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Abstract

This paper deals with the global asymptotic stabilization problem for a class of bilinear systems. A state feedback controller solving this problem is obtained uniting a local controller, having an interesting behavior in a neighborhood of the origin, and a constant controller valid outside this neighborhood. The approach developed is based on the use of a hybrid loop, and more precisely a hybrid state feedback. This result is extended to the case where the state of the plant is not fully available and only the measured output can be used for control purposes. In this case, a dynamical controller constituted by an observer and a state feedback is built by means of hybrid output feedback framework. In both cases, the conditions are expressed by a set of linear matrix inequalities.

Keywords. Bilinear systems, global stabilization, hybrid state and output feedback.

I. Introduction

The global asymptotic stabilization of an equilibrium point by means of state or output feedback for nonlinear systems has been the topic of many studies in the control community. Over the last three decades, different tools have been developed to achieve state feedback stabilization for some specific class of nonlinear systems (see [18] and references therein). By adding some observability assumptions, in some particular case, it is possible to extend these approaches to the output feedback context [2]. Note moreover that some attempts have been made to construct some hybrid output feedback controller in [29] and in [27]. However no general theory have been developed allowing the design of a controller without specifying a class of nonlinear systems. In this paper, we focus on global asymptotic stabilization of an equilibrium point by means of state or output feedback for bilinear control systems. Bilinear systems are a special class of

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nonlinear systems, which may represent a wide variety of physical phenomena. Indeed, bilinear models are used to represent electrical systems, chemical process, biological model... (see for example [21], [1], [16] and [24] and the references therein). Moreover, a nonlinear system may be approximated by a bilinear model (see [22]).

The stabilization of bilinear systems by means of state feedback has been addressed in [13] (see also [33]) based on some Lyapunov-like Assumptions. This result has been extended in the output feedback context by restricting the class of bilinear systems in [15]. Moreover, it is important to point out that in [13], the practical stabilization problem is considered. Hence, the origin of the closed-loop system is not globally asymptotically stable but a neighborhood containing the origin is made globally asymptotically stable. Such a neighborhood can be made arbitrarily small (but different from the origin) by changing the controller.

In the current paper, we consider the global asymptotic stabilization problem for a class of bilinear systems for which there exists a constant feedback (see Assumption 1) making the trajectories of the closed-loop system bounded and converging to an equilibrium point (which is not the origin). From the knowledge of this constant feedback, the problem under investigation is to modify this controller in order to make the origin a globally asymptotically stable equilibrium. More precisely, the idea of the design is to rely on two different controllers: A global one (the constant feedback) and a linear one (synthesized via an LMI based approach inspired from [31]). With these two controllers in hand, the problem becomes an uniting controller problem as introduced in [32] and in [26] (see also [3]). Employing hybrid state feedback framework, it is possible to give sufficient conditions allowing us to design such a suitable uniting controller. Due to the fact that the constant feedback does not depend on the state of the system, this one can be also used in the output feedback context. Hence, the case where the state of the plant is not fully available for feedback is tackled. In this context, the hybrid state feedback framework is employed with a hybrid observer in order to obtain a hybrid output feedback which stabilizes globally asymptotically the origin of the hybrid closed-loop system. The approach developed in the paper can be viewed as an alternative technique to those published in the literature as, for example, in [13], [9], [15], [17], [7].

The paper is organized as follows. In Section II the class of systems considered in this paper and the stabilization problem we intend to solve are defined. Some discussion on the existence of the constant feedback are also given. Based on a switching strategy, the design of a hybrid state feedback making the origin a globally asymptotically stable equilibrium is also presented. The output feedback stabilization is considered in Section III. A numerical example is also presented to illustrate the effectiveness of the technique. Finally, in Section IV, concluding remarks are given.

II. Problem statement

A. Class of systems

The class of bilinear systems under interest in this paper is described by the following ordinary differential equation:

$$\dot{x} = Ax + Bu + \sum_{j=1}^{p} u_j N_j x \ , \ y = Cx \ , \tag{1}$$

where the state x is in $\mathbb{R}^n$, the control input u is in $\mathbb{R}^p$, the measured output y is in $\mathbb{R}^m$ and A, B, C, N_j, $j = 1, \ldots, p$, are matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$. u_j, $j = 1, ..., p$, are the components of the vector u.

The first problem under consideration in this note can be summarized as follows.

Problem 1: Design a state feedback control law $u(x)$ ensuring global stabilization of the origin for the system (1).

Due to the structure of system (1) under strong assumptions between the matrices N_j and A, a controller can be given which ensures global boundedness of the closed-loop trajectories. Actually, we restrict our analysis to the particular case in which there exists $u_\infty = \begin{bmatrix} u_{\infty,1} & \ldots & u_{\infty,p} \end{bmatrix}'$ in $\mathbb{R}^p$ such that the matrix $A + \sum_{j=1}^{p} u_{\infty,j} N_j$ is Hurwitz[1]. In other words, we make the following Assumption.

Assumption 1: There exists a symmetric positive definite matrix P_∞ in $\mathbb{R}^{n\times n}$ and a vector $u_\infty = \begin{bmatrix} u_{\infty,1} & \ldots & u_{\infty,p} \end{bmatrix}'$ in $\mathbb{R}^p$ such that the following inequality is satisfied:

$$P_\infty \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j \right] + \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j \right]' P_\infty < 0 \ . \tag{2}$$

Note that with Assumption 1, the constant control law $u = u_\infty$ does not ensure convergence to the origin of trajectories of the system. Actually, it can be checked that with this control law, the origin is not anymore an equilibrium. However, it can be shown that the trajectories converge toward a new equilibrium point given as[2]:

$$Bu_\infty + \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j \right] x_e = 0 \ .$$

[1]See, for example, [20] to check whether or not Assumption 1 is satisfied. .

[2]It has to be noticed that using this constant control law for stabilization may have some drawbacks especially when the model is uncertain due to lack of robustness properties and control on the stability margin.

The matrix $A + \sum_{j=1}^{p} u_{\infty,j} N_j$ being Hurwitz it is invertible. Hence, we get that the following equilibrium point is a global asymptotic equilibrium point:

$$x_e = -\left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right]^{-1} B u_\infty \ . \tag{3}$$

Note that if we consider the function $x' P_\infty x$, we get along the trajectories of the system (1) with the control law $u = u_\infty$:

$$\dot{\overline{x' P_\infty x}} = x' Q x + 2 x' P B u_\infty \ ,$$

with $Q = P_\infty \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right] + \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right]' P_\infty < 0$.

Hence, we get,

$$\dot{\overline{x' P_\infty x}} < 0 \ , \ \forall x \ \in \ \{x, x' P x > r_\infty\}$$

with

$$r_\infty = 2 \frac{|P B u_\infty|}{\lambda_{\min}\{Q\}}$$

Hence the function $x' P_\infty x$ is a global set-CLF in the sens of [3].

Hence with Assumption 1, designing a controller which ensures boundedness of the trajectory is easy. However, in this note, we investigate how we can modify this controller to ensure that the origin becomes a globally and asymptotically stable equilibrium.

To asymptotically stabilize by means of output feedback the origin of the system we consider an observer controller switching strategy. As we will see in Section II-C and with Theorem 1, we can provide sufficient conditions under which a hybrid state feedback can be designed. In Section III, we combine this state feedback with an observer to obtain a stabilizing output feedback. Before giving these results we first discuss the class of systems considered and more precisely that one satisfying Assumption 1.

B. About Assumption 1

Assumption 1 is a strong assumption. However, in the literature some examples can be found which satisfy this requirement. Indeed, consider the following example which was given in [34].

Example 1: As in [34], consider System (1) with the matrices A, B and N defined as:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ N = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \ . \tag{4}$$

We are looking for u_∞ such that the eigenvalues of the matrix

$$A + u_\infty N = \begin{bmatrix} 0 & 1 \\ -2 & 1 - 0.5 u_\infty \end{bmatrix} \ ,$$

has all its eigenvalues with negative real part. This is indeed the case provided that $u_\infty > 2$.

More generally, for small dimensional systems (i.e. $n = 1, 2, 3$) employing the Routh-Hurwitz sufficient conditions, given the matrices A and the N_j's it may be possible to give explicitly the set of control u_∞ in $\mathbb{R}^p$ guaranteeing the existence of the matrix P_∞ such that Assumption 1 holds (see also [9] for a complete study in the planar case). Employing this criterion, it is indeed possible to show that for the system given in Example 1 of [31] Assumption 1 is not satisfied.

However for systems of larger dimension it may be difficult to check whether or not Assumption 1 is satisfied. It has for instance been studied in [20] in the particular case in which no matrix B is given and when $p = 1$. In [20] the approach is based on the use of change of coordinates in which the matrices A and N are triangular. In that case, the solvability of Assumption 1 can simply be addressed ensuring the solvability of some linear inequalities on the diagonal terms.

Note that inspired by these approaches, when there are several matrices N some sufficient conditions can be given.

Based on a discrete time Euler approximation of the system, an algorithm is given in [14] to obtain a constant controller. Note however, that this one is based on non convex optimization algorithm and may then fail.

Employing the $\mathcal{S}$-Procedure, a simple sufficient condition can be given to guarantee the existence of this control u_∞, as stated below.

Proposition 1 (Sufficient condition for Assumption 1): There exist $v = \begin{bmatrix} v_1 & \dots & v_p \end{bmatrix}'$ in $\mathbb{R}^p$ and a symmetric positive definite matrices P_∞ in $\mathbb{R}^{n\times n}$ such that

1) we have $\sum_{j=1}^{p} v_j \left(N_j' P_\infty + P_\infty N_j\right) \geq 0$;
2) for all $x \neq 0$ such that $x' \left(\sum_{j=1}^{p} v_j \left(N_j' P_\infty + P_\infty N_j\right)'\right) x = 0$ we have $x' \left(A' P_\infty + P_\infty A\right) x < 0$,

then there exists $k > 0$ such that with the control law $u_\infty = -kv$, Assumption 1 is satisfied.

