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Temperature and voltage probes far from equilibrium

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Abstract. We consider an open system of non-interacting electrons consisting of a small sample connected to several reservoirs and temperature or voltage probes. We study the non-linear system of equations that determines the probe parameters. We show that it has a unique solution, which can be computed with a fast converging iterative algorithm. We illustrate our method with two well-known models: the three-terminal system and the open Aharovov-Bohm interferometer.

1 Introduction

Thermodynamic quantities such as entropy, temperature, or chemical potential play a fundamental role in our understanding of equilibrium phenomena. They are given sound microscopic meanings within the framework of equilibrium statistical mechanics. The concept of local thermal equilibrium allows us in principle to define these quantities in interacting systems close to equilibrium. However, extending these definitions far from equilibrium and/or to non-interacting systems where local equilibrium does not make sense is a much more delicate issue (see the discussions in [1, 2, 3]). In this paper, we shall consider an operational point of view, giving to local intensive parameters the values measured by external probes.

Such an experimental approach is well-known in the mesoscopic community: in the description of electric transport in a multi-terminal system, in which all the terminals have the same temperature (typically $T = 0$), one often introduces a *voltage probe* [4] to sense the local electrochemical potential by connecting an additional electronic reservoir under the *zero electric current condition*: the chemical potential of the probe is tuned so that there is no net average electric current into it. In the same spirit, setting all the terminals to the same chemical potential, a *temperature probe* is obtained by requiring that the temperature of the corresponding reservoir is tuned such that there is no average heat current into it [5].

In the scattering approach of Landauer and Büttiker (see Section 2), the existence and uniqueness of such parameters are usually accepted on physical grounds, but we think it is important and interesting to obtain a rigorous mathematical foundation for these fundamental parameters. In the linear response regime, a rigorous proof has recently been given [6]. Here, we shall extend these results to the far-from-equilibrium regime and furthermore provide an efficient numerical method for computing their values. More explicitly, this paper is organized as follows: in Section 2, we describe the framework, in Sections 3 and 4, we present our main results and their proofs. Finally, in Section 5, we illustrate our method by considering two well-known models: the three-terminal system and the open Aharovov-Bohm interferometer.

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2 Framework

We consider a multi-terminal mesoscopic system, that is, a small system $\mathcal{S}$ connected through leads to several infinitely extended particle reservoirs (see Figure 1). We assume that the transport properties of this system can be described within the Landauer-Büttiker framework. More precisely, we consider N reservoirs in equilibrium at inverse temperature $\beta_i = 1/(k_{\mathrm{B}}T_i)$ and chemical potential μ_i ($i = 1, \ldots, N$). The corresponding Fermi-Dirac distributions are

$$f_i(E) = f(E, \beta_i, \mu_i) = \left[1 + \mathrm{e}^{\beta_i(E-\mu_i)}\right]^{-1}. \tag{1}$$

For simplicity, here and in what follows we set the Boltzmann constant k_{B}, the Planck constant h and the elementary charge e to unity.

In the Landauer-Büttiker formalism, one neglects all interactions among the particles and considers the small system as a scatterer for the particles emitted by the reservoirs. Thus, the small system is completely characterized by the one-particle on-shell scattering matrix $S(E) = [S_{ij;mn}(E)]$, where the indices $i, j \in \{1, \ldots, N\}$ label the outgoing/incoming terminals and for each pair (i, j) the indices $m \in \{1, \ldots, M_i(E)\}$ and $n \in \{1, \ldots, M_j(E)\}$ label the open channels in terminals i and j, respectively. The matrix element $S_{ij;mn}(E)$ is the probability *amplitude* for a particle with energy E incident in channel n of terminal j to be transmitted into channel m

of terminal i. The corresponding *total* transmission probability $t_{ij}(E)$ that a particle with energy E goes from terminal j to terminal i is given by [7]

$$t_{ij}(E) = \sum_{m=1}^{M_i(E)} \sum_{n=1}^{M_j(E)} |S_{ij;mn}(E)|^2. \tag{2}$$

The unitarity of the scattering matrix immediately yields the following identities,

$$\sum_{i=1}^{N} t_{ij}(E) = M_j(E), \quad \sum_{j=1}^{N} t_{ij}(E) = M_i(E). \tag{3}$$

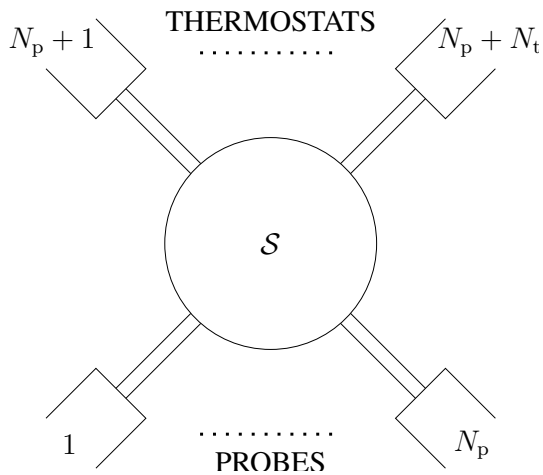

Figure 1: A multi-terminal system: The sample is connected through leads to $N = N_\mathrm{p} + N_\mathrm{t}$ particle reservoirs. The reservoirs $1, \ldots, N_\mathrm{p}$ are probes and the reservoirs $N_\mathrm{p}+1, \ldots, N_\mathrm{p}+N_\mathrm{t}$ are thermostats driving the system out of equilibrium.

