Cautious OWA and Evidential Reasoning for decision under uncertainty

J.M. Tacnet, J. Dezert

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Jean-Marc Tacnet
Cemagref-ETGR
2, rue de la papèterie - B.P. 76
F-38402 Saint Martin d'Hères Cedex, France
Email: jean-marc.tacnet@cemagref.fr

Jean Dezert
Onera - The French Aerospace Lab
F-91761 Palaiseau Cedex, France
Email: jean.dezert@onera.fr

***Abstract*—To make a decision under certainty, multicriteria decision methods aims to choose, rank or sort alternatives on the basis of quantitative or qualitative criteria and preferences expressed by the decision-makers. However, decision is often done under uncertainty: choosing alternatives can have different consequences depending on the external context (or state of the word). In this paper, a new methodology called Cautious Ordered Weighted Averaging with Evidential Reasoning (COWA-ER) is proposed for decision making under uncertainty to take into account imperfect evaluations of the alternatives and unknown beliefs about groups of the possible states of the world (scenarii). COWA-ER mixes cautiously the principle of Yager's Ordered Weighted Averaging (OWA) approach with the efficient fusion of belief functions proposed in Dezert-Smarandache Theory (DSmT).**

Keywords: fusion, Ordered Weighted Averaging (OWA), DSmT, uncertainty, information imperfection, multi-criteria decision making (MCDM)

I. Introduction

A. Decisions under certainty, risk or uncertainty

Decision making in real-life situations are often difficult multi-criteria problems. In the classical Multi-Criteria Decision Making (MCDM) framework, those decisions consist mainly in choosing, ranking or sorting alternatives, solutions or more generally potential actions [17] on the basis of quantitative or qualitative criteria. Existing methods differs on aggregation principles (total or partial), preferences weighting, and so on. In total aggregation multicriteria decision methods such as Analytic Hierarchy Process (AHP) [19], the result for an alternative is a unique value called synthesis criterion. Possible alternatives (A_i) belonging to a given set $A = \{A_1, A_2, \ldots, A_q\}$ are evaluated according to preferences (represented by weights w_j) expressed by the decision-makers on the different criteria (C_j) (see figure 1).

Decisions are often taken on the basis of imperfect information and knowledge (imprecise, uncertain, incomplete) provided by several more or less reliable sources and depending on the states of the world: decisions can be taken in certain, risky or uncertain environment. In a MCDM context, decision under certainty means that the evaluations of the alternative are independent from the states of the world. In other cases, alternatives may be assessed differently depending on the scenarii that are considered.

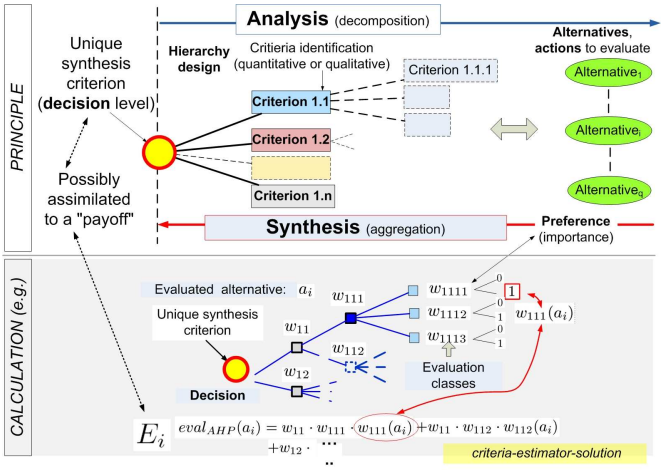

Figure 1. Principle of a multi-criteria decision method based on a total aggregation principle.

In the classical framework of decision theory under uncertainty, Expected Utility Theory (EUT) states that a decision maker chooses between risky or uncertain alternatives or actions by comparing their expected utilities [14]. Let us consider an example of decision under uncertainty (or risk) related to natural hazards management. On the lower parts of torrent catchment basin or an avalanche path, risk analysis consists in evaluating potential damage caused due to the effects of hazard (a phenomenon with an intensity and a frequency) on people and assets at risk. Different strategies (A_i) are possible to protect the exposed areas. For each of them, damage will depend on the different scenarii (S_j) of phenomenon which can be more or less uncertain. An *action* A_i (e.g. building a protection device, a dam) is evaluated through its potential effects r_k to which are associated *utilities* $u(r_k)$ (protection level of people, cost of protection, ...) and *probabilities* $p(r_k)$ (linked to natural events or states of nature S_k). The expected utility $U(a)$ of an action a is estimated through the sum of products of utilities and probabilities of all potential consequences of the action a:

$$U(A_i) = \sum u(r_k) \cdot p(r_k) \tag{1}$$

When probabilities are known, decision is done under risk. When those probabilities becomes subjective, the prospect theory (subjective expected utility theory - SEUT) [12] can apply :

- *the objective utility* (e.g. cost) $u(r_k)$ is replaced by a subjective function (*value*) denoted $v(u(r_k))$;
- the objective weighting $p(r_k)$ is replaced by a subjective function $\pi(p(r_k))$.

