} = 1$ . Since the orbits of parabolic and hyperbolic automorphisms in  $\mathbb{D}$  cluster at a single boundary point (the so-called Denjoy-Wolff fixed point, see [23, p. 78]), the result follows from Theorem 2.1.

Now suppose that there is a universal function  $f$  for  $(\phi_{[n]})$ . Since  $\phi_{[n]}$  is the  $n$ -th iterate of  $\phi$ , we see that  $u \circ \phi_{[n]} = (u \circ \phi_{[n-1]}) \circ \phi$ , and so

$$\{u \circ \phi_{[n]} : u \in \mathcal{B}, n \in \mathbb{N}\} = \{u \circ \phi : u \in \mathcal{B}\}.$$

By assumption  $\{u \circ \phi_{[n]} : u \in \mathcal{B}, n \in \mathbb{N}\}$  is dense in  $\mathcal{B}$ , so  $\{u \circ \phi : u \in \mathcal{B}\}$  is dense, too. Hence there exists  $u_n \in \mathcal{B}$  such that  $u_n \circ \phi \rightarrow z$  uniformly on compacta. Thus

$$(1 - |\phi(0)|^2)(u_n \circ \phi)'(0) = [(1 - |\phi(0)|^2)u_n'(\phi(0))] \frac{\phi'(0)}{1 - |\phi(0)|^2} \rightarrow 1.$$

Since  $(1 - |\phi(0)|^2)|u_n'(\phi(0))| \leq 1$ , we must have  $\frac{|\phi'(0)|}{1 - |\phi(0)|^2} = 1$ . By the Schwarz Lemma,  $\phi$  must be an automorphism. Of course,  $\phi$  cannot have a fixed point in  $\mathbb{D}$  so  $\phi$  is a parabolic or hyperbolic automorphism (see [23]).  $\square$

**4.2. A corollary on hyperbolic distance.** — The following lemma is well known (see e.g. [11, p. 405] or [9, p. 943]).

**Lemma 4.2.** — *Let  $f$  be a holomorphic self-map of  $\mathbb{D}$  such that  $f(0) = 0$  and  $|f'(0)| \geq \delta$ . Then for every  $\eta < \delta$  we have  $D(0, \eta \frac{\delta - \eta}{1 - \eta \delta}) \subseteq f(\eta \mathbb{D})$*

For  $z \in \mathbb{D}$  we let

$$L_a(z) = \frac{|a|}{a} \frac{a - z}{1 - \bar{a}z}$$

and  $D_\rho(a, \varepsilon) = \{z \in \mathbb{D} : \rho(z, a) = |L_a(z)| < \varepsilon\}$  the pseudohyperbolic disk of center  $a$  and radius  $\varepsilon$ . Let the pseudohyperbolic radius of a set  $S$  be denoted by  $r_\rho(S)$ .

**Corollary 4.3.** — *Let  $(\phi_n)$  be a sequence of self-maps such that  $\phi_n(0) \rightarrow 1$ . If  $(C_{\phi_n})$  admits a universal function,  $\limsup_{k \rightarrow +\infty} r_\rho(\phi_k(\eta_k \mathbb{D})) = 1$  for some sequence  $(\eta_k)$ ,  $0 < \eta_k < 1$  with  $\eta_k \rightarrow 1$ .*

*Proof.* — Suppose that  $(C_{\phi_n})$  admits a universal function. By Theorem 2.1 we know that there exists a subsequence, denoted  $(\phi_{n_k})$ , such that

$$\lim_{n \rightarrow \infty} \frac{|\phi_{n_k}'(0)|}{1 - |\phi_{n_k}(0)|^2} = 1.$$