The expected stationary electric and heat currents in lead $i \in \{1, \ldots, N\}$ are given by the celebrated Landauer-Büttiker formulas [8, 9],

$$I_i = \sum_{j=1}^{N} \int [t_{ji}(E) f_i(E) - t_{ij}(E) f_j(E)] \, \mathrm{d}E, \tag{4}$$

$$J_i = \sum_{j=1}^{N} \int [t_{ji}(E) f_i(E) - t_{ij}(E) f_j(E)] \, (E - \mu_i) \mathrm{d}E. \tag{5}$$

From a phenomenological point of view, these expressions can be easily understood: $t_{ji}(E) f_i(E)$ is the average number of particles with energy E that are transmitted from terminal i to terminal j, and $t_{ij}(E) f_j(E)$ is the same but from terminal j to terminal i. Therefore, I_i (J_i) is the *net* average electric (heat) current in lead i, counted positively from the i-th terminal to the system.

Mathematical derivations of these formulas (including existence of a stationary regime) rest on the assumption that the leads are infinitely extended and act as reservoirs [10, 11].

Considerable interest has been devoted to electric transport in which all the terminals have the same temperature. In this context, an important concept emerged: the *voltage probe* [4]. A voltage probe is a large physical component used in mesoscopic experiments to sense the local electrochemical potential. Theoretically, such a probe is modeled as a reservoir under the *zero electric current condition*: the chemical potential of the probe is tuned so that there is no net average electric current into it. If all the terminals have the same temperature, then in general there will also be a heat current into the probe. In this case, we will consider this heat current as dissipation. In the same spirit, setting all the terminals to the same chemical potential, a *temperature probe* is obtained by requiring that the temperature of the corresponding reservoir is tuned such that there is no average heat current into it. Note that in this case, there may be some charge dissipation into the temperature probe.

Let us decompose the N terminals as follows: the first $N_{\rm p}$ reservoirs are temperature or voltage probes and the remaining N_t reservoirs are the thermostats maintaining the system out of equilibrium (see Figure 1). In the *voltage probe configuration*, all the reservoirs are at the same inverse temperature $\beta = \beta_1 = \cdots = \beta_N$, the chemical potentials of the thermostats $\vec{\mu}_{\rm t} = (\mu_{N_{\rm p}+1}, \ldots, \mu_{N_{\rm p}+N_{\rm t}})$ are given and we have to determine the probe parameters $\vec{\mu}_{\rm p} = (\mu_1, \ldots, \mu_{N_{\rm p}})$ such that $\vec{I}_{\rm p} = (I_1, \ldots, I_{N_{\rm p}}) = \vec{0}$. Similarly, in the *temperature probe configuration*, all reservoirs are at the same chemical potential $\mu = \mu_1 = \cdots = \mu_N$, the thermostat inverse temperatures $\vec{\beta}_{\rm t} = (\beta_{N_{\rm p}+1}, \ldots, \beta_{N_{\rm p}+N_{\rm t}})$ are fixed and we have to determine the probe parameters $\vec{\beta}_{\rm p} = (\beta_1, \ldots, \beta_{N_{\rm p}})$ in order to satisfy $\vec{J}_{\rm p} = (J_1, \ldots, J_{N_{\rm p}}) = \vec{0}$.

To our knowledge, no result is available on these two problems beyond the linear approximation around global equilibrium (*i.e.*, linear response theory, see [5]). The same remark applies to other approaches to the determination of local intensive thermodynamic parameters (see *e.g.*, [12, 13]).

3 Results

Note that in both configurations the *self-consistency condition*

$$\vec{I}_{\rm p}(\beta, \vec{\mu}_{\rm t}; \vec{\mu}_{\rm p}) = \vec{0}, \qquad \text{or} \qquad \vec{J}_{\rm p}(\mu, \vec{\beta}_{\rm t}; \vec{\beta}_{\rm p}) = \vec{0},$$

is a system of $N_{\rm p}$ non-linear equations with $N_{\rm p}$ unknown. From a mathematical perspective, it is not at all obvious that such a system admits a solution. Moreover, if a solution exists, it may not be unique. Our main result ensures existence and uniqueness of reasonable solutions to these equations.

We shall make the following general assumptions on the lead Hamiltonians and scattering matrix:

(A) There exists a constant E_0 such that $M_j(E) = 0$ for all $E \leq E_0$ and $j \in \{1, \ldots, N\}$.

(B) $M_j(E) \leq C(1 + |E|)^\eta$ for some constants C and η and all E and $j \in \{1, \ldots, N\}$.

