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# Gradient blow-up in Zygmund spaces for the very weak solution of a linear elliptic equation.

Frédéric ABERGEL

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## Abstract

It is known that the very weak solution of  $-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx$ ,  $\forall \varphi \in C^2(\overline{\Omega})$ ,  $\varphi = 0$  on  $\partial\Omega$ ,  $u \in L^1(\Omega)$  has its gradient in  $L^1(\Omega)$  whenever  $f \in L^1(\Omega; \delta(1 + |\log \delta|))$  with  $\delta(x)$  being the distance of  $u \in \Omega$  to the boundary. Here, we show that if  $f$  is not in this weighted space  $L^1(\Omega; \delta(1 + |\ln \delta|))$ ,  $f \geq 0$ , then its gradient blows up in  $L(\log L)$  at least. Moreover, we show that there exist a domain  $\Omega$  of class  $C^\infty$  and a function  $f \in L^1_+(\Omega, \delta)$  such that the associated very weak solution has its gradient being non integrable on  $\Omega$ .

**Keywords** Very weak solutions; Distance to the boundary; Regularity; Linear PDE; Monotone rearrangement; Lorentz spaces.

## 1 Introduction

In this paper, we state and prove two results related to the behaviour near the boundary of very weak solutions to Laplace's equation. In the first part of the paper, we prove that the very weak solution  $u \in L^1(\Omega)$  of the so-called Brezis weak formulation (see [4, 5, 6])

$$-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega, \quad (1)$$

verifies  $\int_{\Omega} |\nabla u| |\text{Ln } \delta| dx = +\infty$  whenever  $f \geq 0$ ,  $f$  is integrable with respect to the distance function  $\delta(x) = \text{dist}(x, \partial\Omega)$  but  $f \notin L^1(\Omega; \delta(1 + |\text{Ln } \delta|))$ . The result implies in particular, that  $|\nabla u|$  blows up in the Zygmund space  $L(\text{Ln } L)$  :

$$\int_{\Omega_*} |\nabla u|_{**}(t) dt = \infty \text{ with } \frac{1}{t} \int_0^t |\nabla u|_*(\sigma) d\sigma = |\nabla u|_{**}(t), \quad (2)$$

$|\nabla u|_*$  is the decreasing rearrangement of  $|\nabla u|$ ,  $\Omega_* = ]0, \text{meas}(\Omega)[$ .

In the second part, we construct an open bounded smooth set  $\Omega$  and a function  $f \in L^1(\Omega, \delta)$ ,  $f \notin L^1(\Omega, \delta(1 + |\text{Ln } \delta|))$  such that the associate very weak solution  $u$  verifies

$$\int_{\Omega} |\nabla u| dx = +\infty. \quad (3)$$

For the sake of completeness, we recall some general results concerning the equation (1) that we shall summarize in the following

**Proposition 1.** (see [2, 5, 8])

Let  $\Omega$  be an open bounded set of  $C^{2,1}$  in  $\mathbb{R}^N$ ,  $f \in L^1(\Omega, \delta)$ , where  $\delta(x)$  is the distance function of  $x \in \Omega$  to be the boundary  $\partial\Omega$ . Then, there exists a constant  $c > 0$  such that for any solution  $u$  of (1), one has

1.  $|\nabla u|_{L^{1+\frac{1}{N'}, \infty}(\Omega; \delta)} \leq c|f|_{L^1(\Omega; \delta)}$ ,  
 $|u|_{L^{N', \infty}(\Omega)} \leq c|f|_{L^1(\Omega; \delta)}$ ,  $N' = \frac{N}{N-1}$  if  $N \geq 2$ ,  $N' = \infty$  otherwise

2. If  $f \geq 0$  then  $u \geq 0$ .

3. If  $f \in L^1(\Omega; \delta(1 + |\text{Ln } \delta|))$  then  $u \in W_0^{1,1}(\Omega)$  and

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega; \delta(1 + |\text{Ln } \delta|))}$$

