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Gradient blow-up in Zygmund spaces for the very weak solution of a linear elliptic equation

Frédéric ABERGEL

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Abstract

It is known that the very weak solution $u \in L^1(\Omega)$ of $-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx$ for all test functions $\varphi \in C^2(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$, has its gradient in $L^1(\Omega)$ whenever $f \in L^1(\Omega; \delta(1 + |\text{Ln } \delta|))$, $\delta(x)$ being the distance of $x \in \Omega$ to the boundary. In this paper, we show that if $f \geq 0$ is not in this weighted space $L^1(\Omega; \delta(1 + |\text{Ln } \delta|))$, then the gradient of u blows up in $L(\log L)$ at least. Moreover, we show that there exist a domain Ω of class C^∞ and a function $f \in L^1_+(\Omega, \delta)$ such that the associated very weak solution has its gradient being non integrable on Ω .

Keywords Very weak solutions; Distance to the boundary; Regularity; Linear PDE; Monotone rearrangement; Gradient blow-up.

1 Introduction

In this paper, we state and prove two results related to the behaviour near the boundary of very weak solutions to Laplace's equation. In the first part of the paper, we prove that the very weak solution $u \in L^1(\Omega)$ of the so-called Brezis weak formulation (see [4, 5, 6])

$$-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega, \quad (1)$$

verifies $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty$ whenever $f \geq 0$, f is integrable with respect to the distance function $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ but $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$. This result implies in particular that $|\nabla u|$ blows up in the Zygmund space $L(\operatorname{Ln} L)$:

$$\int_{\Omega_*} |\nabla u|_{**}(t) dt = \infty, \quad (2)$$

where $|\nabla u|_{**}(t) = \frac{1}{t} \int_0^t |\nabla u|_*(\sigma) d\sigma$, $|\nabla u|_*$ is the decreasing rearrangement of $|\nabla u|$ and $\Omega_* =]0, \operatorname{meas}(\Omega)[$ (see Section 2 below for the precise definitions).

In the second part, we construct an open bounded smooth set Ω of \mathbb{R}^N , $N \geq 2$, and a function $f \in L^1(\Omega, \delta)$, $f \notin L^1(\Omega, \delta(1 + |\operatorname{Ln} \delta|))$, such that the associate very weak solution u verifies

$$\int_{\Omega} |\nabla u| dx = +\infty. \quad (3)$$

2 Background and notations

The main properties of Lorentz spaces, see e.g. [1, 9], are briefly recalled.

For a Lebesgue measurable set E of Ω , denote by $|E|$ its measure. The **decreasing rearrangement** of a measurable function u is the function u_* defined by

$$u_* : \Omega_* =]0, |\Omega|[\rightarrow \mathbb{R}, \quad u_*(s) = \inf\{t \in \mathbb{R} : |u > t| \leq s\}.$$

In particular, there holds:

$$u_*(0) = \operatorname{ess\,sup}_{\Omega} u, \quad u_*(|\Omega|) = \operatorname{ess\,inf}_{\Omega} u.$$

Introducing

$$|v|_{**}(t) = \frac{1}{t} \int_0^t |v|_*(s) ds \text{ for } t \in \Omega_* =]0, |\Omega|[,$$

the Lorentz spaces can now be defined.

For $1 < p < +\infty$, $1 \leq q \leq +\infty$,

$$L^{p,q}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable, } |v|_{L^{p,q}}^q \equiv \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty \right\}$$

and, for $q = +\infty$,

$$L^{p,\infty}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable}, |v|_{L^{p,\infty}} \equiv \sup_{t \leq |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \right\}.$$

Finally, we define

$$W^1(\Omega, |\cdot|_{p,q}) = \left\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega) \right\}$$

and

$$C_c^m(\Omega) = \left\{ \varphi \in C^m(\Omega), \varphi \text{ has compact support in } \Omega \right\}.$$

For the sake of completeness, we also recall some general results concerning Equation (1).

Proposition 1. (see [2, 5, 8])

Let Ω be an open bounded set of class $C^{2,1}$ in \mathbb{R}^N , $f \in L^1(\Omega, \delta)$, where $\delta(x)$ is the distance function of $x \in \Omega$ to the boundary $\partial\Omega$. Then, there exists a constant $c > 0$ such that for any solution u of (1), one has

1. $|\nabla u|_{L^{1+\frac{1}{N'},\infty}(\Omega;\delta)} \leq c|f|_{L^1(\Omega;\delta)},$
 $|u|_{L^{N',\infty}(\Omega)} \leq c|f|_{L^1(\Omega;\delta)}, N' = \frac{N}{N-1}$ if $N \geq 2, N' = +\infty$ otherwise.