Let  $u_k = \tau_{\phi_{n_k}(0)} \circ \phi_{n_k}$ . A straightforward computation shows that

$$u_k'(0) = \frac{\phi_{n_k}'(0)}{1 - |\phi_{n_k}(0)|^2},$$

so  $\delta_k := |u_k'(0)| \rightarrow 1$ . Choose  $\eta_k = 1 - \sqrt{1 - \delta_k^2}$ . Then  $\eta_k < \delta_k$  and  $r_k := \eta_k \frac{\delta_k - \eta_k}{1 - \delta_k \eta_k} \rightarrow 1$  as  $k \rightarrow \infty$ . By Lemma 4.2,  $D_\rho(0, r_k) \subseteq u_k(\eta_k \mathbb{D})$ . Hence

$$D_\rho(\phi_{n_k}(0), r_k) \subseteq \phi_{n_k}(\eta_k \mathbb{D}).$$

An alternative proof of Corollary 4.3 goes as follows: By Cauchy's integral formula

$$u_k'(0) = \int_0^{2\pi} \frac{u_k(\eta_k e^{i\theta}) e^{-i\theta}}{\eta_k} \frac{d\theta}{2\pi}.$$

It is easy to see that this implies the existence of a  $\theta_0 \in ]0, 2\pi[$  such that  $\frac{|u_k(\eta_k e^{i\theta_0})|}{\eta_k} \geq |u'_k(0)|$ . Let  $z_k := \eta_k e^{i\theta_0}$ . Then  $|u_k(z_k) - u_k(0)| \geq \eta_k |u'_k(0)|$ . Hence

$$\rho(\phi_{n_k}(z_k), \phi_{n_k}(0)) = \rho(u_k(z_k), u_k(0)) = |u_k(z_k) - u_k(0)| \rightarrow 1.$$

□

**4.3. Somewhere dense orbits.** — We now present a (much weaker) analog of a result of Bourdon and Feldman [7] for abstract hypercyclicity phenomena.

**Corollary 4.4.** — *Let  $\phi$  be a holomorphic self-map of  $\mathbb{D}$  and  $\phi_{[n]}$  the  $n$ -th iterate. For  $B_j \in \mathcal{B}$ , let  $E_j = \{B_j \circ \phi_{[n]} : n \in \mathbb{N}\}$ . Suppose that for some  $N \in \mathbb{N} \cup \{\infty\}$  we have  $\bigcup_{j=1}^N \overline{E_j} = \mathcal{B}$ . Then there exists  $j_0$  such that  $B_{j_0}$  is  $\mathcal{B}$ -universal.*

*Proof.* — First we note that our hypothesis implies that  $\phi$  has no fixed point inside  $\mathbb{D}$ , for otherwise each  $f \in \overline{E_j}$  would have the same value  $B_j(p)$  at (the unique) fixed point  $p$  of  $\phi$ . But the set  $\bigcup\{B_j(p) : j = 1, \dots, N\}$  is at most countable, while the set  $\{f(p) : f \in \mathcal{B}\}$  is uncountable. Hence, by the Grand Iteration Theorem (see [23, p. 78]), the iterates  $\phi_{[n]}$  converge locally uniformly to a boundary (fixed) point  $\omega$ . We can assume that  $\omega = 1$ . Since the identity function  $z$  is in the closure of at least one of the sets  $\{B_j \circ \phi_{[n]} : n \in \mathbb{N}\}$  we have that condition (2) of Theorem 3.1 is satisfied. Since  $\phi_{[n]}(0) \rightarrow 1$ , we can apply this theorem to conclude that  $(C_{\phi_{[n]}})$  admits a universal function. On the other hand, since  $\mathcal{B}$  is a Baire set, there exists  $j_0$  such that the set  $\overline{\{B_{j_0} \circ \phi_{[n]} : n \in \mathbb{N}\}}$  contains an open ball  $U$ . Moreover, by Corollary 2.8 the set of  $\mathcal{B}$ -universal functions for  $(\phi_{[n]})$  is dense in  $\mathcal{B}$ , so  $U$  contains a universal function  $f$ . Thus,  $f$  belongs to the closed orbit of  $B_{j_0}$ . In particular, the orbit of  $f$  itself is contained in the closed orbit of  $B_{j_0}$ . Since the former is dense in  $\mathcal{B}$ , the latter must be as well. Thus  $B_{j_0}$  is universal. □

The proof shows that if the orbit of  $B$  under  $C_\phi$  is somewhere dense, then it is everywhere dense and  $B$  is  $\mathcal{B}$ -universal for  $(C_{\phi_{[n]}})$ .

