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QUASINEUTRAL LIMIT OF THE EULER-POISSON SYSTEM FOR IONS IN A DOMAIN WITH BOUNDARIES

DAVID GÉRARD-VARET, DANIEL HAN-KWAN, AND FRÉDÉRIC ROUSSET

Abstract. We study the quasineutral limit of the isothermal Euler-Poisson system describing a plasma made of ions and massless electrons. The analysis is achieved in a domain of $\mathbb{R}^3$ and thus extends former results by Cordier and Grenier [Comm. Partial Differential Equations, 25 (2000), pp. 1099–1113], who dealt with the same problem in a one-dimensional domain without boundary.

1. Introduction

We consider the isothermal Euler-Poisson system which models the behaviour of ions in a background of massless electrons. The massless assumption implies that the electrons follow the classical Maxwell-Boltzmann relation: denoting by n_e their density and after some suitable normalization of the constants, this reads

$$n_e \,=\, e^{-\phi},$$

where $-\phi$ is the electric potential (auto-induced by the charged particles), which satisfies some *Poisson equation*.

Introducing the characteristic observation length L and the Debye length $\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T^i}{N_i e^2}}$ of the plasma (where T^i is the average temperature of the ions, and N_i their average density), we are interested in the study of the behaviour of the system, when

$$\frac{\lambda_D}{L} = \varepsilon \ll 1,$$

which corresponds to the (usual) physical situation where the observation length is much larger than the Debye length (indeed, in usual regimes, $\lambda_D \sim 10^{-5} - 10^{-8} m$). Loosely speaking, the Poisson equation satisfied by $-\phi$ then reads:

$$\varepsilon^2 \Delta_x \phi \,=\, n_i \,-\, n_e.$$

In situations where $\varepsilon \ll 1$, this indicates that the plasma can be considered as being "almost" neutral (that is $n_i \sim n_e$): hence the name quasineutral limit for the limit $\varepsilon \to 0$. However, in a domain with boundaries, it is well-known that there can be some interaction between the plasma and the boundary, so that small scale effects are amplified and boundary layers generally appear. This implies that quasineutrality breaks down near the borders.

In this paper, our goal is to rigorously study this phenomenon of boundary layers by investigating their existence and their stability. More precisely, the system we work on is the following dimensionless isothermal Euler-Poisson system, for $t > 0$ and $x = (x_1, x_2, x_3) = (y, x_3) \in \mathbb{R}^3_+ := \mathbb{R}^2 \times \mathbb{R}_+$:

$$\begin{cases} \partial_t n \,+\, \mathrm{div}\,(nu) \,=\, 0, \\ \partial_t u \,+\, u \cdot \nabla u \,+\, T^i \nabla \ln(n) = \nabla \phi, \\ \varepsilon^2 \Delta \phi \,+\, e^{-\phi} = n, \end{cases} \tag{1.1}$$

where n is the density of ions, $u = (u_1, u_2, u_3) = (u_y, u_3)$ is their velocity field, and $-\phi$ is the electric potential.

We complete this system with the boundary conditions

$$u_3|_{x_3=0} = 0, \quad \phi|_{x_3=0} = \phi_b, \tag{1.2}$$

where ϕ_b is some prescribed potential. We assume that $\phi_b = \phi_b(y)$ is smooth and compactly supported. Otherwise, it is also possible to consider with minor modifications the case where ϕ_b is independent of y, which corresponds to the perfect conductor assumption:

$$\nabla_y \phi = 0.$$

The boundary condition on u corresponds to a non-penetration condition. From the physical point of view, this can be interpreted as a kind of confinement condition. Non homogeneous boundary conditions are also physically pertinent, and some cases could be treated in our framework: we refer to Remark 3 and Section 5.2. Also we shall focus in our analysis on the three-dimensional case and the simplest domain $\mathbb{R}^3_+$. Nevertheless, our analysis can be easily extended to any dimension or more general domains.

The Euler-Poisson system and its asymptotic limits have attracted a lot of attention over the past two decades, possibly due to its many applications to plasmas and semiconductors (see for instance [26, 28]). Among many possibilities, the so-called quasineutral limit, zero-electron-mass limit, zero-relaxation-time limit have been particularly studied.

For the sake of brevity, we only focus here on the quasineutral limit and describe some mathematical contributions. For the corresponding Euler-Poisson system describing electrons (with fixed ions), in a domain without boundary, there have been many mathematical studies of the quasineutral limit (for various pressure laws) and the situation is rather well understood; we refer for instance to [24, 32, 27, 40, 33] and references therein. The quasineutral limit for Two-fluid Euler-Poisson systems (one for ions and one for electrons), still without boundary, has been also recently investigated in [23, 22].

Considering the dynamics of ions with electrons following a Maxwell-Boltzmann law (which is the one we are particularly interested in), Cordier and Grenier studied in [6] the quasineutral limit of the Euler-Poisson system (1.1), in the whole space $\mathbb{R}$. A kinetic version of the problem was also recently studied in [21]. We also mention the work [19] of Guo and Pausader who recently constructed global smooth irrotational solutions to (1.1) with $\varepsilon = 1$ in the whole space $\mathbb{R}^3$.

For quasineutral limits in presence of boundaries, there have been very few rigorous studies, at least to the best of our knowledge. When the Poisson equation is considered alone (the density of the ions being prescribed), a boundary layer analysis was led by Brézis, Golse and Sentis [4] and Ambroso, Méhats and Raviart [2]. For the Poisson equation coupled with some equations for the plasma flow, the are only two main results we are aware of. The first one is due to Slemrod and Sternberg [36], who dealt with the quasineutral limit of some Euler-Poisson models (with massless electrons) in a one-dimensional and stationary case (see also Peng [31] for general boundary conditions). The second is due to Peng and Wang [30] and Violet [39], who studied the quasineutral limit of some Euler-Poisson model (with fixed ions) for stationary and irrotational flows.

To conclude this short review, let us finally mention some other interesting directions which are related to the issue of the quasineutral limit in a domain. In [37], Suzuki studies the asymptotic stability of specific stationary solutions to some Euler-Poisson system, which are interpreted as "boundary layers" (see also the paper of Ambroso [1] for a numerical study). In [20, 10] and references therein, Slemrod et al. study the problem of formation and dynamics of the so-called plasma sheath. We refer also to the work [9] of Crispel, Degond and Vignal for a study of a two-fluid quasineutral plasma, based on a formal expansion.

In this paper, we study the quasineutral limit of (1.1)-(1.2). As far as we know, our results which are described in the next section, give the first description and stability analysis of boundary layers for the quasineutral limit for the full model in a non stationary setting (and we are in a multi-dimensional framework as well).

2. Main results and strategy of the proof

By putting *formally* $\varepsilon = 0$ in the Poisson equation of (1.1), we first directly obtain the *neutrality* condition:

$$n = e^{-\phi},$$

so that one gets from (1.1) in the limit $\varepsilon \to 0$ the following isothermal Euler system:

$$\begin{cases} \partial_t n \, + \, \mathrm{div}\,(nu) \, = \, 0, \\ \partial_t u \, + \, u \cdot \nabla u \, + \, (T^i + 1) \nabla \ln(n) = 0. \end{cases} \tag{2.1}$$

It is natural to expect the convergence to this system. The previous system is well-posed with the only boundary condition

$$u_3|_{x_3=0} = 0.$$

Because of the boundary conditions (1.2), we would like to satisfy also the condition

$$\phi|_{x_3=0} = \phi_b.$$

Nevertheless, the solution (n, u) of (2.1) cannot in general satisfy in addition $n|_{x_3=0} = e^{-\phi_b}$ and hence, we expect the formation of a boundary layer in order to correct this boundary condition.

We shall consider solutions of (2.1) with no vacuum: we fix a reference smooth function n^{ref} which is bounded from below by a positive constant and consider solutions of (2.1) for which n is bounded from below by a positive constant and such that $n - n^{ref}$ is in H^s. The main result contained in this paper is a rigorous proof of the convergence to the isothermal Euler system (2.1).

Theorem 1. *Let (n^0, u^0) a solution to (2.1) such that $(n^0 - n^{ref}, u^0) \in C^0([0,T], H^s(\mathbb{R}^3_+))$ with s large enough. There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists $(n^\varepsilon, u^\varepsilon)$ a solution to (1.1)-(1.2) also defined on $[0,T]$ such that*

$$\sup_{[0,T]} \Big(\|n^\varepsilon - n^0\|_{L^2(\mathbb{R}^3_+)} + \|u^\varepsilon - u^0\|_{L^2(\mathbb{R}^3_+)} \Big) \to_{\varepsilon \to 0} 0. \tag{2.2}$$

Furthermore, the rate of convergence is $O(\sqrt{\varepsilon})$.

Remark 1. *Let us recall that following the classical theory of the initial boundary value problem for the compressible Euler equation (see [35, 18] for example), asking (n^0, u^0) to be a smooth solution to the compressible Euler system imposes some compatibility conditions on the initial data $(n^0|_{t=0}, u^0|_{t=0})$.*

Boundary layers are hidden in the statement of Theorem 1 (recall that they are expected to appear since the boundary conditions do not match). Nevertheless, in order to give a proof, we will have to describe them precisely. Indeed, for the system (1.1) in multi-d, the only existence result that is known is the local existence result of smooth (H^m, $m > 5/2$) solution that can be obtained from the theory of the compressible Euler equation (see [35, 18] for example). A nontrivial part of the statement of Theorem 1 is thus that the solution of (1.1) is defined on an interval of time independent of ε. In $\mathbb{R}^3$ this is a consequence of the fact that (1.1) enjoys H^s estimates that are uniform in ε. These estimates were obtained in [6] by using that (1.1) can be rewritten as a symmetric hyperbolic system perturbed by a large skew-symmetric operator and thus can be cast in the general framework of [13] that generalizes the classical theory of Klainerman and Majda [25] on the incompressible limit. When boundaries are present it seems unlikely to derive directly

uniform H^s estimates of (1.1). Indeed the expected description of the solution $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ of (1.1) is under the form

(2.3)

$$n^\varepsilon = n^0(t,x) + N^0\left(t, y, \frac{x_3}{\varepsilon}\right) + \mathcal{O}(\varepsilon), \quad \phi^\varepsilon = \phi^0(t,x) + \Phi^0\left(t, y, \frac{x_3}{\varepsilon}\right) + \mathcal{O}(\varepsilon), \quad u^\varepsilon = u^0(t,x) + \mathcal{O}(\varepsilon)$$

where (n^0, u^0) is the solution of (2.1), $\phi^0 = -\log n^0$ and the profiles N^0, Φ^0 (the boundary layer profiles) are smooth and fastly decreasing in the last variable and are added in order to match the boundary condition (1.2). In particular, we require

$$\phi^0(t,y,0) + \Phi^0(t,y,0) = \phi^b(y).$$

Note that at leading order, there is no boundary layer for the velocity but that in general, since the solution of (2.1) does not satisfy $n|_{x_3=0} = e^{-\phi_b}$, the profiles Φ^0 and N^0 are not zero for positive times. This yields that n^ε cannot satisfy uniform H^m, $m > 5/2$ estimates.

In order to prove Theorem 1, we shall use a two step argument. We shall first prove that, up to any order, there exists an approximate solution $(n^\varepsilon_{app}, u^\varepsilon_{app}, \phi^\varepsilon_{app})$ for (1.1), in the form of a two-scale asymptotic expansion, and then, we shall prove that we can get a true solution of (1.1) defined on $[0,T]$ by adding a small corrector to the approximate solution. The description of the solution that we get is thus more precise than stated in Theorem 1. We refer to Theorem 3 and Corollary 1 which are stated in Section 4. In particular, as a simple corollary of our estimates, we will get that the behavior in L^∞ is given by:

$$\begin{aligned} \sup_{[0,T_0]} \Big(\Big\| n^\varepsilon - n^0 - N^0\left(t, y, \frac{x_3}{\varepsilon}\right) \Big\|_{L^\infty(\mathbb{R}^3_+)} &+ \|u^\varepsilon - u^0\|_{L^\infty(\mathbb{R}^3_+)} \\ &+ \Big\| \phi^\varepsilon - \phi^0 - \Phi^0\left(t, y, \frac{x_3}{\varepsilon}\right) \Big\|_{L^\infty(\mathbb{R}^3_+)} \Big) \to 0. \end{aligned} \tag{2.4}$$

This two step approach has been very useful to deal with other asymptotic problems involving as above boundary layers of size ε and amplitude $O(1)$ like the vanishing viscosity limit to general hyperbolic systems when the boundary is non-characteristic [12, 15, 17, 29], see also [38], or the highly rotating fluids limit [16, 11, 34]. A common feature of these works is the fact that the true solution of the system remains close to the approximate one on an interval of time that does not shrink when ε goes to zero (i.e. the stability of the approximate solution) requires a spectral assumption on the main boundary layer profile (that is automatically satisfied when the amplitude of the boundary layer is sufficiently weak). One quite striking fact about the quasineutral limit is that we shall establish here the existence and *unconditional stability* of sufficiently accurate approximate solutions involving boundary layers of arbitrary amplitude.

Since the boundary layer profiles solve ordinary differential equations, we shall be able to get their existence (without any restriction on their amplitude) by using classical properties of planar Hamiltonian systems. In order to prove the stability of the approximate solution, there are two difficulties: the fact that the electric field is very large when ε is small (this difficulty is already present when there is no boundary) and the fact that the linearization of (1.1) about the approximate solution contains zero order terms that involve the gradient of the approximate solution and thus that are also singular when ε goes to zero because of the presence of boundary layers. We shall avoid the pseudo-differential approach of [6] which does not seem to be easily adaptable when there is a boundary and get an L^2 type estimate which is uniform in ε by using a suitably *modulated* linearized version of the physical energy of the Euler-Poisson system. We refer for example to [14, 5] for other uses of this approach in singular perturbation problems. In order to estimate higher order derivatives, we shall proceed in a classical way by estimating conormal derivatives of the solution and then by using the equation to recover estimates for normal derivatives. Since the boundary

is characteristic this does not provide any information on the normal derivative of the tangential velocity but we can estimate it by using the vorticity equation.

Theorem 1 (as well as Theorem 3 and Corollary 1) can be generalized to more general settings; this is the purpose of the two following remarks.

Remark 2 (General domains)**.** *Less stringent settings for the domain could also be treated. Instead of the half-space $\mathbb{R}^3_+$, we can as well consider the case of a smooth domain $\Omega \subset \mathbb{R}^3$ whose boundary is an orientable and compact manifold. We refer to Section 5.1 for some elements on how one could adapt the proof.*

Remark 3 (Non-homogeneous boundary condition on u)**.** *Finally, it is possible to adapt the proof in order to study, instead of the non-penetration condition on u, the case of some subsonic outflow condition on the normal velocity u_3. We refer to Section 5.2 for some details on what changes in the proof.*

This paper is organized as follows. Section 3 is dedicated to the construction of a high order boundary layer approximation to a solution of (1.1) (see Theorem 2). In Section 4, we prove the stability of this approximation (*cf* Theorem 3), thus proving the convergence to isothermal Euler. The possible extensions and variations about the result are discussed in the last section.

