



HAL
open science

An analytical solution of Shallow Water system coupled to Exner equation

Christophe Berthon, Stéphane Cordier, Minh H. Le, Olivier Delestre

► **To cite this version:**

Christophe Berthon, Stéphane Cordier, Minh H. Le, Olivier Delestre. An analytical solution of Shallow Water system coupled to Exner equation. *Comptes Rendus. Mathématique*, 2012, 350 (3-4), pp.183-186. 10.1016/j.crma.2012.01.007 . hal-00648343

HAL Id: hal-00648343

<https://hal.science/hal-00648343v1>

Submitted on 7 Dec 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An analytical solution of Shallow Water system coupled to Exner equation

C. Berthon^a, S. Cordier^b, O. Delestre^c and M.H. Le^d

^aLaboratoire de Mathématiques Jean Leray, Université de Nantes – 2 rue de la Houssinière – 44322 Nantes

^bMAPMO UMR CNRS 6628, Université d'Orléans, UFR Sciences, Bâtiment de mathématiques – 45067 Orléans

^cLaboratoire de Mathématiques J.A. Dieudonné & Ecole Polytech Nice - Sophia, Université de Nice - Sophia Antipolis, Parc Valrose – 06108 Nice

^dBRGM – 3 avenue Claude Guillemin - B.P. 36009 – 45060 Orléans Orléans

Received *****; accepted after revision +++++

Presented by

Abstract

In this paper, an exact smooth solution for the equations modeling the bedload transport of sediment in Shallow Water is presented. This solution is valid for a large family of sedimentation laws which are widely used in erosion modeling such as the Grass model or those of Meyer-Peter & Müller. One of the main interest of this solution is the derivation of numerical benchmarks to valid the approximation methods. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Une solution analytique du système de Saint-Venant couplé à l'équation d'Exner. Ce papier présente une solution analytique pour le système modélisant le transport de sédiments par le charriage. Cette solution est valable pour une grande famille de lois sédimentaires comme le modèle de Grass ainsi que celui de Meyer-Peter & Müller. Ce résultat est utile pour la validation des schémas numériques. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1. Introduction

Soil erosion is a consequence of the movements of sediments due to mechanical actions of flows. In the context of bedload transport, a mass conservation law, also called Exner equation [1], is used to update

Email addresses: christophe.berthon@math.univ-nantes.fr (C. Berthon), stephane.cordier@math.cnrs.fr (S. Cordier), Delestre@unice.fr (O. Delestre), mh.le@brgm.fr (M.H. Le).

the bed elevation. This equation is coupled with the shallow water equations describing the overland flows (see [2] and references therein) as follows:

$$\partial_t h + \partial_x(hu) = 0, \quad (1)$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x z_b = 0, \quad (2)$$

$$\partial_t z_b + \partial_x q_b = 0, \quad (3)$$

where h is the water depth, u the flow velocity, z_b the thickness of sediment layer which can be modified by the fluid and g the acceleration due to gravity. The variable hu is also called water discharge and noted by q . Finally, q_b is the volumetric bedload sediment transport rate. Its expressions are usually proposed for granular non-cohesive sediments and quantified empirically [3,4,5].

Many numerical schemes have been developed to solve system (1-3) (see [5] and references therein). The validation of such schemes by an analytical solution is a simple way to ensure their working. Nevertheless, analytical solutions are not proposed in the literature. Up to our knowledge, asymptotic solutions, derived by Hudson in [6], are in general adopted to perform some comparisons with approximated solutions. The solutions are derived for Grass model [3], *i.e.* $q_b = A_g u^3$, when the interaction constant A_g is smaller than 10^{-2} . In this paper, we propose a non obvious analytical solution in the steady state condition of flow.