**4.4. Left composition operators.** — In the introduction we mentioned that the identity function is a universal function for  $(C_{p_n})$ , where  $(p_n)$  is a sequence of finite Blaschke products dense in  $\mathcal{B}$ . Are there any other  $\mathcal{B}$ -universal functions of this trivial type? In other words, what kind of functions  $U \in \mathcal{B}$  are there such that the map  $f \mapsto U \circ f$  is a surjection of  $\mathcal{B}$  onto itself? We may ask (an appropriate version of) the same question about  $V \in \mathcal{S}$ ; when is  $f \mapsto V \circ f$  a surjection of  $\mathcal{B}$  onto  $\mathcal{S}$ ? The answer is given by the following proposition, where  $\mathcal{B}^0$  (resp.  $\mathcal{S}^0$ ) denotes the set of functions in  $\mathcal{B}$  (resp.  $\mathcal{S}$ ) such that  $|f(z)| < 1$  for every  $z \in \mathbb{D}$ .

**Proposition 4.5.** — *a) Let  $U \in \mathcal{B}$ . Then the map  $f \mapsto U \circ f$  is a surjection of  $\mathcal{B}^0$  onto itself if and only if  $U$  is an automorphism of  $\mathbb{D}$ .*

*b) Let  $V \in \mathcal{S}$ . Then the map  $f \mapsto V \circ f$  is a surjection of  $\mathcal{B}^0$  onto  $\mathcal{S}^0$  if and only if  $V = S \circ \tau$  for some automorphism  $\tau$  of  $\mathbb{D}$ .*



*Proof.* — a) Suppose that  $f \mapsto U \circ f$  is a surjection of  $\mathcal{B}^0$  onto  $\mathcal{B}^0$ . Then there exists  $f \in \mathcal{B}^0$  such that  $U \circ f = z$ . By [24, p. 847],  $U$  and  $f$  are inner functions. But every finite Blaschke product of prime degree (in this case  $z$ ) is prime, meaning that  $U$  or  $f$  must be an automorphism. (see [24, p. 847] or [13, p. 254]). In either case  $U$  must be an automorphism. The converse is trivial.

b) Suppose that  $f \mapsto V \circ f$  is a surjection of  $\mathcal{B}^0$  onto  $\mathcal{S}^0$ . Then there exists  $f \in \mathcal{B}^0$  such that  $V \circ f = S$ . As above,  $V$  and  $f$  are inner functions. But  $S$  is semiprime (see [13, p. 255]); since  $V \in \mathcal{S}$ , we conclude that  $f$  is an automorphism and hence  $V = S \circ \tau$ . Again, the converse is trivial.  $\square$

**4.5. Some examples.** — Here are some examples of the sort of behavior we can expect. Our first example is a sequence of self-maps that are not automorphisms, but admit universal Blaschke products. The idea, of looking at conformal maps that take a set slightly larger than  $\mathbb{D}$  onto  $\mathbb{D}$ , can be modified to produce many examples of this type.

**Example 4.6.** — Let  $U_n$  be domains of  $\mathbb{C}$  satisfying  $\mathbb{D} \subseteq U_n \subseteq \{z \in \mathbb{C} : |z| < 1 + \varepsilon_n\}$  where  $(\varepsilon_n)$  is a sequence of positive numbers tending to 0. Let  $(\phi_n)$  be a sequence of conformal maps of  $U_n$  onto  $\mathbb{D}$  with  $\phi_n(0) \rightarrow 1$ . Then  $(C_{\phi_n})$  admits a universal Blaschke product.

