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► **To cite this version:**

Frédéric Abergel, Jean-Michel Rakotoson. Gradient blow up in Zygmund spaces for the very weak solution of a linear elliptic equation. *Discrete and Continuous Dynamical Systems - Series A*, 2013, 33 (5), pp.1809-1818. 10.3934/dcds.2013.33.1809 . hal-00647503v4

**HAL Id: hal-00647503**

**<https://hal.science/hal-00647503v4>**

Submitted on 20 Jan 2012

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# Gradient blow-up in Zygmund spaces for the very weak solution of a linear elliptic equation

Frédéric ABERGEL

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## Abstract

It is known that the very weak solution of  $-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx$ ,  $\forall \varphi \in C^2(\bar{\Omega})$ ,  $\varphi = 0$  on  $\partial\Omega$ ,  $u \in L^1(\Omega)$  has its gradient in  $L^1(\Omega)$  whenever  $f \in L^1(\Omega; \delta(1 + |\ln \delta|))$ ,  $\delta(x)$  being the distance of  $x \in \Omega$  to the boundary. In this paper, we show that if  $f \geq 0$  is not in this weighted space  $L^1(\Omega; \delta(1 + |\ln \delta|))$ , then its gradient blows up in  $L(\log L)$  at least. Moreover, we show that there exist a domain  $\Omega$  of class  $C^\infty$  and a function  $f \in L^1_+(\Omega, \delta)$  such that the associated very weak solution has its gradient being non integrable on  $\Omega$ .

**Keywords** Very weak solutions; Distance to the boundary; Regularity; Linear PDE; Monotone rearrangement; Gradient blow-up.

## 1 Introduction

In this paper, we state and prove two results related to the behaviour near the boundary of very weak solutions to Laplace's equation. In the first part of the paper, we prove that the very weak solution  $u \in L^1(\Omega)$  of the so-called Brezis weak formulation (see [4, 5, 6])

$$-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in C^2(\bar{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega, \quad (1)$$

verifies  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty$  whenever  $f \geq 0$ ,  $f$  is integrable with respect to the distance function  $\delta(x) = \operatorname{dist}(x, \partial\Omega)$  but  $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$ . The result implies in particular, that  $|\nabla u|$  blows up in the Zygmund space  $L(\operatorname{Ln} L)$ :

$$\int_{\Omega_*} |\nabla u|_{**}(t) dt = \infty \quad (2)$$

where  $|\nabla u|_{**}(t) = \frac{1}{t} \int_0^t |\nabla u|_*(\sigma) d\sigma$ ,  $|\nabla u|_*$  is the decreasing rearrangement of  $|\nabla u|$  and  $\Omega_* = ]0, \operatorname{meas}(\Omega)[$  (see Section 2 below).

In the second part, we construct an open bounded smooth set  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$ , and a function  $f \in L^1(\Omega, \delta)$ ,  $f \notin L^1(\Omega, \delta(1 + |\operatorname{Ln} \delta|))$ , such that the associate very weak solution  $u$  verifies

$$\int_{\Omega} |\nabla u| dx = +\infty. \quad (3)$$

## 2 Background and notations

The main properties of Lorentz spaces, see e.g. [1, 10], are briefly recalled.

For a Lebesgue measurable set  $E$  of  $\Omega$ , denote by  $|E|$  its measure.

The **decreasing rearrangement of a measurable function**  $u$  is the function  $u_*$  defined by

$$u_* : \Omega_* = ]0, |\Omega|[ \rightarrow \mathbb{R}, \quad u_*(s) = \inf\{t \in \mathbb{R} : |u > t| \leq s\}.$$

In particular, there holds:

$$u_*(0) = \operatorname{ess\,sup}_{\Omega} u, \quad u_*(|\Omega|) = \operatorname{ess\,inf}_{\Omega} u.$$

Introducing

$$|v|_{**}(t) = \frac{1}{t} \int_0^t |v|_*(s) ds \text{ for } t \in \Omega_* = ]0, |\Omega|[,$$

the Lorentz spaces can now be defined.