Proof: This is a direct consequence of the $\mathcal{S}$-Procedure. Employing the $\mathcal{S}$-Procedure, we get the existence of a positive real number k sufficiently large such that we have

$$P_\infty A + A' P_\infty - k \left[\left(\sum_{j=1}^{p} v_j N_j \right)' P_\infty + P_\infty \sum_{j=1}^{p} v_j N_j \right] < 0 \ .$$

■

Note that in the case where there is only one input (i.e. $p = 1$), item 1 of Proposition 1 means that all the real parts of the eigenvalues of the matrix N have all the same sign. Moreover, when $p = 1$, item 2 is exactly the sufficient condition we would obtain following [13] in order to stabilize the origin of the system without B matrix. However, note that in [13], item 1 is not required and the controller obtained is not a constant controller.

A particular case of Proposition 1 is when there exists v in $\mathbb{R}^p$ such that item 1 holds with a strict inequality. In other words, when there exists a positive definite matrix P_∞ in $\mathbb{R}^{n\times n}$ such that

$$\sum_{j=1}^{p} v_j \left(N_j' P_\infty + P_\infty N_j\right) > 0 \ .$$

In that case, item 2 is trivially satisfied and the conclusion of Proposition 1 holds.

Example 2: Consider Example 2 in [31] in which a system in the form (1) is defined as:

$$A = \begin{bmatrix} 1/6 & 1 \\ 0 & 1/6 \end{bmatrix} \ , \ B = \begin{bmatrix} -4 & 2 \\ -2 & -2 \end{bmatrix} \ , \ N_1 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \ , \ N_2 = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \ . \tag{5}$$

Note that for this system, the matrix N_1 is positive definite. Consequently items 1 and 2 of Proposition 1 hold with $v = \begin{bmatrix} -1 & 0 \end{bmatrix}'$ and $P_\infty = I_2$, and there exists u_∞ in $\mathbb{R}^2$ such that for the system in closed-loop with $u = u_\infty$, the vector x_e defined in (3) is a globally and asymptotically stable equilibrium.

Example 3: Consider Example 3.5 in [13] in which a system in the form (1) is defined as:

$$A = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix} \ , \ B = \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix} \ , \ N_1 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \ , \ N_2 = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \ . \tag{6}$$

Note that for this system, the matrix N_1 is positive definite. Consequently items 1 and 2 of Proposition 1 hold with $v = \begin{bmatrix} -1 & 0 \end{bmatrix}'$ and there exists u_∞ in $\mathbb{R}^2$ such that the closed-loop system is bounded.

Example 4: Consider Example 4.2 in [13] in which a system in the form (1) is defined with:

$$N_1 = \begin{bmatrix} \frac{1}{V_1} & \frac{-1}{V_1} \\ \frac{-1}{V_2} & \frac{1}{V_2} \end{bmatrix} \ , \tag{7}$$

Note that this matrix has all its eigenvalues with strictly positive real part. Hence, items 1 and 2 of Proposition 1 hold with $v = \begin{bmatrix} -1 & 0 \end{bmatrix}'$ and there exists u_∞ in $\mathbb{R}^2$ such that the closed-loop system is bounded. In that case, P_∞ can be simply defined as a solution of the Lyapunov inequality (which is known to exist)

$$N_1' P_\infty + P_\infty N_1 > 0 \ .$$

C. A sufficient condition for state feedback stabilization

With Assumption 1, and the existence of u_∞, the problem is now to design a controller u depending on the state x ensuring global asymptotic stabilization of the origin. This can be performed provided that we are able to design a state feedback ensuring asymptotic stabilization of the origin and with a sufficiently large basin of attraction. If x_e, the attractor of the constant

controller, is included inside the basin of attraction of another controller which asymptotically stabilizes the origin, a switching strategy should solve the problem. Based on the tools given in [12], this switching control can be formulated in terms of hybrid systems and provides a (natural) robustness with respect to small enough measurement noise (see [12, Theorem 15 p.58] or [26]).

From this framework, by considering the new state (x, q) in $\mathbb{R}^n \times \{0, 1\}$, the closed-loop system under study is a hybrid system, that is a system with both continuous and discrete dynamics defined as.

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + B\varphi_q(x) + \sum_{j=1}^{p} (\varphi_q(x))_j N_j x \\ \dot{q} & = & 0 \end{array}\right\} \text{ when } x \in \mathcal{C}_q := \overline{\mathbb{R}^n \setminus \mathcal{D}_q} \tag{8}$$

$$\left.\begin{array}{rcl} x^+ & = & x \\ q^+ & = & 1 - q \end{array}\right\} \text{ when } x \in \mathcal{D}_q \tag{9}$$

where $\varphi_1(x) = u_\infty$, $\varphi_0(x) = F_0 x$, F_0 is a matrix to be designed and $\mathcal{D}_0$ and $\mathcal{D}_1$ are two closed subsets of $\mathbb{R}^n$. Equation (8) defines the continuous dynamics part of the closed-loop system and (9) the discrete dynamics one. In this paper, we consider the notion of solutions of hybrid dynamical system defined on their *hybrid time domain* as described in [12]. Hence, in our framework, the hybrid time domain $S \subset \mathbb{R} \times \mathbb{N}$, is the union of finitely or infinitely many time intervals $[t_j, t_{j+1}] \times \{j\}$, where the sequence $\{t_j\}_{j \geq 0}$ is nondecreasing, with the last interval, if it exists, possibly of the form $[t, T)$ with T finite or $T = \infty$. Given an initial condition (x, q) in $\mathbb{R}^n \times \{0, 1\}$ a solution of system (8) and (9) consists of,

- A hybrid time domain $S \neq \emptyset$;
- A function $x : S \to \mathbb{R}^n$, where $t \mapsto x(t, j)$ is absolutely continuous, for a fixed j, and constant in j for a fixed t over $(t, j) \in S$;
- A function $q : S \to \{0, 1\}$ such that $q(t, j)$ is constant in t, for a fixed j over $(t, j) \in S$;

meeting the conditions

1) $x(0, 0) \in \mathcal{C}_{q(0,0)} \cup \mathcal{D}_{q(0,0)}$;
2) $\forall j \in \mathbb{N}$ and $\forall t$ such that $(t, j) \in S$,

$$\dot{q}(t, j) = 0 \ , \ \dot{x}(t, j) = Ax + B\varphi_q(x) + \sum_{j=1}^{p} (\varphi_q(x))_j N_j x \ , \ x(t, j) \in C_{q(t,j)};$$

3) $\forall (t, j) \in S$ such that $(t, j + 1) \in S$

$$x(t, j + 1) = x(t, j) \ , \ q(t, j + 1) = g_{q(t,j)}(x, j)) \ , \ x(t, j) \in D_{q(t,j)} \ .$$

Note that since the sets $\mathcal{C}_q$ and $\mathcal{D}_q$ for all q in $\{0, 1\}$ are closed and with the local Lipschitz property of φ_q, it yields the existence of non trivial solutions of system (8) and (9) for all initial points (x, q) in $\mathcal{C}_q \cup \mathcal{D}_q \times \{0, 1\}$ (see [12, Prop S2, p.44]). Moreover, the fact that $C_q \cup D_q = \mathbb{R}^n$

for all q in $\{0,1\}$ guarantees that for every initial condition (x,q) in $\mathbb{R}^n \times \{0,1\}$ all maximal solutions of (8) and (9) are either complete or blow up (see [12, Theo S3, p.44]).

Problem 1 under consideration can thus be reformulated as follows.

Problem 2: Find a state feedback described through $\{\{0,1\}, \mathcal{C}_q, \mathcal{D}_q, \varphi_q\}$ such that the set $\{0\} \times \{0,1\} \subset \mathbb{R}^n \times \{0,1\}$ is globally and asymptotically stable[3] for system (8)-(9).

Note that when continuous controllers are considered, this stabilization problem has been addressed for instance in [13]. It has also been addressed in [6], [20], [9] when there are no B matrix. Recently, it has been studied in [31] or in [1] when the local stability is studied. See also the recent works [23], [16] in a power-electronics converters context.

In Section II, it was shown that with Assumption 1, a global controller can be designed to guarantee that the trajectories of the system converge to a point which is not the origin of the system (1). To solve our problem of designing a global asymptotically stable equilibrium we have now to find a way to move this global attractor toward the origin. In order to develop our switching strategy, we consider the problem of designing a local controller ensuring local asymptotic stabilization of the origin and such that x_e is included in the basin of attraction of the origin (associated to the local controller). For instance, the first order approximation of the system (1) is simply given as:

$$\dot{x} = Ax + Bu \ . \tag{10}$$

Hence, we get that provided the couple (A,B) is controllable a matrix F_0 in $\mathbb{R}^{n\times p}$ can be found such that $A + BF_0$ is Hurwitz and ensures that the state feedback $u = F_0 x$ makes the origin of the system (1) a locally asymptotically stable equilibrium.

In other words, there exists a positive definite matrix in $\mathbb{R}^{n\times n}$ denoted P_0 such that:

$$P_0(A + BF_0) + (A + BF_0)' P_0 < 0 \ . \tag{11}$$

Consequently, with this state feedback the origin of the system (1) is a local asymptotic stable equilibrium. In other words, there exists a positive real number R_0 such that along the trajectories of the system (1) with $u = F_0 x$, we have:

$$\dot{\overbrace{x' P_0 x}} < 0 \ , \ \forall \ x \ \in \ \{x \neq 0, x' P_0 x \leq R_0\} \ .$$

Note that in this approach the bilinear terms have not been taken into account. Moreover, in the design nothing is said about x_e and there is no reason to think that x_e is inside the basin of attraction of the local controller. Consequently, in this case, no switching strategy between the two controllers $u = u_\infty$ and $u = F_0 x$ may be considered. However, as shown in [1], [31] it is possible to extend slightly this procedure employing some convex upper approximations of the bilinear terms and by forcing x_e to be inside the basin of attraction of the local controller.

[3]Since $\{0,1\}$ is bounded, the set $\{0\} \times \{0,1\}$ is compact which implies that we rely on the definition of stability as in [11].