$v(\cdot)$ is the felt subjective value in response of the expected cost of the considered action, and $\pi(\cdot)$ is the felt weighting face to the objective probability of the realisation of the result. Prospect theory shows that the function $v(\cdot)$ is asymmetric: loss causes a negative reaction intensity stronger than the positive reaction caused by the equivalent gain. This corresponds to an aversion to risky choices in the area of earnings and a search of risky choices in the area of loss.

In a MCDM context, information imperfection concerns both the evaluation of the alternatives (in any context of certainty, risk or ignorance) and the uncertainty or lack of knowledge about the possible states of the world. Uncertainty and imprecision in multi-criteria decision models has been early considered [16]. Different kinds of uncertainty can be considered: on the one hand the internal uncertainty is linked to the structure of the model and the judgmental inputs required by the model, on the other hand the external uncertainty refers to lack of knowledge about the consequences about a particular choice.

B. Objectives and goals

Several decision support methods exist to consider both information imperfection, sources heterogeneity, reliability, conflict and the different states of the world when evaluating the alternatives as summarized on figure 2. A more complete review can be found in [28]. Here we just remind some recent examples of methods mixing MCDM approaches and Evidential Reasoning[1](ER).

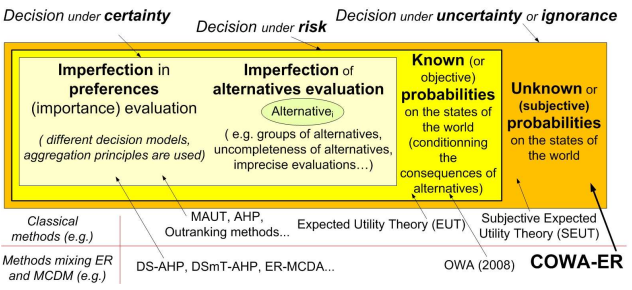

Figure 2. Information imperfection in the different decision support methods

- Dempster-Shafer-based AHP (DS-AHP) has introduced a merging of Evidential Reasoning (ER) with Analytic Hierachy Process (AHP) [19] to consider the imprecision and the uncertainty in evaluation of several alternatives. The idea is to consider criteria as sources [1], [3] and derive weights as discounting factors in the fusion process [5];
- Dezert-Smarandache-based (DSmT-AHP) [8] takes into account the partial uncertainty (disjunctions) between possible alternatives and introduces new fusion rules, based on Proportional Conflict Redistribution (PCR) principle, which allow to consider differences between importance and reliability of sources [23];
- ER-MCDA [28], [29] is based on AHP, fuzzy sets theory, possibility theory and belief functions theory too. This method considers both imperfection of criteria evaluations, importance and reliability of sources.

Introducing ignorance and uncertainty in a MCDM process consists in considering that consequences of actions (A_i) depend of the state of nature represented by a finite set $S = \{S_1, S_2, \ldots, S_n\}$. For each state, the MCDM method provides an evaluation C_{ij}. We assume that this evaluation C_{ij} done by the decision maker corresponds to the choice of A_i when S_j occurs with a given (possibly subjective) probability. The evaluation matrix is defined as $C = [C_{ij}]$ where $i = 1, \ldots, q$ and $j = 1, \ldots, n$.

$$\begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & \cdots & S_j & \cdots & S_n \end{array} \\ \begin{pmatrix} C_{11} & \cdots & C_{11} & \cdots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \cdots & C_{ij} & \cdots & C_{in} \\ & & \vdots & & \\ C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} \end{pmatrix} \end{array} = C \qquad (2)$$

Existing methods using evidential reasoning and MCDM have, up to now, focused on the case of imperfect evaluation of alternatives in a context of decision under certainty. In this paper, we propose a new method for decision under uncertainty that mixes MCDM principles, decision under uncertainty principles and evidential reasoning. For this purpose, we propose a framework that considers uncertainty and imperfection for scenarii corresponding to the state of the world.

This paper is organized as follows. In section II, we briefly recall the basis of DSmT. Section III presents two existing methods for MCDM under uncertainty using belief functions theory: DSmT-AHP as an extension of Saaty's multi-criteria decision method AHP, and Yager's Ordered Weighted Averaging (OWA) approach for decision making with belief structures. The contribution of this paper concerns the section IV where we describe an alternative to the classical OWA, called cautious OWA method, where evaluations of alternatives depend on more or less uncertain scenarii. The flexibility and advantages of this COWA method are also discussed. Conclusions and perspectives are given in section V.

[1]Evidential Reasoning refers to the use of belief functions as theoretical background, not to a specific theory of belief functions (BF) aimed for combining, or conditioning BF. Actually, Dempster-Shafer Theory (DST) [21], Dezert-Smarandache Theory (DSmT) [22], and Smets' TBM [25] are different approaches of Evidential Reasoning.

II. Belief functions and DSmT

Dempster-Shafer Theory (DST) [21] offers a powerful mathematical formalism (the belief functions) to model our belief and uncertainty on the possible solutions of a given problem. One of the pillars of DST is Dempster-Shafer rule (DS) of combination of belief functions. The purpose of the development of Dezert-Smarandache Theory (DSmT) [22] is to overcome the limitations of DST by proposing new underlying models for the frames of discernment in order to fit better with the nature of real problems, and new combination and conditioning rules for circumventing problems with DS rule specially when the sources to combine are highly conflicting. In DSmT, the elements θ_i, $i = 1, 2, \ldots, n$ of a given frame Θ are not necessarily exclusive, and there is no restriction on θ_i but their exhaustivity. Some integrity constraints (if any) can be include in the underlying model of the frame. Instead of working in power-set 2^Θ, we classically work on hyper-power set D^Θ (Dedekind's lattice) - see [22], Vol.1 for details and examples. A (generalized) basic belief assignment (bba) given by a source of evidence is a mapping $m : D^\Theta \to [0, 1]$ such that

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1 \tag{3}$$

The *generalized* credibility and plausibility functions are defined in almost the same manner as within DST, i.e.