4. If  $\Omega$  is a ball,  $f$  is radial then  $u \in W_0^{1,1}(\Omega)$

$$|\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega; \delta)}.$$

5. If  $\Omega = ]a, b[$  then the above estimate holds for all  $f \in L^1(]a, b[, \delta)$ . ◇

In Proposition 1  $L^{p,q}(\Omega)$  denotes the Lorentz space, see e.g. [1, 9]. Their main properties are briefly recalled below. that we recall briefly the properties can be found in [9]

For a Lebesgue measurable set  $E$  of  $\Omega$  we denote by  $|E|$  its measure.

the **decreasing rearrangement of a measurable function**  $u$  is the function  $u_*$ . Then

$$u_* : \Omega_* = ]0, |\Omega|[ \rightarrow \mathbb{R}, \quad u_*(s) = \inf\{t \in \mathbb{R} : |u > t| \leq s\}$$

$$u_*(0) = \operatorname{ess\,sup}_{\Omega} u, \quad u_*(|\Omega|) = \operatorname{ess\,inf}_{\Omega} u.$$

We use the following Lorentz spaces,

for  $1 < p < +\infty$ ,  $1 \leq q \leq +\infty$

$$L^{p,q}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable } |v|_{L^{p,q}}^q = \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty \right\}$$

for  $q = +\infty$

$$L^{p,\infty}(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable } |v|_{L^{p,\infty}} = \sup_{t \leq |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \right\}$$

Finally, we introduce some notations :  $\chi_E$  is the characteristic function of a set  $E \subset \Omega$  and

$$|v|_{**}(t) = \frac{1}{t} \int_0^t |v|_*(s) ds \text{ for } t \in \Omega_* = ]0, |\Omega|.$$

Denoting by  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ , we define the following sets

$$W^1(\Omega, |\cdot|_{p,q}) = \left\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega) \right\}$$

and

$$C_c^m(\Omega) = \left\{ \varphi \in C^m(\Omega), \varphi \text{ has compact support in } \Omega \right\}.$$

## 2 Blow-up in Zygmund space for $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$

The aim of this section is to prove the

### Theorem 1.

Under the same assumptions as for Proposition 1, if  $f \geq 0$  and  $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$  then any solution  $u$  of (1) satisfies

1.  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty;$
2.  $\int_{\Omega_*} |\nabla u|_{**}(t) dt = +\infty,$  (i.e.  $\int_{\Omega} |\nabla u| \max(\operatorname{Ln} |\nabla u|; 0) dx = +\infty$ ).

The proof of Theorem 1 follows

### Lemma 1.

Let  $u$  be a very weak solution of (1). Assume that  $\int_{\Omega} |\nabla u| dx < +\infty$ . Then  $u$  satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega). \quad (4)$$

### Proof

By the density of  $C_c^2(\Omega)$  in  $C_c^1(\Omega)$ , one has

$$\int_{\Omega} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_c^1(\Omega).$$

By usual argument on truncation and convolution (see [10, 7, 3]) one can prove the following approximation result:

### Proposition 2.

Let  $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$ . Then there exists a sequence  $\varphi_n \in C_c^1(\Omega)$  such that

1.  $\exists c > 0, |\nabla \varphi_n|_{\infty} \leq c(|\nabla \varphi|_{\infty} + |\varphi|_{\infty}) \quad \forall n;$
2.  $\varphi_n \rightarrow \varphi$  in  $C(\overline{\Omega})$  (i.e.  $\max_{x \in \Omega} |\varphi_n(x) - \varphi(x)| \xrightarrow{n \rightarrow +\infty} 0$ );

3.  $\nabla\varphi_n \rightharpoonup \nabla\varphi$  in  $L^\infty(\Omega)^N$ -weak star.

(proof omitted).