2. If $f \geq 0$, then $u \geq 0$.

3. If $f \in L^1(\Omega; \delta(1 + |\ln \delta|))$, then $u \in W_0^{1,1}(\Omega)$ and

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega; \delta(1 + |\ln \delta|))}.$$

4. If Ω is a ball and f is radial, then $u \in W_0^{1,1}(\Omega)$ and

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega;\delta)}.$$

5. If $\Omega =]a, b[$ then the above estimate holds for all $f \in L^1(]a, b[, \delta)$.

◇

3 Blow-up in Zygmund space for $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$

The aim of this section is to prove the

Theorem 1.

Under the assumptions of Proposition 1, if $f \geq 0$ and $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$, then any solution u of (1) satisfies

1. $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty$;
2. $\int_{\Omega_*} |\nabla u|_{**}(t) dt = +\infty$ (i.e. $\int_{\Omega} |\nabla u| \max(\operatorname{Ln} |\nabla u|; 0) dx = +\infty$).

Before proving Theorem 1, we state and prove the

Lemma 1.

Let u be a very weak solution of (1) and assume that $\int_{\Omega} |\nabla u| dx < +\infty$. Then, u satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega). \quad (4)$$

Proof

By the density of $C_c^2(\Omega)$ in $C_c^1(\Omega)$, one has

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in C_c^1(\Omega)$$

and the lemma follows.

Using standard truncation and convolution arguments (see [10, 7, 3]), one can also prove the following approximation result:

Proposition 2.

Let $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$. There exists a sequence $\varphi_n \in C_c^1(\Omega)$ such that

1. $\exists c > 0, |\nabla \varphi_n|_{\infty} \leq c \left(|\nabla \varphi|_{\infty} + |\varphi|_{\infty} \right) \quad \forall n,$

2. $\varphi_n \rightarrow \varphi$ in $C(\overline{\Omega})$ (i.e. $\max_{x \in \Omega} |\varphi_n(x) - \varphi(x)| \xrightarrow{n \rightarrow +\infty} 0$),

3. $\nabla \varphi_n \rightharpoonup \nabla \varphi$ in $L^\infty(\Omega)^N$ -weak star.

(proof omitted).

Proof of Theorem 1

Considering $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$ and its approximating sequence as in Proposition 2, there holds

$$\int_{\Omega} \nabla u \cdot \nabla \varphi_n dx = \int_{\Omega} f \varphi_n dx. \quad (5)$$

By Statement 3. of Proposition 2, one also has that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_n dx = \int_{\Omega} \nabla u \cdot \nabla \varphi dx \quad (6)$$

and, using the mean value theorem and Statement 1.,

$$|\varphi_n(x) - \varphi(x)| \leq |\nabla(\varphi_n - \varphi)|_{\infty} \cdot \delta(x) \leq c_{\varphi} \delta(x). \quad (7)$$

Since $f \in L^1(\Omega; \delta)$, the Lebesgue dominated convergence theorem yields

$$\lim_n \int_{\Omega} f \varphi_n dx = \int_{\Omega} f \varphi dx. \quad (8)$$

Combining relations (5) to (8), we obtain the result. \diamond

Next, we want to prove

Lemma 2.

Under the assumptions of Proposition 1, if $f \geq 0$ and u is the very weak solution of (1) such that $\int_{\Omega} |\nabla u| |\ln \delta| dx < +\infty$, then there exists a constant $c(\Omega) > 0$ (independent of u) such that

$$\int_{\Omega} f \delta |\ln \delta| dx \leq c(\Omega) \left(\int_{\Omega} |\nabla u| (1 + |\ln \delta|) dx + \int_{\Omega} f \delta dx \right). \quad (9)$$

Proof Let us first note that, according to Proposition 1, Statement 1, we have

$$\int_{\Omega} |\nabla u| \delta \, dx \leq c \int_{\Omega} f \delta \, dx < +\infty.$$

Therefore, the fact that $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| \, dx < +\infty$ is equivalent to $\int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \delta|) \, dx < +\infty$, since $|\operatorname{Ln} \delta| \geq \beta > 0$ near the boundary.

