Rigidity and Bellows-type Theorem for hedgehogs

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Abstract

We consider rigidity and Gauss infinitesimal rigidity for hedgehogs of $\mathbb{R}^3$ (regarded as Minkowski differences of closed convex surfaces of $\mathbb{R}^3$ with positive Gaussian curvature). Besides, we give a bellows-type theorem for hedgehogs under an appropriate differentiability condition.

Introduction

In 1813, A. L. Cauchy proved (almost rigorously) his famous rigidity theorem: *Any convex polyhedron of* $\mathbb{R}^3$ *is rigid* (that is, no convex polyhedron of $\mathbb{R}^3$ can be continuously deformed so that its faces remain rigid) [2]. First examples of flexible polyhedra were discovered by R. Bricard in 1897 [1], but these « Bricard's flexible octahedra » are self-intersecting. The question of rigidity of embedded non-convex polyhedra remained open until 1977 when R. Connelly discovered a first example of flexible sphere-homeomorphic polyhedron [4]. In the late seventies, R. Connelly and D. Sullivan formulated the so-called « Bellows conjecture » stating that whenever we perform a rigid deformation of a flexible polyhedron P (that is, a continuous deformation of P that changes only its dihedral angles), the volume of P remains constant. This conjecture was proved by I. Sabitov for sphere-homeomorphic polyhedra [14] and by R. Connelly, I. Sabitov, and A. Walz for general orientable 2-dimensional polyhedral surfaces [5].

In 1927, E. Cohn-Vossen proved that smooth closed surfaces of $\mathbb{R}^3$ with everywhere positive Gaussian curvature are rigid [3]. Smooth closed surfaces of $\mathbb{R}^3$ with everywhere positive Gaussian curvature are also Gauss infinitesimally rigid [16], that is rigid with respect to the Gaussian curvature regarded as a function of the outer unit normal [6, Section 2]. In this paper, we consider rigidity and Gauss infinitesimal rigidity for hedgehogs of $\mathbb{R}^3$ (regarded as Minkowski differences of closed convex surfaces of $\mathbb{R}^3$ with positive Gaussian curvature). As noticed by I. Izmestiev, Gauss infinitesimal rigidity can be interpreted as « infinitesimal » uniqueness in the Minkowski problem (prescribing Gauss curvature as a function of the unit normal) of which the author already studied the uniqueness part [12]. In this previous work [12], we presented different ways of constructing pairs of non-congruent hedgehogs that share the same curvature function (i.e., inverse of the Gaussian curvature). This will allow us to give examples of nontrivial (i.e., distinct from a point) hedgehogs that are not Gauss infinitesimally rigid. Assume we have a one parameter family of hedgehogs $(\mathcal{H}_{h_t})_{t\in[0,1]}$, all with the same curvature function. We do not know whether they must be congruent in $\mathbb{R}^3$. However, we shall give a bellows-type theorem for hedgehogs: under an appropriate differentiability condition of the family with respect to the parameter, we shall prove that *all the hedgehogs of the family considered have the same algebraic volume.*

Basic definitions C^2–hedgehogs in $\mathbb{R}^{n+1}$

As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : \mathbb{S}^n \longrightarrow \mathbb{R}$, where $h_K(u)$ is defined by $h_K(u) = \sup\{\langle x, u\rangle \,|x \in K\}$, $(u \in \mathbb{S}^n)$, that is, as the signed distance from the origin to the support hyperplane with normal vector u. In particular, every closed convex hypersurface of class C^2_+ (i.e., C^2-hypersurface with positive Gaussian curvature) is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [15, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u\rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the following system of equations

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, \,.\,\rangle = dh_u(.) \end{cases} .$$

The second equation is obtained from the first by performing a partial differentiation with respect to u. From the first equation, the orthogonal projection of x onto the line spanned by u is $h(u)\,u$ and from the second one, the orthogonal projection of x onto $u^{\perp}$ is the gradient of h at u (cf. Figure 1). Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