Before introducing our approach, let us define the following notation. Given a matrix $\Lambda = (\lambda_{j,i})_{j\in[1,p],i\in[1,n]}$ with $\lambda_{j,i} \geq 0$ in $\mathbb{R}^{p\times n}$, we define the set $\mathcal{N}_\Lambda = \{S_\ell\}_{1\leq\ell\leq 2^{np}}$ of (no more then) 2^{np} matrices in $\mathbb{R}^{p\times n}$ such that for all $1 \leq \ell \leq 2^{np}$, we have[4]: $(S_\ell)_{j,i} = \lambda_{j,i}$ or $(S_\ell)_{j,i} = -\lambda_{j,i}$. Moreover, we rewrite the matrices N_j, j in $\{1,\ldots,p\}$, of system (1), as $N_j = [N_{j,1},\ldots,N_{j,n}]'$. With these definitions and notation in hand, we can now give the following result [5], which is inspired from the result obtained in [31] to solve the state feedback stabilization.

Theorem 1 (State feedback stabilization): Assume Assumption 1 holds. Let $\Lambda = (\lambda_{j,i})$ in $\mathbb{R}^{p\times n}$ be given. If there exist a symmetric positive definite matrix W_0 in $\mathbb{R}^{n\times n}$, and a matrix H_0 in $\mathbb{R}^{p\times n}$ such that the following inequalities hold,

$$\begin{bmatrix} \lambda_{j,i}^2 W_0 & W_0 N_{j,i}' \\ N_{j,i} W_0 & 1 \end{bmatrix} > 0 \ , \ \forall \ \lambda_{j,i} \neq 0 \ , \ \forall \ (j,i) \ \in \ [1,p]\times[1,n] \ , \tag{12}$$

$$AW_0 + W_0 A' + [B + S_\ell]H_0 + H_0'[B + S_\ell]' < 0, \ \forall S_\ell \in \mathcal{N}_\Lambda \ , \tag{13}$$

$$\begin{bmatrix} 1 & x_e' \\ x_e & W_0 \end{bmatrix} > 0 \ . \tag{14}$$

then by taking

$$\mathcal{D}_0 = \left\{x, x'W_0^{-1}x \geq 1\right\}, \mathcal{D}_1 = \left\{x, x'W_0^{-1}x \leq 1-\epsilon\right\}, \epsilon = \tfrac{1-x_e'W_0^{-1}x_e}{2}, F_0 = H_0 W_0^{-1} \tag{15}$$

it follows that the equilibrium $\{0\}\times\{0\} \subset \mathbb{R}^n \times \{0,1\}$ is globally asymptotically stable[6] for the system (8)-(9).

This result is based on the following Lemma which relies on arguments borrowed from [31] (see also [34]).

D. Proof of Theorem 1

Before proving Theorem 1, we need to state and prove the two following lemmas. First of all, let us denote $P_0 = W_0^{-1}$ and $F_0 = H_0 W_0^{-1}$.

Lemma 1 (Local asymptotic stability with the local controller): For the system (1) in closed loop with $u = F_0 x$, the origin is locally asymptotically stable and the following statement are satisfied.

1) $\mathbb{R}^n \setminus \mathcal{D}_0$ and $\mathcal{D}_1$ are forward invariant and included in the basin of attraction of the origin.
2) x_e is included in $\mathcal{D}_1$.

With Lemma 1, the proof of Theorem 1 follows from [12, Example 1 p.51].

[4]The $\lambda_{j,i}$'s are parameters allowing us to estimate the $N_{j,i}x$.

[5]The small difference comes from the fact that $N_j x$ is estimated instead of x as seen in inequality (17).

[6]The definition of global asymptotic stability can be found in [12].

Proof: Pre- and post-multiplying inequality (12) by $\mathtt{Diag}\{P_0, 1\}$, it yields that,

$$\begin{bmatrix} \lambda_{j,i}^2 P_0 & N_{j,i}' \\ N_{j,i} & 1 \end{bmatrix} > 0 \ .$$

With Schur complement it is equivalent to the following inequality,

$$\lambda_{j,i}^2 P_0 - N_{j,i}' N_{j,i} > 0 \ . \tag{16}$$

Hence, it implies that for all x in $\mathbb{R}^n \setminus \{0\}$ and for all (j,i) in $[1,p] \times [1,n]$:

$$|N_{j,i} x|^2 = x' N_{j,i}' N_{j,i} x < \lambda_{j,i}^2 x' P_0 x \ . \tag{17}$$

Consequently, it yields that for all (j,i) in $[1,p] \times [1,n]$

$$-\lambda_{j,i} \leq N_{j,i} x \leq \lambda_{j,i} \ , \ \forall x \ \in \mathbb{R}^n \setminus \mathcal{D}_0 \ .$$

This implies that with the definition of the set of matrix $\mathcal{N}_\Lambda$ that

$$\begin{bmatrix} N_1 x & \dots & N_p x \end{bmatrix} \in \mathtt{Co}\{\mathcal{N}_\Lambda\} \ , \ \forall x \ \in \mathbb{R}^n \setminus \mathcal{D}_0 \ .$$

Consequently, for all x in $\mathbb{R}^n \setminus \mathcal{D}_0$ and all u in $\mathbb{R}^p$ we have:

$$\dot{x} \in \mathtt{Co}\left\{Ax + [B + S_\ell]\, u, S_\ell \in \mathcal{N}_\Lambda\right\} \ .$$

In other words, there exist 2^{np} functions $\mu_\ell : \mathbb{R}^n \to \mathbb{R}_+$ such that for all x in $\mathbb{R}^n \setminus \mathcal{D}_0$ and for all u in $\mathbb{R}^p$:

$$\sum_{\ell=1}^{2^{np}} \mu_\ell(x) = 1 \ , \ \dot{x} = \sum_{\ell=1}^{2^{np}} \mu_\ell(x) \left(Ax + [B + S_\ell]\, u\right) \ .$$

Hence, for all x in $\mathbb{R}^n \setminus \mathcal{D}_0$, it yields that along the trajectories of the system (1) with $u = F_0 x$ we have:

$$\dot{x} = \sum_{\ell=1}^{2^{np}} \mu_\ell(x) \left([A + [B + S_\ell] F_0]\, x\right) \ .$$

Moreover, the time-derivative of the function $x' P_0 x$ along the trajectories of the system (1) satisfies for all x in $\mathbb{R}^n \setminus \mathcal{D}_0$:

$$\dot{\overline{x' P_0 x}} = \sum_{\ell=1}^{2^{np}} \mu_\ell(x) x' \left(P_0 \left(A + [B + S_\ell]\, F_0\right) + \left(A + [B + S_\ell]\, F_0\right)' P_0\right) x \tag{18}$$

On another hand, pre- and post-multiplying inequality (13) by P_0 yields that the following inequality holds:

$$P_0 \left(A + [B + S_\ell]\, F_0\right) + \left(A + [B + S_\ell]\, F_0\right)' P_0 < 0 \ , \ \forall S \in \mathcal{N}_\Lambda \ , \tag{19}$$

Consequently, this implies that along the trajectories of the system (1) with the $u = F_0 x$, we have

$$\dot{\overline{x' P_0 x}} < 0 \ , \ \forall \ x \ \in \ \mathbb{R}^n \setminus \mathcal{D}_0 \ .$$

This implies that the set $\mathbb{R}^n \setminus \mathcal{D}_0$ is forward invariant for the system (1) in closed-loop with $u = F_0 x$ and is included in the basin of attraction of the origin.

Moreover, note that with the Schur complement inequality (14) is equivalent to the following inequality:

$$x_e^T P_0 x_e < 1 \ . \tag{20}$$

Hence, ϵ defined in (15) is positive, $\mathcal{D}_1 \subset (\mathbb{R}^n \setminus \mathcal{D}_0)$ and is also forward invariant for the system (1) in closed-loop with $u = F_0 x$ and is included in the basin of attraction of the origin.

Finally, note that with (20), it yields:

$$x_e' P_0 x_e - 1 + \epsilon = \frac{x_e' P_0 x_e - 1}{2} < 0 \ .$$

Consequently x_e is in $\mathcal{D}_1$ and the last item of Lemma 1 holds. ■

Lemma 2 (Global attractivity with the hybrid controller): For all initial condition (x, q) in $\mathbb{R}^n \times \{0, 1\}$, there exists a hybrid time $(\bar{t}, \bar{j})$ in $\mathtt{dom}(x, q)$ such that $x(\bar{t}, \bar{j})$ is in $\mathcal{D}_1$.

Proof: Consider an initial point (x, q) in $\mathbb{R}^n \setminus \mathcal{D}_1 \times \{0, 1\}$ and assume that there does not exist $(\bar{t}, \bar{j})$ in $\mathtt{dom}(x, q)$ such that $x(\bar{t}, \bar{j})$ is included in $\mathcal{D}_1$. We can distinguish the following three cases.

1) If $q = 0$ and $x \notin \mathcal{D}_0$, then there exists $\delta > 0$ such that for all t in $(0, \delta)$, the couple $(t, 0)$ is included in $\mathtt{dom}(x, q)$ and $(x(t, 0), q(t, 0))$ is inside $\mathcal{C}_0 \times \{0\}$ and outside $\mathcal{D}_0 \times \{0\}$. Consequently, we have $u(t) = H_0 F_0^{-1} x(t, 0)$ for all t in $(0, \delta)$. With Lemma 1, this implies that $(t, 0)$ for all positive time t is included in $\mathtt{dom}(x, q)$ and we get a contradiction with the asymptotic stability of the origin and the fact that $\mathbb{R}^n \setminus \mathcal{D}_0$ is included in its basin of attraction.
2) if $q = 1$ and $x \notin \mathcal{D}_1$, then by assumption, all (t, j) in $\mathtt{dom}(x, q)$ is such that $j = 0$. Consequently, on the time of existence of the solution, we have $u(t) = u_\infty$. Hence, along the trajectories
$$\overbrace{(x - x_e)' P_\infty (x - x_e)}^{\cdot} < 0 \ .$$
This implies that the solution is bounded and thus defined for all t and consequently x_e is an attractor of the solution. However, x_e being in $\mathcal{D}_1$ by Lemma 1 we get a contradiction.
3) if $q = 0$ and $x \in \mathcal{D}_0$, then $(0, 1)$ is inside $\mathtt{dom}(x, q)$ and $x(0, 1)$ is in $\mathcal{C}_1$ and $q(0, 1) = 1$ which implies that we rely on the previous case.