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in D^\Theta}} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in D^\Theta}} m(B) \tag{4}$$

In this paper, we will work with Shafer's model of the frame Θ, i.e. all elements θ_i of Θ are assumed truly exhaustive and exclusive (disjoint). Therefore $D^\Theta = 2^\Theta$ and the generalized belief functions reduces to classical ones. DSmT proposes a new efficient combination rules based on proportional conflict redistribution (PCR) principle for combining highly conflicting sources of evidence. Also, the classical pignistic transformation $BetP(.)$ [26] is replaced by the by the more effective $DSmP(.)$ transformation to estimate the subjective probabilities of hypotheses for classical decision-making. We just recall briefly the PCR fusion rule # 5 (PCR5) and Dezert-Smarandache Probabilistic (DSmP) transformation. All details, justifications with examples on PCR5 and DSmP can be found freely from the web in [22], Vols. 2 & 3 and will not be reported here.

• **The Proportional Conflict Redistribution Rule no. 5**: PCR5 is used generally to combine bba's in DSmT framework. PCR5 transfers the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses, so that the specificity of the information is entirely preserved in this fusion process. Let $m_1(.)$ and $m_2(.)$ be two independent[2] bba's, then the PCR5 rule is defined as follows (see [22], Vol. 2 for full justification and examples): $m_{PCR5}(\emptyset) = 0$ and $\forall X \in 2^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap X = \emptyset}} \left[\frac{m_1(X)^2 m_2(X_2)}{m_1(X) + m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X) + m_1(X_2)}\right] \tag{5}$$

where all denominators in (5) are different from zero. If a denominator is zero, that fraction is discarded. Additional properties of PCR5 can be found in [9]. Extension of PCR5 for combining qualitative bba's can be found in [22], Vol. 2 & 3. All propositions/sets are in a canonical form. A variant of PCR5, called PCR6 has been proposed by Martin and Osswald in [22], Vol. 2, for combining $s > 2$ sources. The general formulas for PCR5 and PCR6 rules are given in [22], Vol. 2 also. PCR6 coincides with PCR5 for the fusion of two bba's.

• **DSmP probabilistic transformation**: $DSmP$ is a serious alternative to the classical pignistic transformation $BetP$ since it increases the probabilistic information content (PIC), i.e. it reduces Shannon entropy of the approximate subjective probability measure drawn from any bba – see [22], Vol. 3, Chap. 3 for details and the analytic expression of $DSmP_\epsilon(.)$. When $\epsilon > 0$ and when the masses of all singletons are zero, $DSmP_\epsilon(.) = BetP(.)$, where the well-known pignistic transformation $BetP(.)$ is defined by Smets in [26].

In the Evidential Reasoning framework, the decisions are usually achieved by computing the expected utilities of the acts using either the subjective/pignistic $BetP\{.\}$ (usually adopted in DST framework) or $DSmP(.)$ (as suggested in DSmT framework) as the probability function needed to compute expectations. Usually, one uses the maximum of the pignistic probability as decision criterion. The maximum of $BetP\{.\}$ is often considered as a balanced strategy between the two other strategies for decision making: the max of plausibility (optimistic strategy) or the max. of credibility (pessimistic strategy). The max of $DSmP(.)$ is considered as more efficient for practical applications since $DSmP(.)$ is more informative (it has a higher PIC value) than $BetP(.)$ transformation. The justification of DSmP as a fair and useful transformation for decision-making support can also be found in [10]. Note that in the binary frame case, all the aforementioned decision strategies yields same final decision.

III. Belief functions and MCDM

Two simple methods for MCDM under uncertainty are briefly presented: DSmT-AHP approach and Yager's OWA approach. The new Cautious OWA approach that we propose will be developed in the next section.

A. DSmT-AHP approach

DSmT-AHP aimed to perform a similar purpose as AHP [18], [19], SMART [30] or DS/AHP [1], [3], etc. that is to find the preferences rankings of the decision alternatives (DA), or groups of DA. DSmT-AHP approach consists in three steps:

- Step 1: we extend the construction of the matrix for taking into account the partial uncertainty (disjunctions) between

[2]i.e. each source provides its bba independently of the other sources.

possible alternatives. If no comparison is available between elements, then the corresponding elements in the matrix is zero. Each bba related to each (sub-) criterion is the normalized eigenvector associated with the largest eigenvalue of the "uncertain" knowledge matrix (as done in standard AHP approach).

- Step 2: we use the DSmT fusion rules, typically the PCR5 rule, to combine bba's drawn from step 1 to get a final MCDM priority ranking. This fusion step must take into account the different importances (if any) of criteria as it will be explained in the sequel.
- Step 3: decision-making can be based either on the maximum of belief, or on the maximum of the plausibility of DA, as well as on the maximum of the approximate subjective probability of DA obtained by different probabilistic transformations.