For  $1 < p < +\infty$ ,  $1 \leq q \leq +\infty$ ,

$$L^{p,q}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable, } |v|_{L^{p,q}}^q \equiv \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty \right\}$$

and, for  $q = +\infty$ ,

$$L^{p,\infty}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable, } |v|_{L^{p,\infty}} \equiv \sup_{t \leq |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \right\}.$$

We recall

$$L_{exp}(\Omega) = \left\{ v \in L^1(\Omega), \quad |v|_{L_{exp}(\Omega)} = \sup_{0 < t \leq |\Omega|} \frac{|v|_{**}(t)}{1 + \text{Ln} \frac{|\Omega|}{t}} < +\infty \right\}$$

and

$$L(\text{Ln } L) = \left\{ v \in L^1(\Omega), \quad |v|_{L(\text{Ln } L)} = \int_{\Omega_*} |v|_{**}(t) dt < +\infty \right\},$$

the dual of  $L(\text{Ln } L)$  is  $L_{exp}(\Omega)$  and one has  $\forall f \in L_{exp}(\Omega), \quad \forall g \in L(\text{Ln } L)$

$$\int_{\Omega} |fg| dx \leq c |f|_{L_{exp}(\Omega)} \cdot |g|_{L(\text{Ln } L)}, \text{ for some constant } c > 0$$

(for more details see [1]).

Finally, we define

$$W^1(\Omega, |\cdot|_{p,q}) = \left\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega) \right\}$$

and

$$\begin{aligned} C_c^m(\Omega) &= \left\{ \varphi \in C^m(\Omega), \varphi \text{ has compact support in } \Omega \right\}. \\ C^{0,1}(\bar{\Omega}) &= \left\{ v : \bar{\Omega} \rightarrow \mathbb{R} \text{ is a Lipschitz function} \right\}, \\ C^{m,1}(\bar{\Omega}) &= \left\{ v \in C^m(\bar{\Omega}) : D^\alpha v \in C^{0,1}(\bar{\Omega}) \text{ for } |\alpha| = m \right\}. \end{aligned}$$

For the sake of completeness, we also recall some general results concerning Equation (1).

**Proposition 1.** (see [2, 5, 9])

Let  $\Omega$  be an open bounded set of class  $C^{2,1}$  in  $\mathbb{R}^N$  (see Gilbarg-Trudinger [8] for a precise definition),  $f \in L^1(\Omega, \delta)$ , where  $\delta(x)$  is the distance function of  $x \in \Omega$  to the boundary  $\partial\Omega$ . Then, there exists a constant  $c > 0$  such that for any solution  $u$  of (1), one has

$$\begin{aligned} 1. \quad & |\nabla u|_{L^{1+\frac{1}{N},\infty}(\Omega;\delta)} \leq c |f|_{L^1(\Omega;\delta)}, \\ & |u|_{L^{N',\infty}(\Omega)} \leq c |f|_{L^1(\Omega;\delta)}, \quad N' = \frac{N}{N-1} \text{ if } N \geq 2, \quad N' = +\infty \text{ otherwise,} \end{aligned}$$

2. If  $f \geq 0$ , then  $u \geq 0$ .

3. If  $f \in L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$ , then  $u \in W_0^{1,1}(\Omega)$  and

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))}.$$

4. If  $\Omega$  is a ball and  $f$  is radial, then  $u \in W_0^{1,1}(\Omega)$  and

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega; \delta)}.$$

5. If  $\Omega = ]a, b[$  then the above estimate holds for all  $f \in L^1(]a, b[, \delta)$ . ◇

### 3 Blow-up in Zygmund space for $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$

The aim of this section is to prove the

#### Theorem 1.

Under the same assumptions as for Proposition 1, if  $f \geq 0$  and  $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$ , then any solution  $u$  of (1) satisfies

1.  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty$ ;
2.  $\int_{\Omega_*} |\nabla u|_{**}(t) dt = +\infty$  (i.e.  $\int_{\Omega} |\nabla u| \max(\operatorname{Ln} |\nabla u|; 0) dx = +\infty$ ).

Before proving Theorem 1, we state and prove the

#### Lemma 1.

Let  $u$  be a very weak solution of (1) and assume that  $\int_{\Omega} |\nabla u| dx < +\infty$ . Then,  $u$  satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega). \quad (4)$$

**Proof**

By the density of  $C_c^2(\Omega)$  in  $C_c^1(\Omega)$ , one has

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in C_c^1(\Omega)$$

and the lemma follows.