3. Construction of boundary layer approximations

We look for approximate solutions of system (1.1) of the form

$$\begin{aligned}(n_a, u_a, \phi_a) \;=\; & \sum_{i=0}^{K} \varepsilon^i \left(n^i(t,x), u^i(t,x), \phi^i(t,x)\right) \\ & + \sum_{i=0}^{K} \varepsilon^i \left(N^i\left(t, y, \frac{x_3}{\varepsilon}\right), U^i\left(t, y, \frac{x_3}{\varepsilon}\right), \Phi^i\left(t, y, \frac{x_3}{\varepsilon}\right)\right)\end{aligned} \tag{3.1}$$

where K is an arbitrarily large integer. The coefficients (n^i, u^i, ϕ^i) of the first sum depend on x. They should model the macroscopic behaviour of the solutions. The coefficients (N^i, U^i, Φ^i) of the second sum depend on t, y, but also on a stretched variable $z = \frac{x_3}{\varepsilon} \in \mathbb{R}_+$. They should model a boundary layer of size ε near the boundary. Accordingly, we shall ensure that

$$(N^i, U^i, \Phi^i) \to 0, \quad \text{as } z \to +\infty \tag{3.2}$$

fast enough. In order for the whole approximation to satisfy (1.2), we shall further impose

$$\begin{aligned} & u_3^i(t,y,0) \,+\, U_3^i(t,y,0) \;=\; 0 \quad \text{for all } i \geq 0, \\ & \phi^0(t,y,0) \,+\, \Phi^0(t,y,0) \;=\; \phi_b, \quad \phi^i(t,y,0) \,+\, \Phi^i(t,y,0) \;=\; 0 \quad \text{for all } i \geq 1. \end{aligned} \tag{3.3}$$

Thus, far away from the boundary, the term (n^0, u^0, ϕ^0) will govern the dynamics of these approximate solutions. As explained in the previous section, we expect (n^0, u^0) to satisfy the following compressible Euler system

$$\begin{cases} \partial_t n \,+\, \text{div}\,(nu) \;=\; 0, \\ \partial_t u \,+\, u \cdot \nabla u \,+\, (T^i + 1)\nabla \ln(n) = 0, \end{cases} \tag{3.4}$$

together with the relation $n^0 = e^{-\phi^0}$. As we will see below, it will also satisfy the non-penetration condition

$$u_3|_{x_3=0} = 0. \tag{3.5}$$

Our results are gathered in the:

Theorem 2. *Let $K \in \mathbb{N}^*, m \in \mathbb{N}^*$ such that $m \geq 3$. For any data (n_0^0, u_0^0) satisfying some compatibility conditions on $\{x_3 = 0\}$ and such that $(n_0^0 - n^{ref}, u_0^0) \in H^{m+2K+3}(\mathbb{R}^3_+)$, there is $T > 0$ and a smooth approximate solution of (1.1) of order K under the form (3.1) such that*

i) *$(n^0 - n^{ref}, u^0) \in C^0([0,T], H^{m+2K+3}(\mathbb{R}^3_+))$, $\phi^0 = -\log n^0$, (n^0, u^0) is a solution to the isothermal Euler system (3.4) with initial data (n_0^0, u_0^0).*

ii) *$\forall\, 1 \leq i \leq K$, $(n^i, u^i, \phi^i) \in C^0([0,T], H^{m+3}(\mathbb{R}^3_+))$.*

iii) *$\forall\, 0 \leq i \leq K$, N^i, U^i, Φ^i are smooth functions that belong together with their derivatives to the set of uniformly exponentially decreasing functions with respect to the last variable.*

iv) *Let us consider $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ a solution to (1.1) and define:*

$$n \,=\, n^\varepsilon \,-\, n_a, \quad u \,=\, u^\varepsilon \,-\, u_a, \quad \phi \,=\, \phi^\varepsilon \,-\, \phi_a.$$

Then (n, u, ϕ) satisfies the system of equations:

$$\begin{cases} \partial_t n + (u_a + u)\cdot\nabla n + n\, div\,(u + u_a) + div\,(n_a u) = \varepsilon^K R_n, \\ \partial_t u + (u_a + u)\cdot\nabla u + u\cdot\nabla u_a + T^i\left(\dfrac{\nabla n}{n_a + n} - \dfrac{\nabla n_a}{n_a}\left(\dfrac{n}{n_a + n}\right)\right) = \nabla\phi + \varepsilon^K R_u, \\ \varepsilon^2\Delta\phi = n - e^{-\phi_a}(e^{-\phi} - 1) + \varepsilon^{K+1} R_\phi. \end{cases} \tag{3.6}$$

where R_n, R_u, R_ϕ are remainders satisfying:

$$\sup_{[0,T]} \|\nabla_x^\alpha R_{n,u,\phi}\|_{L^2(\mathbb{R}^3_+)} \leq C_a \varepsilon^{-\alpha_3}, \quad \forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad |\alpha| \leq m, \tag{3.7}$$

with $C_a > 0$ independent of ε.

The rest of this section is devoted to the proof of this theorem.

3.1. **Collection of inner and outer equations.** First, we derive formally the equations that the leading terms of the expansion (3.1) should satisfy, to make it an approximate solution of (1.1)-(1.2). The solvability of these equations will be discussed in the next subsection.

The starting point of the formal derivation is to plug the expansion (3.1) into (1.1). By considering terms that have the same amplitude (that is the same power of ε in front of it), one obtains a collection of equations on the coefficients (n^i, u^i, ϕ^i) and (N^i, U^i, Φ^i). As usual, we distinguish between two zones:

(1) An *outer zone* (far from the boundary), in which the boundary layer correctors are neglectible.

(2) An *inner zone* (in the boundary layer), in which we use the Taylor expansion

$$n^i(t, y, x_3) \,=\, n^i(t, y, \varepsilon z) \,=\, n^i(t, y, 0) \,+\, \varepsilon\partial_3 n^i(t, y, 0) z \,+\, \ldots \text{ (the same for } u^i \text{ and } \phi^i)$$

to obtain boundary layer equations in variables (t, y, z). For clarity of exposure, for any function $f = f(t, x)$, we will denote by Γf the function $(t, y, Z) \mapsto f(t, y, 0)$.

In the outer zone, by collecting terms that have $O(1)$ amplitude, one sees that (n^0, u^0) satisfies (1.1), and that $n^0 = e^{-\phi^0}$.

In the inner zone, by collecting terms that have $O(\varepsilon^{-1})$ amplitude, one has the relations

$$\begin{cases} \partial_z\left((\Gamma n^0 + N^0)\,(\Gamma u_3^0 + U_3^0)\right) \,=\, 0 \\ (\Gamma u_3^0 + U_3^0)\,\partial_z U_3^0 \,+\, T^i\partial_z \ln(\Gamma n^0 + N^0) = \partial_z \Phi^0 \end{cases}$$

The first line implies that $(\Gamma n^0 + N^0)(\Gamma u_3^0 + U_3^0)$ does not depend on z. From condition (3.3), we deduce

$$(\Gamma n^0 + N^0)\,(\Gamma u_3^0 + U_3^0) = 0. \tag{3.8}$$

Now, the second line also reads

$$(\Gamma n^0 + N^0)\,(\Gamma u_3^0 + U_3^0)\,\partial_z U_3^0 \;+\; T^i \partial_z N^0 = (\Gamma n^0 + N^0)\,\partial_z \Phi^0. \tag{3.9}$$

Combined with the previous equation, this gives

$$T^i \partial_z N^0 = (\Gamma n^0 + N^0)\partial_z \Phi^0$$

or equivalently (still using (3.2))

$$(N^0 + \Gamma n^0) = \Gamma n^0 e^{\Phi^0/T^i} \tag{3.10}$$

Back to (3.9), we obtain that $\Gamma u_3^0 + U_3^0 = 0$, and by (3.2), both $U_3^0 = 0$ and $\Gamma u_3^0 = 0$. We recover as expected that (n^0, u^0) satisfies formally (3.4), together with the non-penetration condition (3.5), and the relation $n^0 = e^{-\phi^0}$.

Still in the inner zone, collecting the $O(1)$ terms in the third line of (1.1), we obtain

$$\partial_z^2 \Phi^0 + e^{-\Gamma\phi^0} e^{-\Phi^0} \;=\; N^0 + \Gamma n^0.$$

As $n^0 = e^{-\phi^0}$, this last equation can be simplified into

$$\partial_z^2 \Phi^0 + \Gamma n^0 e^{-\Phi^0} = N^0 + \Gamma n^0$$

Together with (3.10), we end up with

$$\partial_z^2 \Phi^0 \;+\; \Gamma n^0 \mathcal{S}(\Phi^0) = 0, \quad \mathcal{S}(\Phi) \;:=\; e^{-\Phi} - e^{\Phi/T^i}. \tag{3.11}$$

Note that this is a second order equation in z, so that with the boundary conditions

$$\Phi^0|_{z=0} = \phi_b - \phi^0|_{x_3=0}, \quad \Phi^0|_{z=+\infty} = 0 \tag{3.12}$$

it should determine completely Φ^0. Indeed, the existence of a unique smooth solution of (3.11)-(3.12) with rapid decay at infinity will be established in the next subsection. As Φ^0 is determined, so is N^0 by relation (3.10).

This concludes the formal derivation of (n^0, u^0, ϕ^0), and (U_3^0, Φ^0, N^0). We will now turn to the equations satisfied by (n^1, u^1, ϕ^1) and $(U_3^1, U_y^0, \Phi^1, N^1)$. More generally, we will derive for all $i \geq 1$:

(1) a set of equations on (n^i, u^i, ϕ^i), whose source terms depend on the coefficients (n^k, u^k, ϕ^k) and their derivatives, $k \leq i-1$.
(2) a set of equations on $(U_3^i, U_y^{i-1}, \Phi^i, N^i)$, whose sources depend on (the trace at $x_3 = 0$ of) (n^k, u^k, ϕ^k) and their derivatives, $k \leq i$, as well as on $(U_3^k, U_y^{k-1}, \Phi^k, N^k)$ and their derivatives, $k \leq i-1$.

Indeed, in the outer zone, collecting terms with amplitude $O(\varepsilon^i)$ in (1.1) leads to systems of the type:

$$\begin{cases} \partial_t n^i + \mathrm{div}\,(n^0 u^i) + \mathrm{div}\,(n^i u^0) \;=\; f_n^i \\ \partial_t u^i + u^0 \cdot \nabla u^i + u^i \cdot \nabla u^0 + T^i \nabla \left(\dfrac{n^i}{n^0}\right) = \nabla \phi^i + f_u^i \\ -\,e^{-\phi^0}\,\phi^i = n^i + f_\phi^i, \end{cases} \tag{3.13}$$

where the source terms $f_{n,u,\phi}^i$ depend on (n^k, u^k, ϕ^k) and their derivatives, $k \leq i-1$.

In the inner zone, collecting $O(\varepsilon^i)$ terms in the Poisson equation of (1.1), we get:

$$\partial_z^2 \Phi^i \;-\; \Gamma n^0\, e^{-\Phi^0}\,\left(\Phi^i \;+\; \Gamma \phi^i\right) \;=\; \left(N^i + \Gamma n^i\right) \;+\; F_\Phi^i \tag{3.14}$$

where F^i_Φ depend on (n^k, ϕ^k) and on (N^k, Φ^k), $k \le i-1$. Then, collecting the $O(\varepsilon^{i-1})$ in the equation for ion density n, taking into account that $U^0_3 = 0$ and $\Gamma u^0_3 = 0$, we get

$$\partial_z \left((N^0 + \Gamma n^0)(U^i_3 + \Gamma u^i_3) \right) \; + \; \mathrm{div}_y \left((N^0 + \Gamma n^0)(U^{i-1}_y + \Gamma u^{i-1}_y) \right) \; = \; F^i_N. \tag{3.15}$$

The equation on u_y yields

$$\begin{aligned} \partial_t (U^0_y + \Gamma u^0_y) \; + \; & (U^1_3 + \Gamma u^1_3 + \Gamma \partial_3 u^0_3)\, \partial_z U^0_y \\ & + \; (U^0_y + \Gamma u^0_y) \cdot \nabla_y \left(U^0_y + \Gamma u^0_y \right) \; + \; (T^i + 1) \nabla_y \ln(\Gamma n^0) \; = \; 0 \end{aligned} \tag{3.16}$$

and for $i \ge 2$:

$$\begin{aligned} \partial_t (U^{i-1}_y + \Gamma u^{i-1}_y) \; + \; & (U^i_3 + \Gamma u^i_3 + \Gamma \partial_3 u^{i-1}_3)\, \partial_z U^0_y \; + \; (U^0_y + \Gamma u^0_y) \cdot \nabla_y \left(U^{i-1}_y + \Gamma u^{i-1}_y \right) \\ & + \; \left(U^{i-1}_y + \Gamma u^{i-1}_y \right) \cdot \nabla_y (U^0_y + \Gamma u^0_y) \; = \; F^i_y. \end{aligned} \tag{3.17}$$

Finally, the equation on u_3 yields

$$T^i \partial_z \left(\frac{N^i + \Gamma n^i}{N^0 + \Gamma n^0} \right) \; = \; \partial_z \Phi^i \; + \; F^i_3. \tag{3.18}$$

Again, the source terms F^i_N, F^i_y and F^i_3 depend on (n^k, u^k, ϕ^k), and on $(U^k_3, U^{k-1}_y, \Phi^k, N^k)$, for indices $k \le i-1$. Note that (3.2) requires the following compatibility conditions:

$$\begin{cases} -\, n^0 \phi^i|_{x_3=0} \; = \; n^i|_{x_3=0} + F^i_\Phi|_{z=\infty}, \\ \mathrm{div}_y \left(n^0\, u^{i-1}_y \right)|_{x_3=0} \; = \; F^i_N|_{z=\infty}, \\ \left(\partial_t u^{i-1}_y + u^0_y \cdot \nabla_y u^{i-1}_y + u^{i-1}_y \cdot \nabla_y u^0_y \right)|_{x_3=0} \; = \; F^i_y|_{z=\infty}, \\ F^i_3|_{z=\infty} = 0. \end{cases} \tag{3.19}$$

It remains to show how to deduce the profiles from the equations. First of all, we can derive U^0_y from (3.15) (with $i = 1$) and (3.16). We integrate (3.15) from 0 to z, and with (3.3) deduce

$$(U^i_3 + \Gamma u^i_3) \; = \; -\frac{1}{N^0 + \Gamma n^0} \int_0^z \left(\mathrm{div}_y \left((N^0 + \Gamma n^0)(U^{i-1}_y + \Gamma u^{i-1}_y) \right) \; - \; F^i_N \right) \tag{3.20}$$

For $i = 1$, this last relation allows to express $U^1_3 + \Gamma u^1_3$ in terms of $U^0_y + \Gamma u^0_y$ (and coefficients of lower order). One can then substitute into (3.16) to have a closed equation on U^0_y. We notice that it has the trivial solution $U^0_y = 0$. *In all what follows, we shall restrict to approximate solutions* (3.1) *satisfying* $U^0_y = 0$.

As $U^0_y = 0$, the equation (3.17) simplifies into

$$\partial_t U^{i-1}_y \; + \; \Gamma u^0_y \cdot \nabla_y U^{i-1}_y + \; U^{i-1}_y \cdot \nabla_y \Gamma u^0_y \; = \; H^i \tag{3.21}$$

with

$$H^i \; := \; F^i_y - \left(\partial_t \Gamma u^{i-1}_y \; + \; \Gamma u^0_y \cdot \nabla_y \Gamma u^{i-1}_y \; + \; \Gamma u^{i-1}_y \cdot \nabla_y \Gamma u^0_y \right).$$

One can now derive recursively (n^i, u^i, ϕ^i) and $(U^i_3, U^{i-1}_y, \Phi^i, N^i)$, $i \ge 1$. We take $i \ge 1$, and assume that all lower order profiles (n^k, u^k, ϕ^k) and $(U^k_3, U^{k-1}_y, \Phi^k, N^k)$, $k \le i-1$ are known. From the equations satisfied by the (n^k, u^k, ϕ^k)'s, the last three compatibility conditions in (3.19) are satisfied. In particular, the function H^i in (3.21) decays to zero at infinity. Thus, the linear transport equation (3.21) yields a decaying solution U^{i-1}_y (with the special case $U^0_y = 0$ when $i = 1$). Then, one obtains from (3.20) the expression of $U^i_3 + \Gamma u^i_3$. From the decay condition (3.2), we deduce that

$$u^i_3|_{x_3=0} = \frac{-1}{n^0|_{x_3=0}} \int_0^{+\infty} \left(\mathrm{div}_y \left((N^0 + \Gamma n^0)(U^{i-1}_y + \Gamma u^{i-1}_y) \right) \; - \; F^i_N \right) \tag{3.22}$$

This boundary condition goes with the hyperbolic system on (n^i, u^i) (see (3.13)). Together with an initial condition which is compatible with the boundary condition, it allows to determine (n^i, u^i, ϕ^i). This will be explained rigorously in the next subsection. Note that, as (n^i, u^i, ϕ^i) solves (3.13), the remaining compatibility condition in (3.19) is satisfied. Using again (3.20), one eventually gets U_3^i.

To end up this formal derivation, it remains to handle Φ^i, and N^i. Combining (3.14) and (3.18) (together with the decay condition (3.2)) leads to a second order equation on Φ^i:

$$\partial_z^2 \Phi^i \; + \; \Gamma n^0 \mathcal{S}'(\Phi^0) \; = \; G_\Phi \tag{3.23}$$

where we remind that $\mathcal{S}(\Phi) = e^{-\Phi} - e^{\Phi/T^i}$ and G_Φ is a source term that depends on the lower order terms, decaying to zero as z goes to infinity. Again, the solvability of this system with Dirichlet conditions

$$\Phi^i|_{z=0} = -\phi^i|_{x_3=0}, \quad \Phi^i|_{z=+\infty} = 0 \tag{3.24}$$

will be established later on. Once Φ^i is known, N^i is determined through (3.18).

3.2. **Well-posedness of the reduced models.** To complete the construction of the approximate solutions and get Theorem 2, we must establish the well-posedness of the inner and outer systems derived in the previous paragraph.