The MCDM problem deals with several criteria having different importances and the classical fusion rules cannot be applied directly as in step 2. In AHP, the fusion is done from the product of the bba's matrix with the weighting vector of criteria. Such AHP fusion is nothing but a simple component-wise weighted average of bba's and it doesn't actually process efficiently the conflicting information between the sources. It doesn't preserve the neutrality of a full ignorant source in the fusion. To palliate these problems, we have proposed a new solution for combining sources of different importances in [23]. Briefly, the reliability of a source is usually taken into account with Shafer's discounting method [21] defined by:

$$\begin{cases} m_\alpha(X) = \alpha \cdot m(X), \quad \text{for } X \neq \Theta \\ m_\alpha(\Theta) = \alpha \cdot m(\Theta) + (1-\alpha) \end{cases} \tag{6}$$

where $\alpha \in [0;1]$ is the reliability discounting factor. $\alpha = 1$ when the source is fully reliable and $\alpha = 0$ if the source is totally unreliable. We characterize the importance of a source by an importance factor β in $[0,1]$. β factor is usually not related with the reliability of the source and can be chosen to any value in $[0,1]$ by the designer for his/her own reason. By convention, $\beta = 1$ means the maximal importance of the source and $\beta = 0$ means no importance granted to this source. From this β factor, we define the importance discounting by

$$\begin{cases} m_\beta(X) = \beta \cdot m(X), \quad \text{for } X \neq \emptyset \\ m_\beta(\emptyset) = \beta \cdot m(\emptyset) + (1-\beta) \end{cases} \tag{7}$$

Here, we allow to deal with non-normal bba since $m_\beta(\emptyset) \geq 0$ as suggested by Smets in [24]. This new discounting preserves the specificity of the primary information since all focal elements are discounted with same importance factor. Based on this importance discounting, one can adapt PCR5 (or PCR6) rule for $N \geq 2$ discounted bba's $m_{\beta,i}(.)$, $i = 1, 2, \ldots N$ to get with PCR5$_\emptyset$ fusion rule (see details in [23]) a resulting bba which is then normalized because in the AHP context, the importance factors correspond to the components of the normalized eigenvector $\mathbf{w}$. It is important to note that such importance discounting method cannot be used in DST when using Dempster-Shafer's rule of combination because this rule is not responding to the discounting of sources towards the empty set (see Theorem 1 in [23] for proof). The reliability and importance of sources can be taken into account easily in the fusion process and separately. The possibility to take them into account jointly is more difficult, because in general the reliability and importance discounting approaches do not commute, but when $\alpha_i = \beta_i = 1$. In order to deal both with reliabilities and importances factors and because of the non commutativity of these discountings, two methods have also been proposed in [23] and not reported here.

B. Yager's OWA approach

Let's introduce Yager's OWA approach [33] for decision making with belief structures. One considers a collection of q alternatives belonging to a set $A = \{A_1, A_2, \ldots, A_q\}$ and a finite set $S = \{S_1, S_2, \ldots, S_n\}$ of states of the nature. We assume that the payoff/gain C_{ij} of the decision maker in choosing A_i when S_j occurs are given by positive (or null) numbers. The payoffs $q \times n$ matrix is defined by $C = [C_{ij}]$ where $i = 1, \ldots, q$ and $j = 1, \ldots, n$ as in eq. (2). The decision-making problem consists in choosing the alternative $A^* \in A$ which maximizes the payoff to the decision maker given the knowledge on the state of the nature and the payoffs matrix C. $A^* \in A$ is called the best alternative or the solution (if any) of the decision-making problem. Depending the knowledge the decision-maker has on the states of the nature, he/she is face on different decision-making problems:
1 – Decision-making under certainty: only one state of the nature is known and certain to occur, say S_j. Then the decision-making solution consists in choosing $A^* = A_{i^*}$ with $i^* \triangleq \arg\max_i \{C_{ij}\}$.
2 – Decision-making under risk: the true state of the nature is unknown but one knows all the probabilities $p_j = P(S_j)$, $j = 1, \ldots, n$ of the possible states of the nature. In this case, we use the maximum of expected values for decision-making. For each alternative A_i, we compute its expected payoff $E[C_i] = \sum_j p_j \cdot C_{ij}$, then we choose $A^* = A_{i^*}$ with $i^* \triangleq \arg\max_i \{E[C_i]\}$.
3 – Decision-making under ignorance: one assumes no knowledge about the true state of the nature but that it belongs to S. In this case, Yager proposes to use the OWA operator assuming a given decision attitude taken by the decision-maker. Given a set of values/payoffs c_1, c_2, ..., c_n, OWA consists in choosing a normalized set of weighting factors $W = [w_1, w_2, \ldots w_n]$ where $w_j \in [0,1]$ and $\sum_j w_j = 1$ and for any set of values c_1, c_2, ..., c_n compute OWA$(c_1, c_2, \ldots, c_n)$ as