Using standard truncation and convolution arguments (see [11, 7, 3]), one can also prove the following approximation result:

**Proposition 2.**

Let  $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$ . There exists a sequence  $\varphi_n \in C_c^1(\Omega)$  such that

1.  $\exists c > 0$ ,  $|\nabla \varphi_n|_{\infty} \leq c(|\nabla \varphi|_{\infty} + |\varphi|_{\infty}) \quad \forall n$ ;
2.  $\varphi_n \rightarrow \varphi$  in  $C(\bar{\Omega})$  (i.e.  $\max_{x \in \Omega} |\varphi_n(x) - \varphi(x)| \xrightarrow{n \rightarrow +\infty} 0$ );
3.  $\nabla \varphi_n \rightharpoonup \nabla \varphi$  in  $L^{\infty}(\Omega)^N$ -weak star.

(proof omitted).

**Proof of Theorem 1**

Considering  $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$  and its approximating sequence as in Proposition 2, we have

$$\int_{\Omega} \nabla u \cdot \nabla \varphi_n \, dx = \int_{\Omega} f \varphi_n \, dx. \quad (5)$$

By Statement 3. of Proposition 2, there holds

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_n \, dx = \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx \quad (6)$$

and, using the mean value theorem and Statement 1.

$$|\varphi_n(x) - \varphi(x)| \leq |\nabla(\varphi_n - \varphi)|_{\infty} \cdot \delta(x) \leq c_{\varphi} \delta(x). \quad (7)$$

Since  $f \in L^1(\Omega; \delta)$ , the Lebesgue dominated convergence theorem yields

$$\lim_n \int_{\Omega} f \varphi_n \, dx = \int_{\Omega} f \varphi \, dx. \quad (8)$$

Combining relations (5) to (8), we obtain relation (4) of Lemma 1.  $\diamond$

Next, we want to prove

**Lemma 2.**

*Under the same assumptions as for Proposition 1, if  $f \geq 0$  and  $u$  is the very weak solution (1) such that  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx < +\infty$ , then there exists a constant  $c(\Omega) > 0$  (independent of  $u$ ) such that*

$$\int_{\Omega} f \delta |\operatorname{Ln} \delta| dx \leq c(\Omega) \left( \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \delta|) dx + \int_{\Omega} f \delta dx \right). \quad (9)$$

**Proof** Let us first note that, according to Proposition 1 Statement 1, we have

$$\int_{\Omega} |\nabla u| \delta dx \leq c \int_{\Omega} f \delta dx < +\infty.$$

Therefore, the fact that  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx < +\infty$  is equivalent to  $\int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \delta|) dx < +\infty$ , since  $|\operatorname{Ln} \delta| \geq \beta > 0$  near the boundary.

Fix  $0 < \varepsilon < \frac{1}{2}$  and consider  $\varphi_1 > 0$  the first eigenfunction of  $-\Delta$  with Dirichlet boundary condition:  $-\Delta \varphi_1 = \lambda_1 \varphi_1$  in  $\Omega$ ,  $\varphi_1 = 0$  on  $\partial\Omega$ . Then,  $\varphi \equiv \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$  is a good test function, and there holds

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx = \int_{\Omega} \nabla u \cdot \nabla \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx + \int_{\Omega} \nabla u \cdot \nabla \varphi_1 \frac{\varphi_1 \operatorname{sign}(\operatorname{Ln}(\varphi_1 + \varepsilon))}{\varphi_1 + \varepsilon} dx. \quad (10)$$

Since  $|\nabla \varphi_1|_{\infty} < +\infty$  and  $|\operatorname{Ln}(\varphi_1 + \varepsilon)| \leq |\operatorname{Ln} \varphi_1| + 1$ , we deduce

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|) dx. \quad (11)$$

Letting  $\varepsilon \rightarrow 0$  and using Fatou's lemma yields

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln} \varphi_1| dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|). \quad (12)$$

Since there exist two constants  $c_0 > 0$ ,  $c_1 > 0$  such that  $c_0 \delta \leq \varphi_1 \leq c_1 \delta$ , Relation (9) follows from Relation (12).