(C) For every $j \in \{1, \ldots, N_{\mathrm{p}}\}$ there exists a set $\mathcal{E}_j \subset \mathbb{R}$ of positive Lebesgue measure such that

$$\widetilde{M}_j(E) = \sum_{i=N_{\mathrm{p}}+1}^{N_{\mathrm{p}}+N_{\mathrm{t}}} t_{ij}(E) > 0,$$

for all $E \in \mathcal{E}_j$.

Condition (A) merely asserts that the lead Hamiltonians are bounded below. Condition (B) limit the growth of the number of open scattering channels as function of the energy and is satisfied by any physically reasonable lead Hamiltonian. Finally, Condition (C) can be roughly rephrased as follows: any probe is connected through an open scattering channel to some thermostat.

To formulate our main result, let us denote

$$\underline{\mu} = \min\{\mu_{N_{\mathrm{p}}+1}, \ldots, \mu_{N_{\mathrm{p}}+N_{\mathrm{t}}}\},$$
$$\overline{\mu} = \max\{\mu_{N_{\mathrm{p}}+1}, \ldots, \mu_{N_{\mathrm{p}}+N_{\mathrm{t}}}\},$$

the minimal/maximal chemical potential of the thermostats and define in the same way $\underline{\beta}$ and $\overline{\beta}$.

Theorem 1. *Under the above assumptions, the following hold:*

(1) *The self-consistency condition $\vec{I}_{\mathrm{p}}(\beta, \vec{\mu}_{\mathrm{t}}; \vec{\mu}_{\mathrm{p}}) = \vec{0}$ has a unique solution $\vec{\mu}_{\mathrm{p}} = \vec{\mu}_{\mathrm{p}}(\beta, \vec{\mu}_{\mathrm{t}})$ in the set $\{(\mu_1, \ldots, \mu_{N_{\mathrm{p}}}) \,|\, \mu_j \in [\underline{\mu}, \overline{\mu}]\}$.*

(2) *The self-consistency condition $\vec{J}_{\mathrm{p}}(\mu, \vec{\beta}_{\mathrm{t}}; \vec{\beta}_{\mathrm{p}}) = \vec{0}$ has a unique solution $\vec{\beta}_{\mathrm{p}} = \vec{\beta}_{\mathrm{p}}(\mu, \vec{\beta}_{\mathrm{t}})$ in the set $\{(\beta_1, \ldots, \beta_{N_{\mathrm{p}}}) \,|\, \beta_j \in [\underline{\beta}, \overline{\beta}]\}$.*

(3) *In both cases the solution can be computed by means of a rapidly convergent algorithm (see the next sections for details).*

Remarks. 1. The restriction on the solution is physically reasonable. We do not expect a temperature probe to measure a value below the smallest thermostat temperature or above the highest one. The same remark applies to voltage probes.

2. An alternative approach to probing local intensive parameters is to adjust *both* $\vec{\beta}_{\mathrm{p}}$ and $\vec{\mu}_{\mathrm{p}}$ in such a way that the electric *and* heat currents vanish: $\vec{I}_{\mathrm{p}} = \vec{J}_{\mathrm{p}} = 0$. Such probes thus measure simultaneously the temperature *and* the chemical potential. Note that in this case there is no dissipation at all into the probes. Our method does not apply directly to this situation, basically because the function $f(E, \beta, \mu) - f(E, \beta', \mu')$ does not preserve its sign as E varies if $\beta \neq \beta'$ *and* $\mu \neq \mu'$. To our knowledge, no result is available for such dual probes beyond the linear approximation around global equilibrium (see [14, 6]).

4 Proofs

Let us discuss first the voltage probe configuration. Using the relations (3), the self-consistency condition may be written as

$$I_i = \sum_{j=1}^{N} \int f(E,\beta,\mu_j)[M_j(E)\delta_{ij} - t_{ij}(E)]\mathrm{d}E = 0, \tag{6}$$

for $i = 1,\dots,N_\mathrm{p}$. Under Assumptions (A) and (B),

$$\mu \mapsto X_j(\mu) = \int f(E,\beta,\mu)M_j(E)\mathrm{d}E, \tag{7}$$

defines a strictly increasing continuous function. We shall denote by $X \mapsto \mu_j(X)$ the reciprocal function. The key idea of our approach is to work with the variable $\vec{X} = (X_1(\mu_1),\dots,X_{N_\mathrm{p}}(\mu_{N_\mathrm{p}}))$ instead of $\vec{\mu}_\mathrm{p}$.