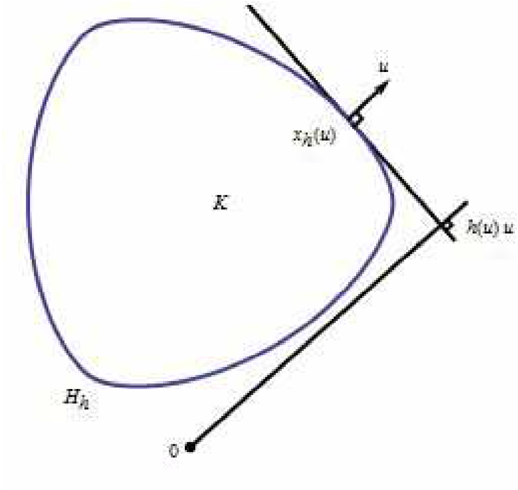

Figure 1. Hedgehogs as envelopes parametrized by their Gauss map

Now, for any C^2-function h on $\mathbb{S}^n$, the envelope $\mathcal{H}_h$ is in fact well-defined (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can be interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h (cf. Figure 2). Note that x_h depends linearly on h.

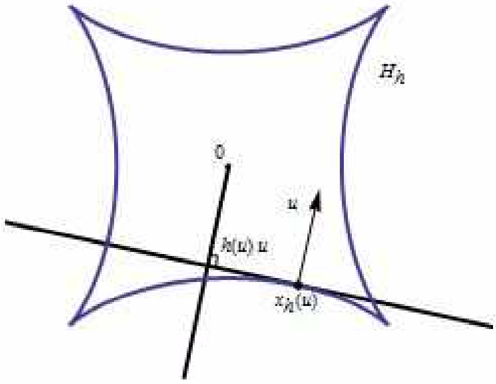

Figure 2. A hedgehog with a C^2-support function

Hedgehogs with a C^2-support function can be regarded as the Minkowski differences of convex hypersurfaces of class C^2_+. Indeed, given any $h \in C^2(\mathbb{S}^n; \mathbb{R})$, for all large enough real constant r, the functions $h+r$ and r are support functions of convex hypersurfaces of class C^2_+ such that $h = (h+r) - r$.

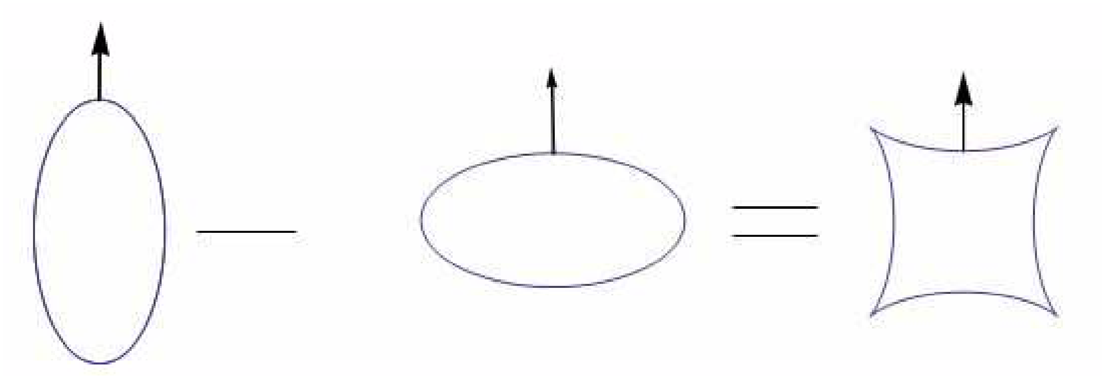

Figure 3. Hedgehogs as differences of convex bodies of class C^2_+

In fact, we can introduce a more general notion of hedgehogs by regarding hedgehogs of $\mathbb{R}^{n+1}$ as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$ [10]. But in the present paper, we shall only consider hedgehogs with a C^2-support function and we will refer to them as 'C^2-hedgehogs'.

Gaussian curvature and algebraic volume of C^2-hedgehogs

Let H_{n+1} denote the $\mathbb{R}$-linear space of C^2-hedgehogs defined up to a translation in the Euclidean linear space $\mathbb{R}^{n+1}$ and identified with their support functions. Analytically speaking, saying that a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is defined up to a translation simply means that the first spherical harmonics of its support function is not specified.