The well-posedness of the inner systems is classical. Indeed, the system (3.4)-(3.5) on (n^0, u^0) is a standard compressible Euler system. For well-posedness, we consider an initial data of the form $(n_0^0 = e^{-\phi_0^0}, u_0^0)$, with $(u_0^0, n_0^0 - n^{ref}) \in H^{m+3+2K}(\mathbb{R}_+^3)^4$. Following [18, 35], if this initial data satisfies standard compatibility conditions at the boundary $\{x_3 = 0\}$, there exists a unique solution $(n^0 = e^{-\phi^0}, u^0)$, with

$$(n^0 - n^{ref}, u^0) \in C^\infty\left([0,T]; H^{m+3+2K}(\mathbb{R}_+^3)^4\right), \quad \text{for some } T > 0.$$

As regards the next systems (3.13)-(3.22), they resume to linear hyperbolic systems on (n^i, u^i), $i \geq 1$, with a source made of some derivatives of previously constructed (n^j, u^j, ϕ^j), $j < i$. For initial data $(n_0^i, u_0^i) \in H^{m+3+2K-2i}(\mathbb{R}_+^3)^4$, chosen in order to match compatibility conditions they have again unique solutions

$$(n^i, u^i) \in C^\infty\left([0,T]; H^{m+3+2K-2i}(\mathbb{R}_+^3)^4\right).$$

The "loss" of regularity is due to the derivatives in the source. Likewise, one shows that:

$$U^i \in C^\infty\left([0,T]; H^{m+3+2K-2i}(\mathbb{R}_+^3)^4\right),$$

The main point is the resolution of the nonlinear boundary layer system (3.11)-(3.12). Note that t and y are only parameters in such a system, involved through the coefficient γn^0 and the boundary data. For each (t, y), (3.11) is an ordinary differential equation in z, of the type

$$\Phi'' + \gamma \mathcal{S}(\Phi) = 0, \quad \gamma > 0, \tag{3.25}$$

and we want to prove that, for any constant ϕ, it has a (unique) solution Φ that connects ϕ to 0, with exponential decay of Φ and its derivatives as z goes to infinity.

Therefore, we rewrite (3.25) as a Hamiltonian system,

$$\frac{d}{dz}\begin{pmatrix} p \\ \Phi \end{pmatrix} = \nabla^\perp H(p, \Phi), \quad p := \Phi', \quad H(p, \Phi) := \frac{p^2}{2} + \mathcal{T}(\Phi) \tag{3.26}$$

where $\mathcal{T}$ is an antiderivative for $\gamma\mathcal{S}$. We choose the one that vanishes at 0:

$$\mathcal{T}(\Phi) := -\gamma\left(e^{-\Phi} + T^i e^{\Phi/T^i}\right) + \gamma(1 + T^i).$$

The Hamiltonian H is of course constant along any trajectory. The linearization of (3.26) at the critical point $(p,\Phi)=(0,0)$ yields

$$\frac{d}{dz}\begin{pmatrix}\dot p\\ \dot\Phi\end{pmatrix} = \begin{pmatrix}0 & \gamma(1+1/T^i)\\ 1 & 0\end{pmatrix}\begin{pmatrix}\dot p\\ \dot\Phi\end{pmatrix}.$$

Hence, we get that $(0,0)$ is a saddle fixed point, and using the stable manifold theorem that its stable manifold is locally a curve, tangent to $\begin{pmatrix}\sqrt{\gamma(1+1/T^i)}\\ -1\end{pmatrix}$. Moreover, as $H(0,0)=0$, the stable manifold is exactly the branch of $\{H(p,\Phi)=0\}$ given by the equations

$$p=-\sqrt{\gamma}\sqrt{e^{-\Phi}+T^ie^{\Phi/T^i}-1-T^i}\ \text{ for }\ \Phi\geq 0,\quad p=\sqrt{\gamma}\sqrt{e^{-\Phi}+T^ie^{\Phi/T^i}-1-T^i}\ \text{ for }\ \Phi<0.$$

In addition, any solution starting on this branch decays exponentially to 0, and by using the equation (3.25), so do all its derivatives. Finally, one needs to check that for any ψ, this branch has a unique point with ordinate ψ. It is indeed the case, as this branch gives p as a decreasing function of Φ, going to infinity at infinity. This concludes the resolution of the o.d.e, and yields in turn the solution Φ^0 of our boundary layer system. Its regularity with respect to (t,y) follows from regular dependence of the solution of (3.25) with respect to the data.

Last step is the well-posedness of the boundary layer systems (3.23)-(3.24) which allow to define Φ^i and N^i. Again, t,y are parameters, and the point is to find a (unique) solution to the inhomogeneous linear ODE

$$\Phi''+\gamma S'(\overline{\Phi})\Phi=F, \tag{3.27}$$

with exponential decay at infinity and a Dirichlet condition $\Phi|_{z=0}=\psi$. Here, $\overline{\Phi}$ and F are arbitrary smooth functions of z, decaying exponentially to 0.

Note that, up to consider $\tilde\Phi(z)=\Phi(z)-\psi\chi(z)$, with $\chi\in C_c^\infty(\mathbb{R}_+)$ satisfying $\chi(0)=0$, we can always assume that $\psi=0$. After this simplification, observing that $S'(\overline{\Phi})\leq 0$, we can apply the Lax-Milgram Lemma: it provides a unique solution $\Phi\in H^1_0(\mathbb{R}_+)$ of (3.27). By Sobolev embedding, it decays to zero at infinity, and it is smooth by elliptic regularity. It remains to obtain the exponential decay of Ψ, hence we write (3.27) as

$$\Phi''+\gamma S'(0)\Phi=G,\quad G:=(\gamma S'(0)-\gamma S'(\overline{\Phi}))\Phi+F$$

Using the Duhamel formula for this constant coefficient ode, and the exponential decay of G, one obtains easily that any bounded solution decays exponentially.

Finally one deduces recursively that U^i also decays exponentially.

Summing up what has been done in this section, we have proved Theorem 2.

4. Stability estimates

This section is devoted to the stability of the boundary layer approximations built in the previous section.

Let us write the solution $(n^\varepsilon,u^\varepsilon,\phi^\varepsilon)$ of (1.1) under the form

$$n^\varepsilon\ =\ n_a\ +n,\quad u^\varepsilon\ =\ u_a\ +u,\quad \phi^\varepsilon\ =\ \phi_a\ +\phi$$

where n_a,u_a,ϕ_a are shorthands for $n^\varepsilon_{app},u^\varepsilon_{app},\phi^\varepsilon_{app}$ (defined in (3.1)). Then, we get for (n,u,ϕ) the system

$$\begin{cases}\partial_t n+(u_a+u)\cdot\nabla n+n\,\mathrm{div}\,(u+u_a)+\mathrm{div}\,(n_au)=\varepsilon^KR_n,\\ \partial_t u+(u_a+u)\cdot\nabla u+u\cdot\nabla u_a+T^i\left(\dfrac{\nabla n}{n_a+n}-\dfrac{\nabla n_a}{n_a}\left(\dfrac{n}{n_a+n}\right)\right)=\nabla\phi+\varepsilon^KR_u,\\ \varepsilon^2\Delta\phi=n-e^{-\phi_a}(e^{-\phi}-1)+\varepsilon^{K+1}R_\phi.\end{cases} \tag{4.1}$$

together with the boundary conditions

$$u_3|_{x_3=0} = 0, \quad \phi|_{x_3=0} = 0 \tag{4.2}$$

and the initial condition

$$u|_{t=0} = \varepsilon^{K+1} u_0, \quad n|_{t=0} = \varepsilon^{K+1} n_0. \tag{4.3}$$

Observe here that in (4.1), $\varepsilon^K R_n, \varepsilon^K R_u$ and $\varepsilon^{K+1} R_\phi$ are remainders that appear because (n_a, u_a, ϕ_a) is not an exact solution of (1.1).

The main result of this section is

Theorem 3. *Let $m \geq 3$, and $(n_0, u_0) \in H^m(\mathbb{R}^3_+)$ some initial data for (4.3), satisfying some suitable compatibility conditions. Let $K \in \mathbb{N}^*, K \geq m$ and (n_a, u_a, ϕ_a) an approximate solution at order K given by Theorem 2 which is defined on $[0, T_0]$. There exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, the solution of (4.1)-(4.2)-(4.3) is defined on $[0, T_0]$ and satisfies the estimate*

$$\varepsilon^{|\alpha|} \|\partial^\alpha (n, u, \phi, \varepsilon \nabla \phi)\|_{L^2(\mathbb{R}^3_+)} \leq C \varepsilon^K, \quad \forall t \in [0, T^0], \quad \forall \alpha \in \mathbb{N}^3, \, |\alpha| \leq m.$$

One can then remark that as a simple rephrase of Theorem 3, we obtain:

Corollary 1. *Let $m \geq 3$. Let $K \in \mathbb{N}^*, K \geq m$ and (n_a, u_a, ϕ_a) an approximate solution at order K given by Theorem 2 which is defined on $[0, T_0]$. There exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, there is a solution $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ to (1.1) which is defined on $[0, T_0]$ and satisfies the estimate:*

$$\left\| \left(n^\varepsilon - n_a, u^\varepsilon - u_a, \phi^\varepsilon - \phi_a \right) \right\|_{H^m(\mathbb{R}^3_+)} \leq C \varepsilon^{K-m}, \quad \forall t \in [0, T^0]. \tag{4.4}$$

In particular, we get the L^2 and L^∞ convergences as $\varepsilon \to 0$:

$$\begin{aligned} &\sup_{[0,T_0]} \left(\left\| n^\varepsilon - n^0 - N^0 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right\|_{L^\infty(\mathbb{R}^3_+)} + \|n^\varepsilon - n^0\|_{L^2(\mathbb{R}^3_+)} \right) \to 0, \\ &\sup_{[0,T_0]} \left(\|u^\varepsilon - u^0\|_{L^\infty(\mathbb{R}^3_+)} + \|u^\varepsilon - u^0\|_{L^2(\mathbb{R}^3_+)} \right) \to 0, \\ &\sup_{[0,T_0]} \left(\left\| \phi^\varepsilon - \phi^0 - \Phi^0 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right\|_{L^\infty(\mathbb{R}^3_+)} + \|\phi^\varepsilon - \phi^0\|_{L^2(\mathbb{R}^3_+)} \right) \to 0. \end{aligned} \tag{4.5}$$

Of course, this contains Theorem 1.

From now on, our goal is to prove Theorem 3.

For $\varepsilon > 0$ fixed, since in the equation (4.1), the term involving ϕ in the second equation can be considered as a semi-linear term, the known local existence results for the compressible Euler equation ([35, 18] for example) can be applied to the system (4.1). Let us assume that $(n_0, u_0) \in H^m(\mathbb{R}^3_+)$ for $m \geq 3$ and that it satisfies suitable compatibility conditions on the boundary, then there exists $T^\varepsilon > 0$ and a unique solution of (4.1) defined on $[0, T^\varepsilon]$ such that $u \in \mathcal{C}([0, T^\varepsilon), H^m)$ and that there exists $M > 0$ satisfying

$$n_a(t,x) + n(t,x) \geq 1/M, \quad e^{-\phi_a}(1 + \min(h_0, h_1)) \geq 1/M, \quad \|\chi(u_a + u)_3\|_{L^\infty} \leq \sqrt{3/4\, T^i}, \quad \forall t \in [0, T^\varepsilon) \tag{4.6}$$

where

$$h_0(\phi) := -\frac{e^{-\phi} - 1 + \phi}{\phi}, \quad h_1(\phi) := e^{-\phi} - 1 \tag{4.7}$$

and $\chi(x_3) := \tilde{\chi}(x_3/\delta)$ where $\tilde{\chi}$ is a smooth compactly supported function, equal to 1 in the vicinity of zero and $\delta > 0$ is chosen so that

$$\|\tilde{\chi}(x_3/\delta)\big((u_a+u)_3\big)|_{t=0}\|_{L^\infty} \leq \sqrt{T^i/2}.$$

Note that we can always chose δ in this way since at $t=0$ we have $\big((u_a+u)_3\big)|_{t=0,x_3=0} = 0$.

The difficulty is thus to prove that the solution actually exists on an interval of time independent of ε. We shall get this result by proving uniform energy estimates combined with the previous local existence result when the initial data and the source term are sufficiently small (i.e. when the approximate solution (n_a, u_a) is sufficiently accurate). Note that there are two difficulties in order to get useful energy estimates. The first one is the singular perturbation coming from the Poisson part, the electric field cannot be considered as a lower order term uniformly in ε. The second one comes from the boundary layer terms contained in the approximate solution which create singular terms, for example, in the second line of (4.1), the zero order term $\frac{\nabla n_a}{n_a}\frac{n}{n_a+n}$ is singular in the sense that when (4.6) is matched, we only have the estimate

$$\left\|\frac{\nabla n_a}{n_a}\frac{n}{n_a+n}\right\|_{L^2} \lesssim \frac{1}{\varepsilon}\|n\|_{L^2}.$$

The main step will be the obtention of uniform estimates for "linearized" systems.

4.1. **Energy for the quasineutral Euler-Poisson system without source.** We start by recalling that the isothermal Euler-Poisson system without source (which corresponds here to the case $\phi_b = 0$) has a conserved physical energy, which is given in the

Proposition 1. *Let $\varepsilon > 0$ and (n, u, ϕ) a strong solution to (1.1) on $[0,T]$ with $\phi_b = 0$. We define the energy functional:*

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^3_+} n|u|^2dx + T^i\int_{\mathbb{R}^3_+} n(\log n - 1)dx + \int_{\mathbb{R}^3_+}(1-\phi)e^{-\phi}dx + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3_+}|\nabla_x\phi|^2dx. \tag{4.8}$$

Then for any $t \in [0,T]$, $\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0)$.

This property will never be used in the sequel; nevertheless, the L^2 and higher order stability estimates which follow are obtained via the study a *modulated* version of this energy. Thus, for clarity of exposure, we briefly present the proof showing that the energy is conserved, as it is much easier to follow but share the same spirit with the subsequent ones.

Proof. We compute the derivative in time of the first term of $\mathcal{E}$, by using the transport equation satisfied by n in (1.1) (which corresponds to the convervation of charge):

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3_+} n|u|^2dx &= \int_{\mathbb{R}^3_+}\partial_t n|u|^2dx + 2\int_{\mathbb{R}^3_+} nu\cdot\partial_t u dx\\ &= -\int_{\mathbb{R}^3_+}\nabla\cdot(nu)|u|^2dx + 2\int_{\mathbb{R}^3_+} nu\cdot\partial_t u dx.\end{aligned}$$

Using the non-penetration boundary condition for u, we have by integration by parts:

$$-\int_{\mathbb{R}^3_+}\nabla\cdot(nu)|u|^2dx = \int_{\mathbb{R}^3_+}(nu)\cdot\nabla|u|^2dx = \int_{\mathbb{R}^3_+}(nu)u\cdot\nabla u dx.$$

Now thanks to the equation satisfied by u in (1.1), we can write:

$$\int_{\mathbb{R}^3_+} nu\cdot\partial_t u dx = -\int_{\mathbb{R}^3_+}(nu)u\cdot\nabla u dx - T^i\int_{\mathbb{R}^3_+} nu\cdot\nabla\log n + \int_{\mathbb{R}^3_+} nu\cdot\nabla\phi dx.$$

By the equation satisfied by n and the non-penetration condition, we infer, by another integration by parts, that

$$T^i \int_{\mathbb{R}^3_+} nu \cdot \nabla \log n = -T^i \int_{\mathbb{R}^3_+} \partial_t n \log n = -\frac{d}{dt} T^i \int_{\mathbb{R}^3_+} n(\log n - 1) dx.$$

Likewise, we get, using once more the conservation of charge,

$$\begin{aligned} \int_{\mathbb{R}^3_+} nu \cdot \nabla \phi dx &= -\int_{\mathbb{R}^3_+} \nabla \cdot (nu) \phi dx \\ &= \int_{\mathbb{R}^3_+} \partial_t n \, \phi dx. \end{aligned}$$

Deriving with respect to time the Poisson equation in (1.1) we obtain

$$\int_{\mathbb{R}^3_+} nu \cdot \nabla \phi dx = \int_{\mathbb{R}^3_+} \varepsilon^2 \partial_t \Delta \phi \phi dx + \int_{\mathbb{R}^3_+} \partial_t e^{-\phi} \phi dx.$$

The last term of the r.h.s. is treated exactly like the pressure term. Considering the first one, by integration by parts, we obtain:

$$\int_{\mathbb{R}^3_+} \partial_t \Delta \phi \, \phi dx = -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3_+} |\nabla_x \phi|^2 dx,$$

which relies on the fact that $\phi = 0$ on $\{x_3 = 0\}$. This proves our claim. □