Fix $0 < \varepsilon < \frac{1}{2}$ and consider $\varphi_1 > 0$ the first eigenfunction of $(-\Delta)$ with Dirichlet boundary condition: $-\Delta \varphi_1 = \lambda_1 \varphi_1$ in Ω , $\varphi_1 = 0$ on $\partial\Omega$. Then, $\varphi \equiv \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$ is a good test function, and there holds

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \, dx = \int_{\Omega} \nabla u \cdot \nabla \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \, dx + \int_{\Omega} \nabla u \cdot \nabla \varphi_1 \frac{\varphi_1 \operatorname{sign}(\operatorname{Ln}(\varphi_1 + \varepsilon))}{\varphi_1 + \varepsilon} \, dx. \quad (10)$$

Since $|\nabla \varphi_1|_{\infty} < +\infty$ and $|\operatorname{Ln}(\varphi_1 + \varepsilon)| \leq |\operatorname{Ln} \varphi_1| + 1$, we deduce

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \, dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|) \, dx. \quad (11)$$

Letting $\varepsilon \rightarrow 0$ and using Fatou's lemma yields

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln} \varphi_1| \, dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|). \quad (12)$$

Since there exist two constants $c_0 > 0$, $c_1 > 0$ such that $c_0 \delta \leq \varphi_1 \leq c_1 \delta$, Relation (9) follows from Relation (12).

End of the proof of Theorem 1

Let $f \geq 0$ be in $L^1(\Omega; \delta)$ and $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$, so that $\int_{\Omega} f(x) \delta |\operatorname{Ln} \delta| \, dx = +\infty$. From Lemma 2, we deduce that

$$\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| \, dx = +\infty, \quad (13)$$

which proves Statement 1.

As for Statement 2., we recall that

$$L_{exp}(\Omega) = \left\{ v \in L^1(\Omega), \quad \sup_{0 < t \leq |\Omega|} \frac{|v|_{**}(t)}{1 + \operatorname{Ln} \frac{|\Omega|}{t}} < +\infty \right\}$$

is the associate space of $L(\text{Ln } L) = \left\{ v \in L^1(\Omega), \int_{\Omega} |v|_{**}(t) dt < +\infty \right\}$ (see [1]). As $|\text{Ln } \delta| \in L_{exp}$ since $\delta^{-\varepsilon} \in L^1(\Omega)$ for $0 < \varepsilon < 1$, there holds

$$\int_{\Omega} |\nabla u| \cdot |\text{Ln } \delta| \leq |\text{Ln } \delta|_{L_{exp}(\Omega)} \cdot |\nabla u|_{L(\text{Ln } L)}. \quad (14)$$

Relation (14) and the fact that $|\text{Ln } \delta|_{L_{exp}(\Omega)} < +\infty$ imply that

$$|\nabla u|_{L(\text{Ln } L)} = \int_{\Omega^*} |\nabla u|_{**}(t) dt = +\infty \quad (15)$$

and Statement 2. is proven. \diamond

4 Existence of a domain Ω and a very weak solution whose gradient blows up in $L^1(\Omega)$

The main result of this section is the

Theorem 2.

There exist a domain Ω of \mathbb{R}^N , $N \geq 2$, of class C^∞ and a function $f \in L^1(\Omega, \delta)$ such that the weak solution u_0 of (1) satisfies

$$|\nabla u_0| \notin L^1(\Omega)$$

(that is: $\int_{\Omega} |\nabla u_0|(x) dx = +\infty$).

The key ingredient in the proof of Theorem 2 is the

Lemma 3.

There exist a domain Ω of \mathbb{R}^N , $N \geq 2$, of class C^∞ and a nonnegative function $g \in L^N(\Omega)$ such that the unique solution $\psi > 0$ of $-\Delta \psi = g$ in Ω , $\psi \in W^{2,N}(\Omega) \cap H_0^1(\Omega)$, satisfies

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty.$$

Let us admit temporarily this lemma (which amounts to saying that $|\nabla\psi(x)|$ is very large near a point of the boundary). Note that, according to Sobolev Embedding Theorem, $W^{2,N}(\Omega)$ is included in $C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1$, but not in $C^{0,1}$ in general.