$$\text{OWA}(c_1, c_2, \ldots, c_n) = \sum_j w_j \cdot b_j \tag{8}$$

where b_j is the jth largest element in the collection c_1, c_2, ..., c_n. As seen in (8), the OWA operator is nothing but a simple weighted average of ordered values of a variable.
Based on such OWA operators, the idea consists for each alternative A_i, $i = 1, \ldots, q$ to choose a weighting vector $W_i = [w_{i1}, w_{i2}, \ldots w_{in}]$ and compute its OWA value $V_i \triangleq \text{OWA}(C_{i1}, C_{i2}, \ldots, C_{in}) = \sum_j w_{ij} \cdot b_{ij}$ where b_{ij} is the

jth largest element in the collection of payoffs C_{i1}, C_{i2},..., C_{in}. Then, as for decision-making under risk, we choose $A^* = A_{i^*}$ with $i^* \triangleq \arg\max_i\{V_i\}$. The determination of W_i depends on the decision attitude taken by the decision-maker. The pessimistic attitude considers for all $i = 1, 2, \ldots, q$, $W_i = [0, 0, \ldots, 0, 1]$. In this case, we assign to A_i the least payoff and we choose the best worst (the max of least payoffs). It is a Max-Min strategy since $i^* = \arg\max_i(\min_j C_{ij})$. The optimistic attitude considers for all $i = 1, 2, \ldots, q$, $W_i = [1, 0, \ldots, 0, 0]$. We commit to A_i its best payoff and we select the best best. It is a Max-Max strategy since $i^* = \arg\max_i(\max_j C_{ij})$. Between these two extreme attitudes, we can define an infinity of intermediate attitudes like the normative/neutral attitude (when or all $i = 1, 2, \ldots, q$, $W_i = [1/n, 1/n, \ldots, 1/n, 1/n]$) which corresponds to the simple arithmetic mean, or Hurwicz attitude (i.e. a weighted average of pessimistic and optimistic attitudes), etc. To justify the choice of OWA method, Yager defines an optimistic index $\alpha \in [0, 1]$ from the components of W_i and proposes to compute (by mathematical programming) the best weighting vector W_i corresponding to a priori chosen optimistic index and having the maximal entropy (dispersion). If $\alpha = 1$ (optimistic attitude) then of course $W_i = [1, 0, \ldots, 0, 0]$ and if $\alpha = 0$ (pessimistic attitude) then $W_i = [0, 0, \ldots, 0, 1]$. I theory, Yager's method doesn't exclude the possibility to adopt an hybrid attitude depending on the alternative we consider. In other words, we are not forced to consider the same weighting vectors for all alternatives.

Example 1: Let's take states $S = \{S_1, S_2, S_3, S_4\}$, alternatives $A = \{A_1, A_2, A_3\}$ and the payoffs matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{pmatrix} 10 & 0 & 20 & 30 \\ 1 & 10 & 20 & 30 \\ 30 & 10 & 2 & 5 \end{pmatrix} \end{array} \tag{9}$$

If one adopts the pessimistic attitude in choosing $W_1 = W_2 = W_3 = [0, 0, 0, 1]$, then one gets for each alternative A_i, $i = 1, 2, 3$ the following values of OWA's: $V_1 = \text{OWA}(10, 0, 20, 30) = 0$, $V_2 = \text{OWA}(1, 10, 20, 30) = 1$ and $V_3 = \text{OWA}(30, 10, 2, 5) = 2$. The final decision will be the alternative V_3 since it offers the best expected payoff.

If one adopts the optimistic attitude in choosing $W_1 = W_2 = W_3 = [1, 0, 0, 0]$, then one gets for each alternative A_i, $i = 1, 2, 3$ the following values of OWA's: $V_1 = \text{OWA}(10, 0, 20, 30) = 30$, $V_2 = \text{OWA}(1, 10, 20, 30) = 30$ and $V_3 = \text{OWA}(30, 10, 2, 5) = 30$. All alternatives offer the same expected payoff and thus the final decision must be chosen randomly or purely ad-hoc since there is no best alternative.

If one adopts the normative attitude in choosing $W_1 = W_2 = W_3 = [1/4, 1/4, 1/4, 1/4]$ (i.e. one assumes that all states of nature are equiprobable), then one gets: $V_1 = \text{OWA}(10, 0, 20, 30) = 60/4$, $V_2 = \text{OWA}(1, 10, 20, 30) = 61/4$ and $V_3 = \text{OWA}(30, 10, 2, 5) = 47/4$. The final decision will be the alternative V_2 since it offers the best expected payoff.

4 – Decision-making under uncertainty: this corresponds to the general case where the knowledge on the states of the nature is characterized by a belief structure. Clearly, one assumes that a priori knowledge on the frame S of the different states of the nature is given by a bba $m(.) : 2^S \to [0, 1]$. This case includes all previous cases depending on the choice of $m(.)$. Decision under certainty is characterized by $m(S_j) = 1$; Decision under risk is characterized by $m(s) > 0$ for some states $s \in S$; Decision under full ignorance is characterized by $m(S_1 \cup S_2 \cup \ldots \cup S_n) = 1$, etc. Yager's OWA for decision-making under uncertainty combines the schemes used for decision making under risk and ignorance. It is based on the derivation of a *generalized expected value* C_i of payoff for each alternative A_i as follows:

$$C_i = \sum_{k=1}^{r} m(X_k) V_{ik} \tag{10}$$

where r is the number of focal elements of the belief structure $(S, m(.))$. $m(X_k)$ is the mass of belief of the focal element $X_k \in 2^S$, and V_{ik} is the payoff we get when we select A_i and the state of the nature lies in X_k. The derivation of V_{ik} is done similarly as for the decision making under ignorance when restricting the states of the nature to the subset of states belonging to X_k only. Therefore for A_i and a focal element X_k, instead of using all payoffs C_{ij}, we consider only the payoffs in the set $M_{ik} = \{C_{ij} | S_j \in X_k\}$ and $V_{ik} = \text{OWA}(M_{ik})$ for some decision-making attitude chosen a priori. Once generalized expected values C_i, $i = 1, 2, \ldots, q$ are computed, we select the alternative which has its highest C_i as the best alternative (i.e. the final decision). The principle of this method is very simple, but its implementation can be quite greedy in computational resources specially if one wants to adopt a particular attitude for a given level of optimism, specially if the dimension of the frame S is large: one needs to compute by mathematical programming the weighting vectors generating the optimism level having the maximum of entropy. As illustrative example, we take Yager's example[3] [33] with a pessimistic, optimistic and normative attitudes.

Example 2: Let's take states $S = \{S_1, S_2, S_3, S_4, S_5\}$ with associated bba $m(S_1 \cup S_3 \cup S_4) = 0.6$, $m(S_2 \cup S_5) = 0.3$ and $m(S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) = 0.1$. Let's also consider alternatives $A = \{A_1, A_2, A_3, A_4\}$ and the payoffs matrix:

$$C = \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \tag{11}$$

The $r = 3$ focal elements of $m(.)$ are $X_1 = S_1 \cup S_3 \cup S_4$, $X_2 = S_2 \cup S_5$ and $X_3 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$. X_1 and X_2 are partial ignorances and X_3 is the full ignorance. One considers the following submatrix (called bags by Yager) for

[3]There is a mistake/typo error in original Yager's example [33].

the derivation of V_{ik}, for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$.

$$M(X_1) = \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \\ M_{41} \end{bmatrix} = \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix}$$

$$M(X_2) = \begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \\ M_{42} \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix}$$

$$M(X_3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \\ M_{43} \end{bmatrix} = \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} = C$$

• Using pessimistic attitude, and applying the OWA operator on each row of $M(X_k)$ for $k = 1$ to r, one gets finally[4]: $V(X_1) = [V_{11}, V_{21}, V_{31}, V_{41}]^t = [7, 5, 3, 6]^t$, $V(X_2) = [V_{12}, V_{22}, V_{32}, V_{42}]^t = [5, 2, 9, 4]^t$ and $V(X_3). = [V_{13}, V_{23}, V_{33}, V_{43}]^t = [5, 2, 3, 4]^t$. Applying formula (10) for $i = 1, 2, 3, 4$ one gets finally the following generalized expected values using vectorial notation:

$$[C_1, C_2, C_3, C_4]^t = \sum_{k=1}^{r=3} m(X_k) \cdot V(X_k) = [6.2, 3.8, 4.8, 5.2]^t$$

According to these values, the best alternative to take is A_1 since it has the highest generalized expected payoff.

• Using optimistic attitude, one takes the max value of each row, and applying OWA on each row of $M(X_k)$ for $k = 1$ to r, one gets: $V(X_1) = [V_{11}, V_{21}, V_{31}, V_{41}]^t = [13, 12, 10, 15]^t$, $V(X_2) = [V_{12}, V_{22}, V_{32}, V_{42}]^t = [6, 10, 13, 9]^t$, and $V(X_3) = [V_{13}, V_{23}, V_{33}, V_{43}]^t = [13, 12, 13, 15]^t$. One finally gets $[C_1, C_2, C_3, C_4]^t = [10.9, 11.4, 11.2, 13.2]^t$ and the best alternative to take with optimistic attitude is A_4 since it has the highest generalized expected payoff.

• Using normative attitude, one takes $W_1 = W_2 = W_3 = W_4 = [1/|X_k|, 1/|X_k|, \ldots, 1/|X_k|]$ where $|X_k|$ is the cardinality of the focal element X_k under consideration. The number of elements in W_i is equal to $|X_k|$. The generalized expected values are $[C_1, C_2, C_3, C_4]^t = [8.91, 8.20, 8.58, 9.25]^t$ and the best alternative with the normative attitude is A_4 (same as with optimistic attitude) since it has the highest generalized expected payoff.

C. Using expected utility theory

In this section, we propose to use a much simpler approach than OWA Yager's approach for decision making under uncertainty. The idea is to approximate the bba $m(.)$ by a subjective probability measure through a given probabilistic transformation. We suggest to use either $BetP$ or (better) $DSmP$ transformations for doing this as explained in [22] (Vol.3, Chap. 3). Let's take back the previous example and compute the $BetP(.)$ and $DSmP_\epsilon(.)$ values from $m(.)$. One gets the same values in this particular example for any $\epsilon > 0$ because we don't have singletons as focal elements of $m(.)$, which is normal. Here $BetP(S_1) = DSmP(S_1) = 0.22$, $BetP(S_2) = DSmP(S_2) = 0.17$, $BetP(S_3) = DSmP(S_3) = 0.22$, $BetP(S_4) = DSmP(S_4) = 0.22$ and $BetP(S_5) = DSmP(S_2) = 0.17$. Based on these probabilities, we can compute the expected payoffs for each alternative as for decision making under risk (e.g. for C_1, we get $7 \cdot 0.22 + 5 \cdot 0.17 + 12 \cdot 0.22 + 13 \cdot 0.22 + 6 \cdot 0.17 = 8.91$). For the 4 alternatives, we finally get:

$$E_{BetP}[C] = E_{DSmP}[C] = [8.91, 8.20, 8.58, 9.25]^t$$

According to these values, one sees that the best alternative with this pignistic or DSm attitude is A_4 (same as with Yager's optimistic or normative attitudes) since it offers the highest pignistic or DSm expected payoff. This much simpler approach must be used with care however because there is a loss of information through the approximation of the bba $m(.)$ into any subjective probability measure. Therefore, we do not recommend to use it in general.