As we saw before, elements of H_{n+1} may be singular hypersurfaces. Since the parametrization x_h can be regarded as the inverse of the Gauss map, the Gaussian curvature K_h of $\mathcal{H}_h$ at $x_h(u)$ is given by $K_h(u) = 1/\det[T_u x_h]$, where $T_u x_h$ is the tangent map of x_h at u. Therefore, singularities are the very points at which the Gaussian curvature is infinite. For every $u \in \mathbb{S}^n$, the tangent map of x_h at the point u is $T_u x_h = h(u)\, Id_{T_u\mathbb{S}^n} + H_h(u)$, where $H_h(u)$ is the symmetric endomorphism associated with the Hessian of h at u. Consequently, if λ is an eigenvalue of the Hessian of h at u then $\lambda + h(u)$ is (up to the sign) one of the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$ and the so-called 'curvature function' $R_h := 1/K_h$ can be given by

$$R_h(u) = \det\left[H_{ij}(u) + h(u)\,\delta_{ij}\right], \qquad (1)$$

where δ_{ij} are the Kronecker symbols and $(H_{ij}(u))$ the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

The case $n = 2$. From (1), the curvature function $R_h := 1/K_h$ of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by $R_h = (\lambda_1 + h)(\lambda_2 + h) = h^2 + h\Delta_2 h + \Delta_{22} h$, where Δ_2 denotes the spherical Laplacian and Δ_{22} the Monge-Ampère operator (respectively the sum and the product of the eigenvalues λ_1, λ_2 of the Hessian of h). So, the equation we shall be dealing with will be the following

$$h^2 + h\Delta_2 h + \Delta_{22} h = 1/K.$$

Note that the so-called 'mixed curvature function' of hedgehogs of $\mathbb{R}^3$, that is,

$$\begin{aligned} R : \mathsf{H}_3^2 &\to C\left(\mathbb{S}^2; \mathbb{R}\right) \\ (f,g) &\mapsto R_{(f,g)} := \tfrac{1}{2}\left(R_{f+g} - R_f - R_g\right) \end{aligned}$$

is bilinear and symmetric:

$(i)\, \forall\, (f,g,h) \in \mathsf{H}_3^3, \forall \lambda \in \mathbb{R}, R_{(f+\lambda g,h)} = R_{(f,h)} + \lambda R_{(g,h)}$;

$(ii)\, \forall\, (f,g) \in \mathsf{H}_3^2, R_{(g,f)} = R_{(f,g)}$.

For any $h \in \mathsf{H}_3$, we have in particular $R_{-h} = R_h$. Note that $R_{(1,f)} = \frac{1}{2}\left(\Delta_2 h + 2h\right)$ is (up to the sign) half the sum of the principal radii of curvature of $\mathcal{H}_h \subset \mathbb{R}^3$. The (algebraic) area and volume of a hedgehog $\mathcal{H}_h$ of $\mathbb{R}^3$ is defined by

$$s(h) = \int_{\mathbb{S}^2} R_h d\sigma \qquad \text{and} \qquad v(h) = \int_{\mathbb{S}^2} h R_h d\sigma.$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$ and $R_h = \det\left(Hess\,(h) + h.Id_2\right)$ the curvature function (that is, the inverse of the Gaussian curvature K_h of $\mathcal{H}_h$) [8]. The (algebraic) area $s(h)$ of $\mathcal{H}_h$ can be interpreted as the difference $s_+(h) - s_-(h)$, where $s_+(h)$ (resp. $s_-(h)$) denotes the total area of the smooth regions of $\mathcal{H}_h$ on which the Gaussian curvature is positive (resp. negative). The (algebraic) volume $v(h)$ of $\mathcal{H}_h$ can be regarded as the integral over $\mathbb{R}^3 - \mathcal{H}_h$ of the index $i_h(x)$ defined as algebraic intersection number of an oriented half-line with origin x with the surface $\mathcal{H}_h$ equipped with its transverse orientation (number independent of the oriented half-line for an open dense set of directions).

Gauss infinitesimal rigidity in the context of hedgehogs

In this work, we shall use the Banach spaces C_m that were introduced by L. Nirenberg in his study of the Minkowski problem in $\mathbb{R}^3$, $(m \in \mathbb{N})$.