Let $\vec{F} : \mathbb{R}^{N_\mathrm{p}} \to \mathbb{R}^{N_\mathrm{p}}$ be defined as

$$\begin{aligned} F_i(\vec{X}) = &\sum_{j=1}^{N_\mathrm{p}} \int f(E,\beta,\mu_j(X_j))t_{ij}(E)\mathrm{d}E \\ &+ \sum_{j=N_\mathrm{p}+1}^{N_\mathrm{p}+N_\mathrm{t}} \int f(E,\beta,\mu_j)t_{ij}(E)\mathrm{d}E. \end{aligned}$$

Then, we can rewrite the self-consistency condition (6) as a fixed point equation

$$\vec{F}(\vec{X}) = \vec{X}. \tag{8}$$

Set $\underline{X}_j = X_j(\underline{\mu})$, $\overline{X}_j = X_j(\overline{\mu})$ and denote

$$\Sigma = \{\vec{X} = (X_1,\dots,X_{N_\mathrm{p}}) \,|\, X_j \in [\underline{X}_j, \overline{X}_j]\}.$$

Notice that the condition $\vec{X} \in \Sigma$ is equivalent to $\mu_j \in [\underline{\mu}, \overline{\mu}]$ for all $j \in \{1,\dots,N_\mathrm{p}\}$.

Lemma 2. $\vec{F}(\Sigma) \subset \Sigma$.

Proof. Let $\vec{X} \in \Sigma$. The monotony of $\mu \mapsto f(E,\beta,\mu)$ implies $f(E,\beta,\mu_j(X_j)) \le f(E,\beta,\overline{\mu})$ for $j = 1,\dots,N_\mathrm{p}$ and $f(E,\beta,\mu_j) \le f(E,\beta,\overline{\mu})$ for $j = N_\mathrm{p}+1,\dots,N_\mathrm{p}+N_\mathrm{t}$. The identities (3) yields

$$F_i(\vec{X}) \le \int f(E,\beta,\overline{\mu})M_i(E)\mathrm{d}E = \overline{X}_i.$$

Proceeding similarly, one shows

$$F_i(\vec{X}) \ge \int f(E,\beta,\underline{\mu})M_i(E)\mathrm{d}E = \underline{X}_i.$$

□

Under Conditions (A) and (B), the function $\vec{F}$ is continuous. Since Σ is compact and convex, it follows from Lemma 2 and the Brouwer fixed point theorem that (8) has a solutions $\vec{X}^\star \in \Sigma$. We shall use Condition (C) to ensure uniqueness of this solution. In the next lemma, we use the norm $\|\vec{X}\| = \sum_{j=1}^{N_\mathrm{p}} |X_j|$.

Lemma 3. *Under Assumptions (A), (B) and (C) there exists a constant $\theta < 1$ such that*

$$\|\vec{F}(\vec{X}) - \vec{F}(\vec{X}')\| \leq \theta \|\vec{X} - \vec{X}'\|, \tag{9}$$

for any $\vec{X}, \vec{X}' \in \Sigma$.

Proof. Denote by $D(\vec{X})$ the derivative of the map $\vec{F}$ at $\vec{X}$. Then one has

$$\vec{F}(\vec{X}) - \vec{F}(\vec{X}') = \int_0^1 D(t\vec{X} + (1-t)\vec{X}')(\vec{X} - \vec{X}')\mathrm{d}t.$$

Since Σ is convex, the estimate (9) holds for any $\vec{X}, \vec{X}' \in \Sigma$ with

$$\theta = \max_{\vec{X} \in \Sigma} \|D(\vec{X})\|,$$

where the matrix norm is given by

$$\|D(\vec{X})\| = \max_{1 \leq j \leq N_\mathrm{p}} \sum_{i=1}^{N_\mathrm{p}} \left| D_{ij}(\vec{X}) \right|.$$

A simple calculation yields

$$D_{ij}(\vec{X}) = \frac{\partial F_i}{\partial X_j}(\vec{X}) = \frac{\int g(E, \beta, \mu_j(X_j)) t_{ij}(E) \mathrm{d}E}{\int g(E, \beta, \mu_j(X_j)) M_j(E) \mathrm{d}E},$$

where the function $g(E, \beta, \mu) = \partial_\mu f(E, \beta, \mu)$ is strictly positive. It follows that

$$\sum_{i=1}^{N_\mathrm{p}} \left| D_{ij}(\vec{X}) \right| \leq 1 - \frac{\int \widetilde{M}_j(E) g(E, \beta, \mu_j(X_j)) \mathrm{d}E}{\int M_j(E) g(E, \beta, \mu_j(X_j)) \mathrm{d}E},$$

and hence

$$\theta \leq 1 - \min_{\substack{\mu \in [\underline{\mu}, \overline{\mu}] \\ 1 \leq j \leq N_\mathrm{p}}} \frac{\int \widetilde{M}_j(E) g(E, \beta, \mu) \mathrm{d}E}{\int M_j(E) g(E, \beta, \mu) \mathrm{d}E}.$$