Proof of Theorem 2

Let us consider the domain Ω constructed in Lemma 3 and assume that, for any $f \in L^1(\Omega, \delta)$, the unique solution u of (1) satisfies $|\nabla u| \in L^1(\Omega)$. Then, define

$$(-\Delta)^{-1} : \begin{array}{ll} L^1(\Omega, \delta) & \rightarrow L^1(\Omega) \\ f & \mapsto u = (-\Delta)^{-1}f, \end{array}$$

u being the unique solution of (1), and set

$$Tf = \nabla(-\Delta)^{-1}f.$$

We start with the

Lemma 4.

If every very weak solution u satisfies $\int_{\Omega} |\nabla u| dx < +\infty$, then

$$\sup_{|f|_{L^1(\Omega,\delta)}=1} |Tf|_{L^1(\Omega)^N} \text{ is finite.}$$

Proof

Choose $0 < \varepsilon \leq \varepsilon_0$ (ε_0 small enough) and set $\Omega_\varepsilon = \{x \in \Omega : \delta(x) > \varepsilon\}$, $T_\varepsilon f = \chi_{\Omega_\varepsilon} \nabla(-\Delta)^{-1}f$, with $\chi_{\Omega_\varepsilon}$ the characteristic function of Ω_ε . If $\int_{\Omega} |\nabla(-\Delta)^{-1}f| dx < +\infty$, then by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla(-\Delta)^{-1}f| dx = \int_{\Omega} |\nabla(-\Delta)^{-1}f| dx$$

and

$$|T_\varepsilon f|_{L^1} \leq \frac{1}{\varepsilon} \int_{\Omega} |\nabla u| \delta dx \leq \frac{c}{\varepsilon} |f|_{L^1(\Omega,\delta)}$$

from Proposition 1.

The Banach-Steinhaus Uniform Boundedness principle then implies that there exists a constant $c(\Omega) > 0$ such that

$$|\nabla u|_{L^1(\Omega)} \leq c(\Omega) |f|_{L^1(\Omega,\delta)} \tag{16}$$

and the lemma is proven. ◇

Going back to the proof of Theorem 2, consider now the sequence

$$u_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega) \text{ (with } p > N)$$

solution of

$$-\Delta u_k = f_k = \min(|f|; k) \text{sign}(f), \quad f_k \rightarrow f \text{ in } L^1(\Omega, \delta).$$

One can show that

$$|\nabla(u_k - u)|_{L^1(\Omega)} \leq c(\Omega) |f - f_k|_{L^1(\Omega, \delta)} \rightarrow 0$$

and therefore

$$u \in W_0^{1,1}(\Omega).$$

Letting ψ be the function introduced in Lemma 3, the fact that $\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty$ implies the existence of a function $f_0 \in L^1(\Omega, \delta)$, $f_0 \geq 0$, such that

$$\int_{\Omega} f_0(x) \psi(x) dx = +\infty.$$

As a matter of fact, the Hopf Maximum Principle ensures the existence of a constant $k_1 > 0$ such that

$$\psi(x) \geq k_1 \delta(x) \quad \forall x \in \Omega.$$

Hence

$$L^1(\Omega, \psi) \subsetneq L^1(\Omega, \delta). \tag{17}$$

If $L^1(\Omega, \psi) = L^1(\Omega, \delta)$, there must exist a constant $c_1(\Omega) > 0$ such that

$$|f|_{L^1(\Omega, \psi)} \leq c_1(\Omega) |f|_{L^1(\Omega, \delta)} \quad \forall f \in L^1(\Omega, \delta). \tag{18}$$

Since the function spaces $L^1(\Omega, \psi)$ and $L^1(\Omega, \delta)$ are Banach spaces, one can deduce this inequality from the properties of Banach spaces, see e.g. [1], Theorem 1.8. Relation (18) would then imply that $\psi(x) \leq c_1(\Omega) \delta(x)$ for all $x \in \Omega$, obviously contradicting the fact that

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty.$$

Let now $f_0 \in L^1(\Omega, \delta)$ be such that $f_0 \notin L^1(\Omega, \psi)$, i.e. such that $\int_{\Omega} |f_0(x)|\psi(x) dx = +\infty$, and assume that $f_0 \geq 0$ (otherwise, simply consider $|f_0|$).