### End of the proof of Theorem 1

Let  $f \geq 0$  be in  $L^1(\Omega; \delta)$  and  $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$ , so that  $\int_{\Omega} f(x)\delta |\operatorname{Ln} \delta| dx = +\infty$ . From Lemma 2, we deduce that

$$\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty, \quad (13)$$

which proves Statement 1.

As for Statement 2., we see that  $|\operatorname{Ln} \delta| \in L_{exp}(\Omega)$  since  $\delta^{-\varepsilon} \in L^1(\Omega)$  for  $0 < \varepsilon < 1$ , there holds

$$\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| \leq c |\operatorname{Ln} \delta|_{L_{exp}(\Omega)} \cdot |\nabla u|_{L(\operatorname{Ln} L)}. \quad (14)$$

Relation (14) and the fact that  $|\operatorname{Ln} \delta|_{L_{exp}(\Omega)} < +\infty$  imply that

$$|\nabla u|_{L(\operatorname{Ln} L)} = \int_{\Omega^*} |\nabla u|_{**}(t) dt = +\infty \quad (15)$$

and Statement 2. is proven.  $\diamond$

## 4 Existence of a domain $\Omega$ and a very weak solution whose gradient blows up in $L^1(\Omega)$

The main result of this section is

### Theorem 2.

*There exist a domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$ , of class  $C^\infty$  and a function  $f \in L^1(\Omega, \delta)$  such that the weak solution  $u_0$  of (1) satisfies*

$$|\nabla u_0| \notin L^1(\Omega)$$

*(that is:  $\int_{\Omega} |\nabla u_0|(x) dx = +\infty$ ).*

The key ingredient in the proof of Theorem 2 is the following

**Lemma 3.**

There exist a domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$ , of class  $C^\infty$  and a nonnegative function  $g \in L^N(\Omega)$  such that the unique solution  $\psi > 0$  of  $-\Delta\psi = g$  in  $\Omega$ ,  $\psi \in W^{2,N}(\Omega) \cap H_0^1(\Omega)$ , satisfies

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty.$$

Let us admit temporarily this lemma ( which merely amounts to saying that  $|\nabla\psi(x)|$  is very large near a point of the boundary). Note that, according to Sobolev Embedding Theorem,  $W^{2,N}(\Omega)$  is included in  $C^{0,\alpha}(\overline{\Omega})$  for all  $\alpha < 1$ , but not in  $C^{0,1}$  in general.

**Proof of Theorem 2**

Let us consider the domain  $\Omega$  constructed in Lemma 3 and assume that, for any  $f \in L^1(\Omega, \delta)$ , the unique solution  $u$  of (1) satisfies  $|\nabla u| \in L^1(\Omega)$ . Then, define

$$\begin{aligned} (-\Delta)^{-1} : L^1(\Omega, \delta) &\rightarrow L^1(\Omega) \\ f &\mapsto u = (-\Delta)^{-1}f, \end{aligned}$$

$u$  being the unique solution of (1), and set

$$Tf = \nabla(-\Delta)^{-1}f.$$

One has the

**Lemma 4.**

If every very weak solution  $u$  satisfies  $\int_{\Omega} |\nabla u| dx < +\infty$ , then

$$\sup_{|f|_{L^1(\Omega, \delta)}=1} |Tf|_{L^1(\Omega)^N} \text{ is finite.}$$

**Proof**

Choose  $0 < \varepsilon \leq \varepsilon_0$  ( $\varepsilon_0$  small enough) and set  $\Omega_\varepsilon = \{x \in \Omega : \delta(x) > \varepsilon\}$ ,  $T_\varepsilon f = \chi_{\Omega_\varepsilon} \nabla(-\Delta)^{-1}f$ , with  $\chi_{\Omega_\varepsilon}$  the characteristic function of  $\Omega_\varepsilon$ . If  $\int_{\Omega} |\nabla(-\Delta)^{-1}f| dx < +\infty$ , then by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \rightarrow 0} |T_\varepsilon f - Tf|_{L^1(\Omega)^N} = 0$$