Condition (C) clearly implies that $\theta < 1$. □

It follows from Lemmas 2, 3 and the Banach fixed point theorem that (8) has a unique solution $\vec{X}^\star$ in Σ. Moreover, the sequence of iterates $\vec{X}_n = \vec{F}(\vec{X}_{n-1})$ converges to $\vec{X}^\star$ for any initial value $\vec{X}_0 \in \Sigma$ with the estimate

$$\|\vec{X}_n - \vec{X}^\star\| \leq \theta^n \|\vec{X}_0 - \vec{X}^\star\|.$$

In the temperature probe configuration, one may proceed in a completely similar way in terms of the functions

$$\beta \mapsto Y_j(\beta) = \int f(E,\beta,\mu)(E-\mu)M_j(E)\mathrm{d}E, \tag{10}$$

their reciprocal $Y \mapsto \beta_j(Y)$ and

$$G_i(\vec{Y}) = \sum_{j=1}^{N_\mathrm{p}} \int f(E,\beta_j(Y_j),\mu)(E-\mu)t_{ij}(E)\mathrm{d}E + \sum_{j=N_\mathrm{p}+1}^{N_\mathrm{p}+N_\mathrm{t}} \int f(E,\beta_j,\mu)(E-\mu)t_{ij}(E)\mathrm{d}E.$$

A natural set Σ can then be defined as before. The crucial observation is that the function $\partial_\beta f(E,\beta,\mu)(E-\mu)$ has a constant sign.

Remark. Strictly speaking, Lemma 3 does not hold at zero temperature because the Fermi function f is not positive in this case. Nevertheless, one easily shows that, under Assumptions (A)–(C), the estimate

$$\|\vec{F}(\vec{X}) - \vec{F}(\vec{X}')\| < \|\vec{X} - \vec{X}'\|,$$

holds for $\vec{X}, \vec{X}' \in \Sigma$, $\vec{X} \neq \vec{X}'$ provided $[\underline{\mu}, \overline{\mu}] \subset \cap_j \mathcal{E}_j$. The uniqueness of the fixed point $\vec{X}^\star$ immediately follows. Moreover, it also follows that the sequence of iterates $\vec{X}_n = \vec{F}(\vec{X}_{n-1})$ converges to $\vec{X}^\star$ for any choice of $\vec{X}_0 \in \Sigma$, although without *a priori* control on the speed of convergence.

5 Examples

As a first example, let us consider the one-channel three-terminal system represented in Figure 2, where two thermostats (2 and 3) drive the system (a perfect lead) out of equilibrium and a probe (1) is connected to the system by a 3×3 scattering matrix S.

Let us consider the energy-independent scattering matrix introduced in [15]:

$$S = \begin{pmatrix} -(a+b) & \sqrt{\varepsilon} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & a & b \\ \sqrt{\varepsilon} & b & a \end{pmatrix}, \tag{11}$$

where $a = \frac{1}{2}(\sqrt{1-2\varepsilon} - 1)$, $b = \frac{1}{2}(\sqrt{1-2\varepsilon} + 1)$ and $\varepsilon \in (0, \frac{1}{2}]$. Here, $\varepsilon = 0$ corresponds to the uncoupled situation (which is excluded) and $\varepsilon = \frac{1}{2}$ to the maximally coupled one. Let us set $T = T_1 = T_2 = T_3$ and define the energy interval in (4)–(5) as $[0, \infty)$. If $T = 0$, then in the linear regime one can compute analytically the self-consistent parameter μ_1^* [7, 14]:

$$\mu_1^*(T=0,\mu_2,\mu_3,\varepsilon) = \frac{\mu_2+\mu_3}{2} + \mathcal{O}(|\mu_2-\mu_3|^2)\,. \tag{12}$$

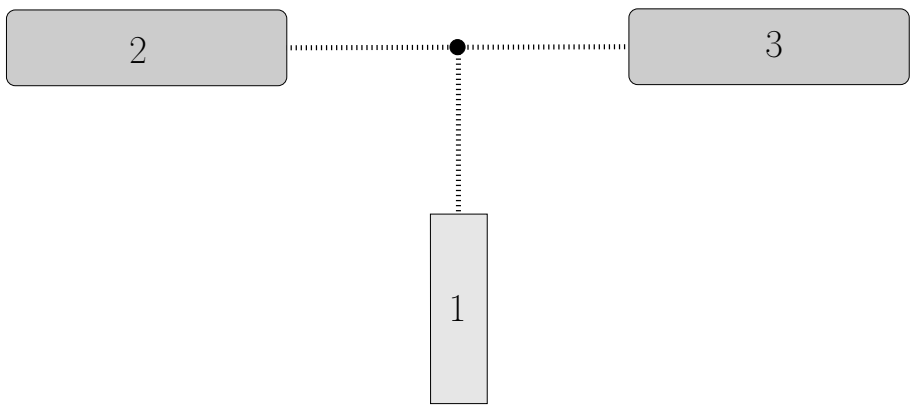

Figure 2: A one-channel system with two thermostats (2 and 3) and one probe (1).