Defining the sequence $f_{0k} = \min(f_0; k)$ and letting \bar{u}_k be the solution of $-\Delta \bar{u}_k = f_{0k}$, one has using Lemma 4 that

$$\begin{aligned} 0 &\leq \int_{\Omega} f_{0k} \psi dx = - \int_{\Omega} \psi \Delta \bar{u}_k dx = - \int_{\Omega} \bar{u}_k \Delta \psi dx = \int_{\Omega} \bar{u}_k g dx \\ &\leq |\bar{u}_k|_{L^{N'}} \cdot |g|_{L^N} \leq c |\nabla \bar{u}_k|_{L^1} \cdot |g|_{L^N} \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N}. \end{aligned} \quad (19)$$

Letting $k \rightarrow +\infty$ in relation (19), we derive from Beppo-Levi's theorem that

$$+\infty = \int_{\Omega} f_0 \psi dx = \lim_{k \rightarrow +\infty} \int_{\Omega} f_{0k} \psi dx \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N} < +\infty,$$

which is absurd. Hence, there exists a function $f_0 \in L^1(\Omega, \delta)$ such that the associate weak solution u_0 satisfies $|\nabla u_0| \notin L^1(\Omega)$. \diamond

Proof of Lemma 3

For the sake of convenience, we start with the case $N = 2$ and generalize the construction in a second step.

Let us first consider the open set

$$\Omega_1 = \left\{ x = (x_1, x_2) : x_1^2 < x_2, x_1^2 + x_2^2 < \frac{1}{e} \right\}$$

and define a preliminary function on Ω_1 by

$$w(x_1, x_2) = (x_2 - x_1^2) \text{Ln} \left(\text{Ln} \frac{1}{x_1^2 + x_2^2} \right).$$

Some remarks are in order:

- In polar coordinates, w can be written as $w = r(\sin(\theta) - r \cos(\theta)) \text{Ln} (|\text{Ln } r|)$.
- Another possibility would be to choose, instead of w , $w_0 = r \sin(\theta) \text{Ln} (|\text{Ln } r|)$ on $\{(r, \theta) : r > 0, 0 < \theta < \pi\}$. In what follows, we can use w or w_0 .

The function w has the following properties:

1. $w > 0$ in Ω_1 ,

2. $w(x_1, x_1^2) = 0$ for $-x_{1c} < x_1 < x_{1c}$ with $x_{1c}^4 + x_{1c}^2 = \frac{1}{e}$, and $w(x_1, x_2) = 0$ for $x_1^2 + x_2^2 = \frac{1}{e}$,

3. $w \in C^\infty(\Omega_1) \cap H^2(\Omega_1)$.

Indeed, it is sufficient to compute $\frac{\partial w}{\partial x_i}$ and $\frac{\partial^2 w}{\partial x_i^2}$ and prove that $\Delta w \in L^2(\Omega)$.

$$w[x_1, x_2] = (-x_1^2 + x_2) \operatorname{Ln} \left[\operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_1} = -\frac{2x_1(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} - 2x_1 \operatorname{Ln} \left[n \left[\frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_2} = -\frac{2x_2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} + \operatorname{Ln} \left[\operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x_1^2} &= -\frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} + \frac{8x_1^2}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} \\ &\quad - \frac{2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} - 2 \operatorname{Ln} \left[\operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right] \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x_2^2} &= -\frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} - \frac{4x_2}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} \\ &\quad - \frac{2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]} \end{aligned}$$

$$\Delta w = \frac{4 \left(x_1^2 - x_2 + (2x_1^2 - x_2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right] \right)}{(x_1^2 + x_2^2) \operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right]^2} - 2 \operatorname{Ln} \left[\operatorname{Ln} \left[\frac{1}{x_1^2 + x_2^2} \right] \right].$$

Using polar coordinates, one can check that

$$\Delta w \in L^2(\Omega), \quad |\nabla w| \in L^p(\Omega) \quad \forall p < +\infty.$$

Consider now $x_{1c} > \eta > 0$ and an open set Ω , of class $C^{2,1}$ at least, such that

$$\Omega \subset [-x_{1c} + \eta; x_{1c} - \eta] \times \left[0, \frac{1}{\sqrt{e}} \right]$$

and

$$\partial\Omega \supset \Gamma_0 = \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_2 = x_1^2 \right\} \cap \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_1^2 + x_2^2 = \frac{1}{e} \right\}.$$

For $a > 0 : 0 < 2a < x_{1c} - \eta$, define a smooth function θ such that

$$\theta \in C_c^\infty(\mathbb{R}^2), \theta \geq 0 \text{ and } \begin{cases} 0 \leq \theta \leq 1, \\ \theta(x_1, x_2) = 1 & \text{if } |x_1| \leq a, \\ \theta(x_1, x_2) = 0 & \text{if } |x_1| > \frac{3a}{2}. \end{cases}$$

In particular, the function θw vanishes on the boundary of Ω . Let us show that $\psi_0 = \theta w$ satisfies the following properties:

1. $-\Delta\psi_0 = f_0 \in L^2(\Omega)$.
2. $\sup_{x \in \Omega} \left\{ \frac{\psi_0(x)}{\delta(x)} \right\} = +\infty$.