4.2. L^2 **estimate for the suitably linearized equations.** We establish here an L^2 estimate for the solution $(\dot{n}, \dot{u}, \dot{\phi})$ of the following linearized system:

$$\begin{cases} \partial_t \dot{n} + (u_a + u) \nabla \dot{n} + (n_a + n) \nabla \cdot \dot{u} + \dot{u} \cdot \nabla (n_a + n) + \dot{n} \operatorname{div} (u_a + u) = r_n, \\ \partial_t \dot{u} + (u_a + u) \cdot \nabla \dot{u} + \dot{u} \cdot \nabla u_a + T^i \left(\dfrac{\nabla \dot{n}}{n_a + n} - \dfrac{\nabla n_a}{n_a} \left(\dfrac{\dot{n}}{n_a + n} \right) \right) = \nabla \dot{\phi} + r_u, \\ \varepsilon^2 \Delta \dot{\phi} = \dot{n} + e^{-\phi_a} \dot{\phi} (1 + h(\phi)) + r_\phi. \end{cases} \tag{4.9}$$

where $r = (r_n, r_u, r_\phi)$ is a given source term, and where $h \in \{h_0, h_1\}$, *cf* (4.6)-(4.7). The reason why we shall consider these two possibilities for h will become clear in view of paragraph 4.3.3. We add to the system the boundary conditions

$$\dot{u}_3|_{x_3=0} = 0, \quad \dot{\phi}|_{x_3=0} = 0. \tag{4.10}$$

The crucial estimate is given by

Proposition 2. *Let (n_a, u_a, ϕ_a) the approximate solution constructed in Theorem 2 and some smooth (n, u, ϕ) such that $u_3|_{x_3=0} = 0$ and*

$$n_a + n \geq 1/M, \quad e^{-\phi_a}(1 + h(\phi)) \geq 1/M, \quad |n| + |u| + |\phi| \leq M, \quad \forall t \in [0, T],\, x \in \mathbb{R}^3_+. \tag{4.11}$$

Then, there exist $C(M)$ and $C(C_a, M)$ independent of ε (C_a only depends on the approximate solution) such that we have on $[0, T]$ the estimate

$$\begin{aligned} \big\| (\dot{n}, \dot{u}, \dot{\phi}, \varepsilon \nabla \dot{\phi})(t) \big\|^2_{L^2(\mathbb{R}^3_+)} &\leq C(M) \Big(\big\| (\dot{n}_0, \dot{u}_0) \big\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^t \big\| (\varepsilon^{-1} r_\phi, \partial_t r_\phi, r_n, r_u) \big\|^2_{L^2(\mathbb{R}^3_+)} \Big) \\ &+ C(C_a, M) \int_0^t \big(1 + \big\| \nabla_{t,x} (n, u, \phi) \big\|_{L^\infty(\mathbb{R}^3_+)} + \varepsilon^{-1} \|n\|_{L^\infty(\mathbb{R}^3_+)} \big) \big\| (\dot{n}, \dot{u}, \dot{\phi}, \varepsilon \nabla \dot{\phi}) \big\|^2_{L^2(\mathbb{R}^3_+)} \Big) \end{aligned}$$

Note that this is indeed a stability estimate for the linearized equation since when we take $(u, n, \phi) = 0$ that is when we linearize exactly on the approximate solution, then we get from the above result the estimate

$$\|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})(t)\|_{L^2}^2 \leq C_a\Big(\|(\dot{n}_0, \dot{u}_0)\|_{L^2}^2 + \int_0^t \Big(\|(\varepsilon^{-1}r_\phi, \partial_t r_\phi, r_n, r_u)\|_{L^2}^2 + \|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})\|_{L^2}^2\Big)\Big)$$

and hence from the Gronwall inequality, we obtain

$$\|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})(t)\|_{L^2}^2 \leq e^{C_a t}\Big(\|(\dot{n}_0, \dot{u}_0)\|_{L^2}^2 + \int_0^t \|(\varepsilon^{-1}r_\phi, \partial_t r_\phi, r_n, r_u)\|_{L^2}^2\Big), \quad \forall t \in [0, T]$$

which is an L^2 type estimate for which the growth rate is uniform in $\varepsilon \in (0, 1]$.

Proof. In order to get this estimate, we shall use a linearized version of the total energy of the system. At first, let us collect a few useful estimates that we shall use for (n_a, u_a). In the proof, we shall denote by C_a a number which may change from line to line but which is uniformly bounded for $\varepsilon \in (0, 1]$ and $T \in (0, T_0]$ where T_0 is the interval of time on which the approximate solution is defined. Since the leading boundary layer term of u_a vanishes, we have

$$\sup_{(0,T)\times\mathbb{R}^3_+} |u_a| + |\nabla_{x,t} u_a| \leq C_a, \quad C_a > 0. \tag{4.12}$$

For n_a, we have

$$\sup_{(0,T)\times\mathbb{R}^3_+} |n_a| + |\nabla_{x_1,x_2,t} n_a| \leq C_a, \qquad \sup_{(0,T)\times\mathbb{R}^3_+} |\partial_3 n_a| \leq \frac{1}{\varepsilon} C_a. \tag{4.13}$$

To make more precise the last estimate, we observe that

$$\partial_3 n_a = \partial_3\left(N_0\left(t, y, \frac{x_3}{\varepsilon}\right)\right) + O(1) = \frac{1}{\varepsilon}\partial_Z N_0\left(t, y, \frac{x_3}{\varepsilon}\right) + O(1), \tag{4.14}$$

where Z stands for the fast variable x_3/ε and hence, from the exponential decay of the boundary layer, we get that

$$\sup_{t\in(0,T), x\in\mathbb{R}^+_3} \frac{1}{\varepsilon}\left|x_3\partial_3 N^0\left(t, y, \frac{x_3}{\varepsilon}\right)\right| \leq \sup_{t\in(0,T), y\in\mathbb{R}^2, z\in\mathbb{R}_+} \left|z\partial_z N^0(t, y, z)\right| \leq C_a, \tag{4.15}$$

Note also that ϕ_a shares similar bounds:

$$\sup_{(0,T)\times\mathbb{R}^3_+} |\phi_a| + |\nabla_{x_1,x_2,t}\phi_a| \leq C_a, \qquad \sup_{(0,T)\times\mathbb{R}^3_+} |\partial_3\phi_a| \leq \frac{1}{\varepsilon} C_a. \tag{4.16}$$

with

$$\partial_3\phi_a = \partial_3\left(\Phi_0\left(t, y, \frac{x_3}{\varepsilon}\right)\right) + O(1) = \frac{1}{\varepsilon}\partial_Z\Phi_0\left(t, y, \frac{x_3}{\varepsilon}\right) + O(1), \tag{4.17}$$

and again, from the exponential decay of the boundary layer,

$$\sup_{t\in(0,T), x\in\mathbb{R}^+_3} \frac{1}{\varepsilon}\left|x_3\partial_3\Phi^0\left(t, y, \frac{x_3}{\varepsilon}\right)\right| \leq \sup_{t\in(0,T), y\in\mathbb{R}^2, z\in\mathbb{R}_+} \left|z\partial_z N^0(t, y, z)\right| \leq C_a, \tag{4.18}$$

Let us now prove the energy estimate. First, multiplying the velocity equation by $(n_a+n)\,\dot{u}$, and performing standard manipulations, we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^3_+}(n_a+n)\frac{|\dot{u}|^2}{2} = \int_{\mathbb{R}^3_+}(n_a+n)\dot{u}\cdot\partial_t\dot{u} + \int_{\mathbb{R}^3_+}\partial_t(n_a+n)\frac{|\dot{u}|^2}{2} \tag{4.19}$$

$$= I_1 + I_2 + I_3 - \int_{\mathbb{R}^3_+}(\dot{u}\cdot\nabla u_a)\cdot(n_a+n)\dot{u} + \int_{\mathbb{R}^3_+}(r_u\cdot((n_a+n)\dot{u})) + \int_{\mathbb{R}^3_+}\partial_t(n_a+n)\frac{|\dot{u}|^2}{2}, \tag{4.20}$$

where

$$I_1 := -T^i\int_{\mathbb{R}^3_+}\left(\frac{\nabla\dot{n}}{n_a+n} - \frac{\nabla n_a}{n_a}\left(\frac{\dot{n}}{n_a+n}\right)\right)\cdot((n_a+n)\dot{u}), \quad I_2 := \int_{\mathbb{R}^3_+}\nabla\dot{\phi}\cdot(n_a+n)\dot{u},$$

$$I_3 = -\int_{\mathbb{R}^3_+}[(u_a+u)\cdot\nabla\dot{u}]\cdot(n_a+n)\dot{u}.$$

The last three terms at the r.h.s. of (4.20) can be easily estimated by using (4.12), (4.13) and (4.11):

$$\begin{aligned}
\int_{\mathbb{R}^3_+}(r_u\cdot((n_a+n)\,\dot{u})) &\le C(C_a,M)\|r_u(t)\|_{L^2(\mathbb{R}^3_+)}\|\dot{u}(t)\|_{L^2(\mathbb{R}^3_+)},\\
\int_{\mathbb{R}^3_+}\partial_t(n_a+n)\frac{|\dot{u}|^2}{2} &\le C(C_a,M)\,(1+\|\partial_t\dot{n}\|_{L^\infty})\,\|\dot{u}(t)\|^2_{L^2(\mathbb{R}^3_+)},\\
\int_{\mathbb{R}^3_+}(\dot{u}\cdot\nabla u_a)\cdot(n_a+n)\dot{u} &\le C(C_a,M)\|\dot{u}(t)\|^2_{L^2(\mathbb{R}^3_+)}.
\end{aligned} \tag{4.21}$$

Let us turn to the treatment of I_1. Integrating by parts, we first have:

$$-\int_{\mathbb{R}^3_+}\left(\frac{\nabla\dot{n}}{n_a+n}\right)\cdot((n_a+n)\dot{u}) = \int_{\mathbb{R}^3_+}\frac{\dot{n}}{n_a+n}\mathrm{div}\,((n_a+n)\dot{u}) - \int_{\mathbb{R}^3_+}\frac{\nabla(n_a+n)}{(n_a+n)^2}(n_a+n)\cdot\dot{n}\,\dot{u}$$

and hence, we obtain that

$$I_1 = T^i\int_{\mathbb{R}^3_+}\frac{\dot{n}}{n_a+n}\mathrm{div}\,((n_a+n)\dot{u}) + T^i\int_{\mathbb{R}^3_+}\Big(\frac{\nabla n_a}{n_a} - \frac{\nabla(n_a+n)}{n_a+n}\Big)\cdot\dot{n}\,\dot{u} := I_1^1 + I_1^2. \tag{4.22}$$

To estimate I_1^2, we observe that

$$\frac{\nabla n_a}{n_a} - \frac{\nabla(n_a+n)}{n_a+n} = \frac{\nabla n_a\, n - n_a\nabla n}{n_a(n_a+n)}.$$

Consequently, we can use (4.12), (4.13) and (4.11) to get that

$$|I_1^2| \le C(C_a,M)\big(\|\nabla n\|_{L^\infty} + \varepsilon^{-1}\|n\|_{L^\infty}\big)\|\dot{u}\|_{L^2(\mathbb{R}^3_+)}\,\|\dot{n}\|_{L^2(\mathbb{R}^3_+)}. \tag{4.23}$$

To estimate I_1^1, we observe that we can write the first line of (4.9) under the form

$$\partial_t\dot{n} + \mathrm{div}\,\big((u_a+u)\dot{n}\big) + \mathrm{div}\,\big((n^a+n)\dot{u}\big) = r_n. \tag{4.24}$$

By using this equation to express div $\big((n^a+n)\dot{u}\big)$, we obtain

$$\begin{aligned}\frac{1}{T^i}I_1^1 &= -\int_{\mathbb{R}^3_+}\frac{\dot{n}}{n_a+n}\big(\partial_t\dot{n}+\text{div }\big((u_a+u)\dot{n}\big)-r_n\big)\\ &= -\partial_t\int_{\mathbb{R}^3_+}\frac{\dot{n}^2}{2(n_a+n)}+\frac{1}{2}\int_{\mathbb{R}^3_+}\dot{n}^2\Big(\partial_t+(u^a+u)\cdot\nabla\Big)\big(\frac{1}{n^a+n}\big)\\ &\quad -\frac{1}{2}\int_{\mathbb{R}^3_+}\frac{\dot{n}^2\,\text{div }(u^a+u)}{n_a+n}+\int_{\mathbb{R}^+_3}r_n\frac{\dot{n}}{n_a+n}.\end{aligned}\tag{4.25}$$

In the above expression, for the two last terms, we use (4.11) to get

$$\Big|-\frac{1}{2}\int_{\mathbb{R}^3_+}\frac{\dot{n}^2\,\text{div }(u_a+u)}{n_a+n}+\int_{\mathbb{R}^+_3}r_n\frac{\dot{n}}{n_a+n}\Big|\le C(C_a,M)\Big(\|r_n(t)\|_{L^2}\,\|\dot{n}(t)\|_{L^2}+(1+\|\nabla u\|_{L^\infty})\,\|\dot{n}(t)\|^2_{L^2}\Big).$$

Next, we observe that

$$\begin{aligned}&\int_{\mathbb{R}^3_+}\dot{n}^2\Big(\partial_t+(u^a+u)\cdot\nabla\Big)\big(\frac{1}{n^a+n}\big)\\ &=-\int_{\mathbb{R}^3_+}\frac{\dot{n}^2}{(n_a+n)^2}\Big(\partial_t(n_a+n)+(u_a+u)\cdot\nabla\big(n_a+n\big)\Big).\end{aligned}$$

From the equation satisfied by n_a and (4.12), we get that

$$|\partial_t n_a+u_a\cdot\nabla n_a|\le C_a.$$

Also, by using that $(u)_3$ vanishes on the boundary, we have $|(u)_3|\le x_3\|\nabla u\|_{L^\infty_T}$ and hence by using (4.15), we also have

$$\|u\cdot\nabla n_a\|_{L^2}\le C_a\|\nabla u\|_{L^\infty}.\tag{4.26}$$

Consequently, we obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3_+}\dot{n}^2\Big(\partial_t+(u^a+u)\cdot\nabla\Big)\big(\frac{1}{n^a+n}\big)\Big|\\ &\le C(C_a,M)(1+\|\nabla_{t,x}n\|_{L^\infty}+\|\nabla u\|_{L^\infty})\|\dot{n}\|^2_{L^2}.\end{aligned}$$

We have thus proven that

(4.27)

$$\begin{aligned}&\Big|I_1+\frac{d}{dt}\int_{\mathbb{R}^3_+}\frac{T^i\dot{n}^2}{2(n_a+n)}\Big|\\ &\le C(C_a,M)\Big(\big(1+\|\nabla_{t,x}n\|_{L^\infty}+\|\nabla u\|_{L^\infty}\big)\|\dot{n}\|^2_{L^2}+\big(\varepsilon^{-1}\|n\|_{L^\infty}+\|\nabla n\|_{L^\infty}\big)\|\dot{n}\|_{L^2}\,\|\dot{u}\|_{L^2}\Big)+\|r_n\|^2_{L^2}.\end{aligned}$$

As regards I_2, since

$$I_2=-\int_{\mathbb{R}^3_+}\dot{\phi}\,\text{div }\big((n_a+n)\dot{u}\big),$$

we use once again (4.24) to write it as

$$I_2=\int_{\mathbb{R}^3_+}\dot{\phi}\,\partial_t\dot{n}\;+\;\int_{\mathbb{R}^3_+}\dot{\phi}\,\text{div }\big((u_a+u)\dot{n}\big)-\int_{\mathbb{R}^3_+}r_n\dot{\phi}.\tag{4.28}$$

By differentiating with respect to time the Poisson equation in (4.9), we can express $\partial_t n$ in terms of $\dot\phi$, and substitute into the first term at the r.h.s of (4.28):

$$\begin{aligned}\int_{\mathbb{R}^3_+} \dot\phi\, \partial_t \dot n &= \int_{\mathbb{R}^3_+} \dot\phi \left(\varepsilon^2 \partial_t \Delta \dot\phi - \partial_t (e^{-\phi_a}(1+h(\phi))\dot\phi) - \partial_t r_\phi\right) \\ &= -\varepsilon^2 \partial_t \int_{\mathbb{R}^3_+} \frac{1}{2}|\nabla\dot\phi|^2 \;-\; \varepsilon\partial_t \int_{\mathbb{R}^3_+} \frac{1}{2}\dot\phi^2 e^{-\phi_a}(1+h(\phi)) \;-\; \frac{1}{2}\int_{\mathbb{R}^3_+} \partial_t(e^{-\phi_a}(1+h(\phi)))|\dot\phi|^2 \\ &\quad - \int_{\mathbb{R}^3_+} \dot\phi\, \partial_t r_\phi.\end{aligned}$$