We have checked that our numerical results are consistent with the relation (12). In the non-linear regime, we made the following observations: Let $T > 0$, $\mu_2, \mu_3 \in \mathbb{R}$ be fixed, then the sequence $\{F^n(X_0)\}_{n=0}^{\infty}$, with $X_0 \in \Sigma$, converges and gives rise to a value μ_1^* independent of ε and conveniently written as

$$\mu_1^*(T, \mu_2, \mu_3, \varepsilon) = \frac{\mu_2 + \mu_3}{2} + \mathcal{N}(T, \mu_2, \mu_3) \ , \tag{13}$$

where the function $\mathcal{N}(T, \mu_2, \mu_3)$ measures the non-linearity. Note, in particular, that the weak coupling limit $\varepsilon \to 0$ does not lead to a different value of μ_1^*. Since, by Theorem 1,

$$\mu_1^*(T, \mu_2, \mu_3, \varepsilon) \in [\min\{\mu_2, \mu_3\}, \max\{\mu_2, \mu_3\}],$$

one deduces that $\mathcal{N}(T, \mu_2, \mu_3) \in [-\Delta\mu/2, \Delta\mu/2]$, with $\Delta\mu = |\mu_2 - \mu_3|$. In Figures 3 and 4, we have plotted the temperature and potential dependence of $\mathcal{N}(T, \mu_2, \mu_3)$, respectively. Let us recall that $\mathcal{N}(T, \mu_2, \mu_3) = 0$ corresponds to the linear case.

Note that the curve in Figure 4 reaches a constant value $\mathcal{N}_\infty(T) = \lim_{\mu_3 \to \infty} \mathcal{N}(T, \mu_2 = 0, \mu_3)$ as μ_3 increases. Interestingly, this is also the case for other values of T and we observed the following scaling law:

$$\mathcal{N}_\infty(\lambda T) = \lambda \mathcal{N}_\infty(T) \ , \quad \forall \lambda > 0. \tag{14}$$

If we attach more probes to the lead and describe all the connection points in terms of the *same* scattering matrix S (see [14] for the construction of the global scattering matrix), then we found that all the probes measure the *same* value, as if all the probes were connected to the same point, but in general this value does not coincide with the one-probe measurement (since adding more probes somehow perturbs the system). This phenomenon can be easily understood: for example, if two probes are attached to the lead, then one can compute analytically the global 4×4 transmission matrix $\{t_{ij}(E)\}$, and one finds that it is symmetric and that $t_{31}(E) = t_{32}(E) = t_{41}(E) = t_{42}(E)$. This means that the two probes are equally coupled to the left and right thermostats and, consequently, that $\mu_1^* = \mu_2^*$. Note, however, that this is not true in general.

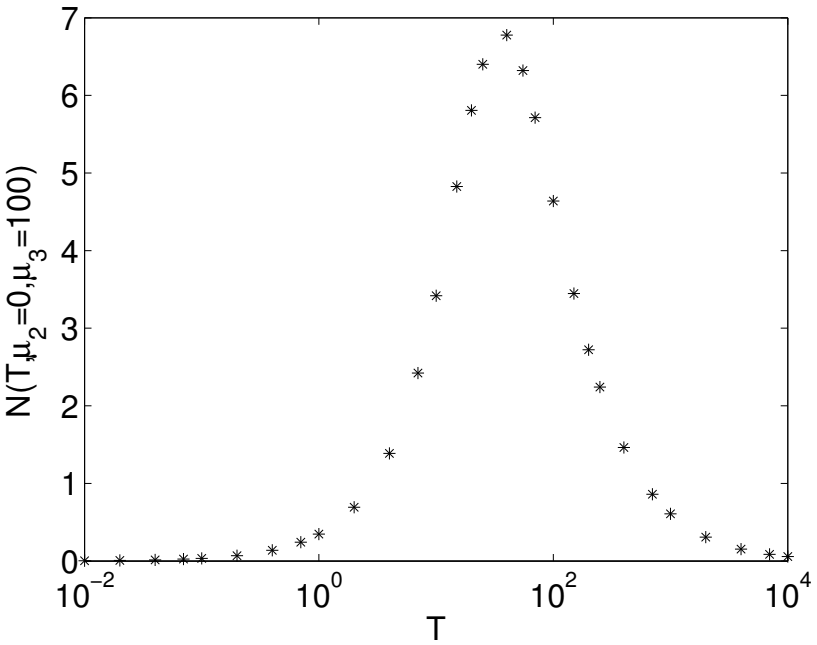

Figure 3: The temperature dependence of $\mathcal{N}(T, \mu_2 = 0, \mu_3 = 100)$.