■

E. Discussion and example

Note that once the parameter Λ in $\mathbb{R}^{p \times n}$ is selected the sufficient condition of Theorem 1 is given in terms of solutions to linear matrix inequalities for which some powerful LMI solvers (see [30] for instance) may be used as illustrated by the numerical example given in the following.

In order to apply Theorem 1, the first step is to select the matrix Λ in $\mathbb{R}^{p\times n}$. It can be shown that a necessary condition for inequalities (12), (13) and (14) to have a solution is that $|N_{j,i}x_e|^2 < \lambda_{j,i}$. Consequently, the $\lambda_{j,i}$'s have to be selected at least larger than $|N_{j,i}x_e|$.

In the case in which x_e is such that $N_{j,i}x_e = 0$, this necessary condition is trivially satisfied with any $\lambda_{j,i}$ positive (see the following example).

On another hand, from inequality (13), we see that if A is not Hurwitz the $\lambda_{j,i}$ have to be selected sufficiently small such that [7] $0_{p\times n} \notin \mathtt{Co}_{S_\ell\in\mathcal{N}_\Lambda}\{B+S_\ell\}$ where $\mathtt{Co}$ denotes the convex hull. Note however that no general strategy exists to select these parameters.

Example 5: As in [34], consider system (1) with the matrices A, B and N defined as:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ N = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \ . \tag{21}$$

First of all, it can be shown that this system doesn't satisfy the assumption of [13, Theorem 3.1]. Consequently, this shows that no state feedback approach leading to a quadratic Lyapunov function can be performed and consequently the approach of [13] cannot be applied. Indeed, consider the matrix P given as

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Assume that P is a Lyapunov matrix associated to the system for a given stabilizing continuous controller $u = \varphi(x)$. More precisely, assume that the following inequality is satisfied along the solution of the system for all $x \neq 0$:

$$\dot{\overline{x'Px}}\Big|_{u=\varphi(x)} < 0 \ .$$

Since P is positive definite it implies that

$$a > 0 \ , \ c > 0 \ , \ ac - b^2 > 0 \ .$$

Moreover, we know that Artstein condition is satisfied. More precisely, we know that[8]:

$$x'(A'P + PA)x < 0 \ , \ \forall x \ \left(1 - \frac{1}{2}x_2\right) B'Px = 0 \ .$$

Note that $B'Px = bx_1 + cx_2$, hence the previous condition can be rewritten

$$x'(A'P + PA)x < 0 \ , \ \forall x \in \{x = (x_1, x_2) \ , \quad x_2 = 2 \text{ or } x_2 = -\frac{b}{c}x_1\}$$

Note that when $x_2 = 2$ it yields,

$$\frac{1}{2}x'(A'P + PA)x = -2bx_1^2 + [2(a+b) - 4c]x_1 + 4(b+c)$$

[7]Otherwise, one will obtain $W_0A' + AW_0 < 0$, which contradicts the assumption on the fact that A is not Hurwitz.

[8]This is in fact exactly Gutman's condition in [13, Theorem 3.1].

Hence to have $x'(A'P+PA)x<0$ when $x_2=2$, it implies that $b+c<0$ (to see this, it suffices to take $x_1=0$ in the previous equality). Since we have $c>0$, this implies that $b<0$.

On another hand, we have when $x_2=-\frac{b}{c}x_1$

$$\frac{1}{2}x'(A'P+PA)x=-x_1^2\frac{b}{c^2}(ac-b^2)$$

Hence to have $x'(A'P+PA)x<0$ when $x_2=-\frac{b}{c}x_1$, it implies that $\frac{b}{c^2}(ac-b^2)>0$. Since we have $ac-b^2>0$, this implies that $b>0$. Thus, one arrives to a contradiction. Consequently for the system studied in the paper, there does not exist a state feedback ensuring global asymptotic stabilization with a quadratic Lyapunov function. Consequently, the approach of [13] fails to be satisfied. The considered system satisfies Assumption 1 with $u_\infty=3$. The first step is to select the $\lambda_{i,j}$'s. We select $\lambda_{1,1}=0.1$ and $\lambda_{1,2}=0.5$. In this case, the set of matrices $\mathcal{N}_\Lambda$ is given as,

$$S_1=\begin{bmatrix}0.1\\0.5\end{bmatrix},S_2=\begin{bmatrix}0.1\\-0.5\end{bmatrix},S_3=\begin{bmatrix}-0.1\\0.5\end{bmatrix},S_4=\begin{bmatrix}-0.1\\-0.5\end{bmatrix}.$$

Hence, we get the following solution: $W_0=\begin{bmatrix}2.5091 & -0.4861\\-0.4861 & 1.0000\end{bmatrix}$, $H_0=\begin{bmatrix}5.6732 & -6.8629\end{bmatrix}$ .. Consequently, the controller obtained from Theorem 1 makes the origin of the system (1) globally asymptotically stable with the data $\epsilon=0.005$, $F_0=\begin{bmatrix}1.0283 & -6.3633\end{bmatrix}$.

III. Output feedback design

We suppose now that only the measurement $y=Cx$ is available for a feedback in order to ensure global asymptotic stabilization of the origin. The output feedback stabilization of bilinear systems has already been addressed in [15] where a dead-beat observer is used. However, in [15] there is no B matrices and similar approach cannot be employed in the present context. The idea of our design will be to follow an observer controller approach. More precisely we assume Assumptions of Theorem 1 hold and we will solve this output feedback problem by designing a hybrid observer that asymptotically estimates the state of the system. This strategy differs from the one in [27] where a hybrid output feedback is obtained based on a norm observer (see also [29] for a result on hybrid output feedback).

With this hybrid output feedback framework, by considering the new state $(x,\hat{x},\tau,q)$ in $\mathbb{R}^n\times\mathbb{R}^n\times[0,2]\times\{0,1\}$, the closed-loop system under study is a hybrid system described by

$$\left.\begin{array}{rcl}\dot{x} & = & Ax + B\varphi_q(\hat{x}) + \sum_{j=1}^{p} (\varphi_q(\hat{x}))_j N_j x \\ \dot{\hat{x}} & = & A\hat{x} + B\varphi_q(\hat{x}) + \sum_{j=1}^{p} (\varphi_q(\hat{x}))_j N_j \hat{x} + \psi_q(Cx, \hat{x}) \\ \dot{\tau} & = & h(\tau) \\ \dot{q} & = & 0\end{array}\right\} \text{ if } (x, (\hat{x}, \tau), q) \in \mathbb{R}^n \times \hat{\mathcal{C}}_q \times \{0, 1\} \ , \tag{22}$$

$$\left.\begin{array}{rcl}x^+ & = & x \\ \hat{x}^+ & = & \hat{x} \\ \tau^+ & = & 0 \\ q^+ & = & 1 - q\end{array}\right\} \text{ when } (x, (\hat{x}, \tau), q) \in \mathbb{R}^n \times \hat{\mathcal{D}}_q \times \{0, 1\} \ , \tag{23}$$

where $\hat{\mathcal{C}}_q = \overline{\mathbb{R}^n \times [0, 2] \setminus \hat{\mathcal{D}}_q}$ where ψ_0 and ψ_1 are the correction terms associated to the observer. Note that to integrate this closed-loop system, only the knowledge of $(\hat{x}, \tau, q)$ is required to decide between jump and flow along the trajectories of the closed-loop system. Hence, to implement this feedback, only the knowledge of y is required.

With the constant control u_∞, we consider the following observability assumption.

Assumption 2: The vector u_∞ in Assumption 1 is such that $\left(C, A + \sum_{j=1}^{p} u_{\infty,j} N_j\right)$ is observable.

Given W_0 obtained from Theorem 1, we can define $\Gamma = \{T_1, \ldots, T_{2^p}\}$ a finite set of real vectors in $\mathbb{R}^p$ such that

$$H_0 W_0^{-1} x \in \mathtt{Co}\{T_\ell, \ell = 1 \ldots, 2^p\} \ , \forall\, x \ \in \ \{x, x' W_0^{-1} x \leq 1\}. \tag{24}$$

We have the following theorem.

Theorem 2 (Output feedback): Assume Assumptions 1 and 2 hold. Assume there exist a matrix Λ in $\mathbb{R}^{p \times n}$, a symmetric positive definite matrix W_0 in $\mathbb{R}^{n \times n}$, and a matrix H_0 in $\mathbb{R}^{p \times n}$ such that inequalities (12), (13), (14) are satisfied[9]. Assume there exist a symmetric positive definite matrix Q_0 in $\mathbb{R}^{n \times n}$ and a matrix D_0 in $\mathbb{R}^{m \times n}$ such that

$$\left[A + \sum_{j=1}^{p} N_j (T_\ell)_j\right]' Q_0 + Q_0 \left[A + \sum_{j=1}^{p} N_j (T_\ell)_j\right] + C' D_0 + D_0 C < 0 \ , \forall T_\ell \in \Gamma \ . \tag{25}$$

Then there exist K_∞ in $\mathbb{R}^{m \times n}$, a function h and a positive real number u_0 such that the output feedback controller defined with the data

$$\hat{\mathcal{D}}_0 = \left\{(\hat{x}, \tau), \hat{x}' W_0^{-1} \hat{x} \geq 1, \tau \geq 1\right\} \ , \ \hat{\mathcal{D}}_1 = \left\{\hat{x}, \hat{x}' W_0^{-1} \hat{x} \leq 1 - \epsilon, \tau \geq 1\right\} \ , \epsilon = \frac{1 - x_e' W_0^{-1} x_e}{2} \ ,$$

[9]In that case, Theorem 1 applies and there exists a stabilizing state feedback.

$$\varphi_0(\hat{x}) = \texttt{sat}_{u_0}(H_0 W_0^{-1}\hat{x}) \ , \ \varphi_1(\hat{x}) = u_\infty \ , \tag{26}$$

$$\psi_0(\hat{x}, y) = Q_0^{-1} D_0(C\hat{x} - y) \ , \ \psi_1(\hat{x}, y) = K_\infty(C\hat{x} - y) \ , \tag{27}$$

where $\texttt{sat}_{u_0}$ is the saturation function of positive level u_0[10], makes the set $\{0\} \times \{0\} \times [0, 2] \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n \times [0, 2] \times \{0, 1\}$ a globally asymptotically stable set for system (22)-(23).