*Proof.* — To see this, let  $B$  be a finite Blaschke product. For  $n$  sufficiently large,  $B$  will have no poles in  $U_n$  and we can define the function  $u_n$  by

$$u_n = \frac{B \circ \phi_n^{-1}}{\max\{|B(z)| : z \in U_n\}}.$$

Then  $u_n \in \mathcal{B}$  and  $u_n \circ \phi_n \rightarrow B$  on compacta, since  $\max\{|B(z)| : z \in U_n\}$  tends to 1 by the assumption on  $U_n$ . Hence  $\{u \circ \phi_n : u \in \mathcal{B}, n \in \mathbb{N}\}$  is dense in  $\mathcal{B}$ . The existence of the universal Blaschke product now follows from Theorem 2.1.  $\square$

Our next example describes the sequences  $(f_{\mu_n})$  of normalized singular inner functions that admit universal functions.

**Example 4.7.** — Let  $f$  be a normalized singular inner function. Then

$$f(z) = f_{\mu}(z) := \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right),$$

where  $\mu$  is a Borel measure on the unit circle  $\mathbb{T}$  given by  $\mu = -\log|f(e^{it})|\frac{dt}{2\pi} + \mu_s$  for some positive Borel measure  $\mu_s$  singular with respect to the Lebesgue measure on  $\mathbb{T}$ . Let  $\hat{\mu}(j) = \int_0^{2\pi} e^{-ijt} d\mu(t)$  denote the  $j$ -th Fourier coefficient of  $\mu$ . Then if the sequence of functions  $(f_{\mu_n})$  is such that  $f_{\mu_n}(0) \rightarrow 1$ , it admits a  $\mathcal{B}$ -universal function if and only if

$$(4.1) \quad \limsup_{n \rightarrow +\infty} \frac{|\hat{\mu}_n(1)|}{\sinh \hat{\mu}_n(0)} = 1.$$

*Proof.* — Since  $f'_{\mu}(z) = f_{\mu}(z) \int_0^{2\pi} \frac{-2e^{it}}{(e^{it} - z)^2} d\mu(t)$ , we see that  $f_{\mu}(0) = e^{-\hat{\mu}(0)}$  and  $f'_{\mu}(0) = -2f_{\mu}(0)\hat{\mu}(1)$ . Hence

$$\frac{|f'_{\mu_n}(0)|}{1 - |f_{\mu_n}(0)|^2} = \frac{2e^{-\hat{\mu}_n(0)}|\hat{\mu}_n(1)|}{1 - e^{-2\hat{\mu}_n(0)}} = \frac{|\hat{\mu}_n(1)|}{\sinh \hat{\mu}_n(0)}.$$

The result follows from Theorem 2.1.  $\square$

As an example of a sequence of such functions that admits a universal function we mention the sequence of roots of the atomic inner function,  $\phi_n = S^{1/n}$ . Also note that if (4.1) holds with  $\lim$  replaced by  $\limsup$  then, automatically,  $f_{\mu_n}(0) \rightarrow 1$ , since  $|f'_{\mu_n}(0)| \leq 2/e$  by Lemma 2.6.

Now we turn to an example of a sequence that does not admit universal functions.

**Example 4.8.** — Let  $\phi_n(z) = \alpha_n z + (1 - \alpha_n)$ , where  $0 < \alpha_n < 1$  and  $\alpha_n \rightarrow 0$ . Then

$$\frac{\phi'_n(0)}{1 - |\phi_n(0)|^2} \rightarrow \frac{1}{2}.$$

By Theorem 2.1, this family of self-maps does not admit universal functions. This tells us that the geometric condition in Corollary 4.3 is not sufficient for guaranteeing the existence of universal functions.