and

$$|T_\varepsilon f|_{L^1(\Omega)^N} \leq \frac{1}{\varepsilon} \int_\Omega |\nabla u| \delta \, dx \leq \frac{c}{\varepsilon} |f|_{L^1(\Omega, \delta)},$$

from Proposition 1., This last inequality shows that  $T_\varepsilon$  is continuous linear operator from  $L^1(\Omega, \delta)$  into  $L^1(\Omega)^N$ . One obtains by the Banach-Steinhaus boundedness principle  $\sup_{\varepsilon > 0} \|T_\varepsilon\| < +\infty$ . This implies that there exists a constant  $c(\Omega) > 0$  such that

$$\int_\Omega |\nabla(-\Delta)^{-1} f| \, dx = |\nabla u|_{L^1(\Omega)} \leq c(\Omega) |f|_{L^1(\Omega, \delta)}. \quad (16)$$

◇

Considering the sequence

$$u_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega) \text{ (with } p > N)$$

solution of

$$-\Delta u_k = f_k = \min(|f|; k) \text{sign}(f), \quad f_k \rightarrow f \text{ in } L^1(\Omega, \delta).$$

By the estimate (16), we deduce

$$|\nabla(u_k - u)|_{L^1(\Omega)} \leq c(\Omega) |f - f_k|_{L^1(\Omega, \delta)} \rightarrow 0$$

and therefore

$$u \in W_0^{1,1}(\Omega).$$

Since  $\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty$ , there exists a function  $f_0 \in L^1(\Omega, \delta)$ ,  $f_0 \geq 0$ , such that

$$\int_\Omega f_0(x) \psi(x) \, dx = +\infty.$$

Indeed, the Hopf Maximum Principle ensures the existence of a constant  $k_1 > 0$  such that

$$\psi(x) \geq k_1 \delta(x) \quad \forall x \in \Omega.$$

Hence

$$L^1(\Omega, \psi) \subsetneq L^1(\Omega, \delta). \quad (17)$$

If  $L^1(\Omega, \psi) = L^1(\Omega, \delta)$ , there must exist a constant  $c_1(\Omega) > 0$  such that

$$|f|_{L^1(\Omega, \psi)} \leq c_1(\Omega) |f|_{L^1(\Omega, \delta)} \quad \forall f \in L^1(\Omega, \delta). \quad (18)$$

This is due to Banach principle. Equivalently, since the function spaces  $L^1(\Omega, \psi)$  and  $L^1(\Omega, \delta)$  are Banach spaces, one can deduce this inequality from the properties of Banach spaces, see e.g. [1], Theorem 1.8.

Relation (18) would then imply that  $\psi(x) \leq c_1(\Omega)\delta(x)$  for all  $x \in \Omega$ . This contradicts the fact that

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty.$$

Therefore, there exists a function  $f_0 \in L^1(\Omega, \delta)$  such that  $f_0 \notin L^1(\Omega, \psi)$ , i.e. such that  $\int_{\Omega} |f_0(x)|\psi(x) dx = +\infty$ . We may obviously assume that  $f_0 \geq 0$  (otherwise, simply consider  $|f_0|$ ).

Defining the sequence  $f_{0k} = T_k(f_0) = \min(f_0; k)$  and  $\bar{u}_k$  the solution of  $-\Delta \bar{u}_k = f_{0k}$ , one has using relation (16) that

$$\begin{aligned} 0 &\leq \int_{\Omega} f_{0k} \psi dx = - \int_{\Omega} \psi \Delta \bar{u}_k dx = - \int_{\Omega} \bar{u}_k \Delta \psi dx = \int_{\Omega} \bar{u}_k g dx \\ &\leq |\bar{u}_k|_{L^{N'}} \cdot |g|_{L^N} \leq c |\nabla \bar{u}_k|_{L^1} \cdot |g|_{L^N} \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N}. \end{aligned} \quad (19)$$

Letting  $k \rightarrow +\infty$  in relation (19), we derive from Beppo-Levi's theorem

$$+\infty = \int_{\Omega} f_0 \psi dx = \lim_{k \rightarrow +\infty} \int_{\Omega} f_{0k} \psi dx \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N} < +\infty,$$

which is absurd. Hence, there exists a function  $f_0 \in L^1(\Omega, \delta)$  such that the associate weak solution  $u_0$  satisfies  $|\nabla u_0| \notin L^1(\Omega)$ .  $\diamond$

### Proof of Lemma 3

For the sake of convenience, we shall start with the case  $N = 2$  and generalize the construction in a second step.