Proof: Let h be a locally Lipschitz function such that:

$$h(\tau) = \begin{cases} v_\tau & \tau \leq 1 \\ > 0 & 1 \leq \tau < 2 \\ 0 & \tau = 2 \end{cases}$$

where v_τ is any positive real number. Also, the positive real number u_0 is defined as

$$u_0 = \max_{\{x, x' W_0^{-1} x \leq 1\}, 1 \leq j \leq p} (H_0 W_0^{-1} x)_j \tag{28}$$

First of all, note that the control $u = \varphi_q(\hat{x})$ is bounded for all $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2] \times \{0, 1\}$. The system under consideration being bilinear, this implies that the continuous part of closed-loop system is globally Lipschitz. Consequently, for all initial conditions, the corresponding trajectories do not blow up at infinity in finite time. This implies that for all solutions initiated from $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2] \times \{0, 1\}$, their time domain $\texttt{dom}(x, \hat{x}, \tau, q)$ is an unbounded set.

The rest of the proof of Theorem 2 is decomposed in three Lemmas, which proofs are given at the end of this proof. The first one establishes asymptotic convergence of the estimate $\hat{x}$ toward the state of the system.

Lemma 3 (Observer convergence): There exists K_∞ such that for all initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2] \times \{0, 1\}$, we have that $|\hat{x}(t, \ell) - x(t, \ell)|$ is bounded and

$$\lim_{t+\ell \to +\infty} |\hat{x}(t, \ell) - x(t, \ell)| = 0 \ .$$

With the previous Lemma, we can now establish the following result concerning boundedness of solutions.

Lemma 4 (Boundedness of solutions): For all initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2] \times \{0, 1\}$, we have that $\hat{x}(t, \ell)$ and $x(t, \ell)$ are bounded.

With the boundedness of solution, with [28, Lemma 3.3], we get the existence of a non empty ω-limit set denoted $\Omega(x, \hat{x}, \tau, q)$ which is weakly invariant. In other words, for all $(x, \hat{x}, \tau, q)$ in $\Omega(x, \hat{x}, \tau, q)$ there exists a complete solution to the closed-loop system such that for all (t, j) in its time domain $(x(t, j), \hat{x}(t, j), \tau(t, j), q(t, j))$ is in $\Omega(x, \hat{x}, \tau, q)$. Also as stated in [28, Lemma 3.3], the distance from $(x(t, j), \hat{x}(t, j), \tau(t, j), q(t, j))$ to $\Omega(x, \hat{x}, \tau, q)$ decreases to zero as $t+j \to +\infty$.

[10]For $i = 1, ..., p$, each component of $\texttt{sat}_{u_0}(u)$ is defined by $\texttt{sat}_{u_0}(v_i) = sign(v_i) \min(u_0, |v_i|)$.

Moreover, as stated in [28, Lemma 3.3], this set is the smallest closed set with this property. Hence, with Lemma 3, we get that for all $(x, \hat{x}, \tau, q)$ in $\Omega(x, \hat{x}, \tau, q)$ we have $\hat{x} = x$. Hence all solutions starting in $\Omega(x, \hat{x}, \tau, q)$ satisfy the hybrid system with continuous dynamics defined as

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + B\varphi_q(x) + \sum_{j=1}^{p} (\varphi_q(x))_j N_j x \\ \dot{\tau} & = & h(\tau) \\ \dot{q} & = & 0 \end{array}\right\} \text{ if } ((x,\tau),q) \in \hat{\mathcal{C}}_q \times \{0,1\} \ , \tag{29}$$

and discrete dynamics with φ_q defined in (26),

$$\left.\begin{array}{rcl} x^+ & = & x \\ \tau^+ & = & 0 \\ q^+ & = & 1-q \end{array}\right\} \text{ when } ((x,\tau),q) \in \hat{\mathcal{D}}_q \times \{0,1\} \ . \tag{30}$$

Note that this system (29)-(30) is similar to the one given in (8)-(9) with the data obtained from Theorem 1 but with two differences:

1) There is an extra variable corresponding to the timer τ.
2) The function $\varphi_0(x) = \mathtt{sat}_{u_0}(H_0 W_0^{-1} x)$ instead of $\varphi_0(x) = H_0 W_0^{-1} x$.

The next step in the proof, is to show that these differences do not modify the behavior of the trajectories and that the origin of the system (29)-(30) is globally asymptotically stable.

Lemma 5 (Asymptotic stability of the system (29)-(30)): The set $\{0\} \times [0,2] \times \{0\}$ in $\mathbb{R}^n \times [0,2] \times \{0,1\}$ is a globally asymptotically stable set for the system (29)-(30).

With Lemma 5, we get that the ω-limit is simply $\{0\} \times [0,2] \times \{0\}$ in $\mathbb{R}^n \times [0,2] \times \{0,1\}$. Since all the trajectories converge toward its ω-limit set (see [28, Lemma 3.3]) we obtain that the set $\{(0,0)\} \times [0,2] \times \{0\}$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$ is a global attractor for the system (22)-(23). To finish the proof, we need to show that local asymptotic stability of this set is also obtained. With inequality (13), there exists ρ_0 a positive real number such that[11],

$$AW_0 + W_0 A' + [B + S_\ell]H_0 + H_0'[B + S_\ell]' < -\rho_0 W_0 \ , \forall S_\ell \in \mathcal{N}_\Lambda \ , \tag{31}$$

Pre- and post-multiplying this inequality by $P_0 = W_0^{-1}$ yields

$$P_0(A + [B + S_\ell]H_0 P_0) + (A + [B + S_\ell]H_0 P_0)' P_0 < -\rho_0 P_0 \ , \forall S_\ell \in \mathcal{N}_\Lambda \ .$$

Consider now an initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$ with $|\hat{x}|$ and $|x - \hat{x}|$ sufficiently small and $q = 0$. This implies that there exists μ such that for all $0 < s < \mu$, $(s, 0)$ is in $\mathtt{dom}(x, \hat{x}, \tau, q)$. For all $s < \mu$, and following equation (18) in the proof of Lemma 1, we have

$$\frac{d}{ds}\hat{x}(s,0)' P_0 \hat{x}(s,0) \leq -\rho_0 \hat{x}(s,0)' P_0 \hat{x}(s,0) + 2\hat{x}(s,0)' P_0 Q_0^{-1} D_0 C[\hat{x}(s,0) - x(s,0)] \ .$$

[11]This one can simply be computed employing LMI tools.

Note that from this inequality, we can introduce two positive real numbers c_1 and c_2 such that

$$\frac{d}{ds}\hat{x}(s,0)'P_0\hat{x}(s,0) \leq -c_1\hat{x}(s,0)'P_0\hat{x}(s,0) + c_2|\hat{x}(s,0) - x(s,0)|^2 \ .$$

On another hand, there exists $\lambda_2 > 0$ such that (this will be formally proven later in (37))

$$\frac{d}{ds}(x(s,0) - \hat{x}(s,0))'Q_0(x(s,0) - \hat{x}(s,0)) \leq -\lambda_2(x(s,0) - \hat{x}(s,0))'Q_0(x(s,0) - \hat{x}(s,0)) \ .$$

Hence, there exists a positive real number κ such that

$$\frac{d}{ds}\hat{x}(s,0)'P_0\hat{x}(s,0) + \kappa(x(s,0) - \hat{x}(s,0))'Q_0(x(s,0) - \hat{x}(s,0)) < 0 \ .$$

This function being proper and positive definite in x and $\hat{x}$ we get the local asymptotic stability of the set $\{(0,0)\} \times [0,2] \times \{0\}$. This concludes the proof of Theorem 2. ■

In the remaining part of this Section we give the proofs of Lemmas 3, 4 and 5.

Proof of Lemma 3. Consider a positive real number λ_1 sufficiently large such that:

$$(A + \sum_{j=1}^{p} N_j u_j + Q_0^{-1}D_0C)'Q_0 + Q_0(A + \sum_{j=1}^{p} N_j u_j + Q_0^{-1}D_0C) \leq \lambda_1 Q_0 \ , \tag{32}$$

for all $|u_j| \leq u_0$. Note that we have the following Lemma, which constructive proof based on high-gain techniques (see [10]) is given in Section A.