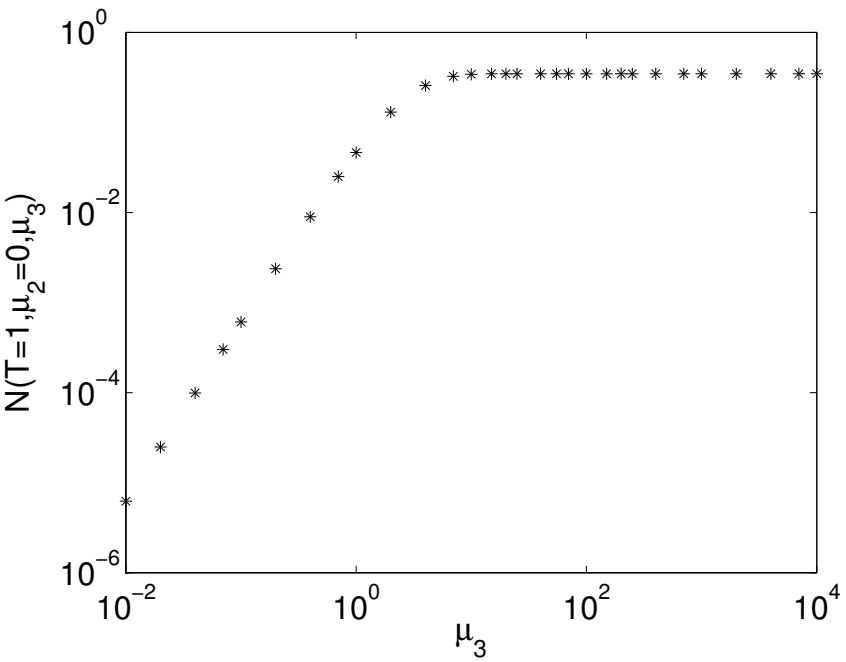

Figure 4: The potential dependence of $\mathcal{N}(T = 1, \mu_2 = 0, \mu_3)$.

As a second example, we consider an Aharonov-Bohm (AB) ring threaded by a magnetic flux ϕ and with a quantum dot (QD) embedded in one of its arms. This system has been subject to intensive investigations both in the independent electron approximation [12, 16] and including interaction effects [17, 19, 18]. We shall study a discrete (tight binding) independent electron model closely related to the work by Aharony *et al.* [20] (see Figure 5). This theoretical model is supposed to imitate an experimental setup [21]. It is assumed that a gate voltage V is applied on the QD, allowing to vary the energies of its eigenstates. Let us write $t_{\mathrm{QD}} = \sqrt{T_{\mathrm{QD}}}e^{i\alpha_{\mathrm{QD}}}$ for the

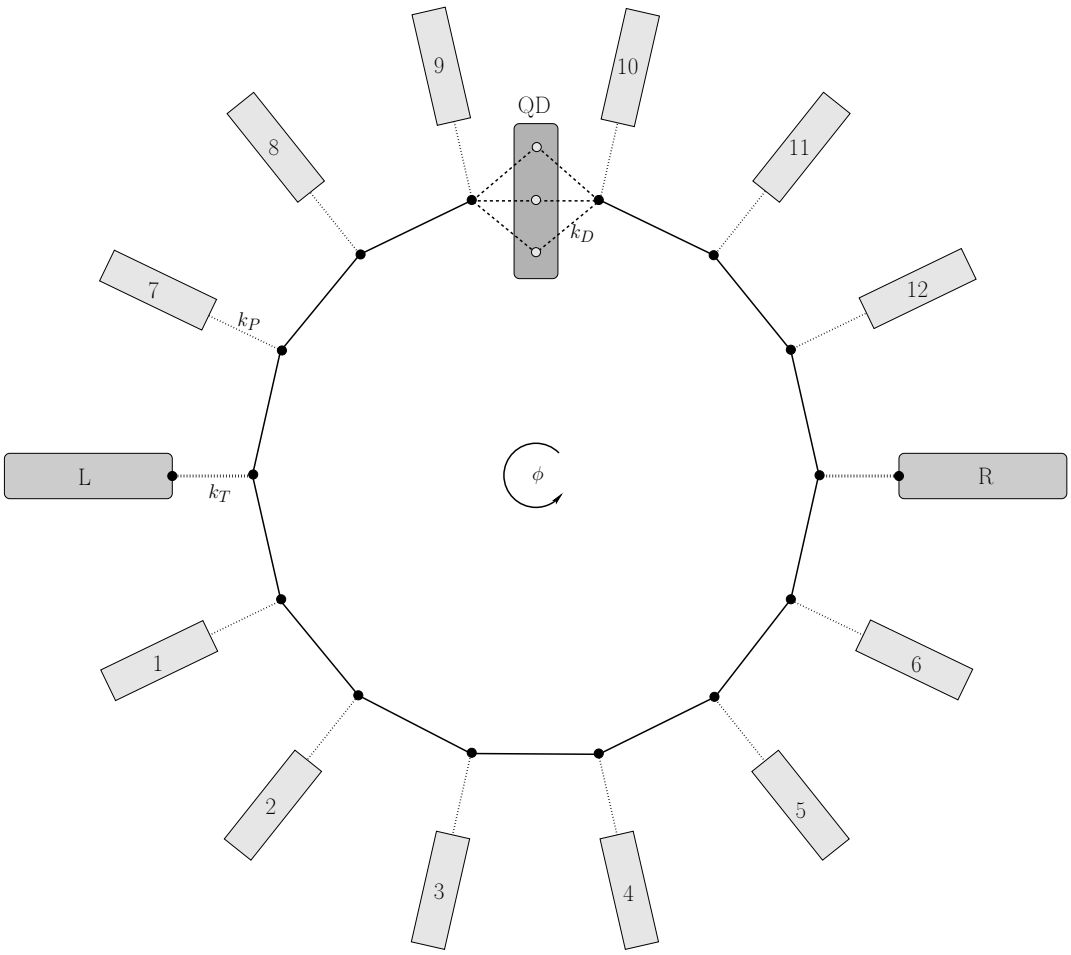

Figure 5: The open AB interferometer. A magnetic flux ϕ crosses the ring and a QD is placed in the upper branch of the ring. The terminals 1 to 12 are voltage probes and the terminals L and R are thermostats. k_T, k_P and k_D are the hopping constants coupling the ring to the thermostats, the probes and the QD. The energy levels of the dot are V, $V + U$ and $V + 2U$.