## 5. Universality on $H(\mathbb{D})$

We shall now derive a necessary and sufficient condition for a sequence of self-maps of  $\mathbb{D}$  to have the associated composition operators  $C_{\phi_n}$  admit universal functions in the Fréchet space  $H(\mathbb{D})$ . The case in which  $\phi_n$  is an automorphism is well known (see, for example, [4, 5, 20]). The situation here is easier than in the case of  $\mathcal{B}$ -universal functions, because we do not need to be as concerned about the norm of the functions under construction. We state separately the following straightforward lemma:

**Lemma 5.1.** — *Let  $(f_n)$  be a sequence of holomorphic functions on a disk  $D(0, R)$ . Suppose that the sequence  $(f_n)$  converges uniformly to an injective function  $f$  on  $D(0, R)$ . Let  $0 < r < R$ . Then  $f_n$  is injective on  $K := \overline{D}(0, r)$  for sufficiently large  $n$ .*

*Proof.* — Let  $z_0$  be a point of  $\overline{D}(0, r)$  and  $r < r' < R$ . For every  $n$  and every  $z \in \partial D(0, r')$ , we have  $|f(z) - f(z_0) - (f_n(z) - f_n(z_0))| \leq 2 \|f - f_n\|_{\overline{D}(0, r')}$  which tends to zero as  $n$  goes to infinity. Since  $f$  is injective on  $\overline{D}(0, r')$ ,  $\delta = \inf_{z \in \partial D(0, r')} |f(z) - f(z_0)|$  is positive, so that for  $n$  large enough,  $|f(z) - f(z_0) - (f_n(z) - f_n(z_0))| < |f(z) - f(z_0)|$  for every  $z \in \partial D(0, r')$ . Hence  $f - f(z_0)$  and  $f_n - f_n(z_0)$  have the same number of zeroes in  $D(0, r')$  by Rouché's Theorem, and  $f_n$  is injective on  $\overline{D}(0, r)$ .  $\square$

**Theorem 5.2.** — *Let  $(\phi_n)$  be a sequence of holomorphic self-maps of  $\mathbb{D}$  such that  $\phi_n(0) \rightarrow 1$ . Then the following assertions are equivalent:*

1. *the sequence of composition operators  $(C_{\phi_n})$  admits an  $H(\mathbb{D})$ -universal function*
2. *there exists a subsequence  $(n_j)$  such that for all compact sets  $K \subseteq \mathbb{D}$  there exists an integer  $j_0$  such that  $\phi_{n_j}$  is injective on  $K$  for all  $j \geq j_0$ .*

*Proof.* — We first prove that (2)  $\Rightarrow$  (1). Choose a sequence of compact sets  $K_n$  (which can be chosen to be disks centered at 0,  $K_l = D(0, 1 - 2^{-l})$  for instance) such that  $\mathbb{D} = \bigcup K_l$  and  $K_l \subseteq K_{l+1}^\circ$ . We note that  $\phi_{n_j} \rightarrow 1$  uniformly on every compact  $K \subseteq \mathbb{D}$ . Let  $(p_l)$  be a dense sequence in  $H(\mathbb{D})$ . We claim that there exist functions  $f_l \in H(\mathbb{D})$  and a subsequence  $k_l$  of  $n_j$  so that

1.  $f_l(1) = 0$
2.  $\|f_l\| < 2^{-l}$  on  $K_l \cup \bigcup_{j=1}^{l-1} \phi_{k_j}(K_j)$
3.  $\|f_l - p_l \circ \phi_{k_l}^{-1}\|_{\phi_{k_l}(K_l)} < 2^{-nl}$

The proof will be done by induction. Fix  $l$ , and then choose  $k_l$  so large that

- i)  $K_l \cap \phi_{k_l}(K_l) = \emptyset$
- ii) for all  $j < l$   $|f_j| < 2^{-j-l}$  on  $\phi_{k_l}(K_l)$
- iii)  $\phi_{k_l}$  is injective on  $K_l$ .

Then  $\phi_{k_l}^{-1}$  is well defined on  $\phi_{k_l}(K_l)$ . By Runge's approximation theorem, there is a polynomial  $q_l$  such that  $|q_l(1)| \leq 4^{-l}$  and (2) and (3) are satisfied with  $q_l$  instead of  $f_l$ . By considering  $f_l = q_l - q_l(1)$  we get assertions (1) to (3).