Definition 1 *Let $\mathcal{H}_h$ be a C^2-hedgehog of $\mathbb{R}^3$. A smooth deformation of $\mathcal{H}_h$ is the data of a differentiable map $h : [0,1] \to C_2$, $t \mapsto h_t := h(t,.)$ such that $h_0 = h$.*

Definition 2 *Let $\mathcal{H}_f$ be a C^2-hedgehog of $\mathbb{R}^3$. An infinitesimal isogauss deformation of $\mathcal{H}_f$ is the data of a family $\left(\mathcal{H}_{f+tg}\right)_{t\in\mathbb{R}}$ of hedgehogs of $\mathbb{R}^3$,*

$$\begin{aligned} x_{f+tg} : \mathbb{S}^2 &\to \mathcal{H}_{f+tg} \subset \mathbb{R}^3 \\ u &\mapsto x_f(u) + t x_g(u) \end{aligned}$$

where $\mathcal{H}_g$ is a hedgehog of $\mathbb{R}^3$ such that the mixed curvature function $R_{(f,g)} := \frac{1}{2}\left(R_{f+g} - R_f - R_g\right)$ be identically zero on $\mathbb{S}^2$.

Definition 3 *Let $\mathcal{H}_f$ be a C^2-hedgehog of $\mathbb{R}^3$. If every infinitesimal isogauss deformation $(\mathcal{H}_{f+tg})_{t\in\mathbb{R}}$ of $\mathcal{H}_f$ is trivial, that is such that $\mathcal{H}_g$ is reduced to a single point, then the hedgehog $\mathcal{H}_f$ will be said to be Gauss infinitesimally rigid.*

Remark 1. The hedgehog $\mathcal{H}_g$ is reduced to a single point if, and only if, its support function g is the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$, which amounts to saying that its curvature function R_g is identically zero on $\mathbb{S}^2$ [7, Theorem 1]. Therefore, the hedgehog $\mathcal{H}_f$ is Gauss infinitesimally rigid if, and only if,we have:

$$\forall g \in C^2\left(\mathbb{S}^2;\mathbb{R}\right), \quad \left(R_{(f,g)}=0\right) \Longrightarrow \left(R_g=0\right).$$

Remark 2. If the hedgehog $\mathcal{H}_f \subset \mathbb{R}^3$ is trivial (that is, reduced to a point), then $\mathcal{H}_f$ is not Gauss infinitesimally rigid. Indeed, for every regular C^2-hedgehog $\mathcal{H}_g \subset \mathbb{R}^3$, we have $R_{(f,g)}=0$ although R_g is not identically zero on $\mathbb{S}^2$.

Gauss infinitesimal rigidity of regular C^2-hedgehogs of $\mathbb{R}^3$

Let us recall the **proof of the Gauss infinitesimal rigidity** (with respect to the curvature function) **of regular C^2-hedgehogs of $\mathbb{R}^3$** (that are closed convex surfaces of class C^2_+ in $\mathbb{R}^3$). It is essentially a rewriting of the proof by J. Stoker [16]: Let $\mathcal{H}_f$ be a regular C^2-hedgehog of $\mathbb{R}^3$. Clearly, the regularity of $\mathcal{H}_f$ is equivalent to the strict positivity of its curvature function $R_f := 1/K_f$. If $(\mathcal{H}_{f+tg})_{t\in\mathbb{R}}$ defines an isogauss deformation of $\mathcal{H}_f$, then we have [12, Lemma 5]:

$$R^2_{(f,g)} \geq R_f.R_g$$

and hence $R_g \leq 0$ on $\mathbb{S}^2$. By taking the origin to be an interior point of the convex body bounded by $\mathcal{H}_f$ in $\mathbb{R}^3$, we may assume without loss of generality that $f>0$ so that $fRg \leq 0$ on $\mathbb{S}^2$. Now, by symmetry of the mixed volume of hedgehogs of $\mathbb{R}^3$ [9], we get:

$$0=\int_S gR_{(f,g)}d\sigma=\int_S fR_{(g,g)}d\sigma=\int_S fR_g d\sigma,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$. Therefore, R_g is identically zero on $\mathbb{S}^2$ which implies that $\mathcal{H}_g$ is reduced to a single point by Remark 1. □

Relation to Minkowski problem

There is a close connection between Gauss infinitesimal rigidity and the uniqueness question in the Minkowski problem extended to hedgehogs. This is due to the following equivalence:

$$\forall (f,g) \in C^2\left(\mathbb{S}^2;\mathbb{R}\right)^2, \quad (R_f = R_g) \Longleftrightarrow \left(R_{(f+g,f-g)} = 0\right).$$