Let us first consider the open set

$$\Omega_1 = \left\{ x = (x_1, x_2) : x_1^2 < x_2, x_1^2 + x_2^2 < \frac{1}{e} \right\}$$

and define the preliminary function on  $\Omega_1$  by

$$w(x_1, x_2) = (x_2 - x_1^2) \operatorname{Ln} \left( \operatorname{Ln} \frac{1}{x_1^2 + x_2^2} \right).$$

Note that, in polar coordinates,  $w$  can be written as  $w = r(\sin(\theta) - r \cos^2(\theta)) \operatorname{Ln}(\operatorname{Ln}(\frac{1}{r^2}))$ .

$w$  has the following properties:

1.  $w > 0$  in  $\Omega_1$ ,
2.  $w(x_1, x_1^2) = 0$  for  $-x_{1c} < x_1 < x_{1c}$  with  $x_{1c}^4 + x_{1c}^2 = \frac{1}{e}$ , and  $w(x_1, x_2) = 0$  for  $x_1^2 + x_2^2 = \frac{1}{e}$ ,
3.  $w \in C^\infty(\Omega_1) \cap H^2(\Omega_1)$ .

Indeed it is sufficient to compute  $\frac{\partial w}{\partial x_i}$  and  $\frac{\partial^2 w}{\partial x_i^2}$  and prove that  $\Delta w \in L^2(\Omega)$ .

$$w[x_1, x_2] = (-x_1^2 + x_2) \operatorname{Ln} \left[ \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_1} = -\frac{2x_1(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]} - 2x_1 \operatorname{Ln} \left[ \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_2} = -\frac{2x_2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]} + \operatorname{Ln} \left[ \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x_1^2} &= -\frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]} + \frac{2(5x_1^2 - x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]} \\ &\quad - 2 \operatorname{Ln} \left[ \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right] \end{aligned}$$

$$\frac{\partial^2 w}{\partial x_2^2} = -\frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]} + \frac{2(x_1^2 - 3x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]}$$

$$\Delta w = \frac{4 \left( x_1^2 - x_2 + (2x_1^2 - x_2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} - 2 \operatorname{Ln} \left[ \operatorname{Ln} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right].$$

Using polar coordinates, one can check that

$$\Delta w \in L^2(\Omega) \text{ and } |\nabla w| \in L^p(\Omega) \text{ for all } p < +\infty.$$

Consider now  $x_{1c} > \eta > 0$  and an open set  $\Omega$ , of class  $C^{2,1}$  at least, such that

$$\Omega \subset [-x_{1c} + \eta; x_{1c} - \eta] \times \left[0, \frac{1}{\sqrt{e}}\right].$$

$$\partial\Omega \text{ contains } \Gamma_0 = \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_2 = x_1^2 \right\} \cap \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_1^2 + x_2^2 = \frac{1}{e} \right\}.$$

For  $a > 0 : 0 < 2a < x_{1c} - \eta$ , define a smooth function  $\theta$  such that

$$\theta \in C_c^\infty(\mathbb{R}^2), \theta \geq 0 \text{ and } \begin{cases} 0 \leq \theta \leq 1, \\ \theta(x_1, x_2) = 1 & \text{if } |x_1| \leq a, \\ \theta(x_1, x_2) = 0 & \text{if } |x_1| > \frac{3a}{2}. \end{cases}$$

In particular, the function  $\theta w$  vanishes on the boundary of  $\Omega$ . Let us show that  $\psi_0 = \theta w$  satisfies the following properties:

1.  $-\Delta\psi_0 = f_0 \in L^2(\Omega)$ .
2.  $\text{Max} \left\{ \frac{\psi_0(x)}{\delta(x)}, x \in \Omega \right\} = +\infty$ .