### Proof of Theorem 1

Considering  $\varphi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$  and its approximating sequence in Proposition 2, we have

$$\int_{\Omega} \nabla u \cdot \nabla \varphi_n dx = \int_{\Omega} f \varphi_n dx. \quad (5)$$

By statement 3. of Proposition 2, we have

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_n dx = \int_{\Omega} \nabla u \cdot \nabla \varphi dx \quad (6)$$

and by the mean value theorem, we have from statement 1.

$$|\varphi_n(x) - \varphi(x)| \leq |\nabla(\varphi_n - \varphi)|_{\infty} \cdot \delta(x) \leq c_{\varphi} \delta(x). \quad (7)$$

Since  $f \in L^1(\Omega; \delta)$ , we derive from the Lebesgue dominated convergence theorem,

$$\lim_n \int_{\Omega} f \varphi_n dx = \int_{\Omega} f \varphi dx. \quad (8)$$

Combining relations (5) to (8), we obtain the result.  $\diamond$

Next we want to prove

### Lemma 2.

*Under the same assumption as for Proposition 1, if  $f \geq 0$  and  $u$  is the very weak solution (1) such that  $\int_{\Omega} |\nabla u| |\text{Ln } \delta| dx < \infty$ ., then there exists a constant  $c(\Omega) > 0$  (independent of  $u$ ) such that*

$$\int_{\Omega} f \delta |\text{Ln } \delta| dx \leq c(\Omega) \left( \int_{\Omega} |\nabla u| (1 + |\text{Ln } \delta|) dx + \int_{\Omega} f \delta dx \right). \quad (9)$$

**Proof** Let us first note that according to Proposition 1 statement 1, we have

$$\int_{\Omega} |\nabla u| \delta dx \leq c \int_{\Omega} f \delta dx < +\infty.$$

Therefore,  $\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx < +\infty$  is equivalent to  $\int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \delta|) dx < +\infty$ , since  $|\operatorname{Ln} \delta| \geq \beta > 0$  near the boundary.

For  $0 < \varepsilon < \frac{1}{2}$ , let us consider  $\varphi_1 > 0$  the first eigenfunction of  $-\Delta$  with the Dirichlet condition,  $-\Delta \varphi_1 = \lambda_1 \varphi_1$  in  $\Omega$ ,  $\varphi_1 = 0$  on  $\partial\Omega$ . Then,  $\varphi \equiv \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$  is a good test function. We then have

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx = \int_{\Omega} \nabla u \cdot \nabla \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx + \int_{\Omega} \nabla u \cdot \nabla \varphi_1 \frac{\varphi_1 \operatorname{sign}(\operatorname{Ln}(\varphi_1 + \varepsilon))}{\varphi_1 + \varepsilon} dx. \quad (10)$$

Since  $|\nabla \varphi_1|_{\infty} < +\infty$  and  $|\operatorname{Ln}(\varphi_1 + \varepsilon)| \leq |\operatorname{Ln} \varphi_1| + 1$  we deduce

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln}(\varphi_1 + \varepsilon)| dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|) dx. \quad (11)$$

Letting  $\varepsilon \rightarrow 0$  Fatou's lemma yields

$$\int_{\Omega} f \varphi_1 |\operatorname{Ln} \varphi_1| dx \leq c \int_{\Omega} |\nabla u| (1 + |\operatorname{Ln} \varphi_1|). \quad (12)$$

Since there exist two constants  $c_0 > 0$ ,  $c_1 > 0$  such that  $c_0 \delta \leq \varphi_1 \leq c_1 \delta$  we deduce relation (9) from relation (12).

### End of the proof of Theorem 1

Let  $f \geq 0$  be  $L^1(\Omega; \delta)$  and  $f \notin L^1(\Omega; \delta(1 + |\operatorname{Ln} \delta|))$  then  $\int_{\Omega} f(x) \delta |\operatorname{Ln} \delta| dx = +\infty$ . Therefore, from Lemma 2, we deduce that

$$\int_{\Omega} |\nabla u| |\operatorname{Ln} \delta| dx = +\infty. \quad (13)$$

While for the second statement 2., we recall that

$$L_{exp}(\Omega) = \left\{ v \in L^1(\Omega) \quad \sup_{0 < t \leq |\Omega|} \frac{|v|_{**}(t)}{1 + \operatorname{Ln} \frac{|\Omega|}{t}} < +\infty \right\}$$

is the associate space of  $L(\operatorname{Ln} L) = \left\{ v \in L^1(\Omega) \quad \int_{\Omega} |v|_{**}(t) dt < +\infty \right\}$  (see [1]). Now we have that  $|\operatorname{Ln} \delta| \in L_{exp}$  since  $\delta^{-\varepsilon} \in L^1(\Omega)$  for  $0 < \varepsilon < 1$ , and we then deduce

$$\int_{\Omega} |\nabla u| \cdot |\operatorname{Ln} \delta| \leq |\operatorname{Ln} \delta|_{L_{exp}(\Omega)} \cdot |\nabla u|_{L(\operatorname{Ln} L)}. \quad (14)$$