Next we claim that  $f = \sum_{l=1}^{\infty} f_l$  is a universal function for  $(C_{\phi_n})$ . To check this, we note first that, due to (2), the series converges locally uniformly on  $\mathbb{D}$ . Now fix  $n$ . On  $K_l$ , use (ii), (2), and (3) to conclude that

$$\begin{aligned} |f \circ \phi_{k_l} - p_l| &\leq \sum_{j < l} |f_j \circ \phi_{k_l}| + \sum_{j > l} |f_j \circ \phi_{k_l}| + |f_l \circ \phi_{k_l} - p_l| \\ &\leq \sum_{j < l} 2^{-j-l} + \sum_{j > l} 2^{-j} + 2^{-l} \rightarrow 0 \text{ as } l \rightarrow \infty, \end{aligned}$$

and this proves that  $f$  is a universal function for  $(C_{\phi_n})$ .

To prove that (1)  $\Rightarrow$  (2), let  $f$  be a  $H(\mathbb{D})$ -universal function for  $C_{\phi_n}$ . Hence there is a sequence  $n_j$  for which  $\phi_{n_j}(0) \rightarrow 1$  and  $f \circ \phi_{n_j} \rightarrow z$ . By Lemma 5.1, for every compact set  $K \subseteq \mathbb{D}$  and for  $j$  sufficiently large, the functions  $f \circ \phi_{n_j}$  must be injective on  $K$  for large  $j$ . Consequently, the same must hold for  $\phi_{n_j}$ .  $\square$

Applying Theorem 5.2 to the sequence defined with  $\phi_n = \alpha_n z + 1 - \alpha_n$ ,  $\alpha_n \rightarrow 0$ , which did not admit a  $\mathcal{B}$ -universal function, we see that this sequence does admit  $H(\mathbb{D})$ -universal functions. Now consider a nonautomorphic, but injective holomorphic self-map  $\phi$  of  $\mathbb{D}$  whose Denjoy-Wolff point is located on the unit circle. In contrast to Corollary 4.1, there exists, by Theorem 5.2, an  $H(\mathbb{D})$ -universal function for the sequence of iterates of  $\phi$ .

The following should be compared to Corollary 4.1.

**Corollary 5.3.** — *Let  $\phi$  be a holomorphic self-map of  $\mathbb{D}$ . Then the sequence of iterates  $(C_{\phi_{[n]}})$  of  $C_\phi$  admits a  $H(\mathbb{D})$ -universal function if and only if  $\phi$  is injective with Denjoy-Wolff fixed point located on the boundary.*

*Proof.* — An elliptic automorphism or a self-map with fixed point in  $\mathbb{D}$  cannot yield  $H(\mathbb{D})$ -universal functions. Thus, it follows from iteration theory (see [23]), that  $(\phi_{[n]})$  converges locally uniformly to a boundary fixed point (the Denjoy-Wolff point) whenever  $(C_{\phi_{[n]}})$  admits a  $H(\mathbb{D})$ -universal function. The injectivity follows as in Corollary 4.1. The converse follows from Theorem 5.2.  $\square$

## References

- [1] R. ARON, P. GORKIN, An infinite dimensional vector space of universal functions for  $H^\infty$  on the ball, *Canad. Math. Bull.*, to appear.