2. Solution of the equations

In order to obtain an analytic solution, we consider q_b as a function of the dimensionless bottom shear stress τ_b^* (see [5]). By using the friction law of Darcy-Weisbach, τ_b^* is given by

$$\tau_b^* = \frac{f u^2}{8(s-1)gd_s},$$

where f is the friction coefficient, $s = \rho_s/\rho$ the relative density of sediment in water and d_s the diameter of sediment. The formulæ of q_b is usually expressed under the form

$$q_b = \kappa(\tau_b^* - \tau_{cr}^*)^p \sqrt{(s-1)gd_s^3}, \quad (4)$$

where τ_{cr}^* is the threshold for erosion, κ an empirical coefficient and p an exponent which is often fixed to $3/2$ in many applications. The expression (4) can be written in the simple form

$$q_b = Au_e^{2p}, \quad (5)$$

where the effective velocity u_e and the interaction coefficient A are defined by

$$\begin{cases} u_e^2 = u^2 - u_{cr}^2, \\ u_{cr}^2 = \tau_{cr}^* \left[\frac{f}{8(s-1)gd} \right]^{-1}, \\ A = \kappa \left[\frac{f}{8(s-1)gd} \right]^p \sqrt{(s-1)gd_s^3}. \end{cases} \quad (6)$$

Remark. The Grass model [3] is one of the simplest case by using $p = 3/2$, $\tau_{cr}^* = 0$ and an empirical coefficient A_g instead of A . The Meyer-Peter & Müller model [4] is one of the most applied by using $p = 3/2$, $\kappa = 8$, $\tau_{cr}^* = 0.047$. The following result is valid for all models rewriting in form (5-6).

Proposition 2.1 *Assume that q_b is defined by (4). For a given uniform discharge q such that $\tau_b^* > \tau_{cr}^*$, system (1-3) has the following analytical unsteady solution*

$$\begin{cases} u_e^2 = \left[\frac{\alpha x + \beta}{A} \right]^{1/p}, \\ u = \sqrt{u_e^2 + u_{cr}^2}, \quad h = q/u, \\ z_b^0 = -\frac{u^3 + 2gq}{2gu} + C, \\ z_b = -\alpha t + z_b^0. \end{cases} \quad (7)$$

where α , β , C are constants and A , u_{cr} are defined by (6).

Proof. We are here concerned by the smooth solution. In view of the assumption $hu = q = \text{cst}$, equations (1-3) reduce to

$$\begin{aligned} \partial_t h &= 0, \\ \partial_x(q^2/h) + gh\partial_x H &= 0, \end{aligned} \quad (8)$$

$$\partial_t H + \partial_x q_b = 0, \quad (9)$$

where $H = h + z$ is the free surface elevation. Differentiating equation (8) with respect to t and then equation (9) with respect to x , we obtain

$$\begin{aligned} \partial_{xt} H &= 0, \\ \partial_x^2 q_b &= 0. \end{aligned} \quad (10)$$

Note that we can write $q_b = q_b(h, q)$ to have $\partial_t q_b = \partial_h q_b \partial_t h + \partial_q q_b \partial_t q = 0$, so q_b is not time-depending. Thank to (10), the expression of q_b is obtained under the form

$$q_b = \alpha x + \beta, \quad (11)$$

where α and β are constant. From (3), we obtain $\partial_t z_b = -\partial_x q_b = -\alpha$ to write

$$z_b = -\alpha t + z_b^0(x). \quad (12)$$

Moreover, from (5) we deduce the effective velocity as follows:

$$u_e^2 = \left[\frac{\alpha x + \beta}{A} \right]^{1/p}.$$

Plugging (12) into the momentum equation (8) and using a direct calculation, we have

$$\partial_x z_b^0 = \left[\frac{q}{u^2} - \frac{u}{g} \right] \partial_x u \Rightarrow z_b^0 = -\frac{u^3 + 2gq}{2gu} + C$$

which concludes the proof.

Remark. As h and u are stationary, the initial condition of (7) is (h, u, z_b^0) . Moreover, the solution (h, u) applied to the Grass model is also an analytical solution of the Shallow Water Equations with the variable topography z_b^0 . Concerning the shallow-water model, other solutions can be found in [7].