Relation (13) and the fact that  $|\operatorname{Ln} \delta|_{L_{exp}(\Omega)} < +\infty$  implies that

$$|\nabla u|_{L(\operatorname{Ln} L)} = \int_{\Omega_*} |\nabla u|_{**}(t) dt = +\infty. \quad (15)$$

◇

### 3 Existence of a domain $\Omega$ and a very weak solution whose gradient blows up in $L^1(\Omega)$

The main result of this section is

#### Theorem 2.

*There exist a domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$  of class  $C^\infty$  and a function  $f \in L^1(\Omega, \delta)$  such that the weak solution  $u_0$  of (1) satisfies*

$$|\nabla u_0| \notin L^1(\Omega)$$

*(that is  $\int_{\Omega} |\nabla u_0|(x) dx = +\infty$ ).*

The key lemma in the proof of Theorem 2 is the following result

#### Lemma 3.

*There exist a domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$  of class  $C^\infty$  and a nonnegative function  $g \in L^N(\Omega)$  such that the unique solution  $\psi > 0$  of  $-\Delta\psi = g$  in  $\Omega$ ,  $\psi \in W^{2,N}(\Omega) \cap H_0^1(\Omega)$  satisfies*

$$\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty.$$

Let us admit temporarily this lemma ( which merely amounts to saying that  $|\nabla\psi(x)|$  is very large near a point of the boundary). Note that according to Sobolev embedding  $W^{2,N}(\Omega)$  is included in  $C^{0,\alpha}(\overline{\Omega})$  for all  $\alpha < 1$  but not in  $C^{0,1}$  in general.



**Proof of Theorem 2**

Let us consider  $\Omega$  the domain constructed in Lemma 3. Assume that for any  $f \in L^1(\Omega, \delta)$  the unique solution  $u$  of (1) satisfies  $|\nabla u| \in L^1(\Omega)$ . Then define

$$(-\Delta)^{-1} : \begin{array}{ll} L^1(\Omega, \delta) & \rightarrow L^1(\Omega) \\ f & \mapsto u = (-\Delta)^{-1}f. \end{array}$$

$u$  being the unique solution of (1), and set

$$Tf = \nabla(-\Delta)^{-1}f.$$

Then one has

**Lemma 4.**

If for all very weak solution  $u$ , we have  $\int_{\Omega} |\nabla u| dx < +\infty$ . Then

$$\sup_{|f|_{L^1(\Omega, \delta)}=1} |Tf|_{L^1(\Omega)^N} \text{ is finite.}$$

**Proof**

Let  $0 < \varepsilon \leq \varepsilon_0$  ( $\varepsilon_0$  small enough).  $\Omega_\varepsilon = \{x \in \Omega : \delta(x) > \varepsilon\}$  and  $T_\varepsilon f = \chi_{\Omega_\varepsilon} \nabla(-\Delta)^{-1}f$  with  $\chi_{\Omega_\varepsilon}$  being the characteristic function of  $\Omega_\varepsilon$ . If  $\int_{\Omega} |\nabla(-\Delta)^{-1}f| dx < +\infty$  then by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla(-\Delta)^{-1}f| dx = \int_{\Omega} |\nabla(-\Delta)^{-1}f| dx;$$

and

$$|T_\varepsilon f|_{L^1} \leq \frac{1}{\varepsilon} \int_{\Omega} |\nabla u| \delta dx \leq \frac{c}{\varepsilon} |f|_{L^1(\Omega, \delta)}.$$