transmission amplitude of the QD. At fixed energy E, the total transmission probability t_{LR} from the reservoirs L to R depends on the gate voltage V and is a periodic function of the AB-flux ϕ.

Expanding this function as a Fourier series one gets:

$$t_{LR}(V,\phi) = A(V) + B(V)\cos(\phi + \beta(V)) + \cdots . \tag{15}$$

It is well known that in the absence of dissipation (*i.e.*, for a closed interferometer in the terminology of [20]) the Onsager-Casimir reciprocity relations [12] imply that the phase $\beta(V)$ can only take the values 0 and π. Hence, as the gate voltage V varies, the phase $\beta(V)$ makes abrupt jumps between these two values. However, dissipation can change this picture. By adding purely absorbing reservoirs (*i.e.*, allowing only outgoing currents) along the branches of the ring Aharony et al. [20] found criteria as to when the "experimental" phase $\beta(V)$, which depends on the details of the opening (*i.e.*, the coupling the absorbing reservoirs), is a good approximation of the "intrinsic" phase $\alpha_{\rm QD}(V)$ of the QD. Here we present some numerical results showing that one may capture the main properties of $\alpha_{\rm QD}$ without introducing any charge dissipation in the absorbing reservoirs, *i.e.*, that β behaves essentially as $\alpha_{\rm QD}$ even if one replaces the absorbing reservoirs of [20] by voltage probes, which we recall allow only heat dissipation. However, instead of consid-

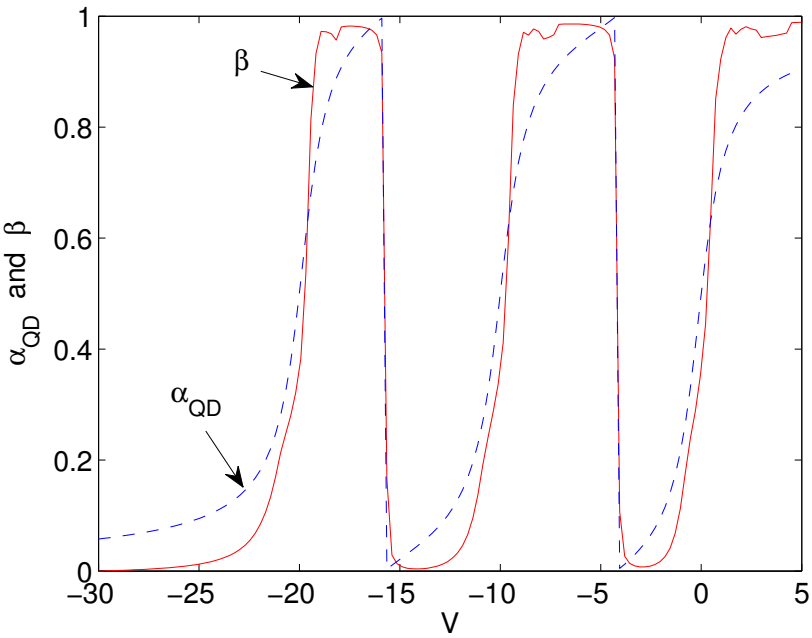

Figure 6: The "intrinsic" phase $\alpha_{\rm QD}$ (blue dashed line) and the "experimental" phase β (red solid line) as a function of the gate voltage V applied on the QD. This corresponds to the setup of Figure 5 with a three-level QD, all temperatures equal to zero, the chemical potential of the thermostats are $\mu_L = 0$ and $\mu_R = 0.2$. The couplings to the thermostats, probes and QD are $k_T = 0.5$, $k_P = 0.5\ k_T$, and $k_D = 0.01\ k_T$, respectively (the parameter k_P plays a role similar to J_x in [20]). All remaining parameters are set as in [20].

ering the expansion (15), we tried to be closer to actual experimental measurements by extracting

the "experimental" phase β from the Fourier expansion of the steady electric current between the two thermostats:

$$I_L = -I_R = \hat{I}_0(V) + \hat{I}_1(V)\cos(\phi + \beta(V)) + \cdots .$$

The results are shown on Figure 6. One sees that the curve $\beta(V)$ follows relatively closely $\alpha_{\mathrm{QD}}(V)$, and in particular reproduces accurately the successive jumps of $\alpha_{\mathrm{QD}}(V)$ from 1 to 0 (the values have been normalized, thus 1 corresponds to π in the paper [20]).

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