IV. The new COWA-ER approach

Yager's OWA approach is based on the choice of given attitude measured by an optimistic index in $[0, 1]$ to get the weighting vector W. How is chosen such an index/attitude ? This choice is ad-hoc and very disputable for users. What to do if we don't know which attitude to adopt ? The rational answer to this question is to consider the results of the two extreme attitudes (pessimistic and optimistic ones) jointly and try to develop a new method for decision under uncertainty based on the imprecise valuation of alternatives. This is the approach developed in this paper and we call it Cautious OWA with Evidential Reasoning (COWA-ER) because it adopts the cautious attitude (based on the possible extreme attitudes) and ER, as explained in the sequel.

Let's take back the previous example and take the pessimistic and optimistic valuations of the expected payoffs. The expected payoffs $E[C_i]$ are imprecise since they belong to interval $[C_i^{\min}, C_i^{\max}]$ where bounds are computed with extreme pessimistic and optimistic attitudes, and one has

$$E[C] = \begin{bmatrix} E[C_1] \\ E[C_2] \\ E[C_3] \\ E[C_4] \end{bmatrix} \subset \begin{bmatrix} [6.2; 10.9] \\ [3.8; 11.4] \\ [4.8; 11.2] \\ [5.2; 13.2] \end{bmatrix}$$

Therefore, one has 4 sources of information about the parameter associated with the best alternative to choose. For decision making under imprecision, we propose to use here again the belief functions framework and to adopt the following very simple COWA-ER methodology based on the following four steps:

- Step 1: normalization of imprecise values in $[0, 1]$;
- Step 2: conversion of each normalized imprecise value into elementary bba $m_i(.)$;
- Step 3: fusion of bba $m_i(.)$ with an efficient combination rule (typically PCR5);

[4]where X^t denotes the transpose of X.

- Step 4: choice of the final decision based on the resulting combined bba.

Let's describe in details each step of COWA-ER. In step 1, we divide each bound of intervals by the max of the bounds to get a new normalized imprecise expected payoff vector $E^{Imp}[C]$. In our example, one gets:

$$E^{Imp}[C] = \begin{bmatrix} [6.2/13.2; 10.9/13.2] \\ [3.8/13.2; 11.4/13.2] \\ [4.8/13.2; 11.2/13.2] \\ [5.2/13.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [0.47; 0.82] \\ [0.29; 0.86] \\ [0.36; 0.85] \\ [0.39; 1.00] \end{bmatrix}$$

In step 2, we convert each imprecise value into its bba according to a very natural and simple transformation [7]. Here, we need to consider as frame of discernment, the finite set of alternatives $\Theta = \{A_1, A_2, A_3, A_4\}$ and the sources of belief associated with them obtained from the normalized imprecise expected payoff vector $E^{Imp}[C]$. The modeling for computing a bba associated to the hypothesis A_i from any imprecise value $[a; b] \subseteq [0; 1]$ is very simple and is done as follows:

$$\begin{cases} m_i(A_i) = a, \\ m_i(\bar{A}_i) = 1 - b \\ m_i(A_i \cup \bar{A}_i) = m_i(\Theta) = b - a \end{cases} \quad (12)$$

where $\bar{A}_i$ is the complement of A_i in Θ. With such simple conversion, one sees that $Bel(A_i) = a$, $Pl(A_i) = b$. The uncertainty is represented by the length of the interval $[a; b]$ and it corresponds to the imprecision of the variable (here the expected payoff) on which is defined the belief function for A_i. In the example, one gets:

Alternatives A_i	$m_i(A_i)$	$m_i(\bar{A}_i)$	$m_i(A_i \cup \bar{A}_i)$
A_1	0.47	0.18	0.35
A_2	0.29	0.14	0.57
A_3	0.36	0.15	0.49
A_4	0.39	0	0.61

Table I
BASIC BELIEF ASSIGNMENTS OF THE ALTERNATIVES

In step 3, we need to combine bba's $m_i(.)$ by an efficient rule of combination. Here, we suggest to use the PCR5 rule proposed in DSmT framework since it has been proved very efficient to deal with possibly highly conflicting sources of evidence. PCR5 has been already applied successfully in all applications where it has been used so far [22]. We call this COWA-ER method based on PCR5 as COWA-PCR5. Obviously, we could replace PCR5 rule by any other rule (DS rule, Dubois& Prade, Yager's rule, etc and thus define easily COWA-DS, COWA-DP, COWA-Y, etc variants of COWA-ER. This is not the purpose of this paper and this has no fundamental interest in this presentation. The result of the combination of bba's with PCR5 for our example is given in of Table II.