Property 1 is obvious, since

$$\begin{cases} -\Delta\psi_0 = -(\Delta\theta w + 2\nabla w \nabla\theta + \Delta w \theta) \in L^2(\Omega), \\ \psi_0 \in H_0^1(\Omega). \end{cases}$$

To prove Property 2, consider $x = (x_1, \alpha x_1)$, $0 < \alpha < 1$, x_1 small enough so that $x \in \Omega$. Then there holds

$$\frac{\psi_0(x)}{\delta(x)} = \frac{\psi_0(x_1, \alpha x_1)}{\delta(x)} \geq \frac{(\alpha - x_1) \text{Ln}((- \text{Ln}(1 + \alpha^2)x_1^2))}{\sqrt{1 + \alpha^2}} \xrightarrow{x_1 \rightarrow 0} +\infty$$

which shows that

$$\sup_{x \in \Omega} \left\{ \frac{\psi_0(x)}{\delta(x)} \right\} = +\infty.$$

Setting $g = |f_1|$ and considering $\psi > 0$ solution of $-\Delta\psi = g$, $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$, one has by the maximum principle that $\psi \geq \psi_0$, so that $\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty$. \diamond

The construction above can be generalized to \mathbb{R}^N , let us outline the main steps of the procedure.

For $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$, set

$$|x'|^2 = x_1^2 + \dots + x_{N-1}^2, \quad \sigma(x) = x_1^2 + \dots + x_N^2.$$

We first consider the open set

$$\Omega_1 = \left\{ x = (x', x_N), |x'|^2 < x_N, \sigma(x) < \frac{1}{e} \right\}$$

and define on Ω_1 the nonnegative function

$$w(x) = (x_N - |x'|^2) \operatorname{Ln} \left(\operatorname{Ln} \frac{1}{\sigma(x)} \right), \quad x \in \Omega_1.$$

w satisfies properties similar to those used in the two-dimensional case. For a small $a > 0$, consider $\theta \in C_c^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} 0 \leq \theta \leq 1, \\ \theta(x', x_N) = 1 & \text{if } |x'| \leq a, \\ \theta(x', x_N) = 0 & \text{if } |x'| > \frac{3a}{2} \end{cases}$$

and an open set Ω of class $C^{2,1}$ with $\operatorname{supp} \theta \cap \Omega_1 \subset \Omega$ and $\theta w = 0$ on $\partial\Omega$.

Then, the function $\psi_0(x) = \theta w(x)$ satisfies $-\Delta\psi_0 \in L^N(\Omega)$, since $\Delta w \in L^N(\Omega)$ thanks to a straightforward computation. Setting

$$e_{N-1} = (1, \dots, 1), \quad x_\alpha = x_N(\alpha, \dots, \alpha, 1) \in \Omega, \quad \alpha > 0,$$

for $x_N > 0$, x_N small enough, there holds

$$\frac{\psi_0(x_\alpha)}{\delta(x_\alpha)} \geq \frac{(1 - \alpha^2 |e_{N-1}|^2 x_N)}{\sqrt{1 + \alpha^2 |e_{N-1}|^2}} \operatorname{Ln} \left(\operatorname{Ln} \left(\frac{1}{x_N^2 (1 + \alpha^2 (N-1))} \right) \right) \xrightarrow{x_N \rightarrow 0} +\infty.$$

Therefore

$$\sup_{x \in \Omega} \left\{ \frac{\psi_0(x)}{\delta(x)} \right\} = +\infty.$$

Finally, considering the solution ψ of

$$\begin{cases} -\Delta\psi = |\Delta\psi_0| = g \in L^N(\Omega) \\ \psi \in W^{1,N}(\Omega) \cap W^{2,N}(\Omega), \end{cases}$$

and by the same argument as in the two-dimensional case, there also holds that

$$\sup_{x \in \Omega} \left\{ \frac{\psi(x)}{\delta(x)} \right\} = +\infty,$$

which ends the proof of Lemma 3. ◇

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