For the last two terms, one has by using (4.16) and (4.11) the straightforward estimate:

(4.29)

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^3_+} \partial_t(e^{-\phi_a}(1+h(\phi)))|\dot\phi|^2 \;-\; \int_{\mathbb{R}^3_+} \dot\phi\, \partial_t r_\phi \\ \leq C(C_a, M)\big(1+\|\partial_t\phi\|_{L^\infty}\big)\|\dot\phi\|^2_{L^2(\mathbb{R}^3_+)} + \|\partial_t r_{\dot\phi}\|_{L^2(\mathbb{R}^3_+)}\|\dot\phi\|_{L^2(\mathbb{R}^3_+)}.\end{aligned}$$

As regards the second term at the r.h.s. of (4.28), we integrate by parts and use the Poisson equation to express n in terms of $\dot\phi$. We get

$$\begin{aligned}\int_{\mathbb{R}^3_+} \dot\phi\, \mathrm{div}\,(\dot n(u_a+u)) &= -\int_{\mathbb{R}^3_+} \nabla\dot\phi\cdot(\dot n(u_a+u)) \\ &= -\int_{\mathbb{R}^3_+} \nabla\dot\phi\cdot(\varepsilon^2\Delta\dot\phi(u_a+u)) \;+\; \int_{\mathbb{R}^3_+} \nabla\dot\phi\cdot(e^{-\phi_a}(1+h(\phi))\dot\phi(u_a+u)) \\ &\quad + \int_{\mathbb{R}^3_+} \nabla\dot\phi\cdot(r_\phi(u_a+u)) = J_1+J_2+J_3.\end{aligned}$$

One has

$$J_2 = \frac{1}{2}\int_{\mathbb{R}^3_+} \nabla|\dot\phi|^2\cdot(e^{-\phi_a}(1+h(\phi))(u_a+u)) \;=\; -\frac{1}{2}\int_{\mathbb{R}^3_+} |\dot\phi|^2\,\mathrm{div}\,(e^{-\phi_a}(1+h(\phi))(u_a+u)). \tag{4.30}$$

Once again, one has to be careful due to the boundary layer part of ϕ_a. Nevertheless, proceeding as for estimate (4.26), we obtain that $(u_a+u)\cdot\nabla\phi_a$ is uniformly bounded in ε and hence, we find:

$$J_2 \;\leq\; C(C_a, M)(1+\|\nabla\phi\|_{L^\infty}+\|\nabla u\|_{L^\infty})\|\dot\phi\|^2_{L^2}.$$

Straightforwardly, thanks to (4.12) one has also

$$J_3 \;\leq\; \frac{1}{\varepsilon}(C_a \;+\; \|u\|_{L^\infty})\,\|r_\phi\|_{L^2(\mathbb{R}^3_+)}\,\|\varepsilon\nabla\dot\phi\|_{L^2(\mathbb{R}^3_+)}.$$

Finally, we compute

(4.31)

$$\begin{aligned}J_1 =\; & \varepsilon^2\int_{\mathbb{R}^3_+} ((\nabla\dot\phi\cdot\nabla)(u_a+u))\cdot\nabla\dot\phi \;+\; \varepsilon^2\int_{\mathbb{R}^3_+}\left(\nabla\frac{|\nabla\dot\phi|^2}{2}\right)\cdot(u_a+u) \\ \leq\; & \varepsilon^2\int_{\mathbb{R}^3_+} |\nabla\dot\phi|^2\,|\nabla(u_a+u)| \;-\; \varepsilon^2\int_{\mathbb{R}^3_+}\frac{|\nabla\dot\phi|^2}{2}\mathrm{div}\,(u_a+u) \;\leq\; C(C_a, \|\nabla_x u\|_{L^\infty})\,\varepsilon^2\int_{\mathbb{R}^3_+}|\nabla\dot\phi|^2\end{aligned}$$

where the last bound comes again from (4.12).

Combining the previous inequalities, we get the following bound:

(4.32)
$$I_2 + \frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}^3_+} \varepsilon^2|\nabla\dot\phi|^2 + e^{\phi_a}(1+h(\phi))\dot\phi^2\Big)$$
$$\leq C(C_a,M)\big(1+\|\nabla_{t,x}\phi\|_{L^\infty} + \|\nabla_x u\|_{L^\infty}\big)\big(\|\dot\phi\|^2_{L^2} + \varepsilon^2\|\nabla\dot\phi\|^2_{L^2}\big) + \varepsilon^{-2}\|r_\phi\|^2_{L^2} + \|\partial_t r_\phi\|^2_{L^2} + \|r_n\|^2_{L^2}.$$

Finally, to estimate I_3 defined after (4.20), we write:

$$I_3 = -\int_{\mathbb{R}^3_+}(n_a+n)(u_a+u)\cdot\nabla\frac{|\dot u|^2}{2} = \int_{\mathbb{R}^3_+}\mathrm{div}\,((n_a+n)(u_a+u))\frac{|\dot u|^2}{2}.$$

Relying once again on (4.12) and proceeding like for (4.26), we infer that:

$$I_3 \leq C(C_a,M)\,(1+\|\nabla(n,u)\|_{L^\infty})\,\|\dot u\|^2_{L^2(\mathbb{R}^3_+)}. \tag{4.33}$$

Eventually, combining (4.20) with (4.21)-(4.27)-(4.32)-(4.33), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3_+}&\left((n_a+n)\frac{|\dot u|^2}{2} + T^i\,\frac{\dot n^2}{2(n_a+n)} + \varepsilon^2\frac{|\nabla\dot\phi|^2}{2} + \frac{1}{2}|\dot\phi|^2e^{-\phi_a}(1+h(\phi))\right)\\ &\leq C(C_a,M)\big(1+\|\nabla_{t,x}(u,n,\phi)\|_{L^\infty} + \varepsilon^{-1}\|n\|_{L^\infty}\big)\big(\|\dot u\|^2_{L^2} + \|\dot n\|^2_{L^2} + \varepsilon^2\|\nabla\dot\phi\|^2_{L^2} + \|\dot\phi\|^2_{L^2}\big)\\ &\quad + \varepsilon^{-2}\|r_\phi\|^2_{L^2} + \|\partial_t r_\phi\|^2_{L^2} + \|r_n\|^2_{L^2} + \|r_u\|^2_{L^2}.\end{aligned} \tag{4.34}$$

We end the proof by integrating in time and by using (4.11). □

4.3. **Nonlinear stability.** We shall now work on the nonlinear system (4.1) in order to get Theorem 3. Thanks to the well-posedness in H^m for $m\geq 3$ of the system (4.1), we can define

$$T^\varepsilon = \sup\big\{T\in[0,T_0],\quad \forall t\in[0,T], \|(n,u,\phi,\varepsilon\nabla\phi)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \leq \varepsilon^r,\ \text{and (4.6) is verified}\big\}$$

where r is chosen such that

$$5/2 < r < K \tag{4.35}$$

and the H^m_ε norm is defined by

$$\|f\|_{H^m_\varepsilon(\mathbb{R}^3_+)} = \sum_{|\alpha|\leq m}\varepsilon^{|\alpha|}\|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}f\|_{L^2(\mathbb{R}^3_+)}.$$

We shall also use the norms:

$$\|f(t)\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} = \sum_{|\alpha|\leq m}\|\mathcal{Z}_0^{\alpha_0}\mathcal{Z}_1^{\alpha_1}\mathcal{Z}_2^{\alpha_2}\mathcal{Z}_3^{\alpha_3}f(t)\|_{L^2(\mathbb{R}^3_+)} \tag{4.36}$$

where the vector fields $\mathcal{Z}_i$ are defined by

$$\mathcal{Z}_0 = \varepsilon\partial_t,\quad \mathcal{Z}_i = \varepsilon\partial_i,\quad i=1,\,2,\quad \mathcal{Z}_3 = \varepsilon\frac{x_3}{1+x_3}\partial_3 \tag{4.37}$$

and

$$\|f(t)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} = \sum_{|\alpha|\leq m}\varepsilon^{|\alpha|}\|\partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}f(t)\|_{L^2(\mathbb{R}^3_+)}.$$

For the sake of brevity, we will also use the following notation:

$$\mathcal{Z}^\alpha = \mathcal{Z}_0^{\alpha_0}\mathcal{Z}_1^{\alpha_1}\mathcal{Z}_2^{\alpha_2}\mathcal{Z}_3^{\alpha_3},\quad \text{for } \alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3).$$

Finally, we set

$$Q_m(t) = \|(n,u,\phi,\varepsilon\nabla\phi)(t)\|_{\mathcal{H}^m_{co,\varepsilon}} + \|\omega(t)\|_{\mathcal{H}^{m-1}_\varepsilon}$$

where we have set $\omega = \varepsilon \operatorname{curl} u$ i.e:

$$\omega = \varepsilon \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}. \tag{4.38}$$

We shall first prove that Q_m is the important quantity to control for the continuation of the solution and then we shall estimate $Q_m(t)$ by using Proposition 2.

4.3.1. *Estimate of the $\mathcal{H}^m_\varepsilon$ norm.* Let us recall a classical estimate for products in dimension 3:

$$\|uv\|_{L^2(\mathbb{R}^3_+)} \lesssim \|u\|_{H^{s_1}(\mathbb{R}^3_+)} \|v\|_{H^{s_2}(\mathbb{R}^3_+)}, \tag{4.39}$$

with $s_1 + s_2 = 3/2$, $s_1 \neq 0$, $s_2 \neq 0$. More generally,

$$\|u_1 \dots u_K\|_{L^2(\mathbb{R}^3_+)} \lesssim \|u_1\|_{H^{s_1}(\mathbb{R}^3_+)} \dots \|u_K\|_{H^{s_K}(\mathbb{R}^3_+)} \tag{4.40}$$

for $s_1 + \dots + s_K = \frac{3}{2}(K-1)$, $s_1 \neq 0, \dots, s_p \neq 0$.

As an application, we state

Lemma 1. *For any u, v and $\varepsilon \in (0,1]$, we have uniformly in ε*

$$\begin{aligned} \|(\varepsilon\partial_t)^k (uv)\|_{H^l_\varepsilon(\mathbb{R}^3_+)} &\lesssim \|u\|_{L^\infty} \|(\varepsilon\partial_t)^k v\|_{H^l_\varepsilon(\mathbb{R}^3_+)} + \|v\|_{L^\infty} \|(\varepsilon\partial_t)^k u\|_{H^l_\varepsilon(\mathbb{R}^3_+)} \\ &+ \varepsilon^{-\frac{3}{2}} \left(\|\langle\varepsilon\partial_t\rangle^k u\|_{H^l_\varepsilon(\mathbb{R}^3_+)} + \|\langle\varepsilon\partial_t\rangle^{k-1} u\|_{H^{l+1}_\varepsilon(\mathbb{R}^3_+)} \right) \left(\|\langle\varepsilon\partial_t\rangle^k v\|_{H^l_\varepsilon(\mathbb{R}^3_+)} + \|\langle\varepsilon\partial_t\rangle^{k-1} v\|_{H^{l+1}_\varepsilon(\mathbb{R}^3_+)} \right) \end{aligned} \tag{4.41}$$

with the notation $\langle\varepsilon\partial_t\rangle^k f = (f, \varepsilon\partial_t f, \cdots, (\varepsilon\partial_t)^k f)$ if $k \geq 0$, 0 if $k < 0$. Moreover for any smooth function F with $F(0) = 0$, we have

$$\|(\varepsilon\partial_t)^k F(u)\|_{H^l_\varepsilon(\mathbb{R}^3_+)} \leq C \left(\|\langle\varepsilon\partial_t\rangle^k u\|_{H^l_\varepsilon(\mathbb{R}^3_+)} + \|\langle\varepsilon\partial_t\rangle^{k-1} u\|_{H^{l+1}_\varepsilon(\mathbb{R}^3_+)} \right), \quad l \geq 1, \tag{4.42}$$

with

$$C = C\left[\|u\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-\frac{3}{2}} \|\langle\varepsilon\partial_t\rangle^k u\|_{H^l_\varepsilon(\mathbb{R}^3_+)} + \varepsilon^{-\frac{3}{2}} \|\langle\varepsilon\partial_t\rangle^{k-1} u\|_{H^{l+1}_\varepsilon(\mathbb{R}^3_+)} \right].$$

Proof. Let us first emphasize that a simple rescaling ($t' = t/\varepsilon$, $x' = x/\varepsilon$) allows to restrict to the case $\varepsilon = 1$.

To prove (4.41), it suffices to use the Leibnitz formula. When all the derivatives are on u or on v, we estimate the other term in L^∞. For the remaining terms which are under the form $\partial_t^{k_1} \partial_x^{\alpha_1} u \, (\partial_t)^{k_2} \partial_x^{\alpha_2} v$ with $k_1 + |\alpha_1| \leq k + l - 1$ and $k_2 + |\alpha_2| \leq k + l - 1$, we use (4.39) to write

$$\|\partial_t^{k_1} \partial_x^{\alpha_1} u \, \partial_t^{k_2} \partial_x^{\alpha_2} v\|_{L^2} \leq \|\partial_t^{k_1} \partial_x^{\alpha_1} u\|_{H^{s_1}(\mathbb{R}^3_+)} \|\partial_t^{k_2} \partial_x^{\alpha_2} v\|_{H^{s_2}(\mathbb{R}^3_+)}$$

with $s_1 + s_2 = 3/2$. Therefore, by taking s_1 and s_2 smaller than 1, we obtain (4.41).

The proof of (4.42) (still for $\varepsilon = 1$) relies on the Faa Di Bruno formula for the quantity $\partial_t^k \partial_x^\alpha F(u)$, $|\alpha| \leq l$, and on the product rule (4.40). For brevity, we just treat the case $l = 1$.

If $|\alpha| = 0$, $\partial_t^k \partial_x^\alpha F(u) = \partial_t^k F(u)$ can be decomposed thanks to the Faa Di Bruno formula as a sum of terms of the type

$$F^{(p)}(u) \prod_{i=1}^p \partial_t^{k_i} u, \quad \text{with} \quad \sum k_i \leq k.$$

Either there is zero factor in the product (that is $k = 0$), and we use the bound

$$\|F(u)\|_{L^2(\mathbb{R}^3_+)} = \|F(u) - F(0)\|_{L^2(\mathbb{R}^3_+)} \leq C\left[\|u\|_{L^\infty}\right] \|u\|_{L^2(\mathbb{R}^3_+)}$$

or there is one factor in the product, which leads to

$$\|F'(u)\partial_t^k u\|_{L^2(\mathbb{R}^3_+)} \leq C\left[\|u\|_{L^\infty(\mathbb{R}^3_+)}\right] \|\partial_t^k u\|_{L^2(\mathbb{R}^3_+)}$$

or there are several factors (which means $k_i < k$ for all i), and we have by inequality (4.40) (with $s_1, \dots, s_{p-1}$ close to 3/2 and $s_p = 1$)

$$\|F^{(p)}(u) \prod_{i=1}^{p} \partial_t^{k_i} u\|_{L^2(\mathbb{R}^3_+)} \le C \left[\|u\|_{L^\infty(\mathbb{R}^3_+)}, \|\langle \partial_t \rangle^{k-1} u\|_{H^{3/2}(\mathbb{R}^3_+)} \right] \|\langle \partial_t \rangle^{k-1} u\|_{H^1(\mathbb{R}^3_+)}$$

If $|\alpha| = 1$, for instance $\alpha = (1,0,0)$, $\partial_t^k \partial_x^\alpha F(u)$ can be decomposed as a sum of terms of the form

$$F^{(p+1)}(u) \prod_{i=1}^{p} \partial_t^{k'_i} u \left(\partial_t^{k''} \partial_{x_1} u \right), \quad \text{with} \quad \sum k'_i + k'' \le k.$$

Either $k'' = k$, and we use the bound

$$\|F'(u) \partial_t^k \partial_{x_1} u\|_{L^2(\mathbb{R}^3_+)} \le C \left[\|u\|_{L^\infty} \right] \|\langle \partial_t \rangle^k u\|_{H^1(\mathbb{R}^3_+)}$$

or $k'' \le k-1$. In this latter case, we apply again (4.40): if one of the k'_i's is k, we get

$$\|F^{(2)}(u)\, \partial_t^k u \;\, \partial_{x_1} u\|_{L^2(\mathbb{R}^3_+)} \le C \left[\|u\|_{L^\infty}, \|\langle \partial_t \rangle^k u\|_{H^1(\mathbb{R}^3_+)} \right] \|\langle \partial_t \rangle^{k-1} u\|_{H^2(\mathbb{R}^3_+)}$$

or all k_i's are less than $k-1$, so that

$$\|F^{(p+1)}(u) \prod_{i=1}^{p} \partial_t^{k'_i} u \;\, \left(\partial_t^{k''} \partial_{x_1} u \right) \|_{L^2(\mathbb{R}^3_+)} \le C \left[\|u\|_{L^\infty}, \|\langle \partial_t \rangle^{k-1} u\|_{H^{3/2}(\mathbb{R}^3_+)} \right] \|\langle \partial_t \rangle^{k-1} u\|_{H^2(\mathbb{R}^3_+)}.$$

The last inequality comes from (4.40) with $s_1 = \dots = s_p$ close to 3/2 and $s_{p+1} = 1$ (for the term $\partial_t^{k''} \partial_{x_1} u$). Combining above inequalities is enough to obtain (4.42).