We derive (the Banach-Steinhaus boundedness principle) that there exists a constant  $c(\Omega) > 0$  such that

$$|\nabla u|_{L^1(\Omega)} \leq c(\Omega) |f|_{L^1(\Omega, \delta)}. \tag{16}$$

◇

Considering the sequence

$$u_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega), \quad p > N, \quad \text{solution of } -\Delta u_k = f_k = \min(|f|; k) \text{sign}(f), \quad f_k \rightarrow f \text{ in } L^1(\Omega, \delta),$$

then one has

$$|\nabla(u_k - u)|_{L^1(\Omega)} \leq c(\Omega)|f - f_k|_{L^1(\Omega, \delta)} \rightarrow 0.$$

Thus

$$u \in W_0^{1,1}(\Omega).$$

Since  $\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty$  there exists a function  $f_0 \in L^1(\Omega, \delta)$ ,  $f_0 \geq 0$  such that

$$\int_{\Omega} f_0(x)\psi(x) dx = +\infty.$$

Indeed one has, using the Hopf maximum principle, the existence of a constant  $k_1 > 0$  such that

$$\psi(x) \geq k_1\delta(x), \quad \forall x \in \Omega.$$

Therefore,

$$L^1(\Omega, \psi) \subsetneq L^1(\Omega, \delta). \quad (17)$$

If  $L^1(\Omega, \psi) = L^1(\Omega, \delta)$  then necessarily, there exists a constant  $c_1(\Omega) > 0$  such that

$$|f|_{L^1(\Omega, \psi)} \leq c_1(\Omega)|f|_{L^1(\Omega, \delta)} \quad \forall f \in L^1(\Omega, \delta). \quad (18)$$

(This is due to Banach principle or equivalently, since the function space  $L^1(\Omega, \psi)$  and  $L^1(\Omega, \delta)$  are Banach function space, one can deduce this inequality from one of properties related to Banach function space [1], Theorem 1.8).

Relation (18) would imply  $\psi(x) \leq c_1(\Omega)\delta(x)$  for all  $x \in \Omega$ . This contradicts the fact that

$$\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty.$$

Therefore, there exists a function  $f_0 \in L^1(\Omega, \delta)$  such that  $f_0 \notin L^1(\Omega, \psi)$ , i.e.  $\int_{\Omega} |f_0(x)|\psi(x) dx = +\infty$ .

We may obviously assume that  $f_0 \geq 0$  (otherwise, we consider  $|f_0|$ ).

Defining the sequence  $f_{0k} = T_k(f_0) = \min(f_0; k)$  and  $\bar{u}_k$  the solution of  $-\Delta\bar{u}_k = f_{0k}$  one has using relation (16)

$$0 \leq \int_{\Omega} f_{0k}\psi dx = - \int_{\Omega} \psi\Delta\bar{u}_k dx = - \int_{\Omega} \bar{u}_k\Delta\psi dx = \int_{\Omega} \bar{u}_k g dx$$

$$\leq |\bar{u}_k|_{L^{N'}} \cdot |g|_{L^N} \leq c |\nabla \bar{u}_k|_{L^1} \cdot |g|_{L^N} \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N}. \quad (19)$$

Letting  $k \rightarrow +\infty$  in relation (19), we derive from Beppo-Levi's theorem

$$+\infty = \int_{\Omega} f_0 \psi \, dx = \lim_{k \rightarrow +\infty} \int_{\Omega} f_{0k} \psi \, dx \leq c |f_0|_{L^1(\Omega, \delta)} |g|_{L^N} < +\infty,$$

which is absurd. Hence, there exists a function  $f_0 \in L^1(\Omega, \delta)$  such that the associate weak solution  $u_0$  is such  $|\nabla u_0| \notin L^1(\Omega)$ .  $\diamond$

### Proof of Lemma 3

For the sake of convenience, we shall start with the case  $N = 2$  and we shall generalize the construction in a second step.