In [11, 12], the author gave examples of pairs of non-congruent hedgehogs of $\mathbb{R}^3$ having the same curvature function. From each of these examples, we can deduce examples of nontrivial hedgehogs that are not Gauss infinitesimally rigid. It is for instance the case of the pair of hedgehogs of $\mathbb{R}^3$ given by:

$$f(u) := \begin{cases} 0 & \text{if} \quad z \leq 0 \\ \exp(-1/z^2) & \text{if} \quad z > 0 \end{cases} \quad \text{and} \quad g(u) := \begin{cases} \exp(-1/z^2) & \text{if} \quad z < 0 \\ 0 & \text{if} \quad z \geq 0, \end{cases}$$

where $u = (x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Indeed, we have clearly $R_{(f,g)} = 0$. Therefore, these two nontrivial hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ are not Gauss infinitesimally rigid. Only nonanalytic examples are known. The question of knowing whether there exists a pair of noncongruent analytic hedgehogs of $\mathbb{R}^3$ with the same curvature function remains open (by 'analytic hedgehogs', we mean 'hedgehogs with an analytic support function'). As a consequence, the question of knowing whether there exist examples of nontrivial analytic hedgehogs that are not Gauss infinitesimally rigid is also open.

A bellows-type theorem for hedgehogs

Lemma 4 *The curvature function $R : C_2 \to C_0$, $h \mapsto R_h$ is differentiable on C_2, and:*

$$\forall (f,g) \in C_2 \times C_2, \qquad dR_f(g) = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{R_{f+tg} - R_f}{t} = 2R_{(f,g)}.$$

Proof of the lemma. Indeed, we have:

$$\forall t \in \mathbb{R}_+^*, \qquad R_{f+tg} - R_f = R_f + 2tR_{(f,g)} + t^2 R_g - R_f$$
$$= t\left(2R_{(f,g)} + tR_g\right),$$

and hence

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \frac{R_{f+tg} - R_f}{t} = \lim_{\substack{t \to 0 \\ t \neq 0}} \left(2R_{(f,g)} + tR_g\right) = 2R_{(f,g)}.$$

□

Theorem 5 *Let $\mathcal{H}_h$ be a C^2-hedgehog of $\mathbb{R}^3$. If a smooth deformation of $\mathcal{H}_h$, say*

$$h : [0,1] \to C_2, t \mapsto h_t := h(t, .),$$

preserves the curvature function (that is, is such that $R_{h_t} = R_h$ for all $t \in [0,1]$), then it also preserves the algebraic volume:

$$\forall t \in [0,1], \quad v(h_t) = v(h).$$

Proof of Theorem 5. By assumption, the map $R \circ h : [0,1] \to C_0$ is constant. Since h is differentiable by assumption and R by Lemma 4, $R \circ h$ is differentiable and the chain rule gives:

$$\forall t \in [0,1], \qquad (R \circ h)'(t) = 2R_{\left(h(t), \left(\frac{\partial h}{\partial t}\right)(t)\right)}.$$

Therefore, a differentiation yields:

$$\forall t \in [0,1], \qquad R_{\left(h(t), \left(\frac{\partial h}{\partial t}\right)(t)\right)} = 0. \tag{2}$$

Now, for every $t_0 \in [0,1]$, we have :

$$\forall t \in [0,1] - \{t_0\}, \qquad \frac{v(h(t)) - v(h(t_0))}{t - t_0} = \frac{1}{3} \int_{\mathbb{S}^2} \frac{h(t) - h(t_0)}{t - t_0} R_{h(t_0)} d\sigma$$

and hence:

$$\lim_{\substack{t \to t_0 \\ t \neq t_0}} \frac{v(h(t)) - v(h(t_0))}{t - t_0} = \frac{1}{3} \int_{\mathbb{S}^2} \left(\frac{\partial h}{\partial t}\right)(t_0) R_{h(t_0)} d\sigma.$$

Besides, by symmetry of the mixed volume of hedgehogs, we have:

$$\frac{1}{3} \int_{\mathbb{S}^2} \left(\frac{\partial h}{\partial t}\right)(t_0) R_{h(t_0)} d\sigma = \frac{1}{3} \int_{\mathbb{S}^2} h(t_0) R_{\left(h(t_0), \left(\frac{\partial h}{\partial t}\right)(t_0)\right)} d\sigma.$$

From (2), we then deduce that:

$$\forall t_0 \in [0,1], \qquad (v \circ h)'(t_0) = \lim_{\substack{t \to t_0 \\ t \neq t_0}} \frac{v(h(t)) - v(h(t_0))}{t - t_0} = 0,$$

and thus all the hedgehogs of the family $(\mathcal{H}_{h_t})$ have the same (algebraic) volume. $\square$