Lemma 6 (Observer with prescribed convergence speed): There exist a matrix K_∞ in $\mathbb{R}^{n\times m}$, a symmetric positive definite matrix Q_∞ in $\mathbb{R}^{n\times n}$ and a positive scalar λ_∞ such that the following matrix inequality is satisfied:

$$\left(A + \sum_{j=1}^{p} u_{\infty,j}N_j + K_\infty C\right)' Q_\infty + Q_\infty \left(A + \sum_{j=1}^{p} u_{\infty,j}N_j + K_\infty C\right) < -\lambda_\infty Q_\infty \ , \tag{33}$$

and such that,

$$\exp\left(\frac{\lambda_1 - \lambda_\infty}{\nu_\tau}\right)\frac{\lambda_{\max}(Q_\infty)\lambda_{\max}(Q_0)}{\lambda_{\min}(Q_\infty)\lambda_{\min}(Q_0)} < 1 \ . \tag{34}$$

The first inequality of Lemma 6 is a classical Lyapunov matrix inequality for observer design. The second one imposes a specific convergence rate to the observer. Note that by writing $e = \hat{x} - x$, the closed-loop system (22)-(23) can be rewritten, with continuous dynamics

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + B\varphi_q(x+e) + \sum_{j=1}^{p}\left(\varphi_q(x+e)\right)_j N_j x \\ \dot{e} & = & \left[A + \sum_{j=1}^{p}\left(\varphi_q(x+e)\right)_j N_j\right] e + \psi_q(Cx, x+e) \\ \dot{\tau} & = & h(\tau) \\ \dot{q} & = & 0 \end{array}\right\} \text{ when } (x, x+e, \tau, q) \in \mathbb{R}^n \times \hat{\mathcal{C}}_q \times \{0,1\} \tag{35}$$

and discrete dynamics,

$$\left.\begin{array}{rcl} x^+ & = & x \\ e^+ & = & e \\ \tau^+ & = & 0 \\ q^+ & = & g_q(x, x+e) \end{array}\right\} \text{ when } (x, x+e, \tau, q) \in \mathbb{R}^n \times \hat{\mathcal{D}}_q \times \{0,1\} \tag{36}$$

To analyze the behavior of the trajectories of this model consider an initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$ and $(t,0)$ in $\mathtt{dom}(x, \hat{x}, \tau, q)$ with $t \geq 0$. Two cases can be distinguished. 1) Assume the initial condition is such that $(x, (\hat{x}, \tau), q)$ is in $\mathbb{R}^n \times \hat{\mathcal{C}}_0 \times \{0\}$. Since no jump occurs, it follows that $(x(s,0), (\hat{x}(s,0), \tau(s,0)), q(s,0))$ is in $\mathbb{R}^n \times \hat{\mathcal{C}}_0 \times \{0\}$ for all s in $[0,t]$. Note that for all s in $[0,t)$, we have,

$$\frac{d}{ds} e(s,0) = \left(A + \sum_{j=1}^{p} N_j \mathtt{sat}_{u_0}(HW_0^{-1}\hat{x}(s,0)) + Q_0^{-1} D_0 C \right) e(s,0) \ .$$

From the definition of λ_1 in (32) we get,

$$\frac{d}{ds} Z(s,0) \leq \lambda_1 Z(s,0) \ , \ \forall \ s \in [0,t) \ ,$$

where $Z(s,\ell)$ is the function defined on the hybrid time domain as $Z(s,\ell) = e(s,\ell)' Q_0 e(s,\ell)$. Hence, this implies that:

$$Z(s,0) \leq \exp(\lambda_1 s) Z(0,0) \ , \ \forall \ s \in [0,t) \ .$$

Moreover, if $t > \frac{1}{v_\tau}$ with the definition of $\dot{\tau}$ in (29), it implies $\tau(s,0) \geq 1$ for all s in $[\frac{1}{v_\tau}, t]$. Since there was no jump, $\hat{x}(s,0)$ is in the subset of $\mathbb{R}^n$ defined as $\{x, x'W_0^{-1}x < 1\}$. Consequently, with (24) and with the definition of u_0 in (28), it follows that

$$\varphi_0(\hat{x}(s,0)) \in \mathtt{Co}\{T_\ell, \ell = 1 \ldots, 2^p\} \ , \forall \ s \in \left[\frac{1}{v_\tau}, t\right] \ .$$

This gives the existence of 2^p positive functions $\mu_\ell : \mathbb{R}^n \to \mathbb{R}_+$ with $\displaystyle\sum_{\ell=1}^{2^p} \mu_\ell(\hat{x}) = 1$ for all $\hat{x}$ in $\mathbb{R}^n$ and such that $\varphi_0(\hat{x}(s,0)) = \sum_{\ell=1}^{2^p} \mu_\ell(\hat{x}(s,0)) T_\ell$. Consequently

$$\frac{d}{ds} e(s,0) = \sum_{\ell=1}^{2^p} \mu_\ell(\hat{x}(s,0)) \left(A + \sum_{j=1}^{p} N_j (T_\ell)_j + Q_0^{-1} D_0 C \right) e(s,0) \ , \forall \ s \in \left[\frac{1}{v_\tau}, t\right] \ .$$

Hence with (25) it yields

$$\frac{d}{ds} Z(s,0) \leq -\lambda_2 Z(s,0) \ , \forall \ s \in \left[\frac{1}{v_\tau}, t\right] \ , \tag{37}$$

where λ_2 is a positive real number such that

$$Q_0 \left(A + \sum_{j=1}^{p} N_j (T_\ell)_j \right) + \left(A + \sum_{j=1}^{p} N_j (T_\ell)_j \right)' Q_0 + D_0 C + C' D_0 \leq -\lambda_2 Q_0 \ , \ell = 1, \ldots, 2^p \ .$$

Consequently, it implies:

$$Z(t,0) \leq \exp\left(\frac{\lambda_1}{v_\tau} - \lambda_2 \max\left\{t - \frac{1}{v_\tau}, 0\right\}\right) Z(0,0) \ . \tag{38}$$

2) Assume the initial condition is such that $(x,(\hat{x},\tau),q)$ is in $\mathbb{R}^n \times \hat{\mathcal{C}}_1 \times \{1\}$. Since there is no jump, it yields that $(x(s,0),(\hat{x}(s,0),\tau(s,0)),q(s,0))$ is in $\mathbb{R}^n \times \hat{\mathcal{C}}_1 \times \{1\}$ for all s in $[0,t]$. With (33), it yields that we have for all $s \leq t$

$$\frac{d}{ds} e(s,0)' Q_\infty e(s,0) \leq -\lambda_\infty e(s,0)' Q_\infty e(s,0) \ .$$

Consequently, for all s such that $(s,0)$ is in $\mathtt{dom}(x,(\hat{x},\tau),q)$, it yields,

$$|e(s,0)|^2 \leq \exp(-\lambda_\infty s) \frac{\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_\infty)} |e(0,0)|^2$$

Since, for all e in $\mathbb{R}^n$, we have $\frac{e'Q_0 e}{\lambda_{\max}(Q_0)} \leq |e|^2 \leq \frac{e'Q_0 e}{\lambda_{\min}(Q_0)}$, this implies that,

$$Z(t,0) \leq \frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)} \exp(-\lambda_\infty t) Z(0,0) \ . \tag{39}$$

For the asymptotic behavior of the trajectories, note that three possibilities have to be considered: a) after finitely many switching, $\hat{x}$ stays in $\hat{\mathcal{C}}_0$; b) after finitely many switching, $\hat{x}$ stays in $\hat{\mathcal{C}}_1$; c) there are infinitely many switching.

For the a) and b) cases, because the system does not blowup, finitely many transitions may be ignored, and without loss of generality, one may assume that $\hat{x}$ is always in $\hat{\mathcal{C}}_q$ ($q = 0$ or 1). Then, by (38) or (39), $\lim_{t\to+\infty} Z(t,0) = 0$.

For c) case, by omitting the first transition if necessary, without loss of generality, one may assume that $\hat{x}$ starts from $\hat{\mathcal{C}}_0$. For all $0 \leq k$ there exists t_k such that (t_k,k) and $(t_k,k+1)$ are in $\mathtt{dom}(x,\hat{x},\tau,q)$, $\tau(t_k,k) \geq 1$ and $\tau(t_k,k+1) = 0$. Moreover, since $\frac{\partial}{\partial s}\tau(s,k) \leq v_\tau$ for all (s,k) in the time domain, this implies that $t_k - t_{k-1} \geq \frac{1}{v_\tau}$,$\forall k > 1$, which shows that there is a strictly positive dwell time between two successive jumps. Since between $Z(t_k,k)$ and $Z(t_{k+2},k+2)$ two jumps occur this implies that both the previous cases have to be considered. Employing the two first items of this analysis, we get for all $2 \leq k \leq \ell$ that,

$$Z(t_k,k) \leq \exp\left(\frac{\lambda_1 - \lambda_\infty}{v_\tau}\right) \frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)} Z(t_{k-2},k-2)$$

Consequently,

$$Z(t_{2k},2k) \leq \left[\exp\left(\frac{\lambda_1 - \lambda_\infty}{v_\tau}\right) \frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)}\right]^k Z(t_0,0) \ .$$

Note that with the definition of λ_∞ in equation (34), we get that

$$\exp\left(\frac{\lambda_1 - \lambda_\infty}{v_\tau}\right) \frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)} < 1 \ .$$

Consequently, this implies that we have $\lim_{t+\ell\to+\infty} Z(t,\ell) = 0$. This function being proper in e, it follows that e is bounded and goes to zero along the solution. □

Proof of Lemma 4. Consider now an initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$. Assume there exists a trajectory initialized from this point which is unbounded. Since from Lemma 3, we have that $|\hat{x} - x|$ is bounded this implies that $|\hat{x}|$ is unbounded. Two cases may be distinguished: a) after finitely many switchings, $\hat{x}, \tau$ remain in $\hat{\mathcal{C}}_1$; b) there is a infinite number of switching. In case a), this implies that possibly after a finite number of switchings, $u = u_\infty$. Hence, this implies that the function defined by $V_1(t,\ell) = \hat{x}(t,\ell)' P_\infty \hat{x}(t,\ell)$ satisfies

$$\frac{\partial}{\partial t} V_1(t,\ell) \leq -\rho_\infty V_1(t,\ell) + 2\hat{x}(t,\ell)' P_\infty K_\infty C(\hat{x}(t,\ell) - x(t,\ell)) \ ,$$

where ρ_∞ is solution of the following linear matrix inequality (a solution exists since equation (2) is satisfied in Assumption 1):

$$P_\infty \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j \right] + \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j \right]' P_\infty < -\rho_\infty P_\infty \ .$$

Hence we can introduce two positive real numbers c_3 and c_4 such that

$$\frac{\partial}{\partial t} V_1(t,\ell) \leq -c_3 V_1(t,\ell) + c_4 |\hat{x}(t,\ell) - x(t,\ell)|^2 \ . \tag{40}$$

From Lemma 3, this implies that V_1 is bounded. The function V_1 being proper in $\hat{x}$, this contradicts the fact that $|\hat{x}|$ is unbounded.