3. Numerical experiments

In this section, we consider the analytical solution (7) applied to the Grass model with $q = 1$, $A_g = \alpha = \beta = 0.005$ and $C = 1$. A relaxation solver is applied to approximate the solution of the model. We will not give here the details of the relaxation solver (for details see [8]), but just the relaxation model for the equations (1-3). Thus, we solve the following relaxation system:

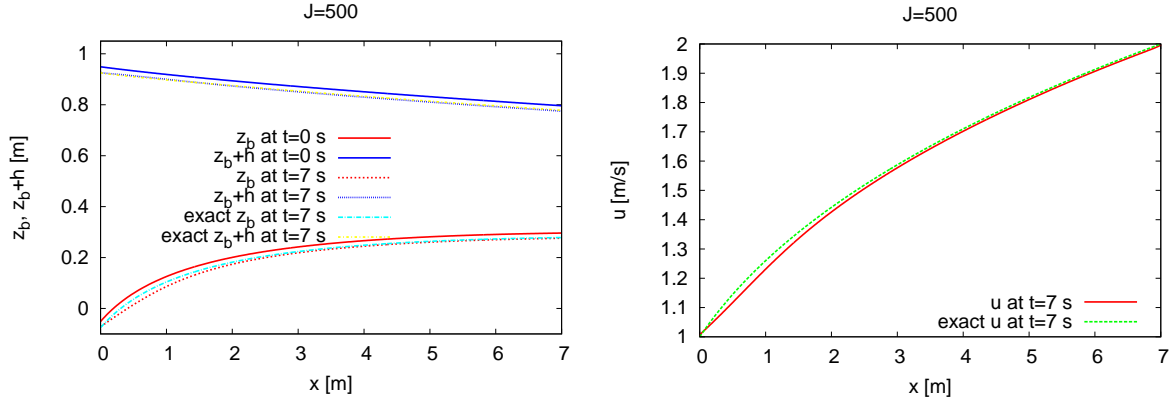


Figure 1. Comparison between the exact solution and the relaxation method for : the water height and the topography (left) and the velocity (right).

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + \pi) + gh\partial_x z_b = 0, \\ \partial_t \pi + u\partial_x \pi + \frac{a^2}{h}\partial_x u = 0, \\ \partial_t z_b + \partial_x q_r = 0, \\ \partial_t q_r + \left(\frac{b^2}{h^2} - u^2\right)\partial_x z_b + 2u\partial_x q_r = 0, \end{cases}$$

that is completed with $\pi = gh^2/2$ and $q_r = q_b$ at the equilibrium. Figure 1 presents the numerical result with $J = 500$ space cells, a CFL fix condition of 1 and $T = 7s$. We only notice little difference on the velocity, near the inflow boundary.

References

- [1] F. Exner. Über die wechselwirkung zwischen wasser und geschiebe in flüssen, Sitzungsber., Akad. Wissenschaften pt. IIa; 1925. Bd. 134.
- [2] M.J. Castro Díaz, E.D. Fernández-Nieto, and A.M.Ferreiro. Sediment transport models in shallow water equations and numerical approach by high order finite volume methods. *Computers & Fluids*, 37(3):299–316, March 2008.
- [3] A.J. Grass. Sediment transport by waves and currents. *SERC London Cent. Mar. Technol*, Report No. FL29, 1981.
- [4] E. Meyer-Peter and R. Müller. Formulas for bed-load transport. In *2nd meeting IAHSR, Stockholm, Sweden*, pages 1–26, 1948.
- [5] S. Cordier, M.H. Le, and T. Morales de Luna. Bedload transport in shallow water models: Why splitting (may) fail, how hyperbolicity (can) help. *Advances in Water Resources*, 34(8):980 – 989, 2011.
- [6] J. Hudson. *Numerical technics for morphodynamic modelling*. PhD thesis, University of Whiteknights, 2001.
- [7] O. Delestre, C. Lucas, P.-A. Ksinant, F. Darboux, C. Laguerre, T.N.T. Vo, F. James and S. Cordier. SWASHES: a library of Shallow Water Analytic Solutions for Hydraulic and Environmental Studies (*Submitted*), <http://arxiv.org/abs/1110.0288>
- [8] E. Audusse, C. Chalons, O. Delestre, N. Goutal, M. Jodeau, J. Sainte-Marie and B. Spinewine. Sediment transport modelling : Three layer models and relaxation schemes. *In preparation : CEMRACS 2011*.