Let us first consider the open set

$$\Omega_1 = \left\{ x = (x_1, x_2) : x_1^2 < x_2, \, x_1^2 + x_2^2 < \frac{1}{e} \right\}.$$

We define the preliminary function on  $\Omega_1$  by

$$w(x_1, x_2) = (x_2 - x_1^2) \operatorname{Ln} \left( \operatorname{Ln} \frac{1}{x_1^2 + x_2^2} \right).$$

(Remarks :

- in polar coordinates  $w$  can be written as  $w = r(\sin(\theta) - r \cos(\theta)) \operatorname{Ln} (|\operatorname{Ln} r|)$ .
  - another possibility for the choice of  $w$  is  $w_0 = r \sin(\theta) \operatorname{Ln} (|\operatorname{Ln} r|)$  on  $\{(r, \theta) : r > 0, 0 < \theta < \pi\}$ ,
- in what follows, we can use  $w$  or  $w_0$  ).

One has the following properties for  $w$ :

1.  $w > 0$  in  $\Omega_1$ ,
2.  $w(x_1, x_1^2) = 0$ ,  $-x_{1c} < x_1 < x_{1c}$  with  $x_{1c}^4 + x_{1c}^2 = \frac{1}{e}$ ,  $w(x_1, x_2) = 0$  for  $x_1^2 + x_2^2 = \frac{1}{e}$ .
3.  $w \in C^\infty(\Omega_1) \cap H^2(\Omega_1)$ .

Indeed it is sufficient to compute  $\frac{\partial w}{\partial x_i}$  and  $\frac{\partial^2 w}{\partial x_i^2}$  and prove that  $\Delta w \in L^2(\Omega)$ .

$$w[x_1, x_2] = (-x_1^2 + x_2) \operatorname{Log} \left[ \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_1} = -\frac{2x_1(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} - 2x_1 \operatorname{Log} \left[ \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\frac{\partial w}{\partial x_2} = -\frac{2x_2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} + \operatorname{Log} \left[ \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x_1^2} &= -\frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_1^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} + \frac{8x_1^2}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} \\ &\quad - \frac{2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} - 2 \operatorname{Log} \left[ \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x_2^2} &= -\frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} + \frac{4x_2^2(-x_1^2 + x_2)}{(x_1^2 + x_2^2)^2 \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} - \frac{4x_2}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} \\ &\quad - \frac{2(-x_1^2 + x_2)}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]} \end{aligned}$$

$$\Delta w = \frac{4 \left( x_1^2 - x_2 + (2x_1^2 - x_2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right)}{(x_1^2 + x_2^2) \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right]^2} - 2 \operatorname{Log} \left[ \operatorname{Log} \left[ \frac{1}{x_1^2 + x_2^2} \right] \right]$$

Using polar coordinates, one can check that

$$\Delta w \in L^2(\Omega) \text{ and } |\nabla w| \in L^p(\Omega) \text{ for all } p < +\infty.$$

One can consider  $x_{1c} > \eta > 0$  and an open set  $\Omega$  of class  $C^{2,1}$  at least such that

$$\Omega \subset [-x_{1c} + \eta; x_{1c} - \eta] \times \left[ 0, \frac{1}{\sqrt{e}} \right].$$

$$\partial\Omega \text{ contains } \Gamma_0 = \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_2 = x_1^2 \right\} \cap \left\{ (x_1, x_2) : -x_{1c} + \eta < x_1 < x_{1c} - \eta, x_1^2 + x_2^2 = \frac{1}{e} \right\}.$$

For  $a > 0 : 0 < 2a < x_{1c} - \eta$ , we define  $\theta \in C_0^\infty(\mathbb{R}^2)$  such that

$$\theta \in C_c^\infty(\mathbb{R}^2), \theta \geq 0 \text{ such that } \begin{cases} 0 \leq \theta \leq 1, \\ \theta(x_1, x_2) = 1 & \text{if } |x_1| \leq a, \\ \theta(x_1, x_2) = 0 & \text{if } |x_1| > \frac{3a}{2}. \end{cases}$$

Therefore, the function  $\theta w$  vanishes on the boundary of  $\Omega$ . Let us show that  $\psi_0 = \theta w$  satisfies the following properties

1.  $-\Delta\psi_0 = f_0 \in L^2(\Omega)$ .
2.  $\text{Max} \left\{ \frac{\psi_0(x)}{\delta(x)}, x \in \Omega \right\} = +\infty$ .