□

We shall first prove that by using the equation, we can estimate the $\mathcal{H}^m_\varepsilon$ norm of the solution of (4.1) on $[0, T^\varepsilon)$:

Proposition 3. *For $m \ge 3$, we have for every $\varepsilon \in (0,1]$, for $t \in [0, T^\varepsilon)$*

$$\|(n, u, \phi, \varepsilon \nabla \phi)(t)\|_{L^\infty(\mathbb{R}^3_+)} \le C \varepsilon^{r-3/2}, \quad \|\nabla \big(n, u, \phi, \varepsilon \nabla \phi\big)(t)\|_{L^\infty(\mathbb{R}^3_+)} \le C \varepsilon^{r-5/2} \tag{4.43}$$

for some $C > 0$ independent of ε and

$$\|\big(n, u, \phi, \varepsilon \nabla \phi\big)(t)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \le C[C_a, M] \big(\varepsilon^{K+1} + \varepsilon^r\big) \tag{4.44}$$

where C stands for a continuous non-decreasing function with respect to all its arguments which does not depend on ε.

Proof. The first set of estimates can be obtained by using the Sobolev inequality in dimension 3 and the definition of T^ε.

To prove (4.44), we proceed by induction on the number k of time derivatives. For $k = 1$, we proceed as follows. At first, by using the evolution equations on n and u in system (4.1), we can compute $\varepsilon \partial_t (n, u)$. We claim that, for $t \in [0, T^\varepsilon)$

$$\begin{aligned} \|\varepsilon \partial_t (n,u)(t)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} \le C \Big[C_a, M, \|(u,n)\|_{L^\infty(\mathbb{R}^3_+)}, \|\varepsilon \nabla (u,n)\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-\frac{3}{2}} \|(u,n)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \Big] \\ \Big(\|(u,n)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \; + \; \|\varepsilon \nabla \phi\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} + \varepsilon^{K+1} \Big). \end{aligned} \tag{4.45}$$

Indeed, the expression of $\varepsilon \partial_t (n, u)$ involves four kinds of terms (besides the "easy ones" $\varepsilon \nabla \phi$ and $\varepsilon^{K+1} R_{n,u}$):

i) terms that are linear in $\varepsilon(\nabla n, \nabla u)$, with coefficients depending on u_a, n_a. They are bounded by $C_a \|\varepsilon \nabla (u,n)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)}$, where C_a depends on the L^∞ norms of $(\varepsilon \nabla_x)^\alpha (u_a, n_a)$, with $|\alpha| \le m-1$.

ii)terms that are linear in (n,u), with coefficients depending on $n_a, \varepsilon\nabla n_a$, ∇u_a. They are bounded by $C[C_a, M]\,\|(u,n)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)}$.

iii) quadratic terms, involving products of n or u with $\varepsilon\nabla n$ or $\varepsilon\nabla u$. By (4.41), Lemma 1, we can bound them by

$$C\left[\|(u,n)\|_{L^\infty} + \|\varepsilon\nabla(u,n)\|_{L^\infty(\mathbb{R}^3_+)} + \varepsilon^{-\frac{3}{2}}\|(u,n)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)}\right] \left(\|(u,n)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} + \|\varepsilon\nabla(u,n)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)}\right)$$

iv) a fully nonlinear term, coming from the pressure in the Euler equation for the ions. It reads

$$\varepsilon\nabla F(n_a,n), \quad \text{with } F(n_a,n) := T^i\left(\frac{\ln(n_a+n)}{n_a+n} - \frac{\ln(n_a+n)}{n_a}\right)$$

In particular, $F(n_a,0)=0$. The nonlinear term can then be evaluated in H^{m-1}_ε using (4.42) (with $k=0, l=m$). It is bounded by

$$C\left[M, C_a, \|n\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-\frac{3}{2}}\|n\|_{H^m_\varepsilon(\mathbb{R}^3_+)}\right]\|n\|_{H^m_\varepsilon(\mathbb{R}^3_+)}.$$

Combining the previous bounds yields (4.45). Therefore, by using (4.43) and the definition of T^ε, we obtain

$$\|\varepsilon\partial_t(n,u)(t)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} \le C\left[C_a, M, \varepsilon^{r-3/2}\right]\left(\varepsilon^{K+1}+\varepsilon^r\right). \tag{4.46}$$

Moreover, by applying $\varepsilon\partial_t$ to the Poisson equation in (4.1), we get that

$$\left(\varepsilon^2\Delta - e^{-(\phi_a+\phi)}\right)\varepsilon\partial_t\phi = \varepsilon\partial_t n + \varepsilon\partial_t\left(e^{-\phi_a}\right)\left(e^{-\phi}-1\right) + \varepsilon^{K+1}\varepsilon\partial_t R_\phi, \quad \varepsilon\partial_t\phi_{/x_3=0} = 0$$

and by applying next $(\varepsilon\partial_x)^\alpha$,

$$\begin{aligned}\left(\varepsilon^2\Delta - e^{-(\phi_a+\phi)}\right)&(\varepsilon\partial_x)^\alpha\varepsilon\partial_t\phi \\ &= (\varepsilon\partial_x)^\alpha\varepsilon\partial_t n + (\varepsilon\partial_x)^\alpha G_1(\cdot,\phi) \;+\; [(\varepsilon\partial_x)^\alpha; G_2(\cdot,\phi)]\varepsilon\partial_t\phi + \varepsilon^{K+1}(\varepsilon\partial_x)^\alpha\varepsilon\partial_t R_\phi,\end{aligned} \tag{4.47}$$

where $G_1(\cdot,\phi) = \varepsilon\partial_t\left(e^{-\phi_a}\right)\left(e^{-\phi}-1\right)$, $\quad G_2(\cdot,\phi) = e^{-(\phi_a+\phi)}$. From there, one can perform standard energy estimates, recursively on $|\alpha|$. The nonlinearities are handled thanks to (4.42). This leads to

$$\begin{aligned}\|\varepsilon\partial_t(\phi,\varepsilon\nabla\phi)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} \le C&\left[C_a, M, \|\phi\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-3/2}\|\phi\|_{H^m_\varepsilon(\mathbb{R}^3_+)}\right] \\ &\left(\|\varepsilon\partial_t n\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} + \|\phi\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} + \varepsilon^{K+1}\right).\end{aligned} \tag{4.48}$$

Consequently, by combining the last estimate and (4.46) and by using that $r>s$, we get that

$$\left\|\varepsilon\partial_t(n,u,\phi,\varepsilon\nabla\phi)\right\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M](\varepsilon^{K+1}+\varepsilon^r).$$

Now let us assume that we have proven that

$$\|(\varepsilon\partial_t)^k(n,u,\phi,\varepsilon\nabla\phi)\|_{H^{m-k}_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M]\left(\varepsilon^{K+1}+\varepsilon^r\right). \tag{4.49}$$

By applying $\varepsilon\,(\varepsilon\partial_t)^k$ to the evolution equations for n and u in (4.1), we obtain an expression for $(\varepsilon\partial_t)^{k+1}(n,u)$. This expression invoves k ε-derivatives with respect to time of linear, quadratic

and fully nonlinear terms. These terms are evaluated again thanks to Lemma 1. We get that for $k+1 \leq m$, $k \geq 1$

$$\begin{aligned}
&\|(\varepsilon\partial_t)^{k+1}(n,u)(t)\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} \\
&\leq C\left[C_a, M, \|(u,n)\|_{L^\infty(\mathbb{R}^3_+)}, \|\varepsilon\nabla(u,n)\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-\frac{3}{2}}\sum_{j=0}^{1}\|\langle\varepsilon\partial_t\rangle^{k-j}(u,n)\|_{H^{m-k+j}_\varepsilon(\mathbb{R}^3_+)}\right] \\
&\left(\sum_{j=0}^{1}\|\langle\varepsilon\partial_t\rangle^{k-j}(u,n)\|_{H^{m-k+j}_\varepsilon(\mathbb{R}^3_+)} + \|(\varepsilon\partial_t)^k\varepsilon\nabla\phi\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} + \varepsilon^{K+1}\right)
\end{aligned}$$

and hence from the induction assumption, we obtain

$$\|(\varepsilon\partial_t)^{k+1}(n,u)(t)\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, M]\big(\varepsilon^{K+1} + \varepsilon^r\big). \tag{4.50}$$

Finally, we can use the Poisson equation to control $(\varepsilon\partial_t)^{k+1}(\varepsilon\nabla\phi, \phi)$. More precisely, applying $(\varepsilon\partial_t)^{k+1}$ to the Poisson equation, we get

$$\big(\varepsilon^2\Delta - e^{-(\phi_a+\phi)}\big)(\varepsilon\partial_t)^{k+1}\phi = (\varepsilon\partial_t)^{k+1}n + \mathcal{C}^k(\phi) + \varepsilon^{K+1}(\varepsilon\partial_t)^{k+1}R_\phi, \quad (\varepsilon\partial_t)^{k+1}\phi_{/x_3=0} = 0$$

where $\mathcal{C}^k(\phi)$ is a commutator. Thanks to Lemma 1 and the induction assumption, we have for this commutator the estimate

$$\begin{aligned}
&\|\mathcal{C}^k(\phi)\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} \\
&\leq C\Big[C_a, M, \|\phi\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-3/2}\sum_{j=0}^{1}\|\langle\varepsilon\partial_t\rangle^{k-j}\phi\|_{H^{m-k+j}_\varepsilon(\mathbb{R}^3_+)}\Big]\sum_{j=0}^{1}\|\langle\varepsilon\partial_t\rangle^{k-j}\phi\|_{H^{m-k+j}_\varepsilon(\mathbb{R}^3_+)} \\
&\leq C[C_a, M]\big(\varepsilon^{K+1} + \varepsilon^r\big).
\end{aligned}$$

Consequently, the standard a priori estimates for the elliptic equation yield

$$\|(\varepsilon\partial_t)^{k+1}(\phi, \varepsilon\nabla\phi)\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} \leq C\,[C_a, M]\left(\|(\varepsilon\partial_t)^{k+1}n\|_{H^{m-k-1}_\varepsilon(\mathbb{R}^3_+)} + \varepsilon^{K+1} + \varepsilon^r\right)$$

and hence we get by combining (4.50) and the last estimate that (4.49) is verified for k changed in $k+1$.

□

By using similar arguments as above, we shall also get:

Lemma 2. *We have for $\varepsilon \in (0,1]$, the estimate*

$$\|(n,u,\phi,\varepsilon\nabla\phi)(0)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, M]\varepsilon^K.$$

Proof. The proof follows the same lines as the previous Proposition. The main difference is that because of the choice of the initial data, the estimate (4.46) is replaced by

$$\|\varepsilon\partial_t(n,u)(0)\|_{H^{m-1}_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, \varepsilon^{r-3/2}]\varepsilon^K.$$

The end of the induction can be performed in the same way.

□

4.3.2. *Normal derivatives estimates.* The aim of this subsection is to prove that we can replace one normal derivative by tangential derivatives thanks to the equation and hence that $\|\varepsilon\,\partial_3(n,u_3)(t)\|_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)}$ can be estimated in terms of Q_m. We shall again need some product estimates:

Lemma 3. *For any u, v and $\varepsilon\in(0,1]$, we have uniformly in ε*

$$\|(\varepsilon\partial_3)^k\mathcal{Z}^\alpha(uv)\|_{L^2(\mathbb{R}^3_+)}\lesssim\|u\|_{L^\infty}\|(\varepsilon\partial_3)^kv\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}+\|v\|_{L^\infty}\|(\varepsilon\partial_3)^ku\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}$$
$$+\varepsilon^{-\frac{3}{2}}\big(\|\langle\varepsilon\nabla\rangle^{k+1}u\|_{\mathcal{H}^{|\alpha|-1}_{co,\varepsilon}(\mathbb{R}^3_+)}+\|\langle\varepsilon\nabla\rangle^k\ u\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}\big)\big(\|\langle\varepsilon\nabla\rangle^{k+1}v\|_{\mathcal{H}^{|\alpha|-1}_{co,\varepsilon}(\mathbb{R}^3_+)}+\|\langle\varepsilon\nabla\rangle^kv\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}\big).\tag{4.51}$$

Moreover for any smooth function F with $F(0)=0$, we have

$$\|(\varepsilon\partial_3)^k\mathcal{Z}^\alpha F(u)\|_{L^2(\mathbb{R}^3_+)}\le C_1\,\|(\varepsilon\partial_3)^ku\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}$$
$$+\ C_2\sum_{1\le|\beta|\le2}\left(\|\langle\varepsilon\partial_3\rangle^{k-|\beta|}(\varepsilon\nabla)^\beta u\|_{\mathcal{H}^{|\alpha|}_{co,\varepsilon}(\mathbb{R}^3_+)}+\|\langle\varepsilon\partial_3\rangle^{k+1-|\beta|}(\varepsilon\nabla)^\beta u\|_{\mathcal{H}^{|\alpha|-1}_{co,\varepsilon}(\mathbb{R}^3_+)}\right)\tag{4.52}$$

where

$$C_1=C_1\left[\|u\|_{L^\infty(\mathbb{R}^3_+)}\right],\quad C_2=C\left[\|u\|_{L^\infty(\mathbb{R}^3_+)},\varepsilon^{-3/2}\|u\|_{\mathcal{H}^{k+|\alpha|}_\varepsilon(\mathbb{R}^3_+)}\right]$$

are two continuous, non-decreasing functions independent of ε. Moreover $C_2[\cdot,0]=0$.

Note that the last statement of the lemma implies that C_2 is small when its second argument is small. This will be used in the sequel to absorb some error terms in the estimates.

Proof. To prove (4.51), it suffices to use again the Leibnitz formula. When all the derivatives are on u or on v, we estimate the other term in L^∞. For the remaining terms which are under the form $(\varepsilon\partial_3)^{k_1}\mathcal{Z}^{\alpha_1}u\,(\varepsilon\partial_3)^{k_2}\mathcal{Z}^{\alpha_2}v$ with $k_1+|\alpha_1|\le k+|\alpha|-1$ and $k_2+|\alpha_2|\le k+|\alpha|-1$, we use (4.39) to write

$$\|(\varepsilon\partial_3)^{k_1}\mathcal{Z}^{\alpha_1}u\,(\varepsilon\partial_3)^{k_2}\mathcal{Z}^{\alpha_2}v\|_{L^2}\le\varepsilon^{-3/2}\|(\varepsilon\partial_3)^{k_1}\mathcal{Z}^{\alpha_1}u\|_{H^{s_1}(\mathbb{R}^3_+)}\|(\varepsilon\partial_3)^{k_2}\mathcal{Z}^{\alpha_2}v\|_{H^{s_2}(\mathbb{R}^3_+)}$$

with $s_1+s_2=3/2$ and we get the result by taking s_1 and s_2 smaller than 1.