In case b), for all j there exists t_j such that (t_j, j) and $(t_j, j+1)$ is in $\mathtt{dom}(x, \hat{x}, \tau, q)$. Consider the function $V_0(s,\ell) = \hat{x}(s,\ell)' P_0 \hat{x}(s,\ell)$. The control input being bounded and e being bounded and going to zero, we get for all t in $\mathtt{dom}(x, \hat{x}, \tau, q)$ the existence of two positive real numbers c_5 and c_6 such that

$$\frac{\partial V_0}{\partial t}(t,\ell) \leq c_5 V_0(t,\ell) + c_6$$

Without loss of generality, we may assume that $\hat{x}(t_0, 0)$ is in $\hat{\mathcal{C}}_0$. Let $t_0' < t_0''$ be two positive real numbers in $[t_0, t_1]$ such that $V_0(t_0', 0) = 1$ and $V_0(s, 0) > 1$ for all s in (t_0', t_0''). Note that we have $t_0'' - t_0' \leq \frac{1}{\nu_\tau}$. Hence, it yields,

$$V_0(s,0) \leq \exp\left(\frac{c_5}{\nu_\tau}\right) + \frac{\exp\left(\frac{c_5}{\nu_\tau}\right) - 1}{c_5} \ , \ \forall s \in [t_0', t_0''] \ .$$

The function V_0 being proper in $\hat{x}$, this implies that $\hat{x}$ is bounded between t_0 and t_1. Note that for all t is $[t_1, t_2]$ V_1 satisfies equality (40) and consequently $\hat{x}$ is bounded. □

Proof of Lemma 5. Note that with this definition of u_0 in equation (28) for all x in the subset of $\mathbb{R}^n$ defined as $\{x, x' W_0^{-1} x \leq 1\}$ we have $\varphi_0(x) = H_0 W_0^{-1} x$. Consequently, we recover the data of Theorem 1. The fact that the set $\{0\} \times [0,2] \times \{0\}$ is locally asymptotically stable follows the same line as in the proof of Theorem 1. To show global attractivity, consider an initial point

(x, τ, q) in $\mathbb{R}^n \times [0, 2] \times \{0, 1\}$. Note that, the solutions being complete and due to the structure of the timer, there exists (t_0, ℓ_0) in $\mathtt{dom}(x, \tau, q)$ such that $\tau(t_0, \ell_0) \geq 1$. Due to the fact that no more than one jump happens in the proof of Theorem 1, we employ the same arguments to obtain global attractivity. □

A. Discussion and example

The state component τ introduced in the system is a timer which enables us to ensure a sufficiently large dwell time between two switches. This is commonly used when dealing with switching controller (for instance this can be found in [5] and [28]). It allows to ensure observer convergence.

The use of saturated control for output feedback design was first introduced in [25]. As in [25], it allows to estimate the evolution speed of the estimate along the trajectories of the closed-loop system. This is for instance used to tune the observer convergence (see the proof of Lemma 3).

If Assumption 1 is a strong assumption this is not the case for Assumption 2. Indeed, as it is well known, giving a generic constant control u_∞, this assumption should be satisfied.

Note that as in Theorem 1, the sufficient condition given by Theorem 2 is a linear matrix inequality. Consequently some powerful LMI solvers (see [30] for instance) may be used as illustrated by the numerical examples given at the end of this Section.

To compute the set of matrices $\Gamma = \{T_1, \ldots, T_{2^p}\}$ the following approach can be employed.

Proposition 2: Let υ_i, $i = 1, \ldots, p$, be defined, for all i as the smallest positive real number such that

$$\begin{bmatrix} W_0 & e_i' H_0 \\ H_0' e_i & \upsilon_i \end{bmatrix} > 0 \ , \tag{41}$$

where e_i is the ith vector of the canonical base of $\mathbb{R}^p$. Then the set of 2^p vectors T_ℓ defined as

$$(T_\ell)_j = \sqrt{\upsilon_j} \text{ or } (T_\ell)_j = -\sqrt{\upsilon_j} \ ,$$

satisfies inequality (24).

Proof: Employing Schur complement, inequality (41) implies that the following inequality holds,

$$H_0' e_i e_i' H_0 - \upsilon_i W_0 \leq 0 \ . \tag{42}$$

It can be checked that by pre and post multiplying by W_0^{-1} inequality (42), it yields for all x in $\mathbb{R}^n$

$$|(H_0 W_0^{-1} x)_i|^2 = |e_i' H_0 W_0^{-1} x|^2 = x' W_0^{-1} H_0' e_i e_i' H_0 W_0^{-1} x \leq \upsilon_i x' W_0^{-1} x \ .$$

Consequently, for all $x = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}'$ such that $x' W_0^{-1} x \leq 1$, it yields that $|(H_0 W_0^{-1} x)_i| \leq \sqrt{\upsilon_i}$. Hence,

$$H_0 W_0^{-1} x \in \mathtt{Co}\{T_\ell, \ell = 1 \ldots, 2^p\} \ , \forall\, x \in \{x, x^T W_0^{-1} x \leq 1\}.$$

■

Inequality (41) being linear in H_0 and W_0, this implies that this inequality and the minimization of the cost υ_i can be included in the design of W_0 and H_0 in Theorem 1.

Example 6: As seen in Section II, Theorem 1 applies and we can construct a stabilizing state feedback. Assume now that the measurement available for feedback is given as $y = Cx$ where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. On this example, we can take $\upsilon = 49.5048$. Hence, from Assumption 2, $\Gamma = \{6.9137, -6.9137\}$. Moreover $u_\infty = 3$ satisfies Assumption 2 (in this particular case any u_∞ satisfies Assumption 2). Now, employing the solver Sedumi and Yalmip (see [19]), we get that the sufficient condition (25) is satisfied with,

$$Q_0 = \begin{bmatrix} 4.0484 & -0.2247 \\ -0.2247 & 0.0219 \end{bmatrix} \ , \ D_0 = \begin{bmatrix} -2.3251 \\ -3.5102 \end{bmatrix} \ .$$

We select the data, $\nu_\tau = 10$; With these data we obtain $u_0 = 7.0360$. Following the design described in the proof of Theorem 2, we get, $K_\infty = \begin{bmatrix} -115 \\ -1757 \end{bmatrix}$. With Matlab and employing an Euler discretization with discretization stepsize equal to 0.001 Figures 1 and 2 are obtained for the initial data: $x(0) = \begin{bmatrix} 0 \\ 4.5 \end{bmatrix}$, $\hat{x}(0,0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, $\tau(0,0) = 0$, $q(0,0) = 0$. The evolution of a solution to the closed-loop system with this initial data is given on Figures 1, 2-a and 2-b. Consequently, the hybrid output feedback controller obtained from Theorem 2 makes the origin a globally asymptotically stable equilibrium for the system (1).

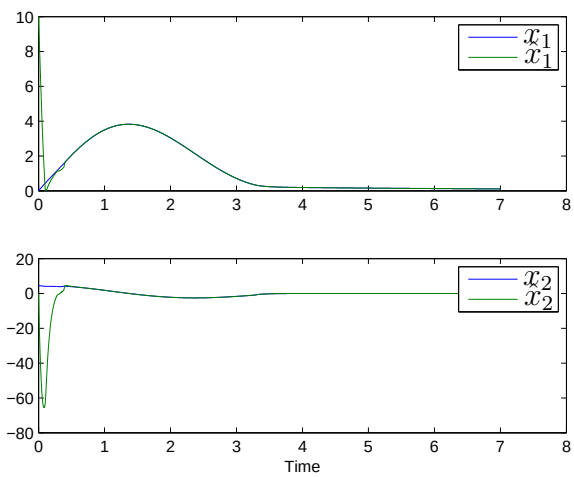
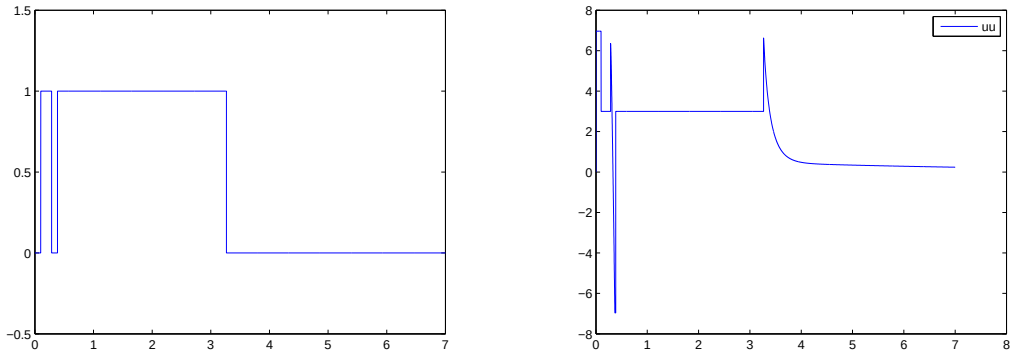


Fig. 1. Evolution of the state x and the observer state $\hat{x}$



a b

Fig. 2. a - Evolution of variable q. b - Evolution of the control u.

IV. Conclusion

We considered a particular class of bilinear systems for which there exists a constant feedback (see Assumption 1) making the trajectories of the closed-loop system bounded and converging to an equilibrium point (which is not the origin). From the knowledge of this constant feedback, a modification of this controller in order to make the origin a globally asymptotically equilibrium point has been proposed by relying on two different controllers, namely a global one (the constant feedback) and a linear one (synthesis via an LMI-based approach). Employing hybrid state feedback framework, it was possible to give some sufficient conditions in terms of LMI allowing us to design an uniting controller. Two cases were carried out: a state feedback controller and an output feedback controller when only the measured output can be used for control purposes. In this last context, the hybrid state feedback framework has been augmented with a hybrid observer to obtain an output feedback globally stabilizing the origin of the hybrid closed-loop system.