- [2] F. BAYART, Universal inner functions on the ball, *Canad. Math. Bull.*, to appear.
- [3] F. BAYART, P. GORKIN, How to get universal inner functions, *Math Ann.*, to appear.
- [4] L. BERNAL-GONZALEZ, A. MONTES-RODRIGUEZ, Universal functions for composition operators, *Complex Var.*, **27** (1995), 47 – 56.
- [5] L. BERNAL-GONZALEZ, A. MONTES-RODRIGUEZ, Non-finite dimensional closed vector spaces of universal functions for composition operators, *J. Approx. Theory*, **82** (1995), 375 – 391.
- [6] L. BERNAL-GONZALEZ, K.-G. GROSSE-ERDMANN, The hypercyclicity criterion for sequences of operators, *Studia Math.* **157** (2003), 17 – 32.
- [7] P. S. BOURDON, N. S. FELDMAN, Somewhere dense orbits are everywhere dense, *Indiana Univ. Math. J.*, **52** (2003), 811 – 819.
- [8] P. S. CHEE, Universal functions in several complex variables, *J. Austral. Math. Soc. Ser., A*, **28** (1979), no. 2, 189 – 196.
- [9] J. CIMA, A. MATHESON, Weakly compact composition operators on VMO, *Rocky Mountain J. Math.*, **32** (2002), 937 – 951.
- [10] T. W. GAMELIN, Uniform Algebras, Chelsea Publishing Company, New York, N. Y., 1984.
- [11] J. B. GARNETT, Bounded analytic functions, Academic Press, Inc., New York-London, 1981.
- [12] P. M. GAUTHIER, J. XIAO, The existence of universal inner functions on the unit ball of  $\mathbb{C}^n$ , *Canad. Math. Bull.*, **48** (2005), 409 – 413.
- [13] P. GORKIN, L. LAROCO, R. MORTINI, R. RUPP, Composition of inner functions, *Results in Math.* **25** (1994), 252 – 269.
- [14] P. GORKIN, R. MORTINI, Radial limits of interpolating Blaschke products, *Math. Ann.*, **331** (2005), 417 – 444.
- [15] P. GORKIN, R. MORTINI, Asymptotic interpolating sequences in uniform algebras, *J. London Math. Soc.*, **67** (2003) , 481 – 498.
- [16] P. GORKIN, R. MORTINI, Universal Blaschke products, *Math. Proc. Cambridge Philos. Soc.*, **136** (2004), 175 – 184.
- [17] P. GORKIN, R. MORTINI, Universal singular inner functions, *Canadian Math. Bull.*, **47** (2004), 17 – 21.
- [18] P. GORKIN, F. LEON-SAAVEDRA, R. MORTINI, Bounded universal functions in one and several complex variables, preprint.
- [19] M. HEINS, A universal Blaschke product, *Arch. Math.*, **6** (1954), 41 – 44.
- [20] A. MONTES-RODRIGUEZ, A note on Birkhoff open sets, *Complex Var.*, **30** (1996), 193 – 198.
- [21] R. MORTINI, Infinite dimensional universal subspaces generated by Blaschke products, *Proc. Amer. Math. Soc.*, to appear.
- [22] M.S. ROBERTSON, Solution to exercise 4468, proposed by H.S. Shapiro, *American Math. Monthly*, **60** (1953), 131.
- [23] J. H. SHAPIRO, Composition operators and classical function theory, Springer-Verlag, 1993, New-York, Berlin, Heidelberg.
- [24] K. STEPHENSON, Analytic functions and hypergroups of function pairs, *Indiana Univ. Math. J.*, **31** (1982), 843 – 884.

FRÉDÉRIC BAYART, Université Bordeaux 1, 351 Cours de la Libération, 33400 Bordeaux, France  
*E-mail* : [bayart@math.u-bordeaux1.fr](mailto:bayart@math.u-bordeaux1.fr)

PAMELA GORKIN, Department of Mathematics, Bucknell University PA 17837 Lewisburg, USA  
*E-mail* : [pgorkin@bucknell.edu](mailto:pgorkin@bucknell.edu)

SOPHIE GRIVAUX, Laboratoire Paul Painlevé, UMR 8524, Université des Sciences et Technologies de Lille,  
Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France • *E-mail* : [grivaux@math.univ-lille1.fr](mailto:grivaux@math.univ-lille1.fr)

RAYMOND MORTINI, Département de Mathématiques, Université Paul Verlaine, Ile du Saulcy, 57045 Metz,  
France • *E-mail* : [mortini@math.univ-metz.fr](mailto:mortini@math.univ-metz.fr)