The proof of (4.52) is very similar to the one of (4.42): broadly, $(\varepsilon\partial_3)$ substitutes to $\varepsilon\nabla$, whereas $\mathcal{Z}$ substitutes to $\varepsilon\partial_t$. The full gradient terms $(\varepsilon\nabla)^\beta$, $|\beta|=1,2$, are still connected to the use of (4.40), with $s_i\in[1,3/2]$. We leave the details to the reader. □

With this technical lemma in hand, we now prove the

Proposition 4. *There exists ε_0 such that for every $\varepsilon\in(0,\varepsilon_0]$, $t\in[0,T^\varepsilon)$, $0\le k\le m$,*

$$\|(\varepsilon\partial_3)^k(n,u,\phi,\varepsilon\nabla\phi)(t)\|_{\mathcal{H}^{m-k}_{co,\varepsilon}(\mathbb{R}^3_+)}\le C[C_a,M]\big(\varepsilon^{K+1}+Q_m(t)\big).$$

Note that we can reformulate the above Proposition into

$$\|(n,u,\phi,\varepsilon\nabla\phi)(t)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\le C[C_a,M]\big(\varepsilon^{K+1}+Q_m(t)\big),\quad\forall t\in[0,T^\varepsilon).\tag{4.53}$$

Proof. By using the definition of the vorticity, we already have:

$$\|\varepsilon\partial_3(u_1,u_2)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}\lesssim\|\omega\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}+\|u\|_{\mathcal{H}^m_{co,\varepsilon}}\lesssim Q_m(t).$$

Next, we notice that

$$\|\varepsilon\partial_3\nabla\phi\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}\lesssim\|\varepsilon\nabla\phi\|_{\mathcal{H}^m_{co,\varepsilon}}+\|(\varepsilon\partial_3)^2\phi\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}.$$

By using the Poisson equation to express $\varepsilon^2\partial_{33}\phi$, together with Lemma 3 to control the nonlinearity, we get that

$$\begin{aligned}\|\varepsilon\partial_3(\varepsilon\nabla\phi)(t)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}} &\leq C\left[C_a, M, \|\phi\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-3/2}\|\phi\|_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)}\right]\left(\varepsilon^{K+1}+Q_m(t)\right)\\ &\leq C\left[C_a, M, \varepsilon^{r-3/2}\right]\left(\varepsilon^{K+1}+Q_m(t)\right).\end{aligned}$$

Finally, by using the equations on n and u_3 in (4.1), we get that

(4.54)

$$A_n\begin{pmatrix}\partial_3 u_3\\ \partial_3 n\end{pmatrix} =$$

$$\begin{pmatrix}\partial_t n + (u_a+u)_{1,2}\cdot\nabla_{1,2}n + (1-\chi)(u_a+u)_3\partial_3 n + u\cdot\nabla n_a + n\,\mathrm{div}\ u_a + (n+n_a)\nabla_{1,2}\cdot(u_1,u_2) - \varepsilon^K R_n\\ \partial_t u_3 + (u_a+u)_{1,2}\cdot\nabla_{1,2}u_3 + (1-\chi)(u_a+u)_3\partial_3 u_3 - T^i\frac{(\partial_3 n_a n)}{n_a(n_a+n)} + \partial_3\phi - \varepsilon^K R_u\end{pmatrix}$$

where A_n is defined by

$$A_n = -\begin{pmatrix} n+n_a & \chi(u_a+u)_3\\ \chi(u_a+u)_3 & T^i(n^a+n)^{-1}\end{pmatrix}.$$

Note that thanks to (4.6), we get that A_n is invertible, its inverse being given by:

$$A_n^{-1} = \frac{1}{T^i - [\chi(u_a+u)_3]^2}\begin{pmatrix} T^i(n^a+n)^{-1} & -\chi(u_a+u)_3\\ -\chi(u_a+u)_3 & n_a+n\end{pmatrix}.$$

Hence, we can use this system to estimate $\|\varepsilon\,\partial_3(n,u_3)(t)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)}$. Note also that the field $\varepsilon(1-\chi)\partial_3$ is equivalent to $\mathcal{Z}_3$. One can again decompose the terms into

i) linear terms, whose coefficients depend on $u_a, \nabla u_a, n_a, \varepsilon\nabla n_a$ and involve (u,n) or $\mathcal{Z}_i(u,n)$. They are bounded by $C[C_a,M]\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}}$.

ii) nonlinear terms: they all involve products of the type $F(x,n,u)\,\mathcal{Z}_i(n,u)$, where the function F satisfies $F(x,0,0)=0$. They can be estimated with Lemma 3 ($k=0$, $|\alpha|=m-1$).

We obtain

$$\begin{aligned}\|\varepsilon\,\partial_3(n,u_3)(t)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} \leq C_1\left(\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}} + \|\varepsilon\nabla\phi\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} + \varepsilon^{K+1}\right)\\ + C_2\left(\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}} + \|(\varepsilon\partial_3)(n,u)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}\right)\end{aligned} \tag{4.55}$$

with

$$C_1 = C_1\left[C_a, M, \|(n,u)\|_{L^\infty(\mathbb{R}^3_+)}, \|\mathcal{Z}(n,u)\|_{L^\infty(\mathbb{R}^3_+)}\right]$$

and with

$$C_2 = C_2\left[C_a, M, \|(n,u)\|_{L^\infty(\mathbb{R}^3_+)}, \|\mathcal{Z}(n,u)\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-3/2}\|(n,u)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\right]$$

that vanishes with its last argument. We used here the notation $\mathcal{Z}f = (\mathcal{Z}_i f)_{0\leq i\leq 3}$.

Now, from the L^∞ estimates and (4.44) in Proposition 3 and from the definition of T_ε, it follows that

$$\begin{aligned}\|\varepsilon\,\partial_3(n,u_3)(t)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} \leq C_1'[\varepsilon^{r-3/2}]\left(\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} + \|\varepsilon\nabla\phi\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} + \varepsilon^{K+1}\right)\\ + C_2'[\varepsilon^{r-3/2}]\left(\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}} + \|(\varepsilon\partial_3)(n,u)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}}\right)\end{aligned}$$

where C_2' vanishes with its argument. As $r>3/2$, we can absorb the last term at the r.h.s in the l.h.s for ε small enough: this yields

$$\|\varepsilon\,\partial_3(n,u_3)(t)\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} \lesssim \left(\|(n,u)\|_{\mathcal{H}^m_{co,\varepsilon}} + \|\varepsilon\nabla\phi\|_{\mathcal{H}^{m-1}_{co,\varepsilon}(\mathbb{R}^3_+)} + \varepsilon^{K+1}\right) \lesssim \varepsilon^{K+1} + Q_m(t). \tag{4.56}$$

We can then prove Proposition 4 by induction on k.

The estimate for $k=1$ is thus already proven. Let us assume that it is proven for $j \leq k$. At first, from the definition of the vorticity, we find that

$$\|(\varepsilon\partial_3)^{k+1}(u_1,u_2)\|_{\mathcal{H}^{m-(k+1)}_{co,\varepsilon}} \lesssim \|(\varepsilon\partial_3)^k\omega\|_{\mathcal{H}^{m-(k+1)}_{co,\varepsilon}} + \|(\varepsilon\partial_3)^k u\|_{\mathcal{H}^{m-k}_{co,\varepsilon}}$$

and the right hand side is already estimated thanks to the definition of Q_m and the induction assumption. In a similar way, by using the Poisson equation, we get that

$$\|(\varepsilon\partial_3)^{k+1}\varepsilon\nabla\phi\|_{\mathcal{H}^{m-(k+1)}_{co,\varepsilon}} \leq C[C_a, M, \varepsilon^{r-3/2}]\big(\varepsilon^{K+1} + \sum_{j=0}^{k}\|(\varepsilon\partial_3)^j(n,\phi,\varepsilon\nabla\phi)\|_{\mathcal{H}^{m-j}_{co,\varepsilon}}\big)$$

and we conclude again by the induction assumption. Finally, to estimate $\|(\varepsilon\partial_3)^{k+1}(n,u_3)\|_{\mathcal{H}^{m-(k+1)}_{co,\varepsilon}}$, we use again (4.54), Lemma 3 and the induction assumption.

□

4.3.3. *Main energy estimate.* We shall finally estimate Q_m by carefully using the system (4.1) and the linear stability estimate of Proposition 2.

Let us before state a useful commutator estimate:

Lemma 4. *For $i=1,2,3$, we have the estimate:*

$$\|[\mathcal{Z}^\alpha, f]\partial_i g\|_{L^2(\mathbb{R}^3_+)} \lesssim C\big[\|(\nabla_{t,x}f,\partial_i g)\|_{L^\infty(\mathbb{R}^3_+)}, \varepsilon^{-\frac{5}{2}}\|f\|_{\mathcal{H}^{|\alpha|}_\varepsilon(\mathbb{R}^3_+)}\big] \big(\|f\|_{\mathcal{H}^{|\alpha|}_\varepsilon(\mathbb{R}^3_+)} + \|g\|_{\mathcal{H}^{|\alpha|}_\varepsilon(\mathbb{R}^3_+)}\big). \tag{4.57}$$

Proof. To get this lemma, it suffices to combine the Leibnitz formula and (4.39) as in the proof of Lemma 1 and Lemma 3. The only term that we handle in a different way is when all the derivatives but one are on $\partial_i g$, in this case, we write

$$\|\mathcal{Z}f\,\mathcal{Z}^{|\alpha|-1}\partial_i g\|_{L^2(\mathbb{R}^3_+)} \leq \|\nabla_{t,x}f\|_{L^\infty(\mathbb{R}^3_+)}\|\varepsilon\mathcal{Z}^{|\alpha|-1}\partial_i g\|_{L^2(\mathbb{R}^3_+)} \lesssim \|\nabla_{t,x}f\|_{L^\infty(\mathbb{R}^3_+)}\|g\|_{\mathcal{H}^{|\alpha|}_\varepsilon(\mathbb{R}^3_+)}.$$

□

Our main energy estimate for (4.1) is the following:

Proposition 5. *There exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$, we have the estimate*

$$Q_m(t)^2 \leq C(C_a,M)\big(\varepsilon^{2K} + T\varepsilon^{2K} + \int_0^t Q_m^2\big), \quad \forall t\in[0,T^\varepsilon).$$

Proof. We shall first estimate $\|(n,u,\phi,\varepsilon\nabla\phi)\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)}$. To this end, we apply the operator $\mathcal{Z}^\alpha$ to the system (4.1). We obtain:

$$\begin{cases} \partial_t\mathcal{Z}^\alpha n + (u_a+u)\cdot\nabla\mathcal{Z}^\alpha n + (n_a+n)\mathrm{div}\ \mathcal{Z}^\alpha u + \mathcal{Z}^\alpha u\cdot\nabla(n_a+n) + \mathcal{Z}^\alpha n\,\mathrm{div}\ (u+u_a) \\ \qquad\qquad = \mathcal{C}_n + \varepsilon^K\mathcal{Z}^\alpha R_n, \\ \partial_t\mathcal{Z}^\alpha u + (u_a+u)\cdot\nabla\mathcal{Z}^\alpha u + \mathcal{Z}^\alpha u\cdot\nabla u_a + T^i\left(\dfrac{\nabla\mathcal{Z}^\alpha n}{n_a+n} - \dfrac{\nabla n_a}{n_a}\left(\dfrac{\mathcal{Z}^\alpha n}{n_a+n}\right)\right) \\ \qquad\qquad = \nabla\mathcal{Z}^\alpha\phi + \mathcal{C}_u + \varepsilon^K\mathcal{Z}^\alpha R_u, \\ \varepsilon^2\Delta\mathcal{Z}^\alpha\phi = \mathcal{Z}^\alpha n + e^{-\phi_a}\mathcal{Z}^\alpha\phi(1+h) + \mathcal{C}_\phi + \varepsilon^{K+1}\mathcal{Z}^\alpha R_\phi. \end{cases} \tag{4.58}$$

One has $h=h_0$ for $|\alpha|=0$ and $h=h_1$ for $|\alpha|\geq 1$. The functions $\mathcal{C}_n, \mathcal{C}_u$ and $\mathcal{C}_\phi$ are remainders due to commutators. One can observe that this corresponds to the "abstract" system that we have studied in Proposition 2. We shall now estimate the remainders in order to be able to apply the stability estimate (4.34).

We first claim that

$$\|(\mathcal{C}_n,\mathcal{C}_u)\|_{L^2} \le C[C_a,M,\varepsilon^{r-s-1}]\|(n,u,\phi,\varepsilon\nabla\phi)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M]\big(\varepsilon^{K+1}+Q_m(t)\big). \tag{4.59}$$

This estimate is in the same spirit as the previous ones. One must distinguish between linear terms and nonlinear terms, the latter being controlled thanks to the product estimates of the previous lemma. For brevity, we only point out two typical terms:

- a typical linear term in $\mathcal{C}_n$ is $[\nabla n_a;\mathcal{Z}^\alpha]\cdot u$. It is *a priori* singular (because of the boundary layer), but noticing that
$$\|[\mathcal{Z},\nabla n_a]f\|_{L^2} \le C_a\|f\|_{L^2}$$
gives the bound $\|[\nabla n_a,\mathcal{Z}^\alpha]\cdot u\|_{L^2(\mathbb{R}^3_+)} \le C_a\|u\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M]\left(\varepsilon^{K+1}+Q_m(t)\right)$.
- a typical nonlinear term in $\mathcal{C}_u$ is the commutator
$$T^i\left[F(n_a,n)\nabla;\mathcal{Z}^\alpha\right]n \;=\; T^i[F(n_a,n);\mathcal{Z}^\alpha]\nabla n \;+\; T^iF(n_a,n)[\mathcal{Z}^\alpha;\nabla]n$$
with $F(n_a,n):=\frac{1}{n_a+n}-\frac{1}{n_a}$. Clearly
$$\|F(n_a,n)[\mathcal{Z}^\alpha;\nabla]n\|_{L^2(\mathbb{R}^3_+)} \;\le\; C_M\|n\|_{\mathcal{H}^{|\alpha|}_\varepsilon(\mathbb{R}^3_+)}$$
As regards the first term at the r.h.s., we use the commutator estimate of Lemma 4:
$$\|[F(n_a,n);\mathcal{Z}^\alpha]\nabla n\|_{L^2(\mathbb{R}^3_+)} \le C\left[\|\nabla_{t,x}F(n_a,n),\nabla_x n\|_{L^\infty(\mathbb{R}^3_+)},\varepsilon^{-5/2}\|F(n_a,n)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\right]\left(\|F(n_a,n)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \;+\; \|n\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\right).$$
The product estimate (4.42) implies that
$$\|F(n_a,n)\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \;\le\; C\left[C_a,M,\|n\|_{L^\infty(\mathbb{R}^3_+)},\varepsilon^{-3/2}\|n\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\right]\|n\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)},$$
so that eventually
$$\|T^i\left[F(n_a,n)\nabla;\mathcal{Z}^\alpha\right]n\|_{L^2(\mathbb{R}^3_+)} \;\le\; C[C_a,M,\varepsilon^{r-5/2}]\|n\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M]\left(\varepsilon^{K+1}+Q_m(t)\right).$$

The treatment of the other terms is left to the reader.