This work leaves some questions open. In particular the best choice of the two controllers we want to unite. A criterion for the selection of the state feedback and the hybrid loop could be used to achieve robustness or performance requirements. Furthermore, another interesting way should consist to study the bilinear systems with multiplicative control inputs only (i.e. system (1) with $B = 0$) as studied, for example, in [8], [17], [7], [4].

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APPENDIX

A. Proof of Lemma 6

To design the vector K_∞ and the matrix Q_∞ in order to get inequalities (33) and (34) we follow a high-gain approach (see [10]).

We introduce the matrices A_∞ in $\mathbb{R}^{n\times n}$ and C_1, ..., C_m in $\mathbb{R}^n$ defined as,

$$A_\infty = A + \sum_{j=1}^{p} u_{\infty,j} N_j \ , \ C = \begin{bmatrix} C_1' \\ \vdots \\ C_m' \end{bmatrix} \ .$$

By Assumption 2, the couple (A_∞, C) is observable. Hence there exists r integers $i_1, \ldots, i_r$ in $\{1, \ldots, m\}$ and r integers $(n_1, \ldots, n_r)$ such that $\sum_{j=1}^{r} n_j = n$, and such that the Kalman matrix $\mathcal{C}$ in $\mathbb{R}^{n\times n}$,

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 \\ \vdots \\ \mathcal{C}_r \end{bmatrix} \ , \ \mathcal{C}_j \in \mathbb{R}^{n_j \times n} \ , \ \mathcal{C}_j = \begin{bmatrix} C_{i_1}' \\ \vdots \\ C_{i_1}' A_\infty^{n_r - 1} \end{bmatrix} \ , \ j = 1, \ldots, r \ .$$

is invertible. Hence we can introduce the two matrices $\bar{A}_\infty$ and $\bar{C}$ in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{r\times n}$,

$$\bar{A}_\infty = \mathcal{C} A_\infty \mathcal{C}^{-1} \ , \ \bar{C} = \tilde{C}\mathcal{C}^{-1} \ , \ \tilde{C} = \begin{bmatrix} C_{i_1}' \\ \vdots \\ C_{i_r}' \end{bmatrix} \ . \tag{43}$$

Note that $\bar{C}_j = e_{n,1+\sum_{\ell=1}^{j-1} n_\ell}$ where $e_{n,s}$ is the s'th vector of the canonical base in $\mathbb{R}^n$. Also,

$$\tilde{C} = HC \ , \ H \in \mathbb{R}^{r\times m} \ , \ H = \begin{bmatrix} e'_{m,i_1} \\ \vdots \\ e'_{m,i_r} \end{bmatrix}$$

Moreover, note that the couple of matrices, $(\bar{A}_\infty, \bar{C})$ is in observability canonical form (in the multi output context). Also, $\bar{A}_\infty$ can be decomposed as, $\bar{A}_\infty = N + R$ with N and R in $\mathbb{R}^{n\times n}$ defined as,

$$N = \texttt{Diag}\left\{N_1, \ldots, N_r\right\} \ , R = \begin{bmatrix} R_1 \\ \vdots \\ R_r \end{bmatrix} \ ,$$

with N_j and R_j respectively in $\mathbb{R}^{n_j\times n_j}$ and in $\mathbb{R}^{n\times n_j}$,

$$N_j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & & \ldots & & 1 \\ 0 & & \ldots & & 0 \end{bmatrix} \ , R_j = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \alpha_{j,1} & \ldots & \alpha_{j,n} \end{bmatrix} \ ,$$

where α_j is in $\mathbb{R}^n$. Note that the couple $(N, \bar{C})$ is also observable. Hence there exists a matrix K in $\mathbb{R}^{n\times r}$ and a symmetric positive definite matrix Q in $\mathbb{R}^{n\times n}$ such that we have,

$$(N + K\bar{C})'Q + Q(N + K\bar{C}) \leq -I_n \ . \tag{44}$$

Now consider the matrix $\mathfrak{L}$ in $\mathbb{R}^{n\times n}$ defined as,

$$\texttt{Diag}\left\{\mathfrak{L}_1, \ldots, \mathfrak{L}_r\right\} \ , \ \mathfrak{L}_j = \texttt{Diag}\left\{L, \ldots, L^{n_j}\right\}$$

where L is a positive real number larger then 1.

Note that we have,

$$\mathfrak{L}_j^{-1} N_j = L N_j \mathfrak{L}_j^{-1} \ , \ \mathfrak{L}^{-1} N = L N \mathfrak{L}^{-1} \ ,$$

and,

$$\mathfrak{L}_j^{-1} R_j = L^{-n_j} R_j \ , \ \mathfrak{L}_j^{-1} R_j = R_j \bar{\mathfrak{L}}^{-1} \ ,$$

where $\bar{\mathfrak{L}} = \texttt{Diag}\{\bar{\mathfrak{L}}_1 \ldots, \bar{\mathfrak{L}}_r\}$, $\bar{\mathfrak{L}}_j = L^{n_j} I_{n_j}$. Hence, this implies the equality,

$$\mathfrak{L}^{-1}(\bar{A}_\infty + L\mathfrak{L}K\bar{C}) = L(N + K\bar{C})\mathfrak{L}^{-1} + R\bar{\mathfrak{L}}^{-1} \ .$$

Consequently, with (44) it yields

$$(\bar{A}_\infty + L\mathfrak{L}K\bar{C})'\mathfrak{L}^{-1}Q\mathfrak{L}^{-1} + \mathfrak{L}^{-1}Q\mathfrak{L}^{-1}(\bar{A}_\infty + L\mathfrak{L}K\bar{C}) \leq -L\mathfrak{L}^{-2} + \mathfrak{L}^{-1}QR\bar{\mathfrak{L}}^{-1} + \bar{\mathfrak{L}}^{-1}R'Q\mathfrak{L}^{-1} \ .$$

By completing the square, this gives,

$$(\bar{A}_\infty + L\mathfrak{L}K\bar{C})'\mathfrak{L}^{-1}Q\mathfrak{L}^{-1} + \mathfrak{L}^{-1}Q\mathfrak{L}^{-1}(\bar{A}_\infty + L\mathfrak{L}K\bar{C}) \leq -\mathfrak{L}^{-1}\left[LI_n - Q^2 - \mathfrak{L}\bar{\mathfrak{L}}^{-1}R'R\bar{\mathfrak{L}}^{-1}\mathfrak{L}\right]\mathfrak{L}^{-1} \ .$$

Note that since $\lambda_{\max}\{\bar{\mathfrak{L}}^{-1}\mathfrak{L}\} = 1$ when $L > 1$, it yields for all $L > 1$,

$$(\bar{A}_\infty + L\mathfrak{L}K\bar{C})'\mathfrak{L}^{-1}Q\mathfrak{L}^{-1} + \mathfrak{L}^{-1}Q\mathfrak{L}^{-1}(\bar{A}_\infty + L\mathfrak{L}K\bar{C}) \leq -\left[L - \lambda_{\max}\{Q^2\} - \lambda_{\max}\{R'R\}\right]\mathfrak{L}^{-2}$$

Pre- and post-multiplying the previous inequality by $\mathcal{C}'$ and $\mathcal{C}$ and employing equalities (43), it yields for all $L > 1$

$$(A_\infty + K_\infty(L)C)'Q_\infty(L) + Q_\infty(L)(A_\infty + K_\infty(L)C) \leq -\left[L - \lambda_{\max}\{Q^2\} - \lambda_{\max}\{R'R\}\right]\mathcal{C}'\mathfrak{L}^{-2}\mathcal{C} \ ,$$

with,

$$K_\infty(L) = \mathcal{C}^{-1}L\mathfrak{L}KH \ , \ Q_\infty(L) = \mathcal{C}'\mathfrak{L}^{-1}Q\mathfrak{L}^{-1}\mathcal{C} \ .$$

Also, since we have,

$$\mathcal{C}'\mathfrak{L}^{-2}\mathcal{C} \geq \frac{Q_\infty(L)}{\lambda_{\max}\{Q\}}$$

we finally get for all $L > \lambda_{\max}\{Q^2\} + \lambda_{\max}\{R'R\}$ inequality (33) with $\lambda_\infty = \lambda_\infty(L)$ and $Q_\infty = Q_\infty(L)$. In other words, we have,

$$(A_\infty + K_\infty(L)C)'Q_\infty(L) + Q_\infty(L)(A_\infty + K_\infty(L)C) \leq -\lambda_\infty(L)Q_\infty(L) \ ,$$

with

$$\lambda_\infty(L) = \frac{L - \lambda_{\max}\{Q^2\} - \lambda_{\max}\{R'R\}}{\lambda_{\max}\{Q\}} \ .$$

Note moreover, that we have,

$$\lambda_{\max}\{Q_\infty(L)\} \leq \lambda_{\max}\{Q\}\lambda_{\max}\{\mathcal{C}'\mathcal{C}\}L^{-2} \ ,$$

and,

$$\lambda_{\min}\{Q_\infty(L)\} \geq \lambda_{\min}\{Q\}\lambda_{\min}\{\mathcal{C}'\mathcal{C}\}L^{-2\max_{j=1,\ldots,r}\{n_j\}} \ .$$

Hence, taking L sufficiently large such that

$$\exp\left(-\frac{L}{\nu_t\lambda_{\max}\{Q\}}\right)L^{2\max_{j=1,\ldots,r}\{n_j\}-2} \tag{45}$$
$$< \exp\left(-\frac{\lambda_2}{\nu_t} + \frac{\lambda_{\max}\{Q^2\} + \lambda_{\max}\{R'R\}}{\nu_t\lambda_{\max}\{Q\}}\right)\frac{\lambda_{\min}\{Q_0\}\lambda_{\min}\{Q\}\lambda_{\min}\{\mathcal{C}'\mathcal{C}\}}{\lambda_{\max}\{Q_0\}\lambda_{\max}\{Q\}\lambda_{\max}\{\mathcal{C}'\mathcal{C}\}} \ ,$$

we get that inequality (34) is satisfied.