The last step 4 is the decision-making from the resulting bba of the fusion step 3. This problem is recurrent in the theory of belief functions and several attitudes are also possible as explained at the end of section II. Table III shows what are the values of credibilities, plausibilities, $BetP$ and $DSmP_{\epsilon=0}$ for each alternative in our example.

Focal Element	$m_{PCR5}(.)$
A_1	0.2488
A_2	0.1142
A_3	0.1600
A_4	0.1865
$A_1 \cup A_4$	0.0045
$A_2 \cup A_4$	0.0094
$A_1 \cup A_2 \cup A_4$	0.0236
$A_3 \cup A_4$	0.0075
$A_1 \cup A_3 \cup A_4$	0.0198
$A_2 \cup A_3 \cup A_4$	0.0374
$A_1 \cup A_2 \cup A_3 \cup A_4$	0.1883

Table II
FUSION OF THE FOUR ELEMENTARY BBA'S WITH PCR5

A_i	$Bel(A_i)$	$BetP(A_i)$	$DSmP(A_i)$	$Pl(A_i)$
A_1	0.2488	0.3126	0.3364	0.4850
A_2	0.1142	0.1863	0.1623	0.3729
A_3	0.1600	0.2299	0.2242	0.4130
A_4	0.1865	0.2712	0.2771	0.4521

Table III
CREDIBITITY AND PLAUSIBILITY OF A_i

Based on the results of Table III, it is interesting to note that, in this example, there is no ambiguity in the decision making whatever the attitude is taken by the decision-maker (the max of Bel, the max of Pl, the max of BetP or the max of DSmP), the decision to take will always be A_1. Such behavior is probably not general in all problems, but at least it shows that in some cases like in Yager's example, the ambiguity in decision can be removed when using COWA-PCR5 instead of OWA which is an advantage of our approach. It is worth to note that Shannon entropy of BetP is $H_{BetP} = 1.9742$ bits is bigger than Shannon entropy of DSmP is $H_{DSmP} = 1.9512$ bits which is normal since DSmP has been developed for increasing the PIC value.

Advantages and extension of COWA-ER: COWA-PCR5 allows also to take easily a decision, not only on a single alternative, but also if one wants on a group/subset of alternatives satisfying a min of credibility (or plausibility level) selected by the decision-maker. Using such approach, it is of course very easy to discount each bba $m_i(.)$ entering in the fusion process using reliability or importance discounting techniques which makes this approach more appealing and flexible for the user than classical OWA. COWA-PCR5 is simpler to implement because it doesn't require the evaluation of all weighting vectors for the bags by mathematical programming. Only extreme and very simple weighting vectors $[1, 0, \ldots, 0]$ and $[0, \ldots, 0, 1]$ are used in COWA-ER. Of course, COWA-ER can also be extended directly for the fusion of several sources of informations when each source can provide a payoffs matrix. It suffices to apply COWA-ER on each matrix to get the bba's of step 3, then combine them with PCR5 (or any other rule) and then apply step 4 of COWA-ER. We can also discount each

source easily if needed. All these advantages makes COWA-ER approach very flexible and appealing for MCDM under uncertainty. In summary, the original OWA approach considers several alternatives A_i evaluated in the context of different uncertain scenarii and includes several ways (pessimistic, optimistic, hurwicz, normative) to interpret and aggregate the evaluations with respect to a given scenario. COWA-ER uses simultaneously the two extreme pessimistic and optimistic decision attitudes combined with an efficient fusion rule as shown on Figure 3. In order to save computational resources (if required), we also have proposed a less efficient OWA approach using the classical concept of expected utility based on DSmP or BetP.

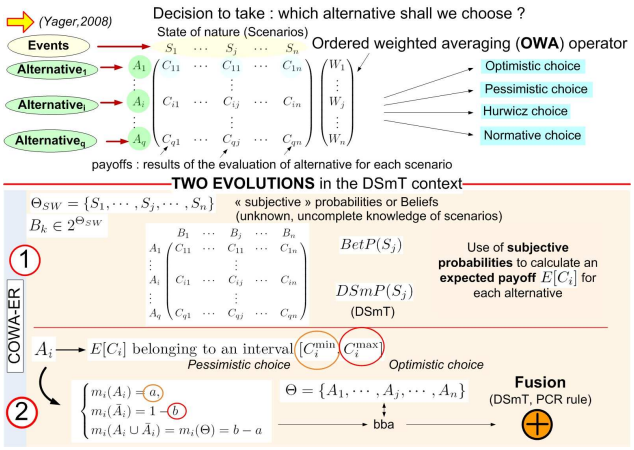

Figure 3. COWA-ER: Two evolutions of Yager's OWA method.

V. Conclusion

In this work, Yager's Ordered Weighted Averaging (OWA) operators are extended and simplified with evidential reasoning (ER) for MCDM under uncertainty. The new Cautious OWA-ER method is very flexible and requires less computational load than classical OWA. COWA-ER improves the existing framework for MCDM since it can deal also with several more or less reliable sources. Further developments are now planned to combine uncertainty about states of the world with the imperfection and uncertainty of alternatives evaluations as previously introduced in the ER-MCDA and DSmT-AHP methods in order to connect them with COWA-ER.

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