For the commutator $\mathcal{C}_\phi$, we get that

$$\|\mathcal{C}_\phi\|_{L^2} \le C[C_a,\varepsilon^{r-3/2}]\varepsilon\|\phi\|_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)} + \varepsilon\left(\|\varepsilon\nabla\phi\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} + \|\varepsilon\nabla\phi\|_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)}\right).$$

The first term at the r.h.s corresponds to the commutator with the semilinear term in the Poisson equation. The second one bounds the commutator $[\varepsilon^2\Delta;\mathcal{Z}^\alpha]\phi$. We use the fact that

$$[\mathcal{Z}_3,\partial_3] = -\partial_3\left(\frac{x_3}{1+x_3}\right)\varepsilon\partial_3.$$

Consequently, by using again Proposition 3 and Proposition 4, we get that

$$\varepsilon^{-1}\|\mathcal{C}_\phi\|_{L^2} \le C(C_a,M,\varepsilon^{r-3/2})\big(\varepsilon^{K+1}+Q_m\big). \tag{4.60}$$

By using similar arguments, we also get that

$$\|\partial_t\mathcal{C}_\phi\|_{L^2} \le C(C_a,\varepsilon^{r-3/2})\|\phi\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} + \|\varepsilon^2\nabla^2\phi\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} + \|\varepsilon\nabla\phi\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}.$$

To estimate the second term, we use the elliptic regularity for the Poisson equation in (4.58). This yields

$$\varepsilon^2\|\nabla^2\phi\|_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} \lesssim \|n\|_{\mathcal{H}^m_\varepsilon} + C[C_a,\varepsilon^{r-3/2}]\|\phi\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} + \|\mathcal{C}_\phi\|_{L^2} + \varepsilon^{K+1}$$

and hence, we find that

$$\|\partial_t\mathcal{C}_\phi\|_{L^2} \le C[C_a,M,\varepsilon^{r-3/2}]\big(\varepsilon^{K+1}+Q_m\big). \tag{4.61}$$

Note that thanks to Proposition 3, we have that on $[0,T^\varepsilon)$,

$$\|(n,u,\phi)\|_{L^\infty(\mathbb{R}^3_+)} \le C\varepsilon^{r-3/2}$$

and hence that the assumption 4.11 in Proposition 2 is matched on $[0,T^\varepsilon)$ for ε sufficiently small. Consequently, we can use Proposition 2 and Lemma 2 for the system (4.58) to get

$$\big\|\big(n,u,\phi,\varepsilon\nabla\phi\big)(t)\big\|^2_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} \le C[M]\Big(\varepsilon^{2K} + \int_0^t \big\|(\varepsilon^{-1}r_\phi, \partial_t r_\phi, r_n, r_u)\big\|^2_{L^2(\mathbb{R}^3_+)}\Big)$$
$$+ C[C_a,M]\int_0^t \big(1+\big\|\nabla_{t,x}\big(n,u,\phi\big)\big\|_{L^\infty(\mathbb{R}^3_+)} + \varepsilon^{-1}\|n\|_{L^\infty(\mathbb{R}^3_+)}\big)\big\|\big(n,u,\phi,\varepsilon\nabla\phi\big)\big)\big\|^2_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)}\Big)$$

where we have set

$$r_n = \mathcal{C}_n + \varepsilon^K\mathcal{Z}^\alpha R_n, \quad r_u = \mathcal{C}_u + \varepsilon^K\mathcal{Z}^\alpha R_u, \quad r_\phi = \mathcal{C}_\phi + \varepsilon^{K+1}\mathcal{Z}^\alpha R_\phi.$$

Consequently, by using the estimates (3.7), (4.59), (4.60), (4.61) and Proposition 3, we get that on $[0,T_\varepsilon]$

$$\big\|\big(n,u,\phi,\varepsilon\nabla\phi\big)(t)\big\|^2_{\mathcal{H}^m_{co,\varepsilon}(\mathbb{R}^3_+)} \le C(C_a,M,\varepsilon^{r-5/2})\Big(\varepsilon^{2K} + \varepsilon^{2K}t + \int_0^t Q_m(t)^2\Big). \tag{4.62}$$

It remains to estimate $\|\omega\|_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)}$ i.e. the second term in the definition of Q_m. By applying the operator $\varepsilon\nabla\times$ to the second equation in the system (4.1), we find that

$$\partial_t\omega + (u_a+u)\cdot\nabla\omega = \varepsilon\mathcal{P}(u_a,\nabla u_a)\big(\nabla u,\nabla u\big) - \varepsilon u\cdot\nabla\omega_a + \varepsilon^K R_\omega \tag{4.63}$$

where $\mathcal{P}(u_a,\nabla u_a)$ is a polynomial of degree less than two whose coefficients depend only on u_a and ∇u_a and thus are uniformly bounded. Note that we have used that

$$\nabla\times\Big(\frac{\nabla n}{n_a+n} - \frac{\nabla n_a}{n_a}\Big(\frac{n}{n_a+n}\Big)\Big) = \nabla\times\nabla\big(\log(n_a+n) - \log n_a\big) = 0.$$

By applying the operator $(\varepsilon\partial)^\alpha = (\varepsilon\partial_{t,x})^\alpha$ to (4.63), we find that

$$\partial_t(\varepsilon\partial)^\alpha\omega + (u_a+u)\cdot\nabla(\varepsilon\partial)^\alpha\omega = \mathcal{C}_\omega \tag{4.64}$$

where by using Lemma 1, we have

$$\|\mathcal{C}_\omega\|_{L^2(\mathbb{R}^3_+)} \le C[C_a,M,\varepsilon^{r-5/2}]\big(\|u\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} + \|[(\varepsilon\partial)^\alpha; u\cdot\nabla]\omega\|_{L^2(\mathbb{R}^3_+)} + \varepsilon^K\big).$$

To estimate the commutator, we use the following variant of Lemma 4: we first write since $\|\omega\|_{\mathcal{H}^{m-1}_\varepsilon} \lesssim \|u\|_{\mathcal{H}^m_\varepsilon}$ that

$$\|[(\varepsilon\partial)^\alpha; u\cdot\nabla]\omega\|_{L^2(\mathbb{R}^3_+)} \le \|\nabla_{t,x}u\|_{L^\infty(\mathbb{R}^3_+)}\|u\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)} + \|(\varepsilon\partial)^\alpha u\cdot\nabla\omega\|_{L^2(\mathbb{R}^3_+)} + \varepsilon^{-\frac{5}{2}}\|u\|^2_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}$$

and to estimate the second term, we use also (4.39), to get since $|\alpha|\le m-1$, $m\ge 3$ that

$$\|(\varepsilon\partial)^\alpha u\cdot\nabla\omega\| \lesssim \varepsilon^{-\frac{3}{2}}\|u\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\|\nabla\omega\|_{H^1_\varepsilon(\mathbb{R}^3_+)} \lesssim \varepsilon^{-\frac{5}{2}}\|u\|^2_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}.$$

We thus get the estimate

$$\|\mathcal{C}_\omega\|_{L^2(\mathbb{R}^3_+)} \le C[C_a,M,\varepsilon^{r-5/2}]\|u\|_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}.$$

Consequently, from a standard L^2 type energy estimate on the equation (4.64) (let us recall that $(u_a+u)_3$ vanishes on the boundary), we get that

$$\frac{d}{dt}\|\omega\|^2_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)} \le C[C_a,M]\big(\varepsilon^{2K} + \int_0^t \|u\|^2_{\mathcal{H}^m_\varepsilon(\mathbb{R}^3_+)}\big)$$

and hence by using Lemma 2 and Proposition 4, we infer that for $t \in [0, T^\varepsilon)$,

$$\|\omega\|^2_{\mathcal{H}^{m-1}_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, M]\big(\varepsilon^{2K} + \varepsilon^{2K} t + \int_0^t Q_m^2\big). \tag{4.65}$$

To end the proof of Proposition 5, it suffices to combine the estimates (4.65) and (4.62).

□

4.3.4. *Proof of Theorem 3.* We can now easily conclude the proof of Theorem 3. By using Proposition 5 and the Gronwall inequality, we get that

$$Q_m(t) \leq C[C_a, M]\varepsilon^K e^{C[C_a,M]t}, \quad \forall t \in [0, T^\varepsilon).$$

This yields thanks to (4.53),

$$\|(n, u, \phi, \varepsilon\nabla\phi)(t)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, M]\big(e^{C[C_a,M]t} + \varepsilon\big)\varepsilon^K, \quad \forall t \in [0, T^\varepsilon).$$

Note that we also have by Sobolev embedding

$$\|(n, \phi)\|_{L^\infty(\mathbb{R}^3_+)} \leq \varepsilon^{K-3/2}\big(e^{C[C_a,M]t} + \varepsilon\big), \quad \forall t \in [0, T^\varepsilon)$$

and

$$\big\|\chi(x_3)u_3\big\|_{L^\infty(\mathbb{R}^3_+)} \leq C[\delta]\varepsilon^{K-5/2}\big(e^{C[C_a,M]t} + \varepsilon\big), \quad \forall t \in [0, T^\varepsilon).$$

In view of the definition of T^ε, we obtain that for ε sufficiently small $T^\varepsilon \geq T_0$ and that

$$\|(n, u, \phi, \varepsilon\nabla\phi)(t)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \leq C[C_a, M]\big(e^{C[C_a,M]t} + \varepsilon\big)\varepsilon^K, \quad \forall t \in [0, T_0].$$

This ends the proof of Theorem 3.

5. Further remarks

5.1. **Generalization to general smooth domains.** In this paragraph, we briefly explain how the work which has been done for the half-space $\mathbb{R}^3_+$ can be adapted to handle the case of a smooth open set $\Omega \subset \mathbb{R}^3_+$ whose boundary $\partial\Omega$ is an orientable compact manifold.

i) *Derivation of boundary layers.*

The principle is to apply the tubular neighborhood theorem in $\mathbb{R}^3$; then, $\partial\Omega$ being orientable, there exist a smooth function $\varphi : \partial\Omega \to \mathbb{R}$ (normalized to have $|\nabla\varphi| = 1$) and a neighborhood $U := \{x \in \Omega, \varphi < \eta\}$ of $\partial\Omega$ in Ω such that for any $x \in U$, there is one and only one $(\tilde{x}, y) \in \partial\Omega \times \mathbb{R}^{+*}$, such that

$$x = \tilde{x} + y\, n_{\tilde{x}}.$$

where $n_{\tilde{x}}$ is the inward-pointing normal at $\tilde{x}$.

Then we can look for approximate solutions of the form:

$$\begin{aligned}(n^\varepsilon_{app}, u^\varepsilon_{app}, \phi^\varepsilon_{app}) &= \sum_{i=0}^K \varepsilon^i \left(n^i(t,x), u^i(t,x), \phi^i(t,x)\right) \\ &+ \sum_{i=0}^K \varepsilon^i \left(N^i\left(t, \tilde{x}, \frac{y}{\varepsilon}\right), U^i\left(t, \tilde{x}, \frac{y}{\varepsilon}\right), \Phi^i\left(t, \tilde{x}, \frac{y}{\varepsilon}\right)\right).\end{aligned} \tag{5.1}$$

and the construction of Section 3 remains the same.

ii) *Stability of the boundary layers.*

The L^2 stability estimate that is based only on integration by parts is still true. For higher order estimates, one can modify the definition of the conormal Sobolev spaces in the usual way by considering a finite set of generators of vector fields tangent to the boundary.

5.2. **Generalization to non homogeneous conditions for u.** Let us now explain how we could generalize our work to the following boundary condition on u:

$$u_3|_{x_3=0} = u_b. \tag{5.2}$$

We shall impose a subsonic outflow condition $-\sqrt{T^i} < u_b < 0$. One can readily check that the boundary $\{x_3 = 0\}$ is *noncharacteristic* in this setting and that this boundary condition is maximal dissipative. From the physical point of view, this boundary condition means that some plasma is somehow absorbed at the boundary.

Let us once again explain what are the main changes in each principal part of the proof. Details are left to the reader.

i) *Derivation of boundary layers.*

Considering the derivation of the approximation, computations get more tedious, but the results remain roughly unchanged. In particular, we can check that there is still no boundary layer term for the velocity at leading order, which means with the same notations as before, that $U^0 = 0$. The principle to show this fact is to obtain closed equations on U^0_y and U^0_3 which have the trivial solution 0, as it was done for U^0_y in Section 3.

ii) *Stability of the boundary layers.*

The main part which is modified due to the new boundary condition is the proof of Proposition 2, in which new boundary contributions appear since $(u_a + u)_3|_{x_3=0} \neq 0$: we have to treat them carefully. Nevertheless, we note that since the boundary condition for the original problem is dissipative, the boundary terms when we integrate by parts in the transport terms have the "good sign". For example, one typical new harmless term comes for instance in (4.25). By integration by parts for the term

$$-\int_{\mathbb{R}^3_+} \frac{\nabla(\dot{n}^2)\cdot(u^a + u)}{n_a + n}$$

the new term that appears is $-\int_{\mathbb{R}^2} \frac{(\dot{n}^2)(u^a+u)\cdot(-e_3)}{n_a+n}|_{x_3=0}$ and one can observe that $(u^a+u)_3|_{x_3=0} \leq 0$, so that this term has indeed the good sign.

One more significant term comes from the computation of J_1 in (4.31). One can check that a first integration by part yields the term $\mathcal{B} := \int_{\mathbb{R}^2} |\nabla\dot{\phi}|^2 (u_a + u)_3|_{x_3=0}$ and a second integration by parts yields the term $-\frac{1}{2}\mathcal{B}$, so that only the sum of these two terms, $\frac{1}{2}\mathcal{B}$, has the good sign.

Furthermore, some terms also have to be estimated more carefully. In (4.29) and (4.30), the two terms

$$-\frac{1}{2}\int_{\mathbb{R}^3_+} \partial_t(e^{-\phi_a}(1 + h(\phi)))|\dot{\phi}|^2 - \frac{1}{2}\int_{\mathbb{R}^3_+} |\dot{\phi}|^2 \operatorname{div}\,(e^{-\phi_a}(1 + h(\phi))(u_a + u))$$

can not be estimated separately as before. They can be treated using the equation satisfied by $e^{-\phi_a}$ (that is the transport equation satisfied by n_a, up to some error term).

The end of the proof of Theorem 3 is unchanged, except for the last part which gets actually simpler. Indeed, the boundary $\{x_3 = 0\}$ is now noncharacteristic and therefore we can estimate all normal derivatives using directly the equations on (n, u), without having to introduce the vorticity.

When $0 < u_b < \sqrt{T^i}$ the boundary $\{x_3 = 0\}$ is still noncharacteristic (this corresponds to some *inflow* condition, which means that some plasma is injected at the boundary) but we still need to add two (in dimension 3) boundary conditions in order to get a well-posed problem. There are many possibilities and some of them do not yield a dissipative problem. In this case, our approach to the L^2 stability estimate of Proposition 2 would break down. We leave this case for future work, as well as the case of supersonic outflow condition, relevant to plasma sheath layer problems.

5.3. **The relative entropy method.** In this paragraph, we focus on the one-dimensional case. First, we recall that Cordier and Peng constructed global entropy solutions to the same isothermal Euler-Poisson system as the one studied in this paper, posed in the whole line $\mathbb{R}$ (see [8]). Although it has never been done, it seems reasonable to believe in the existence of some global entropy solution in the half line $\mathbb{R}^+$, at least for the simplest boundary condition $\phi_b = 0$ (and non-penetration, that is $u_3|_{x_3=0} = 0$).

Assuming the existence of such solutions with weak regularity, one could study the quasineutral limit, using the so-called relative entropy method which was introduced by Brenier in [3]. This strategy was previously briefly evoked in [21]. It consists in considering the following modulated *nonlinear* energy:

$$\begin{aligned}\mathcal{H}_\varepsilon(t) =& \frac{1}{2}\int n|u-u_{app}|^2dx + T^i\int n(\log(n/n_{app}) - 1 + n_{app}/n)dx \\ &+ \int (e^{-\phi}\log\left(e^{-\phi}/n_{app}\right) - e^{-\phi} + n_{app})dx + \frac{\varepsilon^2}{2}\int |\partial_x\phi|^2dx,\end{aligned} \tag{5.3}$$

Then, by an explicit computation (similar to the one given in [21]) of the derivative in time of $\mathcal{H}_\varepsilon$, the principle is to show that this is a Lyapunov function, and thus that it vanishes as $\varepsilon \to 0$, if $\mathcal{H}_\varepsilon(0) \to_{\varepsilon\to 0} 0$. This would roughly proves strong convergence in L^2 for $(n_\varepsilon, u_\varepsilon, \phi_\varepsilon)$ without performing high-orders estimates in order to obtain strong compactness, as in the proof of this paper. Notice furthermore that this seems anyway impossible to justify such high-order estimates for solutions with low regularity.

The convergence would still be local in time, since for this method, one has to consider *strong* solutions (that is with at least Lipschitz regularity) to the limit systems.

5.4. **Two Stream instability.** It seems reasonable to consider the case of several species of ions, say two for the clarity of exposure. This means that each species, described by (n_1, u_1) (resp. (n_2, u_2)) satisfies an isothermal Euler-Poisson system:

$$\begin{cases}\partial_t n_i \,+\, \mathrm{div}\,(n_i u_i) \,=\, 0,\\ \partial_t u_i \,+\, u_i\cdot\nabla u_i \,+\, T^i\,\nabla\ln(n_i) = \nabla\phi,\end{cases} \tag{5.4}$$

and these systems are coupled through the quasineutral Poisson equation:

$$\varepsilon^2\Delta\phi \,+\, e^{-\phi} = n_1 \,+\, n_2. \tag{5.5}$$

One could expect to prove similar results to those proved in this paper. Nevertheless, the situation turns out to be very different in that framework (even without boundary). This is due to the so-called two stream instability. We refer to the work of Cordier, Grenier and Guo [7], who studied that mechanism and showed that any initial data such that

$$u_{1,0} \neq u_{2,0}$$

is non linearly unstable, hence the name two stream instability (actually they do not explicitly treat the case of (5.5), but their study also holds in that case). Instability means here that there exists some $\alpha_0 > 0$ such that, for any $\delta > 0$, there exists an initial condition in a ball of radius δ (for a Sobolev norm with regularity as high as we want) and with center the two-stream initial condition and whose associated solution goes out of the ball of radius α_0 in the L^2 norm, after some time of order $O(|\log\delta|)$. In other words, there are perturbations of the two-stream data which are arbitrarily small in any Sobolev norm (as high as we want) and which are wildly amplified.

One can then observe that it is possible to go from the system (5.4)-(5.5) with any $\varepsilon > 0$ to (5.4)-(5.5) with $\varepsilon = 1$ thanks to the change of variables $(t, x) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$. This means that the quasineutral limit can somehow be seen a long time behaviour limit for the Euler-Poisson system.

By extrapolating the two-stream instability results, one can infer that the formal limit for (5.4)-(5.5) is false in Sobolev regularity for such data.

On the other hand, one can conjecture that the formal limit is true as soon as we avoid two-stream instabilities, that is to say when:

$$u_{1,0} = u_{2,0}.$$

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