Remark 3. Noncongruent hedgehogs that share the same curvature function may of course have different (algebraic) volume. It is for instance the case of the hedgehogs shown on Figure 4 whose support functions f, g are defined on $\mathbb{S}^2$ by

$$f\,(u) := \begin{cases} \exp(-1/z^2) \text{ if } z \neq 0 \\ 0 \quad\quad \text{if} \quad\quad z = 0 \end{cases} \quad \text{and} \quad g\,(u) := \begin{cases} sign\,(z)\,f\,(u) \text{ if } z \neq 0 \\ 0 \quad\quad \text{if} \quad\quad z = 0, \end{cases}$$

where $u = (x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3$.

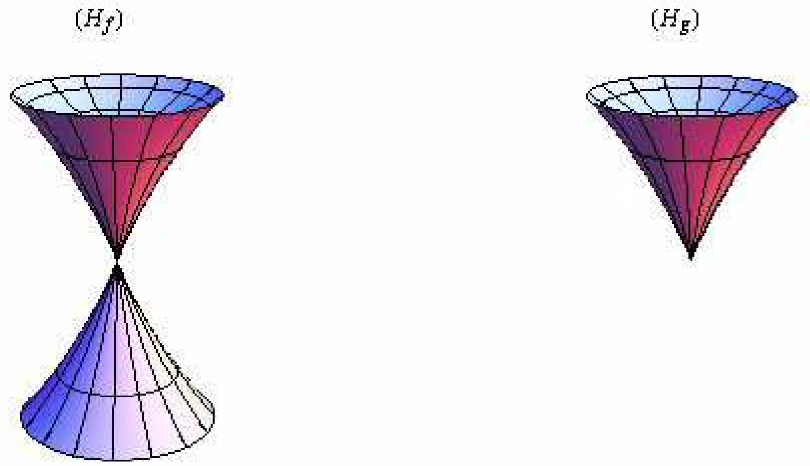

Figure 4. Same curvature function and different algebraic volume

References

[1] R. Bricard, *Mémoire sur la théorie de l'octaèdre articulé*. J. Math. Pures Appl., Vol. 3 (1897), 113-150

[2] A. L. Cauchy, *Sur les polygones et polyèdres, second mémoire*. Journal de l'École Polytechnique 19, (1813), 87-98.

[3] E. Cohn-Vossen, Zwei Satze uber die Starrheit der Eilflchen. Nachr. Ges. Will. Gottingen (1927), 125-134

[4] R.Connelly, *A counterexample to the rigidity conjecture for polyhedra*. Inst. Haut. Etud. Sci., Publ. Math. 47, (1977), 333-338.

[5] R.Connelly, I. Sabitov, A. Walz, *The Bellows conjecture*. Beitr. Algebra Geom. 38 (1997), 1-10.

[6] I. Izmetiev, *Infinitesimal rigidity of convex surfaces through the second derivative of the Hilbert-Einstein functional II: Smooth case.* arXiv:1105.5067.

[7] D. Koutroufiotis, *On a conjectured characterization of the sphere*. Math. Ann. 205, (1973), 211-217.

[8] Y. Martinez-Maure, *De nouvelles inégalités géométriques pour les hérissons.* Arch. Math. 72 (1999), 444-453.
[9] Y. Martinez-Maure, *Hedgehogs and zonoids.* Adv. Math. 158 (2001), 1-17.
[10] Y. Martinez-Maure, *Geometric study of Minkowski differences of plane convex bodies.* Canad. J. Math. 58 (2006), 600-624.
[11] Y. Martinez-Maure, *New notion of index for hedgehogs of* $\mathbb{R}^3$ *and applications.* Eur. J. Comb. 31 (2010), 1037-1049.
[12] Y. Martinez-Maure, *Uniqueness results for the Minkowski problem extended to hedgehogs.* To appear in the Central European Journal of Mathematics.
[13] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large.* Commun. Pure Appl. Math. 6, (1953), 337-394.
[14] I.Kh. Sabitov, *The volume of a polyhedron as a function of length of its edges.* (Russian), Fundam. Prikl. Mat. 2, (1996), 305-307.
[15] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge: Cambridge University Press 1993.
[16] J. J. Stoker, *Differential geometry.* Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

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