Property 1 is obvious, since

$$\begin{cases} -\Delta\psi_0 = -(\Delta\theta w + 2\nabla w \nabla\theta + \Delta w \theta) \in L^2(\Omega), \\ \psi_0 \in H_0^1(\Omega). \end{cases}$$

To prove statement 2.) we consider  $x = (x_1, \alpha x_1)$ ,  $0 < \alpha < 1$ ,  $x_1$  small enough so that  $x \in \Omega$ , then

$$\frac{\psi_0(x)}{\delta(x)} = \frac{\psi_0(x_1, \alpha x_1)}{\delta(x)} \geq \frac{(\alpha - x_1) \text{Ln}((- \text{Ln}(1 + \alpha^2)x_1^2))}{\sqrt{1 + \alpha^2}} \xrightarrow{x_1 \rightarrow 0} +\infty.$$

This show that

$$\sup \left\{ \frac{\psi_0(x)}{\delta(x)} : x \in \Omega \right\} = +\infty.$$

Setting  $g = |f_1|$  and considering  $\psi > 0$  solution of  $-\Delta\psi = g$ ,  $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ , one has by the maximum principle that  $\psi \geq \psi_0$  (so that  $\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty$ ).  $\diamond$

We can generalize the above construction in  $\mathbb{R}^N$ , let us outline the main steps of the procedure.

For  $x = (x', x_N) = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$ , we denote by

$$|x'|^2 = x_1^2 + \dots + x_{N-1}^2, \quad \sigma(x) = x_1^2 + \dots + x_N^2.$$

We first consider the open set

$$\Omega_1 = \left\{ x = (x', x_N), |x'|^2 < x_N, \sigma(x) < \frac{1}{e} \right\}.$$

We define on  $\Omega_1$  the nonnegative function

$$w(x) = (x_N - |x'|^2) \operatorname{Ln} \left( \operatorname{Ln} \frac{1}{\sigma(x)} \right), \quad x \in \Omega_1.$$

$w$  satisfies the same properties as in statements 1., 2., 3.. For  $a > 0$  (small), consider

Consider  $\theta \in C_c^\infty(\mathbb{R}^N)$  such that

$$\begin{cases} 0 \leq \theta \leq 1, \\ \theta(x', x_N) = 1 & \text{if } |x'| \leq a, \\ \theta(x', x_N) = 0 & \text{if } |x'| > \frac{3a}{2}. \end{cases}$$

Let us consider an open set  $\Omega$  of class  $C^{2,1}$  such that  $\operatorname{supp} \theta \cap \Omega_1 \subset \Omega$  and  $\theta w = 0$  on  $\partial\Omega$ .

Then the function  $\psi_0(x) = \theta w(x)$  satisfies  $-\Delta\psi_0 \in L^N(\Omega)$  since  $\Delta w \in L^N(\Omega)$  by a straightforward computation. Setting

$$e_{N-1} = (1, \dots, 1), \quad x_\alpha = x_N(\alpha, \dots, \alpha, 1) \in \Omega, \quad \alpha > 0,$$

for  $x_N > 0$ ,  $x_N$  small enough, there holds and considering  $x_N > 0$  small enough, we have set and

$$\frac{\psi_0(x_\alpha)}{\delta(x_\alpha)} \geq \frac{(1 - \alpha^2 |e_{N-1}|^2 x_N)}{\sqrt{1 + \alpha^2 |e_{N-1}|^2}} \operatorname{Ln} \left( \operatorname{Ln} \left( \frac{1}{x_N^2 (1 + \alpha^2 (N-1))} \right) \right) \xrightarrow{x_N \rightarrow 0} +\infty.$$

Therefore

$$\sup \left\{ \frac{\psi_0(x)}{\delta(x)} : x \in \Omega \right\} = +\infty.$$

Considering

$$\begin{cases} -\Delta\psi = |\Delta\psi_0| = g \in L^N(\Omega) \\ \psi \in W^{1,N}(\Omega) \cap W^{2,N}(\Omega), \end{cases}$$

we have  $\sup \left\{ \frac{\psi(x)}{\delta(x)} : x \in \Omega \right\} = +\infty, g \in L^N(\Omega)$  ◇

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