Property 1 is obvious, since

$$\begin{cases} -\Delta\psi_0 = -(\Delta\theta w + 2\nabla w \nabla\theta + \Delta w \theta) \in L^2(\Omega), \\ \psi_0 \in H_0^1(\Omega). \end{cases}$$

To prove Property 2, consider  $x = (x_1, \alpha x_1)$ ,  $0 < \alpha < 1$ ,  $x_1$  small enough so that  $x \in \Omega$ . Then there holds

$$\frac{\psi_0(x)}{\delta(x)} = \frac{\psi_0(x_1, \alpha x_1)}{\delta(x)} \geq \frac{(\alpha - x_1) \text{Ln}((- \text{Ln}(1 + \alpha^2)x_1^2))}{\sqrt{1 + \alpha^2}} \xrightarrow{x_1 \rightarrow 0} +\infty$$

which shows that

$$\sup \left\{ \frac{\psi_0(x)}{\delta(x)} : x \in \Omega \right\} = +\infty.$$

Setting  $g = |f_1|$  and considering  $\psi > 0$  solution of  $-\Delta\psi = g$ ,  $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ , one has by the maximum principle that  $\psi \geq \psi_0$  (so that  $\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty$ ).  $\diamond$

The construction above can be generalized to  $\mathbb{R}^N$ , let us outline the main steps of the procedure. For  $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$ , set

$$|x'|^2 = x_1^2 + \dots + x_{N-1}^2, \quad \sigma(x) = x_1^2 + \dots + x_N^2.$$

We first consider the open set

$$\Omega_1 = \left\{ x = (x', x_N), |x'|^2 < x_N, \sigma(x) < \frac{1}{e} \right\}$$

and define on  $\Omega_1$  the nonnegative function

$$w(x) = (x_N - |x'|^2) \operatorname{Ln} \left( \operatorname{Ln} \frac{1}{\sigma(x)} \right), \quad x \in \Omega_1.$$

$w$  satisfies properties similar to those stated in Properties 1-3 above for the two-dimensional case.

For a small  $a > 0$ , consider  $\theta \in C_c^\infty(\mathbb{R}^N)$  such that

$$\begin{cases} 0 \leq \theta \leq 1, \\ \theta(x', x_N) = 1 & \text{if } |x'| \leq a, \\ \theta(x', x_N) = 0 & \text{if } |x'| > \frac{3a}{2}. \end{cases}$$

and an open set  $\Omega$  of class  $C^{2,1}$  with  $\operatorname{supp} \theta \cap \Omega_1 \subset \Omega$  and  $\theta w = 0$  on  $\partial\Omega$ .

Then, the function  $\psi_0(x) = \theta w(x)$  satisfies  $-\Delta\psi_0 \in L^N(\Omega)$ , since  $\Delta w \in L^N(\Omega)$  thanks to a straightforward computation. Setting

$$e_{N-1} = (1, \dots, 1), \quad x_\alpha = x_N(\alpha, \dots, \alpha, 1) \in \Omega, \quad \alpha > 0,$$

for  $x_N > 0$ ,  $x_N$  small enough, there holds

$$\frac{\psi_0(x_\alpha)}{\delta(x_\alpha)} \geq \frac{(1 - \alpha^2 |e_{N-1}|^2 x_N)}{\sqrt{1 + \alpha^2 |e_{N-1}|^2}} \operatorname{Ln} \left( \operatorname{Ln} \left( \frac{1}{x_N^2 (1 + \alpha^2 (N - 1))} \right) \right) \xrightarrow{x_N \rightarrow 0} +\infty.$$

Therefore

$$\sup_{x \in \Omega} \left\{ \frac{\psi_0(x)}{\delta(x)} \right\} = +\infty.$$

Finally, considering the solution  $\psi$  of

$$\begin{cases} -\Delta \psi = |\Delta \psi_0| = g \in L^N(\Omega) \\ \psi \in W^{1,N}(\Omega) \cap W^{2,N}(\Omega), \end{cases}$$

there holds that

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty,$$

which ends the proof of Lemma 3. ◇

**Acknowledgment:** the authors thank C. Falaise-Bougant for the realization of this manuscript and for helping with Wolfram Mathematica<sup>®</sup> 8, which was used in the computation of the derivatives of  $w$ .

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